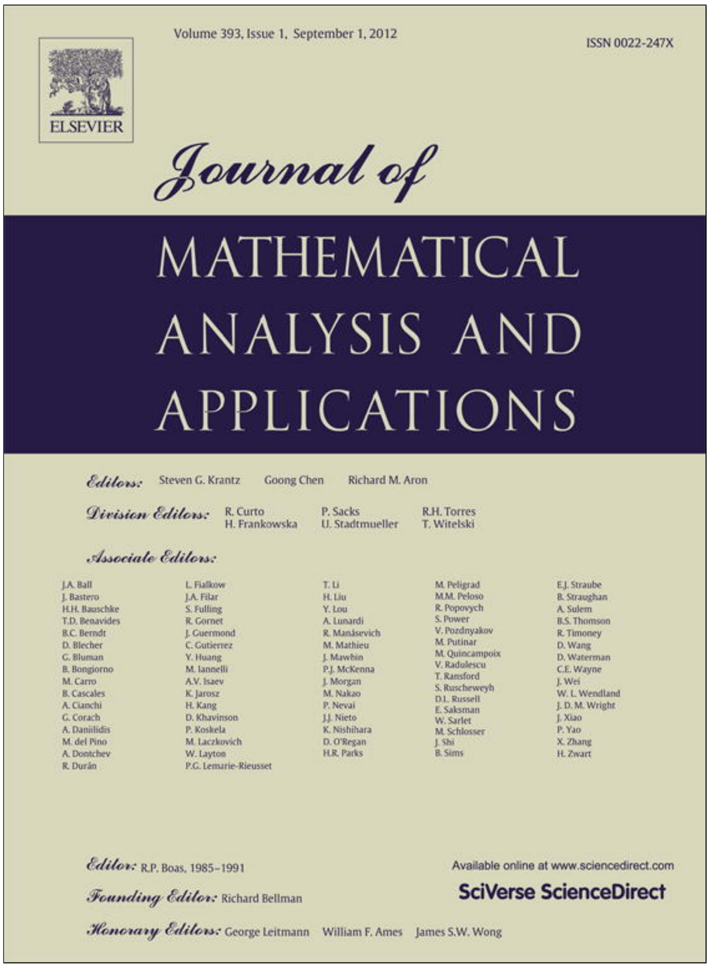
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On frames for Krein spaces

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ABSTRACT

A definition of frames for Krein spaces is proposed, which extends the notion of J-orthonormal bases of Krein spaces. A J-frame for a Krein space $(\mathcal{H}, [\ ,\ ])$ is in particular a frame for $\mathcal{H}$ in the Hilbert space sense. But it is also compatible with the indefinite inner product $[\ ,\ ]$, meaning that it determines a pair of maximal uniformly J-definite subspaces, an analogue to the maximal dual pair associated to a J-orthonormal basis.

Also, each J-frame induces an indefinite reconstruction formula for the vectors in $\mathcal{H}$, which resembles the one given by a J-orthonormal basis.

1. Introduction

In recent years, frame theory for Hilbert spaces has been thoroughly developed; see e.g. [1–4]. For a fixed Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$, a frame for $\mathcal{H}$ is a (generally overcomplete) family of vectors $\mathcal{F} = \{f_i\}_{i\in I}$ in $\mathcal{H}$ which satisfies the inequalities

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2, \quad \text{for every } f \in \mathcal{H}, \tag{1}$$

for positive constants $0 < A \le B$. The (bounded, linear) operator $S : \mathcal{H} \to \mathcal{H}$ defined by

$$Sf = \sum_{i\in I} \langle f, f_i\rangle f_i, \quad f \in \mathcal{H}, \tag{2}$$

is known as the frame operator associated to $\mathcal{F}$. The inequalities in (1) imply that S is a (positive) boundedly invertible operator, and it allows to reconstruct each vector $f \in \mathcal{H}$ in terms of the family $\mathcal{F}$ as follows:

$$f = \sum_{i\in I} \langle f, S^{-1}f_i\rangle f_i = \sum_{i\in I} \langle f, f_i\rangle S^{-1}f_i. \tag{3}$$

The above formula is known as the *reconstruction formula associated to* $\mathcal{F}$. Notice that if $\mathcal{F}$ is a Parseval frame, i.e. if $S = I$, then the reconstruction formula resembles the Fourier series of f associated to an orthonormal basis $\mathcal{B} = \{b_k\}_{k\in K}$ of $\mathcal{H}$:

$$f = \sum_{k\in K} \langle f, b_k\rangle b_k,$$

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but the *frame coefficients* $\{\langle f, f_i \rangle\}_{i\in I}$ given by $\mathcal{F}$ allow to reconstruct f even when some of these coefficients are missing (or corrupted). Indeed, each vector $f \in \mathcal{H}$ may admit several reconstructions in terms of the frame coefficients as a consequence of the redundancy of $\mathcal{F}$. These are some of the advantages of frames over (orthonormal, orthogonal or Riesz) bases in signal processing applications, when noisy channels are involved; e.g., see [5–7].

Given a Krein space $(\mathcal{H}, [\ ,\ ])$ with fundamental symmetry J, a *J-orthonormalized system* is a family $\mathcal{E} = \{e_i\}_{i\in I}$ such that $[e_i, e_j] = \pm\delta_{ij}$, for $i, j \in I$. A J-orthonormal basis is a J-orthonormalized system which is also a Schauder basis for $\mathcal{H}$. If $\mathcal{E} = \{e_i\}_{i\in I}$ is a J-orthonormal basis of $\mathcal{H}$ then the vectors in $\mathcal{H}$ can be represented as follows:

$$f = \sum_{i\in I} \sigma_i [f, e_i] e_i, \quad f \in \mathcal{H}, \tag{4}$$

where $\sigma_i = [e_i, e_i] = \pm 1$.

J-orthonormalized systems (and bases) are intimately related to the notion of dual pair. In fact, each J-orthonormalized system generates a *dual pair*, i.e. a pair $(\mathcal{L}_+, \mathcal{L}_-)$ of subspaces of $\mathcal{H}$ such that $\mathcal{L}_+$ is J-nonnegative, $\mathcal{L}_-$ is J-nonpositive and $\mathcal{L}_+$ is J-orthogonal to $\mathcal{L}_-$, i.e. $[\mathcal{L}_+, \mathcal{L}_-] = 0$. Moreover, if $\mathcal{E}$ is a J-orthonormal basis of $\mathcal{H}$, the dual pair associated to $\mathcal{E}$ is maximal (with respect to the inclusion preorder) and the subspaces $\mathcal{L}_+$ and $\mathcal{L}_-$ are uniformly J-definite, see [8, Chapter 1, Section 10]. Therefore the dual pair $(\mathcal{L}_+, \mathcal{L}_-)$ is a fundamental decomposition of $\mathcal{H}$. Notice that, considering the Hilbert space structure induced by the above fundamental decomposition, the J-orthonormal basis $\mathcal{E}$ turns out to be an orthonormal basis in the associated Hilbert space. Therefore, each J-orthonormal basis can be realized as an orthonormal basis of $\mathcal{H}$ (respect to an appropriate definite inner product).

Given a pair of maximal uniformly J-definite subspaces $\mathcal{M}_+$ and $\mathcal{M}_-$ of a Krein space $\mathcal{H}$, where $\mathcal{M}_+$ is J-positive and $\mathcal{M}_-$ is J-negative, if $\mathcal{F}_\pm = \{f_i\}_{i\in I_\pm}$ is a frame for the Hilbert space $(\mathcal{M}_\pm, \pm[\ ,\ ])$, it is easy to see that

$$\mathcal{F} = \mathcal{F}_+ \cup \mathcal{F}_-,$$

is a frame for $\mathcal{H}$, which produces an *indefinite reconstruction formula*:

$$f = \sum_{i\in I} \sigma_i [f, g_i] f_i = \sum_{i\in I} \sigma_i [f, f_i] g_i, \quad f \in \mathcal{H}, \tag{5}$$

where $\sigma_i = \mathrm{sgn}[f_i, f_i]$ and $\{g_i\}_{i\in I}$ is some (equivalent) frame for $\mathcal{H}$ (see Example 3.4).

The aim of this work is to introduce and characterize a particular family of frames for a Krein space $(\mathcal{H}, [\ ,\ ])$ – hereafter called J-frames – that are compatible with the indefinite inner product $[\ ,\ ]$, in the sense that an indefinite reconstruction formula as in (5) holds (see Proposition 5.4).

Some different approaches to frames for Krein spaces and indefinite reconstruction formulas are developed in [9,10], respectively. As it will be seen along this work, neither of the definitions below is comparable with the J-frame concept introduced here.

In [10], the authors studied when a set of vectors $\{\phi_j\}_{j\in I}$ in a Hilbert space $\mathcal{H}$ can be scaled to obtain a tight frame $\{\alpha_j \phi_j\}_{j\in I}$, and hence a representation of the form

$$f = \sum_{j\in I} c_j \langle f, \phi_j \rangle \phi_j, \quad f \in \mathcal{H}. \tag{6}$$

It turns out that representations as in (6) can exist even when some of the c_j's are negative, and these correspond to what they call "signed frames". Indeed, a Bessel family $\{\psi_j\}_{j\in I}$ in a Hilbert space $\mathcal{H}$ is called a signed frame with signature $\sigma = (\sigma_j)_{j\in I}$, $\sigma_j \in \{-1, 1\}$, if there exist $A, B > 0$ with

$$A\,\|f\|^2 \le \sum_{j\in I} \sigma_j \left|\langle f, \psi_j \rangle\right|^2 \le B\,\|f\|^2 \quad \text{for every } f \in \mathcal{H}.$$

Then, each $f \in \mathcal{H}$ can be represented as

$$f = \sum_{j\in I} \sigma_j \langle f, \psi_j \rangle \varphi_j = \sum_{j\in I} \sigma_j \langle f, \varphi_j \rangle \psi_j,$$

where $\{\varphi_j\}_{j\in I}$ is the dual signed frame (see [10, Theorem 2.4] for the details). Observe that this idea can be interpreted as introducing an indefinite inner product (associated to the signature $\sigma = (\sigma_j)_{j\in I}$) in $\ell_2(I)$. But the sampling space $\mathcal{H}$ does not need to be a Krein space.

On the other hand, in [9] the authors consider Krein spaces as sampling spaces. They say that a family $\{f_n\}_{n\in\mathbb{N}}$ of vectors in $\mathcal{H}$ is a "frame for the Krein space $(\mathcal{H}, [\ ,\ ])$" if there exist constants $A, B > 0$ such that

$$A\,\|f\|_J^2 \le \sum_{n\in\mathbb{N}} |[f, f_n]|^2 \le B\,\|f\|_J^2, \quad \text{for every } f \in \mathcal{H},$$

where $\|\ \|_J$ stands for the norm of the associated Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$. Then, they show that a family $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ is a "frame for the Krein space $(\mathcal{H}, [\ ,\ ])$" if and only if it is a frame (in the usual sense) for the Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$. This is the major difference between J-frames and this concept, because there are frames for the associated Hilbert space $(\mathcal{H}, \langle \ , \ \rangle)$ which are not J-frames for the Krein space $(\mathcal{H}, [\ ,\ ])$ (see Example 3.3).

The paper is organized as follows: Section 2 contains some preliminaries results both in Krein spaces and in frame theory for Hilbert spaces.

Section 3 presents the motivation and what is meant by a J-frame. Briefly, a J-frame for the Krein space $(\mathcal{H}, [\ ,\ ])$ is a Bessel family $\mathcal{F} = \{f_i\}_{i\in I}$ with synthesis operator $T : \ell_2(I) \to \mathcal{H}$ such that the ranges of $T_+ := TP_+$ and $T_- := T(I - P_+)$ are maximal uniformly J-positive and maximal uniformly J-negative subspaces, respectively, where $I_+ = \{i \in I : [f_i, f_i] > 0\}$ and P_+ is the orthogonal projection onto $\ell_2(I_+)$, as a subspace of $\ell_2(I)$. It is immediate that J-orthonormal bases are J-frames, because they generate maximal dual pairs [8, Chapter 1, Section 10.12].

Also, if $\mathcal{F}$ is a J-frame for $\mathcal{H}$, observe that $R(T) = R(T_+) + R(T_-)$ and recall that the sum of a maximal uniformly J-positive and a maximal uniformly J-negative subspace coincides with $\mathcal{H}$ [11, Corollary 1.5.2]. Therefore, each J-frame is in fact a frame for $\mathcal{H}$ in the Hilbert space sense. Moreover, it is shown that $\mathcal{F}_+ = \{f_i\}_{i\in I_+}$ is a frame for the Hilbert space $(R(T_+), [\ ,\ ])$ and $\mathcal{F}_- = \{f_i\}_{i\in I\setminus I_+}$ is a frame for $(R(T_-), -[\ ,\ ])$, i.e. there exist constants $B_- \le A_- < 0 < A_+ \le B_+$ such that

$$A_\pm[f,f] \le \sum_{i\in I_\pm} |[f,f_i]|^2 \le B_\pm[f,f] \quad \text{for every } f \in R(T_\pm). \tag{7}$$

The optimal constants satisfying the above inequalities can be characterized in terms of $T_\pm$ and the Gramian operators of their ranges.

This section ends with a geometrical characterization of J-frames, in terms of the (minimal) angles between the uniformly J-definite subspace $R(T_\pm)$ and the cone of neutral vectors of the Krein space.

Section 4 is devoted to study the synthesis operators associated to J-frames. Given a bounded operator $T : \ell_2(I) \to \mathcal{H}$, it is described under which conditions T is the synthesis operator of a J-frame for the Krein space $\mathcal{H}$.

In Section 5 the J-frame operator is introduced. Given a J-frame $\mathcal{F} = \{f_i\}_{i\in I}$, the *$J$-frame operator* $S : \mathcal{H} \to \mathcal{H}$ is defined by

$$Sf = \sum_{i\in I} \sigma_i\,[f, f_i]\,f_i, \quad f \in \mathcal{H},$$

where $\sigma_i = \operatorname{sgn}([f_i, f_i])$. This operator resembles the frame operator for frames in Hilbert spaces (see (2)), and it has similar properties, in particular $S = TT^+$ where $T : \ell_2(I) \to \mathcal{H}$ is the synthesis operator of $\mathcal{F}$ and T^+ denotes the J-adjoint of T (see Proposition 5.2). Furthermore, each J-frame $\mathcal{F} = \{f_i\}_{i\in I}$ determines an *indefinite reconstruction formula*, which depends on the J-frame operator S:

$$f = \sum_{i\in I} \sigma_i\,[f, S^{-1}f_i]\,f_i = \sum_{i\in I} \sigma_i\,[f, f_i]\,S^{-1}f_i, \quad \text{for every } f \in \mathcal{H}. \tag{8}$$

In this case the family $\{S^{-1}f_i\}_{i\in I}$ turns out to be a J-frame too.

Finally, it will be shown that the J-frame operator of a J-frame $\mathcal{F}$ is intimately related to the projection $Q = P_{R(T_+)//R(T_-)}$ determined by the decomposition $\mathcal{H} = R(T_+) \dotplus R(T_-)$, see Theorem 5.6.

2. Preliminaries

Along this work $\mathcal{H}$ denotes a complex (separable) Hilbert space. If $\mathcal{K}$ is another Hilbert space then $L(\mathcal{H}, \mathcal{K})$ is the algebra of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ and $L(\mathcal{H}) = L(\mathcal{H}, \mathcal{H})$. The groups of linear invertible and unitary operators acting on $\mathcal{H}$ are denoted by $GL(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$, respectively. Also, $L(\mathcal{H})^+$ denotes the cone of positive semidefinite operators acting on $\mathcal{H}$ and $GL(\mathcal{H})^+ = GL(\mathcal{H}) \cap L(\mathcal{H})^+$.

If $T \in L(\mathcal{H}, \mathcal{K})$ then $T^* \in L(\mathcal{K}, \mathcal{H})$ denotes the adjoint operator of T, $R(T)$ stands for its range and $N(T)$ for its nullspace. Also, if $T \in L(\mathcal{H}, \mathcal{K})$ has closed range, $T^\dagger \in L(\mathcal{K}, \mathcal{H})$ denotes the Moore–Penrose inverse of T.

Hereafter, $\mathcal{S} \dotplus \mathcal{T}$ denotes the direct sum of two (closed) subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{H}$. On the other hand, $\mathcal{S} \oplus \mathcal{T}$ stands for the (direct) orthogonal sum of them and $\mathcal{S} \ominus \mathcal{T} := \mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^\perp$. The oblique projection onto $\mathcal{S}$ along $\mathcal{T}$, denoted by $P_{\mathcal{S}//\mathcal{T}}$, is the unique projection with range $\mathcal{S}$ and nullspace $\mathcal{T}$. In particular, $P_{\mathcal{S}} := P_{\mathcal{S}//\mathcal{S}^\perp}$ is the orthogonal projection onto $\mathcal{S}$.

The following result due to Douglas [12], characterizes operator range inclusions. It is quite often used along the paper.

Theorem 2.1. *Given Hilbert spaces $\mathcal{H}$, $\mathcal{K}_1$, $\mathcal{K}_2$ and operators $A \in L(\mathcal{K}_1, \mathcal{H})$ and $B \in L(\mathcal{K}_2, \mathcal{H})$, the following conditions are equivalent:*

(i) *the equation $AX = B$ has a solution in $L(\mathcal{K}_2, \mathcal{K}_1)$;*
(ii) *$R(B) \subseteq R(A)$;*
(iii) *there exists $\lambda > 0$ such that $BB^* \le \lambda AA^*$.*

In this case, there exists a unique $D \in L(\mathcal{K}_2, \mathcal{K}_1)$ such that $AD = B$ and $R(D) \subseteq \overline{R(A^*)}$; moreover, $N(D) = N(B)$ and $\|D\| = \inf\{\lambda > 0 : BB^* \le \lambda AA^*\}$. The operator D is called the reduced solution of $AX = B$.

Corollary 2.2. *Let $T \in L(\mathcal{H})^+$. If $R(T) = R(T^{1/2})$, then $R(T)$ is closed.*

2.1. Krein spaces

In what follows we present the standard notation and some basic results on Krein spaces. For a complete exposition on the subject (and the proofs of the results below) see the books by Azizov and Iokhvidov [8] and Bognár [13] and the monographs by Ando [11] and by Dritschel and Rovnyak [14].

Given a Krein space $(\mathcal{H}, [\ ,\ ])$ with a *fundamental decomposition* $\mathcal{H} = \mathcal{H}_+ \dotplus \mathcal{H}_-$, the direct (orthogonal) sum of the Hilbert spaces $(\mathcal{H}_+, [\ ,\ ])$ and $(\mathcal{H}_-, -[\ ,\ ])$ is denoted by $(\mathcal{H}, \langle\ ,\ \rangle)$.

Observe that the indefinite metric and the inner product of $\mathcal{H}$ are related by means of a *fundamental symmetry*, i.e. a unitary selfadjoint operator $J \in L(\mathcal{H})$ which satisfies:

$$[x, y] = \langle Jx, y\rangle, \quad x, y \in \mathcal{H}.$$

If $\mathcal{H}$ and $\mathcal{K}$ are Krein spaces, $L(\mathcal{H}, \mathcal{K})$ stands for the vector space of linear transformations which are bounded respect to the associated Hilbert spaces $(\mathcal{H}, \langle\ ,\ \rangle_{\mathcal{H}})$ and $(\mathcal{K}, \langle\ ,\ \rangle_{\mathcal{K}})$. Given $T \in L(\mathcal{H}, \mathcal{K})$, the J-adjoint operator of T is defined by $T^+ = J_{\mathcal{H}} T^* J_{\mathcal{K}}$, where $J_{\mathcal{H}}$ and $J_{\mathcal{K}}$ are the fundamental symmetries associated to $\mathcal{H}$ and $\mathcal{K}$, respectively. An operator $T \in L(\mathcal{H})$ is J-selfadjoint if $T = T^+$.

A vector $x \in \mathcal{H}$ is *J-positive* if $[x, x] > 0$. A subspace $\mathcal{S}$ of $\mathcal{H}$ is *J-positive* if every $x \in \mathcal{S}$, $x \neq 0$, is a J-positive vector. A subspace $\mathcal{S}$ of $\mathcal{H}$ is *uniformly J-positive* if there exists $\alpha > 0$ such that

$$[x, x] \geq \alpha \|x\|^2, \quad \text{for every } x \in \mathcal{S},$$

where $\|\ \|$ stands for the norm of the associated Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$.

J-nonnegative, J-neutral, J-negative, J-nonpositive and uniformly J-negative vectors and subspaces are defined analogously.

Remark 2.3. If $\mathcal{S}_+$ is a closed uniformly J-positive subspace of a Krein space $(\mathcal{H}, [\ ,\ ])$, observe that $(\mathcal{S}_+, [\ ,\ ])$ is a Hilbert space. In fact, the forms $[\ ,\ ]$ and $\langle\ ,\ \rangle$ are equivalent inner products on $\mathcal{S}_+$, because

$$\alpha \|f\|^2 \leq [f, f] \leq \|f\|^2, \quad \text{for every } f \in \mathcal{S}_+.$$

Analogously, if $\mathcal{S}_-$ is a closed uniformly J-negative subspace of $(\mathcal{H}, [\ ,\ ])$, $(\mathcal{S}_-, -[\ ,\ ])$ is a Hilbert space.

Proposition 2.4 ([8, Corollary 7.17])**.** *Let $\mathcal{H}$ be a Krein space with fundamental symmetry J and $\mathcal{S}$ a J-nonnegative closed subspace of $\mathcal{H}$. Then, $\mathcal{S}$ is the range of a J-selfadjoint projection if and only if $\mathcal{S}$ is uniformly J-positive.*

Recall that, given a closed subspace $\mathcal{M}$ of a Krein space $\mathcal{H}$, the Gramian operator of $\mathcal{M}$ is defined by:

$$G_{\mathcal{M}} = P_{\mathcal{M}} J P_{\mathcal{M}},$$

where $P_{\mathcal{M}}$ is the orthogonal projection onto $\mathcal{M}$ and J is the fundamental symmetry of $\mathcal{H}$. If $\mathcal{M}$ is J-semidefinite, then $\mathcal{M} \cap \mathcal{M}^{[\perp]}$ coincides with $\mathcal{N} := \{f \in \mathcal{M} : [f, f] = 0\}$. Therefore, it is easy to see that

$$G_{\mathcal{M}} = G_{\mathcal{M} \ominus \mathcal{N}}.$$

Given a subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$, the *J-orthogonal companion* to $\mathcal{S}$ is defined by

$$\mathcal{S}^{[\perp]} = \{x \in \mathcal{H} : [x, s] = 0 \text{ for every } s \in \mathcal{S}\}.$$

A subspace $\mathcal{S}$ of $\mathcal{H}$ is *J-non-degenerated* if $\mathcal{S} \cap \mathcal{S}^{[\perp]} = \{0\}$. Notice that if $\mathcal{S}$ is a J-definite subspace of $\mathcal{H}$ then it is J-non degenerated.

2.2. Angles between subspaces and reduced minimum modulus

Given closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of a Hilbert space $\mathcal{H}$, the cosine of the *Friedrichs angle* between $\mathcal{S}$ and $\mathcal{T}$ is defined by

$$c(\mathcal{S}, \mathcal{T}) = \sup\{|\langle x, y\rangle| : x \in \mathcal{S} \ominus \mathcal{T}, \|x\| = 1, y \in \mathcal{T} \ominus \mathcal{S}, \|y\| = 1\}.$$

It is well known that

$$c(\mathcal{S}, \mathcal{T}) < 1 \Leftrightarrow \mathcal{S} + \mathcal{T} \text{ is closed } \Leftrightarrow c(\mathcal{S}^{\perp}, \mathcal{T}^{\perp}) < 1.$$

Furthermore, if $P_{\mathcal{S}}$ and $P_{\mathcal{T}}$ are the orthogonal projections onto $\mathcal{S}$ and $\mathcal{T}$, respectively, then $c(\mathcal{S}, \mathcal{T}) < 1$ if and only if $(I - P_{\mathcal{S}})P_{\mathcal{T}}$ has closed range. See [15] for further details.

The next definition is due to Kato, see [16, Chapter IV, Section 5].

Definition 2.5. The *reduced minimum modulus* $\gamma(T)$ of an operator $T \in L(\mathcal{H}, \mathcal{K})$ is defined by

$$\gamma(T) = \inf\{\|Tx\| : x \in N(T)^{\perp}, \|x\| = 1\}.$$

Observe that $\gamma(T) = \sup\{C \geq 0 : C\|x\| \leq \|Tx\| \text{ for every } x \in N(T)^{\perp}, \|x\| = 1\}$. It is well known that $\gamma(T) = \gamma(T^*) = \gamma(T^*T)^{1/2}$. Also, it can be shown that an operator $T \neq 0$ has closed range if and only if $\gamma(T) > 0$. In this case, $\gamma(T) = \|T^{\dagger}\|^{-1}$.

If $\mathcal{H}$ and $\mathcal{K}$ are Krein spaces with fundamental symmetries $J_{\mathcal{H}}$ and $J_{\mathcal{K}}$, respectively, and $T \in L(\mathcal{H}, \mathcal{K})$ then

$$\gamma(T^+) = \gamma(J_{\mathcal{H}}T^*J_{\mathcal{K}}) = \gamma(T^*) = \gamma(T),$$

because $J_{\mathcal{H}}$ (resp. $J_{\mathcal{K}}$) is a unitary operator on $\mathcal{H}$ (resp. $\mathcal{K}$).

Remark 2.6. If $\mathcal{M}_+$ is a closed J-nonnegative subspace of a Krein space $\mathcal{H}$ then

$$\gamma(G_{\mathcal{M}_+}) = \alpha^+, \tag{9}$$

where $\alpha^+ \in [0, 1]$ is the supremum among the constants $\alpha \in [0, 1]$ such that $\alpha\|f\|^2 \leq [f, f]$ for every $f \in \mathcal{M}_+$. From now on, the constant α^+ is called the *definiteness bound* of $\mathcal{M}_+$. Notice that α^+ is in fact a maximum for the above set and $\mathcal{M}^+$ is uniformly J-positive if and only if $\alpha^+ > 0$.

Analogously, if $\mathcal{M}_-$ is a J-nonpositive subspace then $\gamma(G_{\mathcal{M}_-}) = \alpha^-$, where α^- is the definiteness bound of $\mathcal{M}_-$, i.e.

$$\alpha^- = \max\{\alpha \in [0, 1] : [f, f] \leq -\alpha\|f\|^2 \text{ for every } f \in \mathcal{M}_-\}.$$

2.3. Frames for Hilbert spaces

The following is the standard notation and some basic results on frames for Hilbert spaces, see [1,2,4].

A *frame* for a Hilbert space $\mathcal{H}$ is a family of vectors $\mathcal{F} = \{f_i\}_{i\in I} \subset \mathcal{H}$ for which there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad \text{for every } f \in \mathcal{H}. \tag{10}$$

The optimal constants (maximal for A and minimal for B) are known, respectively, as the upper and lower frame bounds.

If a family of vectors $\mathcal{F} = \{f_i\}_{i\in I}$ satisfies the upper bound condition in (10), then $\mathcal{F}$ is a *Bessel family*. For a Bessel family $\mathcal{F} = \{f_i\}_{i\in I}$, the *synthesis operator* $T \in L(\ell_2(I), \mathcal{H})$ is defined by

$$Tx = \sum_{i\in I} \langle x, e_i\rangle f_i,$$

where $\{e_i\}_{i\in I}$ is the standard orthonormal basis of $\ell_2(I)$. It holds that $\mathcal{F}$ is a frame for $\mathcal{H}$ if and only if T is surjective. In this case, the operator $S = TT^* \in L(\mathcal{H})$ is invertible and is called the *frame operator*. It can be easily verified that

$$Sf = \sum_{i\in I} \langle f, f_i\rangle f_i, \quad \text{for every } f \in \mathcal{H}. \tag{11}$$

This implies that the frame bounds can be computed as: $A = \|S^{-1}\|^{-1}$ and $B = \|S\|$. From (11), it is also easy to obtain the *canonical reconstruction formula* for the vectors in $\mathcal{H}$:

$$f = \sum_{i\in I} \langle f, S^{-1}f_i\rangle f_i = \sum_{i\in I} \langle f, f_i\rangle S^{-1}f_i, \quad \text{for every } f \in \mathcal{H},$$

and the frame $\{S^{-1}f_i\}_{i\in I}$ is called the *canonical dual frame* of $\mathcal{F}$. More generally, if a frame $\mathcal{G} = \{g_i\}_{i\in I}$ satisfies

$$f = \sum_{i\in I} \langle f, g_i\rangle f_i = \sum_{i\in I} \langle f, f_i\rangle g_i, \quad \text{for every } f \in \mathcal{H}, \tag{12}$$

then $\mathcal{G}$ is called a *dual frame* of $\mathcal{F}$.

3. *J*-frames: definition and basic properties

Let $(\mathcal{H}, \langle\ ,\ \rangle)$ be a separable Hilbert space that models a signal space. A common task in signal processing applications is to take samples of the signals $x \in \mathcal{H}$, for instance to save or to transmit them. Mathematically, taking samples of a signal can be represented as follows: given a frame $\mathcal{G} = \{g_i\}_{i\in K}$ that spans a closed subspace $\mathcal{S}$ (called the sampling subspace), the samples of $x \in \mathcal{H}$ are given by the family of coefficients $\{\langle x, g_i\rangle\}_{i\in K}$, see [17] and the references therein.

Assume that the signals carrying the desired information are those containing only high frequencies or only low frequencies. In order to clarify the idea, suppose that $x \in \mathcal{H}$ is a piece of music and it is intended to discriminate those fragments where high frequencies are predominant (a trumpet) from those fragments where low frequencies are predominant (a bass).

It turns out that some filters for the signals can be modeled as orthogonal projections acting on $\mathcal{H}$. Hence, consider an ideal low pass filter, i.e. an orthogonal projection $P \in L(\mathcal{H})$, and the complementary (ideal high pass) filter $I - P$. Therefore, the signals with the same energy at high and low band frequencies $\{x \in \mathcal{H} : \|Px\| = \|(I-P)x\|\}$ are considered disturbances, see e.g. [18,19].

For this particular application, given an arbitrary signal $x \in \mathcal{H}$, the filtered signals Px and $(I-P)x$ are sampled and x is discarded in case that the modulus of the difference $\|Px\|^2 - \|(I-P)x\|^2$ is small enough. Also, notice that sampling both filtered signals $y_1 = Px$ and $y_2 = (I-P)x$ with frames $\mathcal{G}_1 = \{g_i\}_{i\in I_1}$ and $\mathcal{G}_2 = \{h_i\}_{i\in I_2}$, which span $R(P)$ and $N(P)$ respectively, is equivalent to sampling $y = y_1 + y_2 \in \mathcal{H}$ with the frame $\mathcal{F} = \{f_i\}_{i\in I} = \{g_i\}_{i\in I_1} \cup \{h_i\}_{i\in I_2}$ for $\mathcal{H}$.

The space $\mathcal{H}$ can be endowed with an indefinite inner product (depending on the filters) in order to characterize the set of disturbances as the cone of J-neutral vectors $\mathcal{C}$ of $\mathcal{H}$. Indeed, $J = P - (I-P) = 2P - I$ is a fundamental symmetry which turns $\mathcal{H}$ into a Krein space. Furthermore, a signal is a disturbance if and only if it is J-neutral with respect to the indefinite inner product given by

$$[\,y, z\,] = \langle Py, Pz \rangle - \langle (I-P)y, (I-P)z \rangle ,$$

where $y, z \in \mathcal{H}$ are arbitrary signals.

Observe that the vectors of the frame $\mathcal{F}$ are away from the disturbances set $\mathcal{C}$, i.e. the sampling vectors are not highly correlated with the disturbances (see Remark 3.13 for a precise description of this fact). However, once that the cone of disturbances is determined, the following questions naturally arise: Are there other frames whose sampling vectors are not highly correlated with the disturbances? Given an arbitrary frame $\mathcal{F}' = \{f_i'\}_{i\in I}$ for $\mathcal{H} \times \mathcal{H}$, is $\mathcal{F}'$ good for this sampling scheme? (in the sense that it stays away from the disturbances set). How correlated are the sampling vectors in $\mathcal{F}'$ and the cone of disturbances $\mathcal{C}$?

The above discussion motivates the following definition. Let $\mathcal{H}$ be a Krein space with fundamental symmetry J. Given a Bessel family $\mathcal{F} = \{f_i\}_{i\in I}$ in $\mathcal{H}$ consider the synthesis operator $T \in L(\ell_2(I), \mathcal{H})$. If $I_+ = \{i \in I : [f_i, f_i] \geq 0\}$ and $I_- = \{i \in I : [f_i, f_i] < 0\}$, consider the orthogonal decomposition of $\ell_2(I)$ given by

$$\ell_2(I) = \ell_2(I_+) \oplus \ell_2(I_-), \tag{13}$$

and denote by $P_\pm$ the orthogonal projection onto $\ell_2(I_\pm)$. Also, let $T_\pm = TP_\pm$. If $\mathcal{M}_\pm = \overline{\operatorname{span}\{f_i : i \in I_\pm\}}$, notice that $\operatorname{span}\{f_i : i \in I_\pm\} \subseteq R(T_\pm) \subseteq \mathcal{M}_\pm$ and

$$R(T) = R(T_+) + R(T_-).$$

Definition 3.1. The Bessel family $\mathcal{F} = \{f_i\}_{i\in I}$ is a J-*frame* for $\mathcal{H}$ if $R(T_+)$ is a maximal uniformly J-positive subspace of $\mathcal{H}$ and $R(T_-)$ is a maximal uniformly J-negative subspace of $\mathcal{H}$.

Notice that, in particular, every J-orthogonalized basis of a Krein space $\mathcal{H}$ is a J-frame for $\mathcal{H}$, because it generates a maximal dual pair, see [8, Chapter 1, Section 10.12].

If $\mathcal{F}$ is a J-frame, as a consequence of its maximality, $R(T_\pm)$ is closed. So, $R(T_\pm) = \mathcal{M}_\pm$ and, by [11, Corollary 1.5.2], $\mathcal{M}_+ + \mathcal{M}_- = \mathcal{H}$. Then, it follows that $\mathcal{F}$ is a frame for the associated Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$ because

$$R(T) = R(T_+) + R(T_-) = \mathcal{M}_+ + \mathcal{M}_- = \mathcal{H}.$$

Given a Bessel family $\mathcal{F} = \{f_i\}_{i\in I}$, consider the subspaces $R(T_+)$ and $R(T_-)$ as above. If $K_\pm : \mathcal{D}_\pm \to \mathcal{H}_\mp$ is the angular operator associated to $R(T_\pm)$, the *operator of transition* associated to the Bessel family $\mathcal{F}$ is defined by

$$F = K_+P + K_-(I-P) : \mathcal{D}_+ + \mathcal{D}_- \to \mathcal{H},$$

where $P = \frac{1}{2}(I+J)$ is the J-selfadjoint projection onto $\mathcal{H}_+$ and $\mathcal{D}_\pm$ is a subspace of $\mathcal{H}_\pm$ (the domain of $K_\pm$), see [20].

Proposition 3.2. *Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a Bessel family in $\mathcal{H}$. Then, $\mathcal{F}$ is a J-frame if and only if F is everywhere defined (i.e. $\mathcal{D}_+ + \mathcal{D}_- = \mathcal{H}$) and $\|F\| < 1$.*

Proof. See [20, Proposition 2.6]. □

It follows from the definition that, given a J-frame $\mathcal{F} = \{f_i\}_{i\in I}$ for the Krein space $\mathcal{H}$, $[f_i, f_i] \neq 0$ for every $i \in I$, i.e. $I_\pm = \{i \in I : \pm[f_i, f_i] > 0\}$. This fact allows to endow the coefficients space $\ell_2(I)$ with a Krein space structure. Denote $\sigma_i = \operatorname{sgn}([f_i, f_i]) = \pm 1$ for every $i \in I$. Then, the diagonal operator $J_2 \in L(\ell_2(I))$ defined by

$$J_2\, e_i = \sigma_i\, e_i, \quad \text{for every } i \in I, \tag{14}$$

is a selfadjoint involution on $\ell_2(I)$. Therefore, $\ell_2(I)$ with the fundamental symmetry J_2 is a Krein space.

Now, if $T \in L(\ell_2(I), \mathcal{H})$ is the synthesis operator of $\mathcal{F}$, the J-adjoints of T, T_+ and T_- can be easily calculated, in fact if $f \in \mathcal{H}$:

$$T_\pm^+ f = \pm \sum_{i\in I_\pm} [f, f_i] e_i,$$

and $T^+ f = (T_+ + T_-)^+ f = T_+^+ f + T_-^+ f = \sum_{i\in I_+}[f, f_i]e_i - \sum_{i\in I_-}[f, f_i]e_i = \sum_{i\in I}\sigma_i[f, f_i]e_i$.

Example 3.3. It is easy to see that not every frame of J-nonneutral vectors is a J-frame: given the Krein space obtained by endowing $\mathbb{C}^3$ with the sesquilinear form

$$[(x_1, x_2, x_3), (y_1, y_2, y_3)] = x_1\overline{y_1} + x_2\overline{y_2} - x_3\overline{y_3},$$

consider $f_1 = (1, 0, \frac{1}{\sqrt{2}})$, $f_2 = (0, 1, \frac{1}{\sqrt{2}})$ and $f_3 = (0, 0, 1)$. Observe that $\mathcal{F} = \{f_1, f_2, f_3\}$ is a frame for $\mathbb{C}^3$ because it is a (linear) basis for the space.

On the other hand, $\mathcal{M}_+ = \operatorname{span}\{f_1, f_2\}$ and $\mathcal{M}_- = \operatorname{span}\{f_3\}$. If $(a, b, \frac{1}{\sqrt{2}}(a+b))$ is an arbitrary vector in $\mathcal{M}_+$ then

$$[f, f] = |a|^2 + |b|^2 - \frac{1}{2}|a+b|^2 = \frac{1}{2}|a-b|^2 \geq 0,$$

so $\mathcal{M}_+$ is a J-nonnegative subspace of $\mathbb{C}^3$. But $\mathcal{M}_+$ is not uniformly J-positive, because $(1, 1, \sqrt{2}) \in \mathcal{M}_+$ is a (non-trivial) J-neutral vector. Therefore, $\mathcal{F}$ is not a J-frame for $(\mathbb{C}^3, [\ ,\ ])$.

The following is a handy way to construct J-frames for a given Krein space. Along this section, it will be shown that every J-frame can be realized in this way.

Example 3.4. Given a Krein space $\mathcal{H}$ with fundamental symmetry J, let $\mathcal{M}_+$ (resp. $\mathcal{M}_-$) be a maximal uniformly J-positive (resp. J-negative) subspace of $\mathcal{H}$. If $\mathcal{F}_\pm = \{f_i\}_{i\in I_\pm}$ is a frame for the Hilbert space $(\mathcal{M}_\pm, \pm[\ ,\ ])$ then $\mathcal{F} = \mathcal{F}_+ \cup \mathcal{F}_-$ is a J-frame for $\mathcal{H}$.

Indeed, by Remark 2.3, $\mathcal{F}_+$ and $\mathcal{F}_-$ are Bessel families in $\mathcal{H}$. Hence, $\mathcal{F}$ is a Bessel family and, if $I = I_+ \dot{\cup} I_-$ (the disjoint union of I_+ and I_-), the synthesis operator $T \in L(\ell_2(I), \mathcal{H})$ of $\mathcal{F}$ is given by

$$Tx = T_+x_+ + T_-x_- \quad \text{if } x = x_+ + x_- \in \ell_2(I_+) \oplus \ell_2(I_-) =: \ell_2(I),$$

where $T_\pm : \ell_2(I_\pm) \to \mathcal{M}_\pm$ is the synthesis operator of $\mathcal{F}_\pm$. Then, it is clear that $R(TP_\pm) = \mathcal{M}_\pm$ is a maximal uniformly J-definite subspace of $\mathcal{H}$.

Proposition 3.5. *Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a J-frame for $\mathcal{H}$. Then, $\mathcal{F}_\pm = \{f_i\}_{i\in I_\pm}$ is a frame for the Hilbert space $(\mathcal{M}_\pm, \pm[\ ,\ ])$, i.e. there exist constants $B_- \leq A_- < 0 < A_+ \leq B_+$ such that*

$$A_\pm[f, f] \leq \sum_{i\in I_\pm} |[f, f_i]|^2 \leq B_\pm[f, f] \quad \text{for every } f \in \mathcal{M}_\pm. \tag{15}$$

Proof. If $\mathcal{F} = \{f_i\}_{i\in I}$ is a J-frame for $\mathcal{H}$, then $R(T_+) = \mathcal{M}_+$ is a (maximal) uniformly J-positive subspace of $\mathcal{H}$. So, T_+ is a surjection from $\ell_2(I)$ onto the Hilbert space $(\mathcal{M}_+, [\ ,\ ])$. Therefore, $\mathcal{F}_+$ is a frame for $(\mathcal{M}_+, [\ ,\ ])$. In particular, there exist constants $0 < A_+ \leq B_+$ such that (15) is satisfied for $\mathcal{M}_+$. The assertion on $\mathcal{F}_-$ follows analogously. $\square$

Now, assuming that $\mathcal{F}$ is a J-frame for a Krein space $(\mathcal{H}, [\ ,\ ])$, a set of constants $\{B_-, A_-, A_+, B_+\}$ satisfying (15) is going to be computed. They depend only on the definiteness bounds for $R(T_\pm)$, the norm and the reduced minimum modulus of $T_\pm$.

Suppose that $\mathcal{F}$ is a J-frame for a Krein space $(\mathcal{H}, [\ ,\ ])$ with synthesis operator $T \in L(\ell_2(I), \mathcal{H})$. Since $R(T_+) = \mathcal{M}_+$ is a (maximal) uniformly J-positive subspace of $\mathcal{H}$, there exists $\alpha_+ > 0$ such that $\alpha_+\|f\|^2 \leq [f, f]$ for every $f \in \mathcal{M}_+$. So,

$$\sum_{i\in I_+} |[f, f_i]|^2 = \|T_+^+ f\|^2 \leq \|T_+^+\|^2\|f\|^2 \leq B_+[f, f], \qquad \text{for every } f \in \mathcal{M}_+,$$

where $B_+ = \frac{\|T_+^+\|^2}{\alpha_+} = \frac{\|T_+\|^2}{\alpha_+}$. Furthermore, since $N(T_+^+)^\perp = J(\mathcal{M}_+)$, if $f \in \mathcal{M}_+$,

$$\begin{aligned}\sum_{i\in I_+} |[f, f_i]|^2 &= \|T_+^+ f\|^2 = \|T_+^+ P_{J(\mathcal{M}_+)} f\|^2 \geq \gamma(T_+^+)^2\|P_{J(\mathcal{M}_+)} f\|^2 = \gamma(T_+)^2\|P_{\mathcal{M}_+} Jf\|^2 \\ &= \gamma(T_+)^2\|G_{\mathcal{M}_+} f\|^2 \geq \gamma(T_+)^2\gamma(G_{\mathcal{M}_+})^2\|f\|^2 \geq A_+[f, f],\end{aligned}$$

where $A_+ = \gamma(T_+)^2\gamma(G_{\mathcal{M}_+})^2 = \gamma(T_+)^2\alpha_+^2$, see Remark 2.6.

Analogously, $A_- = -\gamma(T_-)^2\alpha_-^2$ and $B_- = -\frac{\|T_-\|^2}{\alpha_-}$ satisfy Eq. (15) for every $f \in R(T_-) = \mathcal{M}_-$, if α_- is the definiteness bound of the (maximal) uniformly J-negative subspace $\mathcal{M}_-$.

Usually, the bounds $A_\pm = \pm\alpha_\pm^2\gamma(T_\pm)^2$ and $B_\pm = \pm\frac{\|T_\pm\|^2}{\alpha_\pm}$ are not optimal for the J-frame $\mathcal{F}$.

Definition 3.6. Let $\mathcal{F}$ be a J-frame for the Krein space $\mathcal{H}$. The optimal constants $B_- \leq A_- < 0 < A_+ \leq B_+$ satisfying (15) are called the *J-frame bounds of $\mathcal{F}$*.

In order to compute the J-frame bounds associated to a J-frame $\mathcal{F} = \{f_i\}_{i\in I}$, consider the uniformly J-definite subspaces $\mathcal{M}_+$ and $\mathcal{M}_-$. Recall that $\mathcal{F}_+ = \{f_i\}_{i\in I_+}$ is a frame for the Hilbert space $(\mathcal{M}_+, [\ ,\ ])$. Then, if $G_+ = G_{\mathcal{M}_+}|_{\mathcal{M}_+} \in GL(\mathcal{M}_+)$, the frame bounds for $\mathcal{F}_+$ are given by $A_+ = \|(S_{G_+})^{-1}\|_+^{-1}$ and $B_+ = \|S_{G_+}\|_+$, where $S_{G_+} = T_+T_+^*G_+$ is the frame operator of $\mathcal{F}_+$ and $\|f\|_+ = [f,f]^{1/2} = \|G_+^{1/2}f\|, f \in \mathcal{M}_+$, is the operator norm associated to the inner product $[\ ,\ ]$. Therefore,

$$A_+ = \|(S_{G_+})^{-1}\|_+^{-1} = \|G_+^{1/2}(T_+T_+^*G_+)^{-1}\|^{-1} = \|G_+^{-1/2}(T_+T_+^*)^{-1}\|^{-1},$$

and $B_+ = \|S_{G_+}\|_+ = \|G_+^{1/2}T_+T_+^*G_+\|$. Analogously, it follows that $\mathcal{F}_- = \{f_i\}_{i\in I_-}$ is a frame for the Hilbert space $(\mathcal{M}_-, -[\ ,\ ])$. So, the frame bounds for $\mathcal{F}_-$ are given by

$$A_- = \|G_-^{-1/2}(T_-T_-^*)^{-1}\|^{-1} \quad \text{and} \quad B_- = \|G_-^{1/2}T_-T_-^*G_-\|,$$

where $G_- = G_{\mathcal{M}_-}|_{\mathcal{M}_-} \in GL(\mathcal{M}_-)$. Thus, the J-frame bound associated to $\mathcal{F}$ can be fully characterized in terms of $T_\pm$ and the Gramian operators $G_{\mathcal{M}_\pm}$.

3.1. Characterizing J-frames in terms of frame inequalities

Given a Bessel family $\mathcal{F} = \{f_i\}_{i\in I}$ in a Krein space $\mathcal{H}$, the inequalities:

$$A\,[f,f] \le \sum_{i\in I} |[f,f_i]|^2 \le B\,[f,f] \quad \text{for every } f \in \mathcal{M} = \overline{\operatorname{span}\{f_i : i\in I\}}, \tag{16}$$

with $B \ge A > 0$, ensure that $\mathcal{M}$ is a J-nonnegative subspace of $\mathcal{H}$. However, they do not imply that $\mathcal{M}$ is uniformly J-positive, i.e. $(\mathcal{M}, [\ ,\ ])$ is not necessarily a inner product space. See the example below.

Example 3.7. Consider again the Krein space $(\mathbb{C}^3, [\ ,\ ])$ as in Example 3.3. As it was mentioned before, $\mathcal{M} = \operatorname{span}\{f_1 = (1, 0, 1/\sqrt{2}), f_2 = (0, 1, 1/\sqrt{2})\}$ is a J-nonnegative but not uniformly J-positive subspace of $\mathbb{C}^3$.

In this case, the orthogonal basis

$$v_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right), \qquad v_2 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right) \quad \text{and} \quad v_3 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1\right),$$

is a basis of eigenvectors of $G_{\mathcal{M}}$, corresponding to the eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = 0$, respectively. Moreover, $\mathcal{M} = \operatorname{span}\{v_1, v_2\}$. Thus, if $f \in \mathcal{M}$ there exists $\alpha, \beta \in \mathbb{C}$ such that $f = \alpha v_1 + \beta v_2$ and then, since $G_{\mathcal{M}}v_1 = 0 \in \mathbb{C}^3$, it is easy to see that

$$|[f,f_1]|^2 + |[f,f_2]|^2 = |\beta|^2\left(|\langle v_2, f_1\rangle|^2 + |\langle v_2, f_2\rangle|^2\right) = |\beta|^2 = [f,f].$$

Therefore, (16) holds with $A = B = 1$, but $\{f_1, f_2\}$ cannot be extended to a J-frame, since $\mathcal{M}$ is not a uniformly J-positive subspace. □

The next result gives a complete characterization of the families satisfying (16) for $B \ge A > 0$. It is straightforward to formulate and prove analogues of all these assertions for a family satisfying (16) for negative constants $B \le A < 0$.

Proposition 3.8. *Given a Bessel family $\mathcal{F} = \{f_i\}_{i\in I}$ in a Krein space $\mathcal{H}$, let $\mathcal{M} = \overline{\operatorname{span}\{f_i : i\in I\}}$ and $\mathcal{N} = \mathcal{M} \cap \mathcal{M}^{[\perp]}$. If there exist constants $0 < A \le B$ such that*

$$A\,[f,f] \le \sum_{i\in I} |[f,f_i]|^2 \le B\,[f,f] \quad \text{for every } f \in \mathcal{M}, \tag{17}$$

then $\mathcal{M} \ominus \mathcal{N}$ is a (closed) uniformly J-positive subspace of $\mathcal{M}$. Moreover, if $\mathcal{F}$ is a frame for the Hilbert space $(\mathcal{M}, \langle\ ,\ \rangle)$, the converse holds.

Proof. First, suppose that there exist $0 < A \le B$ such that (17) holds. So, $\mathcal{M}$ is a J-nonnegative subspace of $\mathcal{H}$, or equivalently, $(\mathcal{M}, [\ ,\ ])$ is a semi-inner product space.

If $T \in L(\ell_2(I), \mathcal{H})$ is the synthesis operator of the Bessel sequence $\mathcal{F}$ and $C = \|T^*\|^2 > 0$, then $TT^* \le CP_{\mathcal{M}}$. So, using (17) it is easy to see that:

$$A\,\langle G_{\mathcal{M}}f, f\rangle \le \|T^+(P_{\mathcal{M}}f)\|^2 = \left\langle (P_{\mathcal{M}}JTT^*JP_{\mathcal{M}})f, f\right\rangle \le C\left\langle (G_{\mathcal{M}})^2 f, f\right\rangle, \quad f \in \mathcal{H}. \tag{18}$$

Thus, $0 \le G_{\mathcal{M}} \le \frac{C}{A}(G_{\mathcal{M}})^2$. Applying Theorem 2.1 it is easy to see that

$$R((G_{\mathcal{M}})^{1/2}) \subseteq R(G_{\mathcal{M}}) \subseteq R((G_{\mathcal{M}})^{1/2}).$$

Moreover, it follows by Corollary 2.2 that $R(G_{\mathcal{M}})$ is closed because $R(G_{\mathcal{M}}) = R((G_{\mathcal{M}})^{1/2})$.

Let $\mathcal{M}' = \mathcal{M} \ominus \mathcal{N}$ and notice that $\mathcal{M}'$ is a closed uniformly J-positive subspace of $\mathcal{H}$. In fact, since $R(G_{\mathcal{M}})$ is closed, there exists $\alpha > 0$ such that

$$[f, f] = \langle G_{\mathcal{M}} f, f \rangle = \|(G_{\mathcal{M}})^{1/2} f\|^2 \geq \alpha \|f\|^2 \quad \text{for every } f \in N(G_{\mathcal{M}})^{\perp} = \mathcal{M} \ominus \mathcal{N}.$$

Conversely, suppose that $\mathcal{F}$ is a frame for $(\mathcal{M}, \langle\ ,\ \rangle)$, i.e. there exist constants $B' \geq A' > 0$ such that

$$A' P_{\mathcal{M}} \leq TT^* \leq B' P_{\mathcal{M}},$$

where $T \in L(\ell_2(I), \mathcal{M})$ is the synthesis operator of $\mathcal{F}$. If $\mathcal{M}' = \mathcal{M} \ominus \mathcal{N}$ is a uniformly J-positive subspace of $\mathcal{H}$, then there exists $\alpha > 0$ such that $\alpha P_{\mathcal{M}'} \leq G_{\mathcal{M}'} \leq P_{\mathcal{M}'}$. As a consequence of Theorem 2.1, $R((G_{\mathcal{M}'})^{1/2}) = \mathcal{M}' = R(G_{\mathcal{M}'})$. Since $G_{\mathcal{M}} = G_{\mathcal{M}'}$ it is easy to see that

$$A'(G_{\mathcal{M}})^2 = A'(G_{\mathcal{M}'})^2 \leq P_{\mathcal{M}} JTT^* J P_{\mathcal{M}} \leq B'(G_{\mathcal{M}'})^2 = B'(G_{\mathcal{M}})^2.$$

Therefore, $R(P_{\mathcal{M}} JT) = R(G_{\mathcal{M}'}) = R((G_{\mathcal{M}'})^{1/2})$, or equivalently, there exist $B \geq A > 0$ such that

$$AG_{\mathcal{M}} = AG_{\mathcal{M}'} \leq P_{\mathcal{M}} JTT^* J P_{\mathcal{M}} \leq BG_{\mathcal{M}'} = BG_{\mathcal{M}},$$

i.e. $A\,[f, f] \leq \sum_{i \in I} |[f, f_i]|^2 \leq B\,[f, f]$ for every $f \in \mathcal{M}$. $\square$

Theorem 3.9. *Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a frame for $\mathcal{H}$. If $I_{\pm} = \{i \in I : \pm[f_i, f_i] \geq 0\}$ and $\mathcal{M}_{\pm} = \overline{\text{span}\{f_i : i \in I_{\pm}\}}$ then, $\mathcal{F}$ is a J-frame if and only if $\mathcal{M}_{\pm} \cap \mathcal{M}_{\pm}^{[\perp]} = \{0\}$ and there exist constants $B_- \leq A_- < 0 < A_+ \leq B_+$ such that*

$$A_{\pm}\,[f, f] \leq \sum_{i \in I_{\pm}} |[f, f_i]|^2 \leq B_{\pm}\,[f, f] \quad \textit{for every } f \in \mathcal{M}_{\pm}. \tag{19}$$

Proof. If $\mathcal{F}$ is a J-frame, the conditions on $\mathcal{M}_{\pm}$ follow by its definition and by Proposition 3.5. Conversely, if $\mathcal{M}_+$ is J-non degenerated and there exist constants $0 < A_+ \leq B_+$ such that

$$A_+\,[f, f] \leq \sum_{i \in I_{\pm}} |[f, f_i]|^2 \leq B_+\,[f, f] \quad \text{for every } f \in \mathcal{M}_+,$$

then, by Proposition 3.8, $\mathcal{M}_+$ is a uniformly J-positive subspace of $\mathcal{H}$. Therefore, there exist constants $0 < A \leq B$ such that

$$A\,\|P_{\mathcal{M}_+} f\|^2 \leq \|T_+^+ P_{\mathcal{M}_+} f\|^2 \leq B \|P_{\mathcal{M}_+} f\|^2 \quad \text{for every } f \in \mathcal{H}.$$

But these inequalities can be rewritten as

$$A\,P_{\mathcal{M}_+} \leq P_{\mathcal{M}_+} J T_+ T_+^* J P_{\mathcal{M}_+} \leq B\,P_{\mathcal{M}_+}.$$

Then, by Theorem 2.1, $R(P_{\mathcal{M}_+} J T_+) = R(P_{\mathcal{M}_+}) = \mathcal{M}_+$. Furthermore, $P_{J(\mathcal{M}_+)}(R(T_+)) = J(\mathcal{M}_+)$ because

$$J(\mathcal{M}_+) = J(R(P_{\mathcal{M}_+} J T_+)) = R((J P_{\mathcal{M}_+} J) T_+) = R(P_{J(\mathcal{M}_+)} T_+) = P_{J(\mathcal{M}_+)}(R(T_+)).$$

Therefore, taking the counterimage of $P_{J(\mathcal{M}_+)}(R(T_+))$ by $P_{J(\mathcal{M}_+)}$, it follows that

$$\mathcal{H} = R(T_+) \dotplus J(\mathcal{M}_+)^{\perp} \subseteq \mathcal{M}_+ \dotplus \mathcal{M}_+^{[\perp]} = \mathcal{H}.$$

Thus, $R(T_+) = \mathcal{M}_+$ and $\mathcal{F}_+$ is a frame for $\mathcal{M}_+$. Analogously, $\mathcal{F}_- = \{f_i\}_{i \in I_-}$ is a frame for $\mathcal{M}_-$. Finally, since $\mathcal{F}$ is a frame for $\mathcal{H}$,

$$\mathcal{H} = R(T) = R(T_+) + R(T_-),$$

which proves the maximality of $R(T_{\pm})$. Thus, $\mathcal{F}$ is a J-frame for $\mathcal{H}$. $\square$

3.2. *A geometrical characterization of J-frames*

Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a J-frame for $\mathcal{H}$ and consider $\mathcal{F} = \mathcal{F}_+ \cup \mathcal{F}_+$ the partition of $\mathcal{F}$ into J-positive and J-negative vectors. Moreover, let $\mathcal{M}_{\pm}$ be the (maximal) uniformly J-definite subspace of $\mathcal{H}$ generated by $\mathcal{F}_{\pm}$.

The aim of this section is to show that it is possible to bound the correlation between vectors in $\mathcal{F}_+$ (resp. $\mathcal{F}_-$) and vectors in the cone of neutral vectors $\mathcal{C} = \{n \in \mathcal{H} : [n, n] = 0\}$, in a strong sense:

$$|\langle f, n \rangle| \leq c_{\pm}\, \|f\|\, \|n\|, \quad f \in \mathcal{M}_{\pm}, n \in \mathcal{C}, \tag{20}$$

for some constants $\frac{\sqrt{2}}{2} \leq c_{\pm} < 1$. In order to make these ideas precise, consider the notion of minimal angle between a subspace $\mathcal{M}$ and the cone $\mathcal{C}$.

Definition 3.10. Given a closed subspace $\mathcal{M}$ of the Krein space $\mathcal{H}$, consider

$$c_0(\mathcal{M}, \mathcal{C}) = \sup \{ |\langle m, n \rangle| : m \in \mathcal{M}, n \in \mathcal{C}. \|n\| = \|m\| = 1 \}. \tag{21}$$

Then, there exists a unique $\theta(\mathcal{M}, \mathcal{C}) \in [0, \frac{\pi}{4}]$ such that $\cos(\theta(\mathcal{M}, \mathcal{C})) = c_0(\mathcal{M}, \mathcal{C})$. In this case, $\theta(\mathcal{M}, \mathcal{C})$ is the *minimal angle* between $\mathcal{M}$ and $\mathcal{C}$.

Observe that if the subspace $\mathcal{M}$ contains a non-trivial J-neutral vector (e.g. if $\mathcal{M}$ is J-indefinite or J-semidefinite) then $c_0(\mathcal{M}, \mathcal{C}) = 1$, or equivalently, $\theta(\mathcal{M}, \mathcal{C}) = 0$. On the other hand, it will be shown that the minimal angle between a uniformly J-positive (resp. uniformly J-negative) subspace $\mathcal{M}$ and $\mathcal{C}$ is always bounded away from 0.

Proposition 3.11. *Let $\mathcal{M}$ be a J-semidefinite subspace of $\mathcal{H}$ with definiteness bound α. Then,*

$$c_0(\mathcal{M}, \mathcal{C}) = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{1+\alpha}{2}} + \sqrt{\frac{1-\alpha}{2}} \right). \tag{22}$$

In particular, $\mathcal{M}$ is uniformly J-definite if and only if $c_0(\mathcal{M}, \mathcal{C}) < 1$.

Proof. Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be a fundamental decomposition of $\mathcal{H}$ and suppose that $\mathcal{M}$ is a J-nonnegative subspace of $\mathcal{H}$.

Let $m \in \mathcal{M}$ with $\|m\| = 1$. Then, there exist (unique) $m^\pm \in \mathcal{H}_\pm$ such that $m = m^+ + m^-$. In this case,

$$1 = \|m\|^2 = \|m^+\|^2 + \|m^-\|^2 \quad \text{and} \quad \alpha \le [\, m, m \,] = \|m^+\|^2 - \|m^-\|^2. \tag{23}$$

Claim. *For a fixed $m \in \mathcal{M}$ with $\|m\| = 1$, $\sup \{ |\langle m, n \rangle| : n \in \mathcal{C}, \|n\| = 1 \} = \frac{1}{\sqrt{2}}(\|m^+\| + \|m^-\|)$.*

Indeed, consider $n \in \mathcal{C}$ with $\|n\| = 1$. Then, there exist (unique) $n_\pm \in \mathcal{H}_\pm$ such that $n = n^+ + n^-$. In this case,

$$0 = [\, n, n \,] = \|n^+\|^2 - \|n^-\|^2 \quad \text{and} \quad 1 = \|n\|^2 = \|n^+\|^2 + \|n^-\|^2,$$

which imply that $\|n^+\| = \|n^-\| = \frac{1}{\sqrt{2}}$. Therefore,

$$|\langle m, n \rangle| \le \left| \langle m^+, n^+ \rangle \right| + \left| \langle m^-, n^- \rangle \right| \le \frac{1}{\sqrt{2}} (\|m^+\| + \|m^-\|).$$

On the other hand, if $m^- \neq 0$ then let $n_m := \frac{1}{\sqrt{2}}(\frac{m^+}{\|m^+\|} + \frac{m^-}{\|m^-\|})$, otherwise consider $n_m = \frac{1}{\sqrt{2}}(m + z)$, with $z \in \mathcal{H}_-$, $\|z\| = 1$. Now, it is easy to see that $n_m \in \mathcal{C}$ and that $|\langle m, n_m \rangle| = \frac{1}{\sqrt{2}} (\|m^+\| + \|m^-\|)$ which together with the previous facts prove the claim.

Now, let $\mathcal{M}_1 = \{ m = m^+ + m^- \in \mathcal{M} : m^\pm \in \mathcal{H}_\pm, \|m\| = 1 \}$. Using the claim above it follows that

$$c_0(\mathcal{M}, \mathcal{C}) = \frac{1}{\sqrt{2}} \sup_{m \in \mathcal{M}_1} (\|m^+\| + \|m^-\|). \tag{24}$$

If $\alpha = 1$ then $\mathcal{M}$ is a subspace of $\mathcal{H}_+$. Also, it is easy to see that $c_0(\mathcal{M}, \mathcal{C}) = \frac{1}{\sqrt{2}}$. Thus, in this particular case, $c_0(\mathcal{M}, \mathcal{C}) = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{1+\alpha}{2}} + \sqrt{\frac{1-\alpha}{2}} \right)$.

On the other hand, if $\alpha < 1$, let $k_0 \in \mathbb{N}$ be such that $\frac{1-\alpha}{2} > \frac{1}{2k_0}$. Observe that, by the definition of the definiteness bound, for every integer $k \ge k_0$ there exists $m_k = m_k^+ + m_k^- \in \mathcal{M}_1$ such that $\alpha \le \|m_k^+\|^2 - \|m_k^-\|^2 < \alpha + \frac{1}{k}$. Then, it follows that

$$\alpha + 1 \le 2\|m_k^+\|^2 < \alpha + 1 + \frac{1}{k},$$

or equivalently, $\sqrt{\frac{\alpha+1}{2}} \le \|m_k^+\| < \sqrt{\frac{\alpha+1}{2} + \frac{1}{2k}}$. Moreover, $\|m_k^-\| = \sqrt{1 - \|m_k^+\|^2}$ implies that

$$\sqrt{\frac{1-\alpha}{2} - \frac{1}{2k}} < \|m_k^-\| \le \sqrt{\frac{1-\alpha}{2}}.$$

Therefore, for every integer $k \ge k_0$ there exists $m_k \in \mathcal{M}_1$ such that

$$\sqrt{\frac{1-\alpha}{2} - \frac{1}{2k}} + \sqrt{\frac{\alpha+1}{2}} < \|m_k^+\| + \|m_k^-\| < \sqrt{\frac{\alpha+1}{2} + \frac{1}{2k}} + \sqrt{\frac{1-\alpha}{2}}.$$

Thus, $c_0(\mathcal{M}, \mathcal{C}) = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{1+\alpha}{2}} + \sqrt{\frac{1-\alpha}{2}} \right)$.

Assume now that $\mathcal{M}$ is a J-nonpositive subspace of $(\mathcal{H}, [\ ,\ ])$ with definiteness bound α, for $0 \le \alpha \le 1$. Then, $\mathcal{M}$ is a J-nonnegative subspace of the antispace $(\mathcal{H}, -[\ ,\ ])$, with the same definiteness bound α. Furthermore, the cone of J-neutral vectors for the antispace is the same as for the initial Krein space $(\mathcal{H}, [\ ,\ ])$. Therefore, we can apply the previous arguments and conclude that (22) also holds for J-nonpositive subspaces.

Finally, the last assertion in the statement follows from the formula in (22). □

Let $\mathcal{F}$ be a J-frame for $\mathcal{H}$ as above. Notice that (20) holds for some constant $\frac{\sqrt{2}}{2} \le c_\pm < 1$ if and only if $c_0(\mathcal{M}_\pm, \mathcal{C}) < 1$, i.e. that the minimal angles $\theta(\mathcal{M}_\pm, \mathcal{C})$ are bounded away from 0. This is intimately related with the fact that the aperture between the subspaces $\mathcal{M}_+$ (resp. $\mathcal{M}_-$) and $\mathcal{H}_+$ (resp. $\mathcal{H}_-$) is bounded away from $\frac{\pi}{4}$, whenever $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is a fundamental decomposition.

Remark 3.12. Given a Krein space $\mathcal{H}$, fix a fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Then, if $\mathcal{M}$ is a J-nonnegative subspace of $\mathcal{H}$, the minimal angle between $\mathcal{M}$ and $\mathcal{C}$ is related with the *aperture* $\Phi(\mathcal{M}, \mathcal{H}_+)$ between the subspaces $\mathcal{M}$ and $\mathcal{H}_+$, see [21] and Exercises 3–6 to [8, Chapter 1, Section 8]. In fact, if $K \in L(\mathcal{H}_+, \mathcal{H}_-)$ is the angular operator associated to $\mathcal{M}$ then, by [8, Chapter 1, Section 8 Exercise 4],

$$\Phi(\mathcal{M}, \mathcal{H}_+) = \frac{\|K\|}{\sqrt{1 + \|K\|^2}}.$$

Also, if α is the definiteness bound of $\mathcal{M}$ then $\|K\| = \sqrt{\frac{1-\alpha}{1+\alpha}}$, see [8, Chapter 1, Lemma 8.4]. Therefore, $\Phi(\mathcal{M}, \mathcal{H}_+) = \frac{\|K\|}{\sqrt{1+\|K\|^2}} = \sqrt{\frac{1-\alpha}{2}}$. Since $\Phi(\mathcal{M}, \mathcal{H}_+) = \sin\varphi(\mathcal{M}, \mathcal{H}_+)$ for an angle $\varphi(\mathcal{M}, \mathcal{H}_+) \in [0, \frac{\pi}{4}]$ between $\mathcal{M}$ and $\mathcal{H}_+$, it is easy to see that

$$\cos\varphi(\mathcal{M}, \mathcal{H}_+) = \sqrt{1 - \sin^2\varphi(\mathcal{M}, \mathcal{H}_+)} = \sqrt{\frac{1+\alpha}{2}}.$$

Therefore, if $\varphi = \varphi(\mathcal{M}, \mathcal{H}_+)$,

$$\cos\left(\frac{\pi}{4} - \varphi\right) = \frac{\sqrt{2}}{2}(\cos\varphi + \sin\varphi) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{1+\alpha}{2}} + \sqrt{\frac{1-\alpha}{2}}\right) = \cos(\theta(\mathcal{M}, \mathcal{C})),$$

i.e. $\varphi(\mathcal{M}, \mathcal{H}_+) + \theta(\mathcal{M}, \mathcal{C}) = \frac{\pi}{4}$. □

Remark 3.13. Regarding the discussion at the beginning of Section 3, consider any (redundant) J-frame $\mathcal{F} = \{f_i\}_{i\in I}$ for $(\mathcal{H}, [\ ,\ ])$. As usual, denote $\mathcal{M}_+$ and $\mathcal{M}_-$ the maximal uniformly J-definite subspaces generated by $\mathcal{F}$. Since $\mathcal{M}_\pm$ is uniformly J-definite, Proposition 3.11 shows that $c_0(\mathcal{M}_\pm, \mathcal{C}) < 1$. That is, J-frames provide a class of frames for $\mathcal{H}$ with the desired properties, namely the correlation between the sampling vectors and the cone of disturbances is controlled by $c_0(\mathcal{M}_\pm, \mathcal{C})$ because

$$|\langle f_i, n\rangle| \le c_0(\mathcal{M}_\pm, \mathcal{C})\, \|f_i\|\, \|n\| \quad \text{whenever } i \in I_\pm \text{ and } n \in \mathcal{C}. \tag{25}$$

Moreover, later in Proposition 5.4, it will be shown that the J-frame $\mathcal{F}$ admits a (canonical) dual J-frame that induces a linear (indefinite) stable and redundant encoding–decoding scheme in which the correlation between both the sampling and reconstructing vectors and the cone of neutral vectors is bounded from above. These remarks provide a quantitative measure of the advantage of considering J-frames with respect to usual frames in this setting.

4. On the synthesis operator of a J-frame

If $\mathcal{F}$ is a J-frame with synthesis operator T, then $QT = T_+ = TP_+$, where $Q = P_{\mathcal{M}_+//\mathcal{M}_-}$. Therefore,

$$Q = QTT^\dagger = TP_+T^\dagger.$$

So, given a surjective operator $T : \ell_2(I) \to \mathcal{H}$, the idempotency of $TP_+T^\dagger$ is a necessary condition for T to be the synthesis operator of a J-frame.

Lemma 4.1. *Let $T \in L(\ell_2(I), \mathcal{H})$ be surjective. Suppose that $P_\mathcal{S}$ is the orthogonal projection onto a closed subspace $\mathcal{S}$ of $\ell_2(I)$ such that $c(\mathcal{S}, N(T)^\perp) < 1$. Then, $TP_\mathcal{S}T^\dagger$ is a projection if and only if*

$$N(T) = \mathcal{S} \cap N(T) \oplus \mathcal{S}^\perp \cap N(T).$$

Proof. Suppose that $Q = TP_\mathcal{S}T^\dagger$ is a projection. Then, if $P = P_{N(T)^\perp}$, $E = PP_\mathcal{S}P$ is an orthogonal projection because it is selfadjoint and

$$E^2 = (PP_\mathcal{S}P)^2 = PP_\mathcal{S}PP_\mathcal{S}P = T^\dagger(TP_\mathcal{S}T^\dagger)^2T = T^\dagger(TP_\mathcal{S}T^\dagger)T = PP_\mathcal{S}P = E.$$

Therefore, $(PP_{\mathcal{S}})^k = E^{k-1}P_{\mathcal{S}} = EP_{\mathcal{S}} = (PP_{\mathcal{S}})^2$ for every $k \geq 2$. So, by [15, Lemma 18],

$$PP_{\mathcal{S}} = P_{\mathcal{S}} \wedge P = P_{\mathcal{S}}P.$$

Then, since $P_{\mathcal{S}}$ and P commute, it follows that $N(T) = \mathcal{S} \cap N(T) \oplus \mathcal{S}^{\perp} \cap N(T)$ (see [15, Lemma 9]).

Conversely, suppose that $N(T) = \mathcal{S} \cap N(T) \oplus \mathcal{S}^{\perp} \cap N(T)$. Then, $P_{\mathcal{S}}$ and P commute and

$$(TP_{\mathcal{S}}T^{\dagger})^2 = TP_{\mathcal{S}}(T^{\dagger}T)P_{\mathcal{S}}T^{\dagger} = TP_{\mathcal{S}}PP_{\mathcal{S}}T^{\dagger} = TPP_{\mathcal{S}}T^{\dagger} = TP_{\mathcal{S}}T^{\dagger}. \quad \square$$

Hereafter consider the set of possible decompositions of $\mathcal{H}$ as a (direct) sum of a pair of maximal uniformly definite subspaces, or equivalently, the associated set of projections:

$$\mathcal{Q} = \{Q \in L(\mathcal{H}) : Q^2 = Q, R(Q) \text{ is uniformly } J\text{-positive and } N(Q) \text{ is uniformly } J\text{-negative}\}.$$

Proposition 4.2. *Let $T \in L(\ell_2(I), \mathcal{H})$ be surjective. Then, T is the synthesis operator of a J-frame if and only if there exists $I_+ \subset I$ such that $\ell_2(I_+)$ (as a subspace of $\ell_2(I)$) satisfies $c(N(T)^{\perp}, \ell_2(I_+)) < 1$ and*

$$TP_+T^{\dagger} \in \mathcal{Q},$$

where $P_+ \in L(\ell_2(I))$ is the orthogonal projection onto $\ell_2(I_+)$.

Proof. If T is the synthesis operator of a J-frame, the existence of such a subset I_+ has already been discussed before.

Conversely, suppose that there exists such a subset I_+ of I. Then, since $c(N(T)^{\perp}, \ell_2(I_+)) < 1$ and $Q = TP_+T^{\dagger} \in \mathcal{Q}$, it follows from Lemma 4.1 that P_+ and $P = P_{N(T)^{\perp}}$ commute. Therefore,

$$QT = TP_+P = TPP_+ = TP_+,$$

and $(I - Q)T = T(I - P_+)$. Hence, $R(TP_+) = R(Q)$ is (maximal) uniformly J-positive and $R(T(I - P_+)) = N(Q)$ is (maximal) uniformly J-negative. Therefore $\mathcal{F} = \{Te_i\}_{i \in I}$ is by definition a J-frame for $\mathcal{H}$. $\square$

Theorem 4.3. *Given a surjective operator $T \in L(\ell_2(I), \mathcal{H})$, the following conditions are equivalent:*

(i) *There exists $U \in \mathcal{U}(\ell_2(I))$ such that TU is the synthesis operator of a J-frame.*
(ii) *There exists $Q \in \mathcal{Q}$ such that*

$$QTT^*(I - Q)^* = 0. \tag{26}$$

(iii) *There exist closed range operators $T_1, T_2 \in L(\ell_2(I), \mathcal{H})$ such that $T = T_1 + T_2$, $R(T_1)$ is uniformly J-positive, $R(T_2)$ is uniformly J-negative and $T_1T_2^* = T_2T_1^* = 0$.*

Proof. (i) $\Rightarrow$ (ii): Suppose that there exists $U \in \mathcal{U}(\ell_2(I))$ such that $V = TU$ is the synthesis operator of a J-frame. If $I_{\pm} = \{i \in I : \pm[Ve_i, Ve_i] > 0\}$ and $P_{\pm} \in L(\ell_2(I))$ is the orthogonal projection onto $\ell_2(I_{\pm})$, define $V_{\pm} = VP_{\pm}$. Then, $V = V_+ + V_-$ and $\mathcal{M}_{\pm} = R(V_{\pm})$ is a maximal uniformly J-definite subspace. So, considering $Q = P_{\mathcal{M}_+//\mathcal{M}_-} \in \mathcal{Q}$, it is easy to see that $QV = V_+$, $(I - Q)V = V_-$ and

$$QTT^*(I - Q)^* = QVV^*(I - Q)^* = V_+V_-^* = VP_+P_-V^* = 0.$$

(ii) $\Rightarrow$ (iii): Suppose that there exists $Q \in \mathcal{Q}$ such that $QTT^*(I - Q)^* = 0$. Defining $T_1 = QT$ and $T_2 = (I - Q)T$, it follows that $T = T_1 + T_2$, $R(T_1) = R(Q)$ is uniformly J-positive, $R(T_2) = N(Q)$ is uniformly J-negative and

$$T_1T_2^* = T_2T_1^* = 0,$$

because (26) says that $R(T_2^*) = R(T^*(I - Q)^*) \subseteq N(QT) = N(T_1)$.

(iii) $\Rightarrow$ (i): If there exist closed range operators $T_1, T_2 \in L(\ell_2(I), \mathcal{H})$ satisfying the conditions of item 3., notice that $T_1T_2^* = 0$ implies that $N(T_2)^{\perp} \subseteq N(T_1)$, or equivalently, $N(T_1)^{\perp} \subseteq N(T_2)$.

Consider the projection $Q = P_{R(T_1)//R(T_2)} \in \mathcal{Q}$ and notice that $QT = T_1$ and $(I - Q)T = T_2$. If $\mathcal{B}_1 = \{u_i\}_{i \in I_1}$ is an orthonormal basis of $N(T_1)^{\perp}$, consider the family $\{f_i^+\}_{i \in I_1}$ in $\mathcal{H}$ given by $f_i^+ = Tu_i$. But, if $i \in I_1$,

$$f_i^+ = QTu_i + (I - Q)Tu_i = T_1u_i \in R(T_1),$$

because $u_i \in N(T_1)^{\perp} \subseteq N(T_2)$. Therefore, $\{f_i^+\}_{i \in I_1} \subseteq R(T_1)$. Since T_1 is an isomorphism between $N(T_1)^{\perp}$ and $R(T_1)$, it follows that $R(T_1) = \overline{\text{span}\{f_i^+\}_{i \in I_1}}$.

Analogously, if $\mathcal{B}_2 = \{b_i\}_{i \in I_2}$ is an orthonormal basis of $N(T_1)$, the family $\{f_i^-\}_{i \in I_2}$ defined by $f_i^- = Tb_i$ $(i \in I_2)$ lies in $R(T_2)$. Since T_2 is an isomorphism between $N(T_2)^{\perp}$ and $R(T_2)$, it follows that

$$R(T_2) = T_2(N(T_1)) \subseteq \overline{\text{span}\{f_i^-\}_{i \in I_2}} \subseteq R(T_2).$$

Finally, consider $U \in \mathcal{U}(\ell_2(I))$ which turns the standard orthonormal basis $\{e_i\}_{i\in I}$ into $\mathcal{B}_1 \cup \mathcal{B}_2$. Then, if $V = TU$ and $\mathcal{F} = \{Ve_i\}_{i\in I} = \{f_i^+\}_{i\in I_1} \cup \{f_i^-\}_{i\in I_2}$, it is easy to see that

$$I_+ = \{i \in I : [\, Ve_i, Ve_i\,] > 0\} = I_1 \quad \text{and} \quad I_- = \{i \in I : [\, Ve_i, Ve_i\,] < 0\} = I_2.$$

So, $R(V_+) = R(T_1)$ is maximal uniformly J-positive and $R(V_-) = R(T_2)$ is maximal uniformly J-negative. Therefore, $\mathcal{F}$ is a J-frame for $\mathcal{H}$ with synthesis operator $V = TU$. $\square$

5. The *J*-frame operator

Definition 5.1. Given a J-frame $\mathcal{F} = \{f_i\}_{i\in I}$, the *$J$-frame operator* $S : \mathcal{H} \to \mathcal{H}$ is defined by

$$Sf = \sum_{i\in I} \sigma_i [\, f, f_i\,] f_i, \quad \text{for every } f \in \mathcal{H},$$

where $\sigma_i = \operatorname{sgn}([\, f_i, f_i\,])$.

The following proposition compiles some basic properties of the J-frame operator.

Proposition 5.2. *Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a J-frame with synthesis operator $T \in L(\ell_2(I), \mathcal{H})$. Then, its J-frame operator $S \in L(\mathcal{H})$ satisfies:*

(i) $S = TT^+$;
(ii) $S = S_+ - S_-$, *where* $S_+ := T_+T_+^+$ *and* $S_- := -T_-T_-^+$ *are J-positive operators;*
(iii) *S is an invertible J-selfadjoint operator;*
(iv) $\operatorname{ind}_\pm(S) = \dim \mathcal{H}_\pm$, *where* $\operatorname{ind}_\pm(S)$ *are the indices of* S.

Proof. If $\mathcal{F} = \{f_i\}_{i\in I}$ is a J-frame with synthesis operator $T \in L(\ell_2(I), \mathcal{H})$, then $T^+f = \sum_{i\in I} \sigma_i [\, f, f_i\,] e_i$ for $f \in \mathcal{H}$. So,

$$TT^+f = T\left(\sum_{i\in I} \sigma_i [\, f, f_i\,] e_i\right) = \sum_{i\in I} \sigma_i [\, f, f_i\,] f_i = Sf, \quad \text{for every } f \in \mathcal{H}.$$

Furthermore, if $I_\pm = \{i \in I : \pm[\, f_i, f_i\,] > 0\}$, consider $T_\pm = TP_\pm$ as usual. Then,

$$TT^+ = (T_+ + T_-)(T_+ + T_-)^+ = T_+T_+^+ + T_-T_-^+ = T_+T_+^+ - (-T_-T_-^+),$$

because $T_+T_-^+ = T_-T_+^+ = 0$. Therefore, $S = S_+ - S_-$ if $S_\pm := \pm T_\pm T_\pm^+$. Notice that $S_\pm$ is a J-positive operator because

$$S_\pm = \pm T_\pm T_\pm^+ = \pm T_\pm J_2 T_\pm^* J = T_\pm T_\pm^* J.$$

To prove the invertibility of S observe that, if $Sf = 0$ then $S_+f = S_-f$. But $R(S_+) \cap R(S_-) \subseteq R(T_+) \cap R(T_-) = \{0\}$. Thus, S is injective. On the other hand, $R(S) = S(\mathcal{M}_+^{[\perp]}) + S(\mathcal{M}_-^{[\perp]})$ because $\mathcal{H} = \mathcal{M}_+^{[\perp]} \dotplus \mathcal{M}_-^{[\perp]}$. But it is easy to see that $\mathcal{M}_\pm^{[\perp]} \subseteq N(S_\pm)$. So, $S(\mathcal{M}_\pm^{[\perp]}) = S_\mp(\mathcal{M}_\pm^{[\perp]})$ and $R(S) = S_-(\mathcal{M}_+^{[\perp]}) + S_+(\mathcal{M}_-^{[\perp]}) = R(S_-) + R(S_+) = \mathcal{M}_+ + \mathcal{M}_- = \mathcal{H}$. Therefore, S is invertible.

Finally, the identities $\operatorname{ind}_\pm(S) = \dim \mathcal{H}_\pm$ follow from the indices definition. Recall that if $A \in L(\mathcal{H})$ is a J-selfadjoint operator, $\operatorname{ind}_+(A)$ is the supremum of all positive integers r such that there exists a positive invertible matrix of the form $([\, Ax_j, x_k\,])_{j,k=1,\dots,r}$, where $x_1, \dots, x_r \in \mathcal{H}$ (if no such r exists, $\operatorname{ind}_-(A) = 0$). Similarly, $\operatorname{ind}_-(A) = \operatorname{ind}_+(-A)$ is the supremum of all positive integers m such that there exists a negative invertible matrix of the form $([\, Ay_j, y_k\,])_{j,k=1,\dots,m}$, where $y_1, \dots, y_m \in \mathcal{H}$, see [14]. $\square$

Corollary 5.3. *Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a J-frame for $\mathcal{H}$ with J-frame operator $S \in L(\mathcal{H})$. Then, $R(S_\pm) = \mathcal{M}_\pm$ and $N(S_\pm) = \mathcal{M}_\pm^{[\perp]}$. Furthermore, if $Q = P_{\mathcal{M}_+//\mathcal{M}_-}$,*

$$S_+ = QSQ^+ \quad \text{and} \quad S_- = -(I - Q)S(I - Q)^+. \tag{27}$$

Proof. Recall that $S_+ := T_+T_+^+ = T_+(J_2T_+^*J) = T_+T_+^*J$. Then, $R(S_+) = R(T_+T_+^*J) = R(T_+T_+^*) = R(T_+) = \mathcal{M}_+$ because $R(T_+)$ is closed. Since S_+ is J-selfadjoint, it follows that $N(S_+) = R(S_+)^{[\perp]} = \mathcal{M}_+^{[\perp]}$. Analogously, $R(S_-) = \mathcal{M}_-$ and $N(S_-) = \mathcal{M}_-^{[\perp]}$.

Since $S = S_+ - S_-$, if $Q = P_{\mathcal{M}_+//\mathcal{M}_-}$ then

$$QS = Q(S_+ - S_-) = S_+,$$

by the characterization of the range and nullspace of S_+. Therefore, $SQ^+ = QS = QSQ^+$. Analogously, $S(I-Q)^+ = (I-Q)S = (I - Q)S(I - Q)^+$. $\square$

The above corollary states that S is the diagonal block operator matrix

$$S = \begin{pmatrix} S_+ & 0 \\ 0 & -S_- \end{pmatrix}, \tag{28}$$

according to the (oblique) decompositions $\mathcal{H} = \mathcal{M}_-^{[\perp]} \dotplus \mathcal{M}_+^{[\perp]}$ and $\mathcal{H} = \mathcal{M}_+ \dotplus \mathcal{M}_-$ of the domain and codomain of S, respectively.

5.1. The indefinite reconstruction formula associated to a J-frame

Given a J-frame $\mathcal{F} = \{f_i\}_{i\in I}$ with synthesis operator T, there is a duality between $\mathcal{F}$ and the frame $\mathcal{G} = \{g_i\}_{i\in I}$ given by $g_i = S^{-1}f_i$: if $f \in \mathcal{H}$,

$$f = SS^{-1}f = TT^{+}(S^{-1}f) = T\left(\sum_{i\in I}\sigma_i[\, S^{-1}f, f_i\,]e_i\right) = \sum_{i\in I}\sigma_i[\, S^{-1}f, f_i\,]f_i = \sum_{i\in I}\sigma_i[\, f, S^{-1}f_i\,]f_i.$$

Analogously,

$$f = S^{-1}Sf = S^{-1}(TT^{+}f) = S^{-1}\left(\sum_{i\in I}\sigma_i[\, f, f_i\,]f_i\right) = \sum_{i\in I}\sigma_i[\, f, f_i\,]S^{-1}f_i.$$

Therefore, for every $f \in \mathcal{H}$, there is an *indefinite reconstruction formula* associated to $\mathcal{F}$:

$$f = \sum_{i\in I}\sigma_i[\, f, g_i\,]f_i = \sum_{i\in I}\sigma_i[\, f, f_i\,]g_i. \tag{29}$$

The following question arises naturally: is $\mathcal{G} = \{S^{-1}f_i\}_{i\in I}$ also a J-frame for $\mathcal{H}$?

Proposition 5.4. *If $\mathcal{F} = \{f_i\}_{i\in I}$ is a J-frame for a Krein space $\mathcal{H}$ with J-frame operator S, then $\mathcal{G} = \{S^{-1}f_i\}_{i\in I}$ is also a J-frame for $\mathcal{H}$.*

Proof. Given a J-frame $\mathcal{F} = \{f_i\}_{i\in I}$ for $\mathcal{H}$ with J-frame operator S, observe that the synthesis operator of $\mathcal{G} = \{S^{-1}f_i\}_{i\in I}$ is $V := S^{-1}T \in L(\ell_2(I), \mathcal{H})$. Furthermore, by Corollary 5.3, $S(\mathcal{M}_{\mp}^{[\perp]}) = \mathcal{M}_{\pm}$. Then, $S^{-1}(\mathcal{M}_{\pm}) = \mathcal{M}_{\mp}^{[\perp]}$ and it follows that

$$[\, S^{-1}f_i, S^{-1}f_i\,] > 0 \quad \text{if and only if} \quad [\, f_i, f_i\,] > 0.$$

Thus, $V_{\pm} = VP_{\pm} = S^{-1}T_{\pm}$ and $R(V_+)$ (resp. $R(V_-)$) is a maximal uniformly J-positive (resp. J-negative) subspace of $\mathcal{H}$. So, $\mathcal{G}$ is a J-frame for $\mathcal{H}$. $\square$

If $\mathcal{F} = \{f_i\}_{i\in I}$ is a frame for a Hilbert space $\mathcal{H}$ with synthesis operator $T \in L(\ell_2(I), \mathcal{H})$, then the family $\{(TT^*)^{-1}f_i\}_{i\in I}$ is called the *canonical dual frame* because it is a dual frame for $\mathcal{F}$ (see (12)) and it has the following optimal property: Given $f \in \mathcal{H}$,

$$\sum_{i\in I}\left|\left\langle f, (TT^*)^{-1}f_i\right\rangle\right|^2 \le \sum_{i\in I}|c_i|^2, \quad \text{whenever } f = \sum_{i\in I}c_if_i, \tag{30}$$

for a family $(c_i)_{i\in I} \in \ell_2(I)$. In other words, the above representation has the smallest ℓ_2-norm among the admissible frame coefficients representing f (see [22]).

Remark 5.5. If $\mathcal{F} = \{f_i\}_{i\in I}$ is a J-frame for $\mathcal{H}$ then $\mathcal{F}_{\pm} = \{f_i\}_{i\in I_{\pm}}$ is a frame for the Hilbert space $(\mathcal{M}_{\pm}, \pm[\ ,\ ])$. Furthermore, the frame operator associated to $\mathcal{F}_+$ is $S_+ = T_+T_+^{+}$ and its canonical dual frame is given by $\mathcal{G}_+ = \{S_+^{-1}f_i\}_{i\in I_+}$. Analogously, the frame operator associated to $\mathcal{F}_-$ is $S_- = -T_-T_-^{+}$ and its canonical dual frame is given by $\mathcal{G}_- = \{S_-^{-1}f_i\}_{i\in I_-}$.

Then, since $\mathcal{H} = \mathcal{M}_+ \dotplus \mathcal{M}_-$, $\mathcal{H}$ can be seen as the (outer) direct sum of the Hilbert spaces $(\mathcal{M}_+, [\ ,\ ])$ and $(\mathcal{M}_-, -[\ ,\ ])$, i.e. the inner product given by

$$\langle f, g\rangle_{\mathcal{F}} = [f_+, g_+] - [f_-, g_-], \quad f = f_+ + f_-, g = g_+ + g_-, f_+, g_+ \in \mathcal{M}_+, f_-, g_- \in \mathcal{M}_-,$$

turns $(\mathcal{H}, \langle\ ,\ \rangle_{\mathcal{F}})$ into a Hilbert space and the projection $Q = P_{\mathcal{M}_+//\mathcal{M}_-}$ is selfadjoint in this Hilbert space. So, if $f \in \mathcal{H}$,

$$\begin{aligned}\sum_{i\in I}|[\, f, S^{-1}f_i\,]|^2 &= \sum_{i\in I_+}|[\, Qf, S_+^{-1}f_i\,]|^2 + \sum_{i\in I_-}|[\, (I-Q)f, S_-^{-1}f_i\,]|^2 \\ &\le \sum_{i\in I_+}|c_i^+|^2 + \sum_{i\in I_-}|c_i^-|^2,\end{aligned}$$

whenever $f_+ = Qf = \sum_{i\in I_+}c_i^+f_i$ and $f_- = (I-Q)f = \sum_{i\in I_-}c_i^-f_i$, for families $(c_i^{\pm})_{i\in I_{\pm}} \in \ell_2(I_{\pm})$. Therefore,

$$\sum_{i\in I}|[\, f, S^{-1}f_i\,]|^2 \le \sum_{i\in I}|c_i|^2,$$

whenever $f = \sum_{i\in I}c_if_i$ for some $(c_i)_{i\in I} \in \ell_2(I)$. In other words, the J-frame $\mathcal{G} = \{S^{-1}f_i\}_{i\in I}$ is the canonical dual frame of $\mathcal{F}$ in the Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle_{\mathcal{F}})$.

5.2. Characterizing the J-frame operators

In a Hilbert space $\mathcal{H}$, it is well known that every positive invertible operator $S \in L(\mathcal{H})$ can be realized as the frame operator of a frame $\mathcal{F} = \{f_i\}_{i\in I}$ for $\mathcal{H}$, see [4]. Indeed, if $\mathcal{B} = \{x_i\}_{i\in I}$ is an orthonormal basis of $\mathcal{H}$, consider $T : \ell_2(I) \to \mathcal{H}$ given by $Te_i = S^{1/2}x_i$ for $i \in I$. Then, for every $f \in \mathcal{H}$,

$$TT^*f = \sum_{i\in I} \left\langle f, S^{1/2}x_i \right\rangle S^{1/2}x_i = S^{1/2}\left(\sum_{i\in I}\left\langle S^{1/2}f, x_i\right\rangle x_i\right) = Sf.$$

Therefore, $\mathcal{F} = \{S^{1/2}x_i\}_{i\in I}$ is a frame for $\mathcal{H}$ and its frame operator is given by S.

The following paragraphs are devoted to characterize the set of J-frame operators.

Theorem 5.6. *Let $S \in GL(\mathcal{H})$ be a J-selfadjoint operator acting on a Krein space $\mathcal{H}$ with fundamental symmetry J. Then, the following conditions are equivalent:*

(i) *S is a J-frame operator, i.e. there exists a J-frame $\mathcal{F}$ with synthesis operator T such that $S = TT^+$.*
(ii) *There exists a projection $Q \in \mathcal{Q}$ such that QS is J-positive and $(I - Q)S$ is J-negative.*
(iii) *There exist J-positive operators $S_1, S_2 \in L(\mathcal{H})$ such that $S = S_1 - S_2$ and $R(S_1)$ (resp. $R(S_2)$) is a uniformly J-positive (resp. J-negative) subspace of $\mathcal{H}$.*

Proof. (i) → (ii): Follows from Proposition 5.2 and Corollary 5.3.

(ii) → (iii): If there exists a projection $Q \in \mathcal{Q}$ such that QS is J-positive and $(I-Q)S$ is J-negative, consider the J-positive operators $S_1 = QS$ and $S_2 = -(I - Q)S$. Then, $S = S_1 - S_2$ and, by hypothesis, $R(S_1) = R(Q)$ is uniformly J-positive and $R(S_2) = R(I - Q) = N(Q)$ is uniformly J-negative.

(iii) → (i): Suppose that there exist J-positive operators $S_1, S_2 \in L(\mathcal{H})$ such that $S = S_1 - S_2$ and $R(S_1)$ (resp. $R(S_2)$) is a uniformly J-positive (resp. J-negative) subspace of $\mathcal{H}$. Denoting $\mathcal{K}_j = R(S_j)$ for $j = 1, 2$, observe that $A_j = S_jJ|_{\mathcal{K}_j} \in GL(\mathcal{K}_j)^+$. Therefore, there exists a frame $\mathcal{F}_j = \{f_i\}_{i\in I_j} \subset \mathcal{K}_j$ for $\mathcal{K}_j$ such that $A_j = T_jT_j^*$ if $T_j \in L(\ell_2(I_1), \mathcal{K}_j)$ is the synthesis operator of $\mathcal{F}_j$, for $j = 1, 2$.

Then, consider $\ell_2(I) := \ell_2(I_1) \oplus \ell_2(I_2)$ and $T \in L(\ell_2(I), \mathcal{H})$ given by

$$Tx = T_1x_1 + T_2x_2, \quad \text{if } x \in \ell_2(I), x = x_1 + x_2, x_j \in \ell_2(I_j) \text{ for } j = 1, 2.$$

It is easy to see that T is the synthesis operator of the frame $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. Furthermore $\mathcal{F}$ is a J-frame such that $I_+ = I_1$ and $I_- = I_2$.

Finally, endow $\ell_2(I)$ with the indefinite inner product defined by the diagonal operator $J_2 \in L(\ell_2(I))$ given by

$$J_2\, e_i = \sigma_i\, e_i,$$

where $\sigma_i = 1$ if $i \in I_1$ and $\sigma_i = -1$ if $i \in I_2$. Notice that $T_1J_2 = T_1$ and $T_2J_2 = -T_2$. Furthermore, $T_1T_2^* = T_2T_1^* = 0$ because $R(T_2^*) = N(T_2)^\perp \subseteq \ell_2(I_1)^\perp = \ell_2(I_2) \subseteq N(T_1)$. Thus,

$$TT^+ = TJ_2T^*J = (T_1 + T_2)(T_1^* - T_2^*)J = T_1T_1^*J - T_2T_2^*J = A_1J - A_2J = S_1 - S_2 = S. \quad \square$$

Given a J-frame $\mathcal{F} = \{f_i\}_{i\in I}$ for $\mathcal{H}$ with J-frame operator $S \in L(\mathcal{H})$, it follows from Corollary 5.3 that

$$S(\mathcal{M}_-^{[\perp]}) = \mathcal{M}_+ \quad \text{and} \quad S(\mathcal{M}_+^{[\perp]}) = \mathcal{M}_- \tag{31}$$

i.e. S maps a maximal uniformly J-positive (resp. J-negative) subspace into another maximal uniformly J-positive (resp. J-negative) subspace. The next proposition shows under which hypotheses the converse holds.

Proposition 5.7. *Let $S \in GL(\mathcal{H})$ be a J-selfadjoint operator. Then, S is a J-frame operator if and only if the following conditions hold:*

(i) *there exists a maximal uniformly J-positive subspace $\mathcal{T}$ of $\mathcal{H}$ such that $S(\mathcal{T})$ is also maximal uniformly J-positive;*
(ii) *$[\,Sf, f\,] \geq 0$ for every $f \in \mathcal{T}$;*
(iii) *$[\,Sg, g\,] \leq 0$ for every $g \in S(\mathcal{T})^{[\perp]}$.*

Proof. If S is a J-frame operator, consider $\mathcal{T} = \mathcal{M}_-^{[\perp]}$ which is a maximal uniformly J-positive subspace $\mathcal{T}$ of $\mathcal{H}$. Then, $S(\mathcal{T}) = \mathcal{M}_+$ is also maximal uniformly J-positive. Furthermore,

$$[\,Sf, f\,] = [\,SQ^+f, Q^+f\,] = [\,QSQ^+f, f\,] = [\,S_+f, f\,] \geq 0 \quad \text{for every } f \in \mathcal{T},$$

where $Q = P_{\mathcal{M}_+//\mathcal{M}_-}$. Also, $S(\mathcal{T})^{[\perp]} = \mathcal{M}_+^{[\perp]} = N(Q^+) = R((I - Q)^+)$. So,

$$[\,Sg, g\,] = [\,S(I - Q)^+g, (I - Q)^+g\,] = [\,(I - Q)S(I - Q)^+g, g\,] = [\,-S_-g, g\,] \leq 0 \quad \text{for every } g \in S(\mathcal{T})^{[\perp]}.$$

Conversely, suppose that there exists a maximal uniformly J-positive subspace $\mathcal{T}$ satisfying the hypotheses. Let $\mathcal{M} = S(\mathcal{T})$, which is maximal uniformly J-positive. Then, consider $Q = P_{\mathcal{M}//\mathcal{T}^{[\perp]}}$. It is well defined because $\mathcal{T}^{[\perp]}$ is maximal uniformly J-negative, see [11, Corollary 1.5.2]. Moreover, $Q \in \mathcal{Q}$.

Notice that $R(S(I-Q)^+) = S(\mathcal{M}^{[\perp]}) = S(S(\mathcal{T})^{[\perp]}) = S(S^{-1}(\mathcal{T}^{[\perp]})) = \mathcal{T}^{[\perp]}$. Therefore, $QS(I-Q)^+ = 0$ and

$$QS = QSQ^+ + QS(I-Q)^+ = QSQ^+.$$

Furthermore, if $[\,Sf, f\,] \geq 0$ for every $f \in \mathcal{T}$ then QS is J-positive. Analogously, if $[\,Sg, g\,] \leq 0$ for every $g \in S(\mathcal{T})^{[\perp]}$ then $(I-Q)S$ is J-negative. Then, by Theorem 5.6, S is a J-frame operator. $\square$

As it was proved in Proposition 5.2, if an operator $S \in L(\mathcal{H})$ is a J-frame operator then it is an invertible J-selfadjoint operator satisfying $\mathrm{ind}_{\pm}(S) = \dim(\mathcal{H}_{\pm})$. Unfortunately, the converse is not true.

Example 5.8. Consider the Krein space obtained by endowing $\mathbb{C}^2$ with the sesquilinear form

$$[(x_1, x_2), (y_1, y_2)] = x_1\overline{y_1} - x_2\overline{y_2},$$

and the invertible J-selfadjoint operator S, whose matrix in the standard orthonormal basis is given by

$$S = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Then, S satisfies $\mathrm{ind}_{\pm}(S) = \dim(\mathcal{H}_{\pm})$, but it maps each J-positive vector into a J-negative vector. Then, by Proposition 5.7, S cannot be a J-frame operator.

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