SPARSE NULLSTELLENSATZ, RESULTANTS AND DETERMINANTS OF COMPLEXES

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Abstract. We refine and extend a result by Tuitman on the supports of a Bézout identity satisfied by a finite sequence of sparse Laurent polynomials without common zeroes in the toric variety associated to their supports. When the number of these polynomials is one more than the dimension of the ambient space, we obtain a formula for computing the sparse resultant as the determinant of a Koszul type complex.

1. Introduction and statement of the main results

Given a field K, and a finite sequence $f_1, \ldots, f_k$ in $R_K := K[t_1, \ldots, t_n]$, the classical *Hilbert's Nullstellensatz* ([Hil93]) states that the ideal generated by these elements is the whole ring R_K if and only if the variety defined by them in the affine space $\mathbb{A}^n_{\overline{K}}$ over $\overline{K}$, the algebraic closure of K, is empty. The *effective Nullstellensatz* deals with the problem of giving an algorithmic procedure to decide if this is the case, and if so to compute a *Bézout identity*

$$\sum_{i=1}^{k} g_i f_i = 1 \tag{1.1}$$

from the input sequence $f_1, \ldots, f_k$. Note that if degree bounds for the g_i's are provided, then (1.1) turns into an easy to verify Linear Algebra routine. This was the original approach of Grete Hermann in [Her26], and a lot of improvement has been done since then, see for instance [DKS13] and the references therein.

A first attempt to deal with this problem can be made by homogenizing the input, i.e. considering $F_1, \ldots, F_k \in \tilde{R}_K := K[t_0, \ldots, t_n]$, which are the homogenizations of $f_1, \ldots, f_k$ up to their total degrees $d_1, \ldots, d_k$ respectively. We can then study the Koszul complex associated to this sequence:

$$Kos(\tilde{R}_K, F_1, \ldots, F_k): \ 0 \to \ \tilde{R}_K^{\binom{k}{k}} \overset{\Psi_k}{\to} \tilde{R}_K^{\binom{k}{k-1}} \overset{\Psi_{k-1}}{\to} \ldots \overset{\Psi_2}{\to} \tilde{R}_K^{\binom{k}{1}} \overset{\Psi_1}{\to} \tilde{R}_K^{\binom{k}{0}} \to 0, \tag{1.2}$$

where the maps Ψ_i are defined by setting:

$$\Psi_i(e_{j_1,\ldots,j_i}) := \sum_{\ell=1}^{i} (-1)^{\ell+1} F_{j_\ell}\, e_{j_1,\ldots,\check{j_\ell},\ldots,j_i}, \tag{1.3}$$

and we denote the standard basis of the free $\tilde{R}_K$-module $\tilde{R}_K^{\binom{k}{i}}$ with $\left(e_{j_1,\ldots,j_i}\right)_{1\leq j_1<\ldots<j_i\leq k}$. Note that $\Psi_1 = (F_1, \ldots, F_k)$. As this sequence is homogeneous, the maps in the complex (1.2) are also graded as well, and hence we can consider the different graded pieces of (1.2). For a non-negative integer d, the d-th graded part of $Kos(\tilde{R}_K, F_1, \ldots, F_k)$

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(the one which has degree d in $\tilde{R}_K^{\binom{k}{0}}$ above) will be denoted by $Kos(\tilde{R}_K, F_1, \ldots, F_k)_d$. The following result is well-known, see Theorem 5.1 in [DD01] for the case $k = n+1$. The general case can be reduced to this one by using standard properties of Koszul complexes of sets containing regular sequences (cf. [BH93, Corollary 1.6.13]).

Theorem 1.1. *Let $d \geq d_1 + \ldots + d_k - n$. The following are equivalent:*

(1) *there are no common zeroes of $F_1, \ldots, F_k$ in projective space $\mathbb{P}^n_{\overline{K}}$;*
(2) *$Kos(\tilde{R}_K, F_1, \ldots, F_k)_d$ is exact as a complex of K-vector spaces;*
(3) *the rightmost nontrivial map of $Kos(\tilde{R}_K, F_1, \ldots, F_k)_d$ is onto.*

Theorem 1.1 provides an optimal bound (if $d < d_1 + \ldots + d_k - n$ the result does not hold for $k = n+1$, see for instance Theorem 1.5.16 in [CD05a]) on the degrees of the g_i's which implies a Bézout identity as in (1.1) at the cost of requiring the input not to have zeroes not only in affine space but in a compactification of it, the n-th dimensional projective space $\mathbb{P}^n_{\overline{K}}$. Indeed from the exactness of $Kos(\tilde{R}_K, F_1, \ldots, F_k)_d$ one can find homogeneous polynomials $G_1, \ldots, G_k \in \tilde{R}_K$ such that $\sum_i G_i F_i = t_0^d$. By setting $t_0 = 1$, we recover $g_1, \ldots, g_k$ such that $\deg(g_i) \leq d_1 + \ldots + d_k - n - d_i$, satisfying (1.1), and hence solving the effective Nullstellensatz problem.

Remark 1.2. As we are dealing with homogeneous polynomials and their zeroes in projective space, for the sequence $F_1, \ldots, F_k$ to have no common zeroes, by Krull's Hauptidealsatz (cf. Theorem 10.2 in [Eis95]) we are necessarily forced to have $k \geq n+1$.

Remark 1.3. The inequality $d \geq d_1 + \ldots + d_k - n$ is also sharp, as it is also well-known (see for instance Theorem 1 in Section 5.11 of [AGV85]) that if $k = n+1$, the Jacobian of $F_1, \ldots, F_{n+1}$ has degree $d_1 + \ldots + d_{n+1} - n - 1$, and does not belong to the ideal generated by the input polynomials if they define the empty variety in $\mathbb{P}^n_{\overline{K}}$, hence the rightmost nontrivial map of this graded piece of $Kos(\tilde{R}_K, F_1, \ldots, F_{n+1})$ cannot be onto.

Remark 1.4. The strongest hypothesis of requiring $F_1, \ldots, F_k$ not having common zeroes in $\mathbb{P}^n_{\overline{K}}$ actually forces very low bounds for the degrees of the g_i's in the Nullstellensatz. In general it is expected that $\deg(g_j) \leq \prod_{i=1}^k d_i - d_j$, see for instance Theorem 1 in [DKS13], and this bound is sharp. This shows that the approach to the Nullstellensatz via homogenization is far from being enough and sharp, and extra work should be done to deal with this problem in general.

Of special interest is the so-called "codimension one" case, i.e. when $k = n+1$. Here, tools from Elimination Theory like resultants can be used to deal with this problem. To introduce them, consider the following generic homogeneous polynomials in $n+1$ variables $t_0, \ldots, t_n$ of respective positive degrees $d_1, \ldots, d_{n+1}$:

$$F_i^g := \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n+1},\, |\alpha| = d_i} c_{i,\alpha} t^\alpha, \; i = 1, \ldots, n+1, \tag{1.4}$$

with each of the $c_{i,\alpha}$ being a new indeterminate. Set $\mathbb{K} := \mathbb{Q}(c_{i,\alpha})$. We easily note that the sequence $F_1^g, \ldots, F_{n+1}^g$ does not have common zeroes in $\mathbb{P}^n_{\overline{\mathbb{K}}}$. Hence, the complex $Kos(\tilde{R}_{\mathbb{K}}, F_1^g, \ldots, F_{n+1}^g)_d$ is exact for $d \geq d_1 + \ldots + d_{n+1} - n$ thanks to Theorem 1.1. In the special case $n = 1$, $d = d_1 + d_2 - 1$, (1.2) reduces to only one nontrivial map between $\mathbb{K}$-vector spaces of the same dimension $d_1 + d_2$, whose determinant equals the classical Sylvester resultant of the homogeneous bivariate forms F_1 and F_2, see [CLO05, Chapter 3 §1]. This resultant can be generalized for $n > 1$: given positive degrees $d_1, \ldots, d_{n+1}$, there is an irreducible polynomial called $\mathrm{Res}_{d_1,\ldots,d_{n+1}} \in \mathbb{Z}[c_{i,\alpha}]$ called the resultant of $F_1^g, \ldots, F_{n+1}^g$, which vanishes after a specialization of the $c_{i,\alpha}$ in a field K if and only if the specialized system arising from (1.4) after this specialization has a solution in the projective space over $\overline{K}$, see [CLO05, Chapter 3 §2] for its proper definition and basic properties.

In another direction, the determinant of a linear map among spaces of the same dimension with fixed bases can be generalized to the determinant of a finite exact complex of K-vector spaces of finite dimension, fixing bases in each of the nonzero terms of it. In practice, it involves the computation of maximal minors of all the nontrivial matrices representing the differentials in the aforementioned bases. An introduction to this topic can be found in [Del87] but this concept goes also all the way back to Cayley, see for instance [GKZ94, Appendix A]. Computational aspects can be found in [Dem84, Cha93], we refer the reader to Section 2.4 for more.

The following is a classical result which generalizes the computation of the Sylvester resultant of two univariate polynomials by computing the determinant of a matrix. See [Cha93] and also Theorem 5.1 in [DD01] for proofs. In our case, all the bases that we are going to consider to compute determinants are the monomial bases of each of the terms in the complex.

Theorem 1.5. *For $d \geq d_1 + \ldots + d_{n+1} - n$, we have that:*

(1) *Up to a sign, the determinant of the complex $Kos(\tilde{R}_{\mathbb{K}}, F_1^g, \ldots, F_{n+1}^g)_d$ is equal to* $\mathrm{Res}_{d_1,\ldots,d_{n+1}}$.
(2) *This determinant can be computed as the* gcd *of the maximal minors of the d-th graded piece of Ψ_1, the rightmost nontrivial map in $Kos(\tilde{R}_{\mathbb{K}}, F_1^g, \ldots, F_{n+1}^g)_d$.*
(3) *Let K be any field. For homogeneous polynomials $F_1, \ldots, F_{n+1} \in K[t_0, \ldots, t_n]$ of respective degrees $d_1, \ldots, d_{n+1}$, the complex $Kos(\tilde{R}_K, F_1, \ldots, F_{n+1})_d$ is exact if and only if* $\mathrm{Res}_{d_1,\ldots,d_{n+1}}(F_1, \ldots, F_{n+1}) \neq 0$.

In this paper, we are going to present extensions of Theorems 1.1 and 1.5 to the sparse setting. To do this, we start with finite sets $\mathcal{A}_1, \ldots, \mathcal{A}_k \subset \mathbb{Z}^n$. For a field K of characteristc zero, and $i = 1, \ldots, k$, consider polynomials

$$f_i := \sum_{b \in \mathcal{A}_i} f_{i,b} t^b \in K[t_1^{\pm 1}, \ldots, t_n^{\pm 1}], \tag{1.5}$$

with $f_{i,b} \in K^* := K \setminus \{0\}$. We say that $\mathcal{A}_i$ is the *support* of f_i, $i = 1, \ldots, k$. For short we denote with $K[t^{\pm 1}]$ the ring of Laurent polynomials $K[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. The torus $(\overline{K}^*)^n$ will be denoted with $\mathbb{T}^n_{\overline{K}}$. It is the natural environment to look for common zeroes of (1.5).

Set $P_i := \mathrm{conv}(\mathcal{A}_i) \subset \mathbb{R}^n$ for the convex hull of $\mathcal{A}_i$, $i = 1, \ldots, k$. It is an *integral* polytope, meaning that all its vertices are in $\mathbb{Z}^n$. We further set

$$P := P_1 + \ldots + P_k, \tag{1.6}$$

where the operation in the right hand side is the Minkowski sum of polytopes, as defined in [CLO05, Chapter 7]. If P is n-dimensional, there exists a *toric variety* X_P containing $\mathbb{T}^n_{\overline{K}}$ as a dense subset, and a polynomial ring S having as many variables as facets of P, and such that $f_1, \ldots, f_k$ get "homogenized" to $F_1, \ldots, F_k$, elements of a polynomial ring S to be described later. Homogeneous here means with respect to the grading given by the Chow group of X_P (see (2.12) to get the homogenization formula) which satisfies that $\xi \in \mathbb{T}^n_{\overline{K}}$ is a common zero of the f_i's if and only if it is a common zero of the F_i's.

To illustrate our results and help the reader digest the technical background, we consider the following example which we will carry out all along the paper.

Example 1.6. *Set $n = 2$, and $k = 3$. We will simplify the subindexes to avoid an overuse of notation. Let*

$$f_1 := f_{10} + f_{11}t_1t_2, \quad f_2 := f_{20} + f_{21}t_1^2t_2 + f_{22}t_1t_2^2, \quad f_3 := f_{30} + f_{31}t_1^2t_2 + f_{32}t_2.$$

In this case, we have

$$\mathcal{A}_1 = \{(0,0), (1,1)\}, \quad \mathcal{A}_2 = \{(0,0), (2,1), (1,2)\}, \quad \mathcal{A}_3 = \{(0,0), (2,1), (0,1)\},$$

and their convex hulls P_1, P_2 and P_3 are a segment and two triangles in $\mathbb{R}^2$, respectively. The Minkowski sum $P = P_1 + P_2 + P_3$ is the heptagon in $\mathbb{R}^2$ with vertices $(0,0)$, $(0,1)$, $(1,3)$, $(2,4)$, $(4,4)$, $(5,3)$, and $(4,2)$, see Figure 1.

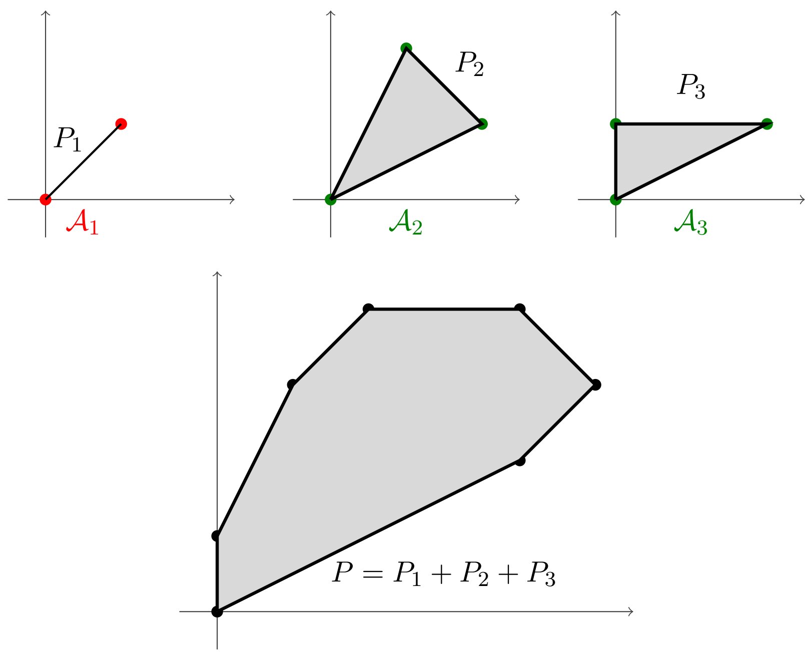


FIGURE 1. Supports, convex hulls and Miknowski sum of the polynomials in Example 1.6.

To deal with a more general situation, assume that we are given another integral polytope $Q \subset \mathbb{R}^n$. Associated to the Minkowski sum $P + Q$ there is a toric variety X_{P+Q} "lying over" X_P, with its corresponding homogeneous coordinate ring S' and hence $f_1, \ldots, f_k$ can also be homogenized to $F'_1, \ldots, F'_k \in S'$. When Q is a single point, we recover the variety X_P so this case covers the previous one and extends to a more general setup.

For each of the facets of $P + Q$ there is an invariant Weil divisor associated to it. We will denote by $D_1, \ldots, D_{n+r}$ these divisors. As in Canny and Emiris' matrix construction for the resultant (cf. [CE93]), we fix a vector $\delta \in \mathbb{Q}^n$. Associated to it we define the divisor $D_\delta := \sum_j -D_j$, the sum being over all the divisors associated to facets whose inner normal η_j satisfies $\langle \eta_j, \delta\rangle > 0$. If there are no such facets, then D_δ is set equal to zero. Let α_δ be the degree of D_δ. Each of the F_i' has a degree α_{P_i} associated to its Newton polytope P_i. Set $\alpha_P = \sum_{i=1}^k \alpha_{P_i}$, and α_Q the degree associated to the polytope Q in X_{P+Q}.

As in the homogeneous case, we can consider the Koszul complex $Kos(S', F_1', \ldots, F_k')$, and its graded pieces determined by the Chow group of X_{P+Q}. The following is our generalization of Theorem 1.1 to the sparse setting:

Theorem 1.7. *Let K be a field of characteristic zero, $\mathcal{A}_1, \ldots, \mathcal{A}_k \subset \mathbb{Z}^n$ finite sets, $f_1, \ldots, f_k$ as in* (1.5)*, and P as in* (1.6)*. If P is an n-dimensional polytope, the following are equivalent:*

1. *there are no common zeroes of $F_1, \ldots, F_k$ in X_P, where F_i is the homogenization of f_i explained above and defined properly in* (2.12)*;*
2. *$Kos(S', F_1', \ldots, F_k')_{\alpha_\delta+\alpha_P+\alpha_Q}$ is exact as a complex of K-vector spaces;*
3. *the rightmost nontrivial map of $Kos(S', F_1', \ldots, F_k')_{\alpha_\delta+\alpha_P+\alpha_Q}$ is onto.*

Remark 1.8. The hypothesis of P being n-dimensional can be dropped by the simple observation that if the dimension of this polytope is smaller, then one can reparametrize (1.5) with fewer than n variables.

From Theorem 1.7 we derive the following sharp sparse Nullstellensatz for polynomials without zeroes in X_P.

Theorem 1.9. *Let K be a field of characteristic zero, $\mathcal{A}_1, \ldots, \mathcal{A}_k \subset \mathbb{Z}^n$ finite sets, $f_1, \ldots, f_k$ as in* (1.5)*, and P as in* (1.6)*. If $\delta \in \big((-1,1)\cap\mathbb{Q}\big)^n$, and P is an n-dimensional polytope, for any integral polytope Q, the following are equivalent:*

1. *there are no common zeroes of $F_1, \ldots, F_k$ in X_P;*
2. *for all $g \in K[t^{\pm 1}]$ supported in $(P+Q+\delta)\cap\mathbb{Z}^n$, there exist $g_1, \ldots, g_k \in K[t^{\pm 1}]$ supported in $(P_1 + \ldots + P_{i-1} + P_{i+1} + \ldots + P_k + Q + \delta) \cap \mathbb{Z}^n$, $1 \le i \le k$ respectively, such that*

$$g = g_1 f_1 + \ldots + g_k f_k. \tag{1.7}$$

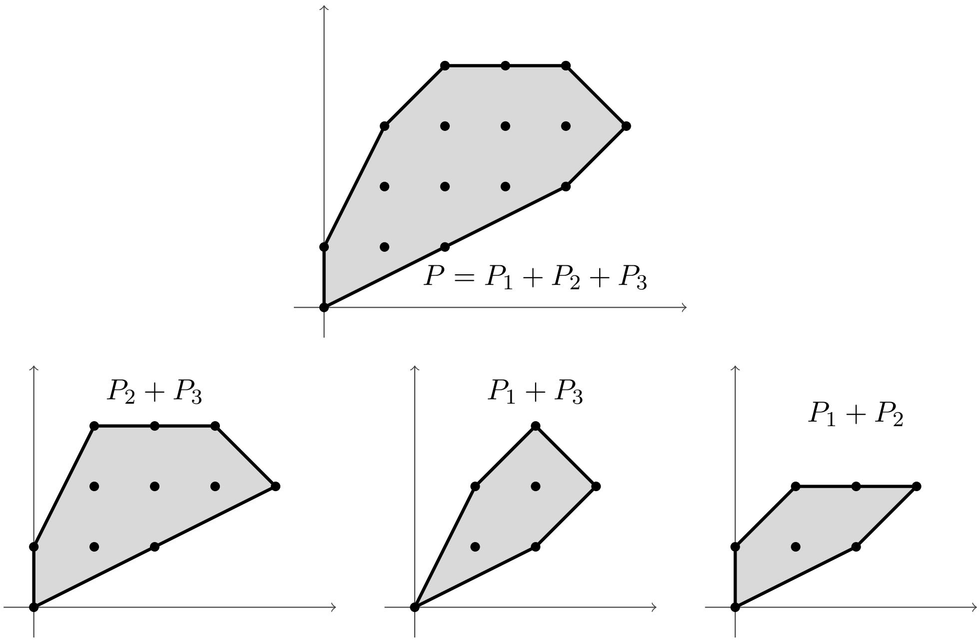
Note that if $\delta \notin \mathbb{Z}^n$, then for any integral polytope R we have that $\#(R+\delta)\cap\mathbb{Z}^n < \#R\cap\mathbb{Z}^n$, so the addition of δ above actually shrinks the supports in the Bezout identity (1.7).

There are several "sparse Nullstellensatze without zeroes in X_P" in the literature with a similar flavour than Theorem 1.9. To mention some, Theorem 3 in [CDV06] presents a Nullstellensatz in the same conditions for $k = n+1$, all the P_i's being equal to the same n-dimensional polytope, $Q = \{\mathbf{0}\}$, and $\delta = (0, \ldots, 0)$. With different motivation and techniques, Theorem 1.2 in [Wul11] deals with the case of polynomials in $\mathbb{C}[t_1, \ldots, t_n]$ having the proper codimension in the toric variety and no components at infinity. Their result holds for a "large poltyope" P_0 containing all the vectors of the standard basis of $\mathbb{Z}^n$, and the support of each of the f_i's. So, in general larger than our P.

The closest to our main result is Theorem 1.3 in [Tui11], where the claim is proven with our hypothesis for the case $Q = \{\mathbf{0}\}$ and $\delta = (0, \ldots, 0)$. Theorem 1.9 above represents an improvement over this result. For instance, in our running Example 1.6, the polynomials g_1, g_2, g_3 in (1.7) have supports of cardinality 11, 7, 7 respectively if one uses Tuitman's results. From our results above, these cardinalities can shrink to $7, 5, 4$ if $\delta = (\frac{1}{2}, \frac{1}{2})$, and 8, 4, 4 if $\delta = (0, -\frac{1}{2})$, see below.

Moreover, the complex $Kos(S', F'_1, \ldots, F'_k)_{\alpha_\delta + \alpha_P + \alpha_Q}$ from Theorem 1.7 has the following shape for $\delta = (0, 0)$, and $Q = \{\mathbf{0}\}$ (this follows also from Tuitman's, see Figure 1):

$$0 \to K \to K^2 \oplus K^4 \oplus K^4 \to K^{11} \oplus K^7 \oplus K^7 \to K^{16} \to 0. \tag{1.8}$$



Theorem 1.7 above shrinks the size of this complex considerably. For instance we have for $\delta = (\frac{1}{2}, \frac{1}{2})$ and $Q = \{\mathbf{0}\}$, that $Kos(S', F'_1, \ldots, F'_k)_{\alpha_\delta + \alpha_P + \alpha_Q}$ is isomorphic to the following complex of K-vector spaces, see Figure 2:

$$0 \to K \oplus K^2 \oplus K \to K^7 \oplus K^5 \oplus K^4 \to K^{12} \to 0, \tag{1.9}$$

while for $\delta = (0, -\frac{1}{2})$, we obtain (see Figure 3):

$$0 \to K^2 \oplus K^2 \to K^8 \oplus K^4 \oplus K^4 \to K^{12} \to 0. \tag{1.10}$$

In all these complexes (1.8), (1.9), and (1.10), the maps are of "Koszul type" as defined in (1.3). For instance, the rightmost nontrivial map in the three of them corresponds to an application of the form $(g_1, g_2, g_3) \mapsto g_1 f_1 + g_2 f_2 + g_3 f_3$.

Our results are sharp also in terms of the size of the set of exponents, as it is known that if one actually replaces D_δ with $\sum_{j=1}^{n+r} -D_j$, then the corresponding graded piece of the last map of $Kos(S', F'_1, \ldots, F'_k)_{\alpha_\delta + \alpha_P + \alpha_Q}$ is never onto except for very degenerate situations. This is known if $k = n+1$ and all the polytopes P_i being $n-$th dimensional (cf. [CD05b]), and can be confirmed also in our running Example 1.6, as by counting the number of lattice points in the interior of the polytopes produces a complex of the form

$$0 \to K \to K^4 \oplus K^2 \oplus K \to K^7 \to 0,$$

which can never be exact as the alternating sum of the dimensions above is not zero.

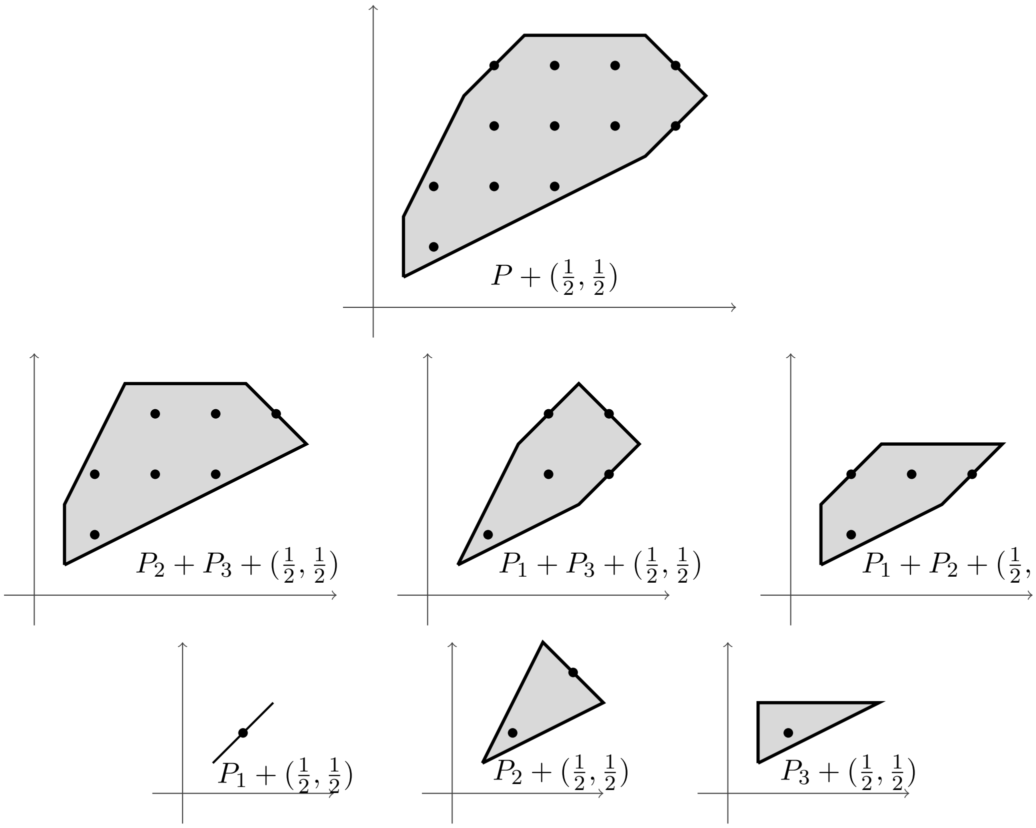


FIGURE 2. Supports of the monomials appearing in Example 1.6 with $\delta = (\frac{1}{2}, \frac{1}{2})$.

It should be emphasized, however, that general bounds for the sparse Nullstellensatz are known (see for instance [Som99]) and they are larger in size and nature than those presented in Theorem 1.9. As in Remark 1.4, imposing stronger conditions on the input system is an advantage, but extra ideas are needed to deal with the general sparse Nullstellensatz than those treated here.

To generalize Theorem 1.5, we need to consider generic families of $n+1$ polynomials with supports in $\mathcal{A}_1, \dots, \mathcal{A}_{n+1}$ respectively:

$$f_i^g := \sum_{b \in \mathcal{A}_i} c_{i,b} t^b, \; 1 \leq i \leq n+1. \tag{1.11}$$

with $c_{i,b}$ being new indeterminates. As before, here we denote with $\mathbb{K}$ the field $\mathbb{Q}(c_{i,b})$, and with $S'_{\mathbb{K}} := \mathbb{K}[x_1, \dots, x_{n+r}]$, the homogeneous coordinate ring of X_{P+Q}.

Associated with these data, there is a *sparse resultant* ([GKZ94, CLO05, DS15]) which we will denote with $\mathrm{Res}_{\mathcal{A}_1,\dots,\mathcal{A}_{n+1}} \in \mathbb{Z}[c_{i,b}]$, and it will be properly defined in Section 2.3. The main property of this polynomial is that if it is not identically 1, then it vanishes under a specialization of $f_i^g \mapsto f_i \in K[t^{\pm 1}]$ if and only if the homogenized system $F_1, \dots, F_{n+1} \in S$ made from $f_1, \dots, f_{n+1}$ with respect to $P_1, \dots, P_{n+1}$ respectively as in (2.12) has a nontrivial solution in X_P.

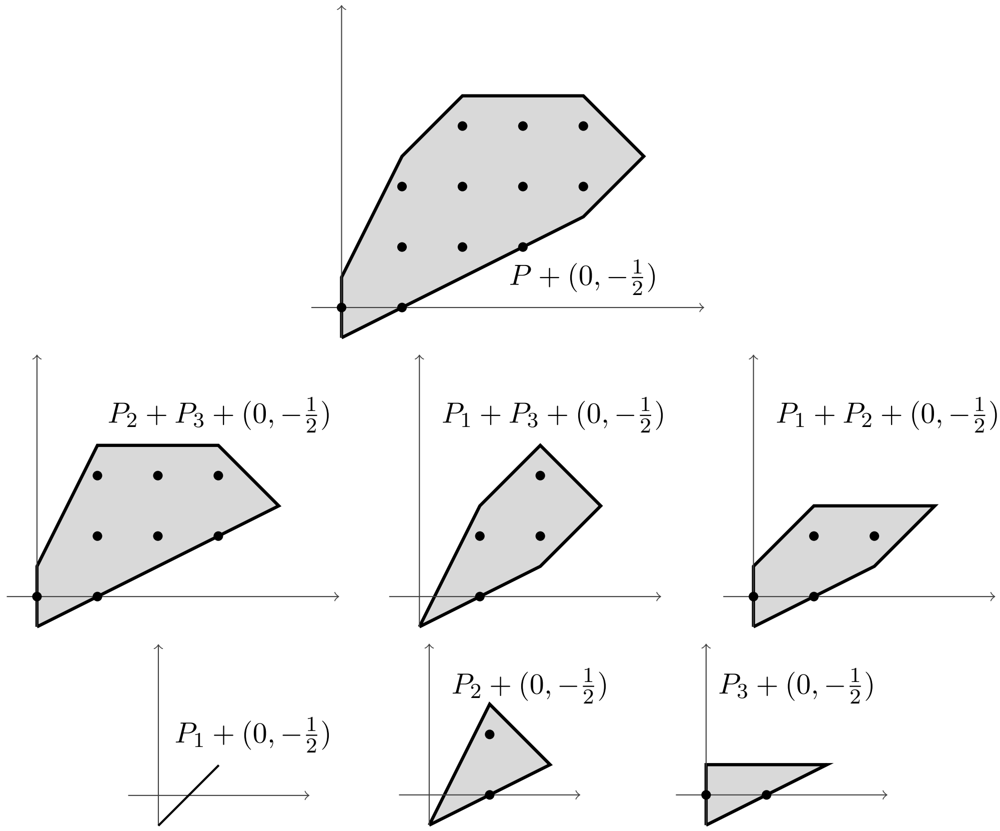


FIGURE 3. Supports of the monomials appearing in Example 1.6 with $\delta = (0, -\frac{1}{2})$.

Example 1.10. *For our running Example 1.6, we have that, up to a sign,*

$$\begin{array}{rcl} \mathrm{Res}_{\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3} & = & -f_{10}f_{11}^3f_{21}f_{22}f_{30}^2 + f_{10}f_{11}^3f_{20}f_{22}f_{30}f_{31} + f_{10}^4f_{22}^2f_{31}^2 - f_{11}^4f_{20}f_{21}f_{30}f_{32} \\ & & +f_{11}^4f_{20}^2f_{31}f_{32} + 2f_{10}^3f_{11}f_{21}f_{22}f_{31}f_{32} + f_{10}^2f_{11}^2f_{21}^2f_{32}^2. \end{array} \tag{1.12}$$

To avoid an excess of notation, we denote with $F_1^g, \ldots, F_{n+1}^g \in S'_{\mathbb{K}}$ the homogeneizations of (1.11) to the coordinate ring of X_{P+Q}.

For any degree β, we have that $Kos(S'_{\mathbb{K}}, F_1^g, \ldots, F_{n+1}^g)_\beta$ is a complex of finite dimensional $\mathbb{K}$-vector spaces, each of them either being trivial or having a monomial basis. When this complex is exact, its determinant will be taken with respect to these monomial bases.

Theorem 1.11. *Let K be a field of characteristic zero, $\mathcal{A}_1, \ldots, \mathcal{A}_k \subset \mathbb{Z}^n$ finite sets, P as in* (1.6)*, $f_1^g, \ldots, f_{n+1}^g$ be as in* (1.11)*, and $F_1^g, \ldots, F_{n+1}^g$ their respective P_i-homogenizations, $1 \leq i \leq n+1$. If P is n-dimensional, Q any integral polytope, and $\delta \in \mathbb{Q}^n$, then $Kos(S'_{\mathbb{K}}, F_1^g, \ldots, F_{n+1}^g)_{\alpha_\delta+\alpha_P+\alpha_Q}$ is exact as a complex of $\mathbb{K}$-vector spaces. Moreover:*

(1) *Up to a nonzero factor in $\mathbb{Q}$, the determinant of $Kos(S'_{\mathbb{K}}, F_1^g, \ldots, F_{n+1}^g)_{\alpha_\delta+\alpha_P+\alpha_Q}$ is equal to* $\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$.

(2) *Up to a sign,* $\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$ *is equal to the* gcd *in* $\mathbb{Z}[c_{i,b}]$ *of the maximal minors of the rightmost nontrivial map of $Kos(S'_{\mathbb{K}}, F_1^g, \ldots, F_{n+1}^g)_{\alpha_\delta+\alpha_P+\alpha_Q}$.*

(3) *Let K be any field. For sparse polynomials $f_1,\dots,f_{n+1}\in K[t^{\pm 1}]$ with supports $\mathcal{A}_1,\dots,\mathcal{A}_{n+1}$ respectively, such that* $\mathrm{Res}_{\mathcal{A}_1,\dots,\mathcal{A}_{n+1}}\neq 1$, *denote with $F'_1,\dots,F'_{n+1}$ the homogeneizations of these polynomials as in* (2.12). *Then, the complex $Kos(S'_K,F'_1,\dots,F'_{n+1})_{\alpha_\delta+\alpha_P+\alpha_Q}$ is exact if and only if* $\mathrm{Res}_{\mathcal{A}_1,\dots,\mathcal{A}_{n+1}}(f_1,\dots,f_{n+1})\neq 0$.

The degree α_δ in the grading of the Koszul complex in Theorem 1.9 generalises the "$-n$" in the bound $d_1+\ldots+d_{n+1}-n$ appearing in Theorem 1.5. In addition, all Canny-Emiris type matrices ([CE93, DJS23]) for the resultant can be regarded as non singular maximal square submatrices of the rightmost nontrivial map in $Kos(S'_{\mathbb{K}},F_1^g,\dots,F_{n+1}^g)_{\alpha_\delta+\alpha_P}$, for a suitable generic $\delta\in\mathbb{Q}^n$. The effective degree α_Q added corresponds to another polytope Q, which also produces a new family of generalized Canny-Emiris type matrices as it was described in [DJS23, Remark 4.28].

Example 1.12. *Continuing with our running Example 1.6, we have that the determinants of the complexes* (1.9) *and* (1.10) *are equal to* $\mathrm{Res}_{\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3}$ *described in* (1.12). *It is well-known that nonzero maximal minors for the rightmost nontrivial map of these two complexes can be obtained "a la Canny-Emiris" as explained in* [CE93, DJS23]. *From Theorem 1.11 we now deduce that* $\mathrm{Res}_{\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3}$ *is the* gcd *of all these maximal minors.*

It should be also mentioned that Theorem 1.11 gives a positive answer to Remark 4.20 in [DJS23] and it is independent of the results in that paper. So, it provides a simplification to the approach of proving Canny-Emiris' conjecture for computing $\mathrm{Res}_{\mathcal{A}_1,\dots,\mathcal{A}_{n+1}}$ as a quotient of two determinants.

The paper is organized as follows: in Section 2 we will review basic results on toric varieties, sparse resultants, and determinants of complexes which will be of use in our proofs. We then turn to Section 3 to prove Theorems 1.7 and 1.9, and finish with the proof of Theorem 1.11 in Section 4.

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2. Background and useful tools

2.1. **Toric Varieties.** All the terminology and most of the results that we are going to use here are borrowed from [CLS11]. For simplicity and abuse of notation, we will assume in this section that K is an algebraically closed field of characteristic zero (what was $\overline{K}$ in the previous one). Let X be an n-dimensional *complete toric variety.* As such, there is an n-dimensional lattice N, and X is determined by a fan Σ in $N_{\mathbb{R}}:=N\otimes\mathbb{R}$ such that $|\Sigma|=N_{\mathbb{R}}$. Let M be the dual lattice of N, and denote with

$\eta_1, \ldots, \eta_{n+r}$ $(r \geq 1)$ the primitive generators in N of the 1-dimensional cones in Σ. We have then that r is the rank of the *Chow group* $A_{n-1}(X)$, which can be represented as the cokernel of the morphism

$$\begin{array}{ccl} M & \to & \mathbb{Z}^{n+r} \\ m & \to & (\langle m, \eta_1 \rangle, \ldots, \langle m, \eta_{n+r} \rangle). \end{array} \tag{2.1}$$

Let $S := K[x_1, \ldots, x_{n+r}]$ be the *homogeneous coordinate ring* of X, as defined in [Cox95]. As its name suggests, one can regard $x_1, \ldots, x_{n+r}$ as (homogeneous) coordinates of X as follows: for each $\sigma \in \Sigma$, we set $x^{\widehat{\sigma}} := \prod_{\eta_j \notin \sigma} x_j$. Set

$$Z = V(x^{\widehat{\sigma}},\, \sigma \in \Sigma) \subset K^{n+r}. \tag{2.2}$$

Each point $\xi \in K^{n+r} \setminus Z$, corresponds to a point $x_\xi \in X$. It can be shown that the map $K^{n+r} \setminus Z \to X$ which sends ξ into x_ξ is such that makes X a "quotient" of $K^{n+r} \setminus Z$ by the action of an r-dimensional group G, see [Cox95] for more details.

As any n-dimensional toric variety, X has the torus $\mathbb{T}_K^n$ as an open dense subvariety, with coordinate ring $K[t^{\pm 1}]$, and the "trivial" fan $\{\mathbf{0}\}$. These coordinates are related to those given by $x_1, \ldots, x_{n+r}$ via

$$t_i = \prod_{j=1}^{n+r} x_j^{\langle \eta_j, e_i \rangle},\ i = 1, \ldots, n. \tag{2.3}$$

From this we mean that given $\boldsymbol{t} = (t_1, \ldots, t_n) \in \mathbb{T}_K^n$, there exist $(x_1, \ldots, x_{n+r}) \in K^{n+r} \setminus Z$ satisfying (2.3). So, there is a point in X, the quotient of this set via the group G, representing $\boldsymbol{t}$. Note that all the x_i's are actually in $K \setminus \{0\}$ so any point of $\mathbb{T}_K^n$ is actually lifted to a point in $\mathbb{T}_K^{n+r} \subset K^{n+r} \setminus Z$.

Each variable x_j of S corresponds to a generator η_j and hence to a torus-invariant irreducible divisor D_j of X. We will denote with $[D_j]$ the class of D_j in $A_{n-1}(X)$, $j = 1, \ldots, n+r$. As in [Cox95] we grade S in such a way that the monomial $\prod_{j=1}^{n+r} x_j^{a_j}$ has degree $\sum_{j=1}^{n+r} a_j [D_j] \in A_{n-1}(X)$. From (2.1) it is clear that $\prod_{j=1}^{n+r} x_j^{a_j}$ and $\prod_{j=1}^{n+r} x_j^{b_j}$ have the same degree if and only if there is $m \in M$ such that

$$b_j = \langle m, \eta_j \rangle + a_j,\ j = 1, \ldots, n+r. \tag{2.4}$$

Set $M_{\mathbb{R}} := M \otimes \mathbb{R}$. Associated to a divisor $D = \sum_{j=1}^{n+r} a_j D_j$, we can define the polytope

$$P_D := \{u \in M_{\mathbb{R}} :\ \langle u, \eta_j \rangle \geq -a_j,\ 1 \leq j \leq n+r\} \subset M_{\mathbb{R}}. \tag{2.5}$$

Note that there may be more than one way of writing D as a linear combination of the D_j's, so the polytope P_D is well defined up to translations by elements in M. By denoting with S_α the graded piece of S of degree $\alpha = [D]$, we have that

$$S_\alpha \simeq H^0(X, \mathcal{O}_X(D)) = \bigoplus_{m \in P_D \cap M} K \cdot \boldsymbol{\chi}^m, \tag{2.6}$$

where $\boldsymbol{\chi}^m \in \mathrm{Hom}(N \otimes_{\mathbb{Z}} K^\times, K^\times)$ is a character in $\mathbb{T}_K^n \simeq N \otimes_{\mathbb{Z}} K^\times$, and $\mathcal{O}_X(D)$ denotes the coherent sheaf on X associated to the divisor D, see [Cox95, Proposition 1.1]. So, any polynomial $F \in S$ of degree α can be written as

$$F = \sum_{m \in P_D \cap M} c_m \prod_{j=1}^{n+r} x_j^{\langle m, \eta_j \rangle + a_j}, \tag{2.7}$$

with $c_m \in K$. A point $x \in X$ is a *zero* of F if there exists $\xi \in K^{n+r}$ such that $F(\xi) = 0$ and $x_\xi = x$. For a finite set $\mathcal{F}$ of homogeneous elements of S, we denote with $V_X(\mathcal{F}) \subset X$ the set of common zeroes of all $F \in \mathcal{F}$. It is a Zariski closed set in X.

A divisor D is *effective* if one can write $D = \sum_{j=1}^{n+r} b_j D_j$ with $b_j \geq 0$. Recall that ([CLS11, Theorem 4.2.8]) a divisor $D = \sum_{j=1}^{n+r} b_j D_j$ is Cartier if and only if for all $\sigma \in \Sigma$ generated by $\{\eta_{j_1}, \ldots, \eta_{j_l}\}$, there exists $m_\sigma \in M$ such that $\langle m_\sigma, \eta_{j_i}\rangle = -b_{j_i}$ for $i = 1, \ldots, l$.

We also have the following results which will be of use in the sequel.

Proposition 2.1. [CLS11, Proposition 6.1.1] *A Cartier divisor D is basepoint free, i.e. $\mathcal{O}_X(D)$ is generated by its global sections, if and only if for all n-dimensional $\sigma \in \Sigma$, $m_\sigma \in P_D$.*

A *very ample* divisor D ([CLS11, Definition 6.1.9]) is one being basepoint free, and also such that the map

$$\begin{array}{rccl} \phi_D : & X & \to & \mathbb{P}(H^0(X, \mathcal{O}_X(D))^\vee) \\ & x & \mapsto & (\boldsymbol{\chi}^m(x))_{m \in P_D \cap M} \end{array} \tag{2.8}$$

is a closed embedding. An *ample* divisor D is one such that ℓD is very ample for some integer $\ell > 0$. A *semi-ample* divisor is one such that $\mathcal{O}_X(D)$ is generated by its global sections, or equivalently (as X is a complete toric variety) basepoint free, see [CLS11, Theorem 6.3.12]. The following criteria for ampleness and semi-ampleness will be of use in the sequel. For a Cartier divisor D, the function $\varphi_D : N \otimes \mathbb{R} \to \mathbb{R}$ such that $\varphi_D(u) := \langle m_\sigma, u\rangle$ if $u \in \sigma$, is well defined, and linear in each cone σ ([CLS11, Theorem 4.2.12]).

Proposition 2.2 (Theorems 6.1.7 and 6.1.14 in [CLS11])**.** *If X is a complete toric variety, and D a Cartier divisor on X, then*

1. *D is semi-ample $\iff$ φ_D is a convex function.*
2. *D is ample $\iff$ φ_D is a strictly convex function.*

Definition 2.3 (Definition 2.2 in [BT22])**.** Given a zero-dimensional subscheme Y of X, a degree α is called *Y-basepoint free* if there exists $H \in S_\alpha$ such that $Y \cap V(H) = \emptyset$.

The following result is going to be useful in the text.

Proposition 2.4. *If Y is zero-dimensional and contained in the torus $\mathbb{T}_K^n$, then any effective α is Y-basepoint free.*

Proof. Let D be such that $[D] = \alpha$. Then any polynomial in S_α is as in (2.7) for suitable $c_m \in K$, $m \in P_D \cap M$. By hypothesis, any point $\boldsymbol{\xi} \in Y$ satisfies $\prod_{j=1}^{n+r} \xi_j^{\langle m, \eta_j\rangle + a_j} \neq 0$. Each of these finitely many points amounts to a linear inequality which can always be fulfilled for a suitable choice of coefficients in K. $\square$

A $\mathbb{Q}$-*divisor* is an element of the form $\sum_{j=1}^{n+r} q_j D_j$, with $q_j \in \mathbb{Q}$. It is called $\mathbb{Q}$-*ample* (resp. $\mathbb{Q}$-*Cartier*) if a positive integer multiple of it is ample (resp. Cartier). We will also need the following version of the generalization of Kawamata and Viehweg vanishing theorem for toric varieties given by Mustaţă. For a $\mathbb{Q}$-divisor of the form $D = \sum_{j=1}^{n+r} q_j D_j$, we define $\lceil D \rceil := \sum_{j=1}^{n+r} \lceil q_j \rceil D_j$.

Theorem 2.5 (Corollary 2.5 in [Mus02])**.** *If D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-ample divisor, then $H^i(X, \mathcal{O}_X(\lceil D \rceil + \sum_{j=1}^{n+r} -D_j)) = 0$ for all $i > 0$.*

2.2. **The toric varieties X_P and X_{P+Q}.** Going back to our original formulation, assume that we have our input system (1.5) with associated supports $\mathcal{A}_1,\ldots,\mathcal{A}_k\subset\mathbb{Z}^n$, and polytopes $P_1,\ldots,P_k$ and P as in (1.6) being $n-$dimensional. Note that in this case we are identifying $M\simeq\mathbb{Z}^n$. Let Q be another integral polytope, and denote with $\eta_1,\ldots,\eta_{n+r}$ the primitive inner normal vectors in $N\simeq\mathbb{Z}^n$ to the $n+r$ facets of $P+Q$. We then have that there are integers $a_1,\ldots,a_{n+r}$ such that

$$P=\{u\in\mathbb{R}^n:\,\langle u,\eta_j\rangle\geq -a_j,\,1\leq j\leq n+r\}\subset\mathbb{R}^n, \tag{2.9}$$

which implies that there are integers a_{ij}, $1\leq i\leq k$, $1\leq j\leq n+r$ such that $a_j=a_{1j}+\ldots+a_{kj}$ for $j=1,\ldots,n+r$, and

$$P_i=\{u\in\mathbb{R}^n\,:\,\langle u,\eta_j\rangle\geq -a_{ij},\ 1\leq j\leq n+r\},\ i=1,\ldots,k. \tag{2.10}$$

We also have that there are integers $b_1,\ldots,b_{n+r}$ such that

$$Q=\{u\in\mathbb{R}^n:\,\langle u,\eta_j\rangle\geq -b_j,\,1\leq j\leq n+r\}\subset\mathbb{R}^n. \tag{2.11}$$

Assume w.l.o.g. that $\eta_1,\ldots,\eta_{n+\ell}$ are the primitive inner normals to the facets of P, and let X_P be the toric variety associated with P, i.e. the one whose fan is the *normal fan* Σ_P defined in Chapter 2 of [CLS11], where the maximal cones of this fan are in correspondence with the vertices of P, and the cone associated to a vertex v is generated by the rays generated by those η_j such that the facet determined by η_j contains v. In the same way the variety X_{P+Q} is defined, by using the fan Σ_{P+Q}. Note that this fan refines Σ_P and hence, thanks to Theorem 3.4.11 in [CLS11], there is a toric proper morphism $X_{P+Q}\to X_P$.

For $i=1,\ldots,k$ we set $D_{P_i}:=\sum_{j=1}^{n+r}a_{ij}D_j$ and $\alpha_{P_i}:=\sum_{j=1}^{n+r}a_{ij}[D_j]$ the divisor associated to P_i in X_{P+Q} and its respective degree in the Chow group of this variety. Note that $D_P=\sum_{i=1}^{k}D_{P_i}$ and $\alpha_P=\sum_{i=1}^{k}\alpha_{P_i}$. Set also $D_Q=\sum_{j=1}^{n+r}b_jD_j$ and $\alpha_Q=[D_Q]$.

As explained in [KS05, §2.2], for any integral polytope P_i, the associated piece-wise linear function $\varphi_{D_{P_i}}:\mathbb{R}^n\simeq N\otimes\mathbb{R}\to\mathbb{R}$ defined by using its normal fan Σ_{P_i} is convex, this is because D_{P_i} is basepoint free in X_{P_i} thanks to Proposition 2.1. The same applies to D_P in X_P and D_Q in X_Q. As the fan Σ_{P+Q} is a common refinement of Σ_{P_i}, Σ_P, Σ_Q, it turns out that the pull-back of all these divisors to X_{P+Q} has the same (convex) piece-wise linear function, and define the same polytope. This implies then that all the divisors D_{P_i}, $1\leq i\leq k$, D_P and D_Q are Cartier and semi-ample in X_{P+Q}. From [CLS11, Proposition 6.1.10], we have that $D_{P+Q}:=D_P+D_Q$ is in addition ample in X_{P+Q}. Note that neither D_P nor D_Q nor any of the D_{P_i} is necessarily ample.

As $\mathcal{A}_i\subset\mathbb{Z}^n\cap P_i$, each of the f_i's from (1.5) can be both "$P-$homogeneized" and "$(P+Q)$-homogeneized" to F_i and F_i' respectively, using (2.6) and (2.7) as follows:
(2.12)

$$\begin{aligned}F_i(x_1,\ldots,x_{n+\ell}) &= \textstyle\sum_{b\in\mathcal{A}_i}f_{i,b}\prod_{j=1}^{n+\ell}x_j^{\langle b,\eta_j\rangle+a_{ij}} = \Big(\prod_{j=1}^{n+\ell}x_j^{a_{ij}}\Big)f_i(t_1,\ldots,t_n),\\ F_i'(x_1,\ldots,x_{n+r}) &= \textstyle\sum_{b\in\mathcal{A}_i}f_{i,b}\prod_{j=1}^{n+r}x_j^{\langle b,\eta_j\rangle+a_{ij}} = \Big(\prod_{j=1}^{n+r}x_j^{a_{ij}}\Big)f_i(t_1,\ldots,t_n),\end{aligned}$$

the right hand side equalities due to (2.3). From here, it is clear that if $\boldsymbol{t}\in\mathbb{T}_K^n$ is such that $f_i(\boldsymbol{t})=0$, then $F_i(x_{\boldsymbol{t}})=F_i'(x_{\boldsymbol{t}}')=0$ for the corresponding points of $\boldsymbol{t}$ in the varieties X_P and X_{P+Q} respectively. It turns out that the closure of the zeroes of f_i in X_P (resp. X_{P+Q}) is the variety of zeroes of F_i (resp. F_i').

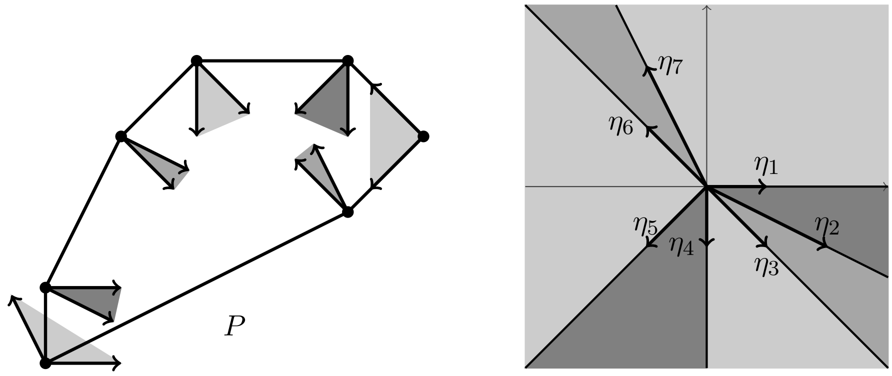

FIGURE 4. Inner normals and normal fan of the polygon P of Example 1.6.

Example 2.6. *For our running Example 1.6, the heptagon P has the following inner normals, numbered clockwise, se Figure 4:*

$$\begin{array}{l}\eta_1=(1,0),\ \eta_2=(2,-1),\ \eta_3=(1,-1),\ \eta_4=(0,-1),\ \eta_5=(-1,-1),\\ \eta_6=(-1,1),\ \eta_7=(-1,2).\end{array}$$

The normal fan associated to this polygon has the seven maximal cones covering $\mathbb{R}^2$ generated by η_1, η_2; η_2, η_3; $\ldots$; η_6, η_7; and η_7, η_1. The toric variety X_P is then a toric surface made by gluing seven affine spaces all of them equivalent to $\mathbb{A}^2_K$ except the one associated to the cone η_7, η_1 which is a quotient of this space via the action of a finite group. So it is not smooth. For more on toric surfaces, we refer the reader to Chapter 10 *of* [CLS11].

Its homogeneous coordinate ring is then $S = K[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$, and one can easily verify that the variety Z of (2.2) *is equal to*

$$V(x_3x_4x_5x_6x_7,\ x_1x_4x_5x_6x_7,\ x_1x_2x_5x_6x_7,\ x_1x_2x_3x_6x_7,\ x_1x_2x_3x_4x_7,\ x_1x_2x_3x_4x_5,\ x_2x_3x_4x_5x_6).$$

By setting $Q = \{\mathbf{0}\}$, the homogenization (2.12) *for this example is:*

$$\begin{array}{rcl} F_1 &=& f_{10}\, x_4x_5^2 + f_{11}\, x_1x_2x_7,\\ F_2 &=& f_{20}\, x_3x_4^2x_5^3x_6 + f_{21}\, x_1^2x_2^3x_3^2x_4 + f_{22}\, x_1x_6^2x_7^3,\\ F_3 &=& f_{30}\, x_2x_3x_4x_5^3x_6 + f_{31}\, x_1^2x_2^4x_3^2 + f_{32}\, x_5^2x_7^2. \end{array}$$

As in the Introduction, pick $\delta \in N \otimes \mathbb{Q} \simeq \mathbb{Q}^n$, and set $D_\delta := \sum_{\langle \delta,\eta_j\rangle>0} -D_j$. Note that for any $r > 0$, we have $D_\delta = D_{r\delta}$.

Proposition 2.7. *For $\delta \in \mathbb{Q}^n$, we have*

$$H^i\big(X_{P+Q},\, \mathcal{O}_{X_{P+Q}}(D_\delta + D_Q + \sum_{j\in J} D_{P_j})\big) = 0\ \forall J \subset \{1,\ldots,k\}\ \forall i > 0.$$

Proof. Assume w.l.o.g. that $D_\delta = \sum_{\langle \delta,\eta_j\rangle>0} -D_j = \sum_{j=1}^{d} -D_j$ for $1 \le d < n+r$. Fix $J \subset \{1,\ldots,k\}$. Recall that $D_P + D_Q$ is an ample divisor in X_{P+Q}, and write

$$D_P + D_Q = \sum_{j=1}^{n+r} c_j D_j. \tag{2.13}$$

As $P+Q$ is an n-dimensional integral polytope, there must be a monomial in $S_{\alpha_P+\alpha_Q}$ of the form $x_1^{c_1}\ldots x_{n+r}^{c_{n+r}}$, with $c_j \ge 0$ and $c_j > 0$ if $\langle \delta, \eta_j\rangle = 0$ (for instance, one can pick a monomial corresponding via (2.6) to any vertex of $P+Q$ not contained in a facet with inner normal η_j such that $\langle\delta,\eta_j\rangle = 0$). We assume w.l.o.g. that the data $(c_j)_{1\leq j\leq n+r}$ satisfies these conditions.

For any $z\in\mathbb{Z}_{<0}$ such that $z\delta\in\mathbb{Z}^n$, thanks to (2.4) we have that $D_P + D_Q = \sum_{j=1}^{n+r}(c_j+\langle z\delta,\eta_j\rangle)D_j$. If we pick $|z|$ large enough, we can assume then w.l.o.g. that $c_j<0$ for $j\leq d$ and $c_j>0$ otherwise. Let $c>0$ be larger than the maximum of the absolute values of all the c_j's, and set

$$D := \frac{1}{c}(D_P + D_Q) + D_Q + \sum_{j\in J} D_{P_j}.$$

Note that D is $\mathbb{Q}$-Cartier, and we clearly have that

$$\lceil D\rceil + \sum_{j=1}^{n+r} -D_j = D_\delta + D_Q + \sum_{j\in J} D_{P_j}.$$

The claim will follow thanks to Theorem 2.5 provided that D is $\mathbb{Q}$-ample. To show this, note that both $D^1 := c\left(\frac{1}{c}(D_P+D_Q)\right)$ and $D^2 := c(D_Q + \sum_{j\in J} D_{P_j})$ are the divisors associated to integral polytopes, the first of them being equal to $P+Q$. From the properties of these divisors explained in Section 2.2, we conclude that D^1 is ample, and D^2 semi-ample in X_{P+Q}. Hence, thanks to Proposition 2.2, φ_{D^1} is strictly convex, and φ_{D^2} convex. We deduce then that $\varphi_{D^1+D^2}$ is the sum of a strictly convex function plus a convex function. So, it is a strictly convex function, which implies that $D^1 + D^2 = c\,D$ is ample thanks again to Proposition 2.2. This concludes with the proof of the claim. $\square$

2.3. **Sparse Resultants.** We review here some results on sparse resultants. We will mainly follow [DS15] and [DJS23]. Given finite sets $\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}\subset\mathbb{Z}^n$ of respective cardinality $r_1,\ldots,r_{n+1}$ consider the generic polynomials $f_1^g,\ldots,f_{n+1}^g$ defined in (1.11). They are elements of $\mathbb{K}[t^{\pm}]$ but also of $\mathbb{Z}[c_{i,b}][t^{\pm 1}]$.

Given an algebraically closed field K of characteristic zero, the *incidence variety* of the family (1.11) is

$$\Omega_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}} = \{(\tilde{c}_{i,b},\xi) :\, f_i^g(\tilde{c}_{i,b},\xi)=0,\, 1\leq i\leq n+1\}\subset \mathbb{P}_K^{r_1-1}\times\ldots\times\mathbb{P}_K^{r_{n+1}-1}\times\mathbb{T}_K^n,$$

which is an irreducible subvariety of codimension $n+1$ defined over $\mathbb{Q}$. Denote with

$$\pi : \mathbb{P}_K^{r_1-1}\times\ldots\times\mathbb{P}_K^{r_{n+1}-1}\times\mathbb{T}_K^n \to \mathbb{P}_K^{r_1-1}\times\ldots\times\mathbb{P}_K^{r_{n+1}-1}, \tag{2.14}$$

the projection which forgets the last factor. If the Zariski closure of the image $\overline{\pi(\Omega_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}})}$ is a hypersurface, the *sparse eliminant* $\mathrm{Elim}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$ is defined as any irreducible polynomial in $\mathbb{Z}[c_{i,b}]$ giving an equation for it. Otherwise, the eliminant is set equal to 1. The sparse resultant, denoted by $\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$, is defined as any primitive polynomial in $\mathbb{Z}[c_{i,b}]$ giving an equation for the direct image $\pi_*(\Omega_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}})$.

Both of these polynomials are well defined up to a sign. Moreover $\mathrm{Elim}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$ is irreducible when it is not equal to 1, and there exists $d\in\mathbb{N}$ such that $\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}} = \mathrm{Elim}^d_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$.

The following properties of eliminants and resultants will be of use in the text.

Proposition 2.8.

(1) *For* $i = 1, \ldots, n+1$, $\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$ *is a homogeneous polynomial in the coefficients of* f_i^g *of degree* $MV(P_1, \ldots, \check{P}_i, \ldots, P_{n+1})$*, where* $P_j = \mathrm{conv}(\mathcal{A}_j)$ *for* $j = 1, \ldots, n$*, and* $MV(\cdot)$ *denotes the* mixed volume *of* n *polytopes as defined in* [DS15, (2.29)].

(2) *Let* $f_1, \ldots, f_{n+1} \in K[t^{\pm 1}]$ *be such that their supports are contained in* $\mathcal{A}_1, \ldots, \mathcal{A}_{n+1}$ *respectively, and* $F_1, \ldots, F_{n+1} \in S$ *(resp.* $F'_1, \ldots, F'_{n+1} \in S$*) their homogeneizations with respect to* $P_1, \ldots, P_{n+1}$ *via* (2.12)*. If* $\mathrm{Elim}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$ *is not equal to* 1*, then,* $\mathrm{Elim}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}(f_1, \ldots, f_{n+1}) = 0 \iff V_{X_P}(F_1, \ldots, F_{n+1}) \neq \emptyset \iff V_{X_{P+Q}}(F'_1, \ldots, F'_{n+1}) \neq \emptyset$.

Proof.

(1) Follows from Proposition 3.4 in [DS15].

(2) If $\overline{\pi(\Omega_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}})}$ is a hypersurface, then we have from Proposition 3.2 in [DS15] and (2.14) in loc. cit. that $\mathrm{Elim}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}(f_1, \ldots, f_{n+1}) = 0$ if and only if the "homogenization" of this system to the toric multiprojective variety $X_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$ defined in [DS15, (2.17)], has a common zero in this set. As there is a proper morphism of toric varieties $\phi : X_P \to \mathbb{P}^{r_1-1} \times \ldots \times \mathbb{P}^{r_{n+1}-1}$ satisfying $\phi(X_P) = X_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$ (cf. Lemma 2.6 in [DS15]), the claim follows straightforwardly for X_P by noting that the F_i's are global sections of divisors in X_P corresponding to the homogenizations of the f_i's in $X_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$. The same argument applies to X_{P+Q}.

□

Remark 2.9. There is a combinatorial criteron on $\mathcal{A}_1, \ldots, \mathcal{A}_{n+1}$ to detect when $\mathrm{Elim}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}} = 1$, see Proposition 3.8 in [DS15] for instance.

2.4. **Determinants of complexes.** We present here the notion of determinants of exact and generically exact complexes, and some of their properties which will be of use to prove Theorem 1.11. We will follow mainly the presentation given in [GKZ94, Appendix A], see also [Dem84, Cha93] for more on this subject.

Let K be any field. A *complex* of K-vector spaces is a graded vector space $\bigoplus_{i\in\mathbb{Z}} V_i$ together with K-linear maps $d_i : V_i \to V_{i-1}$ satisfying $d_i \circ d_{i+1} = 0$. We will assume that all but finitely many of the V_i's are different from zero.

For an n-dimensional K-vector space V of dimension n, we define $\mathrm{Det}(V)$ as the one-dimensional vector space $\wedge^n V$, and for $V = 0$, we set $\mathrm{Det}(0) = K$. Given a one-dimensional vector space V, we denote with V^{-1} its dual space V^*. The *determinantal vector space* of a finite-dimensional complex $\bigoplus_{i\in\mathbb{Z}} V_i$ of vectors spaces having only finitely many of the V_i's different from zero is the one-dimensional K-vector space defined as $\bigotimes_i \mathrm{Det}(V_i)^{\otimes(-1)^i}$. The cohomology spaces of a complex form another graded vector space and there is a natural isomorphism between the determinantal vector spaces associated to both (see [GKZ94, Proposition 3, Appendix A]). If the complex is exact, its determinantal vector space can be naturally identified with K ([GKZ94, Corollary 6, Appendix A]). To make explicit this identification, one chooses bases B_i of all the nontrivial V_i's and, as in the linear case, defines the *determinant of* $\bigoplus_{i\in\mathbb{Z}} V_i$ *with respect to the system of bases* $\{B_i\}$ as the element of $K^\times$ which is the image under this natural identification of the element $\bigotimes_i \mathrm{Det}(B_i)^{(-1)^i} \in \mathrm{Det}(\bigoplus_{i\in\mathbb{Z}} V_i)$, where $\mathrm{Det}(B_i)$ is the wedge product of all the elements in B_i, (cf. [GKZ94, Definition 7, Appendix

A]). In the case corresponding to only two nontrivial vector spaces $0 \to V_2 \stackrel{d_2}{\to} V_1 \to 0$, this notion coincides with the determinant of the matrix of d_2 in the bases B_2 and B_1.

In general, to compute this determinant once the bases are fixed, we can use the so-called "Cayley method": suppose that

$$0 \to V_r \stackrel{d_r}{\to} V_{r-1} \stackrel{d_{r-1}}{\to} \ldots \stackrel{d_2}{\to} V_1 \stackrel{d_1}{\to} V_0 \to 0 \tag{2.15}$$

is an exact complex of finite-dimensional vector spaces $V_0, \ldots, V_r$ with bases $B_0, \ldots, B_r$ respectively and, for $i = 1, \ldots, r$, let M_i be the matrix of d_i with respect to the bases B_i and B_{i-1}. A family of subsets $I_i \subset B_i$, $i = 0, \ldots, r$ is called *admissible* ([GKZ94, Definition 12, Appendix A]) if $I_0 = B_0$, $I_r = \emptyset$, and for any $i = 1, \ldots, r$, we have $\#(B_i \setminus I_i) = \#(I_{i-1})$, and the submatrix of M_i having its columns indexed by $B_i \setminus I_i$, and its rows indexed by I_{i-1} is invertible. We have that admissible sets do exist ([GKZ94, Proposition 13, Appendix A]), and moreover:

Proposition 2.10 (Theorem 14, Appendix A in [GKZ94]). *For any admissible collection $(I_i)_{i=0,\ldots,r}$, for $i = 1, \ldots, r$, denote with Δ_i the determinant of the submatrix of M_i having its columns indexed by $B_i \setminus I_i$, and its rows indexed by I_{i-1}. Then, the determinant of* (2.15) *with respect to the bases $B_0, \ldots, B_r$ is equal to*

$$\prod_{i=1}^{r} \Delta_i^{(-1)^{i+1}}.$$

We will be interested in computing the determinant of complexes of K-vector spaces once these bases are fixed by using the "descending method" ([Dem84], see also [DJ05]), which consists of the following algorithm:

(1) Set $I_0 = B_0$.
(2) For $i = 1, \ldots, r$, choose I_i as any subset of B_i such that the submatrix of M_i indexed by the columns in $B_i \setminus I_i$ and the rows in I_{i-1} is of maximal rank.

As the complex is exact, we deduce straightforwardly that if this algorithm finishes successfully, then $I_r = \emptyset$.

The fact that the algorithm succeeds in producing an admissible sequence follows from the following claim.

Proposition 2.11. *For $i = 0, \ldots, r$, the set I_i obtained by the above algorithm satisfies $\ker(d_i) \oplus \langle B_i \setminus I_i \rangle = V_i$.*

Proof. We proceed by induction. The identity holds trivially for $i = 0$.

Assume $i > 0$ and $\ker(d_{i-1}) \oplus \langle B_{i-1} \setminus I_{i-1} \rangle = V_{i-1}$. Let $\pi_{i-1} : V_{i-1} \to \langle I_{i-1} \rangle$ be the projection with kernel $\langle B_{i-1} \setminus I_{i-1} \rangle$ and $\tilde{d}_i = \pi_{i-1} \circ d_i : V_i \to \langle I_{i-1} \rangle$. Note that the matrix of $\tilde{d}_i$ with respect to the bases B_i and I_{i-1} is the submatrix of M_i with rows indexed by I_{i-1}. We have $\ker(\tilde{d}_i) = \ker(d_i)$, since $\operatorname{Im}(d_i) \cap \langle B_{i-1} \setminus I_{i-1} \rangle = \ker(d_{i-1}) \cap \langle B_{i-1} \setminus I_{i-1} \rangle = \{0\}$ because of the induction assumption and the fact that the complex is exact. Then, $\dim(\operatorname{Im}(\tilde{d}_i)) = \dim(\operatorname{Im}(d_i)) = \dim(\ker(d_{i-1})) = \#I_{i-1}$ and $\tilde{d}_i$ is onto. We conclude that the algorithm chooses a subset I_i of B_i such that $\#(B_i \setminus I_i) = \#I_{i-1}$ and the submatrix of $\tilde{d}_i$ with columns indexed by $B_i \setminus I_i$ is invertible. In particular, $\ker(\tilde{d}_i) \cap \langle B_i \setminus I_i \rangle = \{0\}$ and, comparing dimensions, we deduce that $\ker(d_i) \oplus \langle B_i \setminus I_i \rangle = V_i$. □

Let now $R = A[y_1, \ldots, y_s]$ be a ring of polynomials in the indeterminates $y_1, \ldots, y_s$, with coefficients in a noetherian integral domain A, such that R is factorial and regular (i.e. every projective R-module has a projective finite resolution). Denote with $K(R)$ its field of fractions. A finite *generically exact complex* of R-modules of finite rank is a complex of free R-modules

$$0 \to \mathbf{M}_r \overset{d_r}{\to} \mathbf{M}_{r-1} \overset{d_{r-1}}{\to} \dots \overset{d_1}{\to} \mathbf{M}_0 \to 0 \tag{2.16}$$

such that their tensorization with $K(R)$ makes the resulting complex of finite dimensional $K(R)$-vector spaces exact. Given a system of R-bases $B_0, \dots, B_r$ of the free R-modules involved, the determinant of this complex is then defined as the determinant of (2.16) tensored with $K(R)$. It is an element of $K(R)^\times$. In some situations, one can compute it as the gcd of the maximal minors of the matrix of the map d_1 in the corresponding bases. To state the conditions ensuring that such property holds, as usual we define the *cohomology modules* of (2.16) as $H^i(\bigoplus_j \mathbf{M}_j) := \ker(d_i)/\mathrm{Im}(d_{i+1}), i \in \mathbb{Z}$. We will also need the notion of *multiplicity* of a finitely generated R-module $\mathbf{M}$ along a prime ideal $\mathfrak{p}$ of R, which is defined, via localization at $\mathfrak{p}$, as $\mathrm{mult}_{\mathfrak{p}}(\mathbf{M}) = \sum_{\ell \geq 0} \dim_{k_{\mathfrak{p}}} \big(\mathfrak{m}_{\mathfrak{p}}^{\ell} \mathbf{M}_{\mathfrak{p}} / \mathfrak{m}_{\mathfrak{p}}^{\ell+1} \mathbf{M}_{\mathfrak{p}}\big)$ if the sum is finite or 0 otherwise. Here, $\mathfrak{m}_{\mathfrak{p}}$ denotes the maximal ideal of the local ring $R_{\mathfrak{p}}$ and $k_{\mathfrak{p}} = R_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$ its residual field. We have the following result.

Theorem 2.12. [GKZ94, Theorem 34, Appendix A] *If for any prime element $p \in R$, the multiplicity of $H^i(\bigoplus_j \mathbf{M}_j)$ along the ideal $\langle p \rangle$ of R is equal to zero $\forall i > 0$, then –up to an invertible element in R– the determinant of* (2.16) *with respect to the bases $B_0, \dots, B_r$ is equal to the* gcd *of the maximal minors of the matrix of d_1 in the bases B_1 and B_0.*

3. Proof of Theorems 1.7 and 1.9

To prove Theorem 1.7, assume first that K is algebraically closed.
(1) $\implies$ (2) Suppose first that $V_{X_P}(F_1, \dots, F_k) = \emptyset$, which is equivalent to having $V_{X_{P+Q}}(F'_1, \dots, F'_k) = \emptyset$. As F'_i is a global section of $\mathcal{O}_{X_{P+Q}}(D_{P_i})$, $1 \leq i \leq k$, we get the Koszul complex of sheaves

$$\begin{aligned} 0 \to \mathcal{O}_{X_{P+Q}}\big(-\textstyle\sum_{1 \leq i \leq k} D_{P_i}\big) \to \dots \to \bigoplus_{|J|=\ell} \mathcal{O}_{X_{P+Q}}(-\textstyle\sum_{j \in J} D_{P_j}) \to \dots \\ \dots \to \textstyle\bigoplus_{1 \leq i \leq k} \mathcal{O}_{X_{P+Q}}(-D_{P_i}) \to \mathcal{O}_{X_{P+Q}} \to 0 \end{aligned} \tag{3.1}$$

which is exact since each $\mathcal{O}_{X_{P+Q}}(D_{P_i})$ is locally free and $F'_1, \dots, F'_k$ define the empty variety in X_{P+Q}.

If we now tensor (3.1) with $\mathcal{O}_{X_{P+Q}}(D_\delta + D_Q + D_P)$, and use that each of the divisors appearing in (3.1) is Cartier, we get a new complex

$$\begin{aligned} 0 \to \mathcal{O}_{X_{P+Q}}\big(D_\delta + D_Q\big) \to \dots \to \textstyle\bigoplus_{|J|=\ell} \mathcal{O}_{X_{P+Q}}(D_\delta + D_Q + \sum_{j \notin J} D_{P_j}) \to \dots \\ \dots \to \textstyle\bigoplus_{1 \leq i \leq k} \mathcal{O}_{X_{P+Q}}(D_\delta + D_Q + D_P - D_{P_i}) \to \mathcal{O}_{X_{P+Q}}(D_\delta + D_Q + D_P) \to 0 \end{aligned} \tag{3.2}$$

which is also exact because all sheaf Tor groups vanish when one of the factors is locally free, which is the case with the factors of (3.1), see the proof of Theorem 2.2 in [CD05b] for more. Thanks to Proposition 2.7, we have that, for $j > 0$, and $J \subset \{1, \dots, k\}$, the following cohomology vanish:

$$H^j\Big(X_{P+Q}, \bigoplus_J \mathcal{O}_{X_{P+Q}}(D_\delta + D_Q + \sum_{i \notin J} D_{P_i})\Big) \simeq \bigoplus_J H^j\Big(X_{P+Q}, \mathcal{O}_{X_{P+Q}}(D_\delta + D_Q + \sum_{i \notin J} D_{P_i})\Big).$$

This implies straightforwardly that taking the global sections of (3.2) preserves exactness. We use (2.6) to identify $H^0\left(X_{P+Q}, \mathcal{O}_{X_{P+Q}}\left(D_\delta + D_Q + \sum_{i\notin J} D_{P_i}\right)\right)$ with the elements in S' of degree $\alpha_\delta + \alpha_Q + \sum_{i\notin J}\alpha_{P_i}$, so we have that the following complex of K-vector spaces is exact:

$$0 \to S'_{\alpha_\delta+\alpha_Q} \to \ldots \to \bigoplus_i S'_{\alpha_\delta+\alpha_Q+\alpha_P-\alpha_{P_i}} \to S'_{\alpha_\delta+\alpha_Q+\alpha_P} \to 0. \tag{3.3}$$

This is the $(\alpha_\delta+\alpha_Q+\alpha_P)-$graded piece of the Koszul complex $Kos(S', F'_1, \ldots, F'_k)$, which proves the "if" part.
(2) $\implies$ (3) Trivial.
(3) $\implies$ (1) Let $f_1^g, \ldots, f_k^g$ be the generic polynomials defined in (1.11), and denote with $F_1^{g\prime}, \ldots, F_k^{g\prime}$ their respective homogenizations in $S'_{\mathbb{K}}$ with respect to $P_1, \ldots, P_k$. For a particular specialization of coefficients $c_{i,a} \mapsto y_{i,a}$ in K, we abbreviate with $F_{i,y} \in K[x_1, \ldots, x_{n+r}] =: S^0$ the specialized polynomial. Let $\mathcal{V}$ be the set in the space of coefficients consisting of all $y_{i,a}$ such that the system $F_{1,y}, \ldots, F_{k,y}$ has a common zero in $\mathbb{T}_K^{n+r}$. Then, no monomial in $S^0_{\alpha_\delta+\alpha_Q+\alpha_P}$ can be in the image of $\Psi_{1,\alpha_\delta+\alpha_Q+\alpha_P}$, the rightmost nontrivial map in $Kos(S^0, F_{1,y}, \ldots, F_{k,y})_{\alpha_\delta+\alpha_Q+\alpha_P}$, as otherwise by specializing the variables x_i in the projective coordinates of the common zero we would get a contradiction. Thus, for any system contained in $\mathcal{V}$, $\Psi_{1,\alpha_\delta+\alpha_Q+\alpha_P}$ is not onto. We conclude that $\mathcal{V}$ is contained in the algebraic variety defined by the vanishing of all maximal minors of the rightmost nontrivial map in the complex $Kos(S'_{\mathbb{K}}, F_1^{g\prime}, \ldots, F_k^{g\prime})_{\alpha_\delta+\alpha_Q+\alpha_P}$, and hence the Zariski closure of $\mathcal{V}$ is also contained in this variety. Since $\mathbb{T}_K^{n+r}$ is dense in $K^{n+r}\setminus Z$, where Z has been defined in (2.2) by using the fan Σ_{P+Q}, and the toric variety X_{P+Q} is the quotient of $K^{n+r}\setminus Z$ via the action of a group (cf. [CLS11, §5.1]), we have that the Zariski closure of $\mathcal{V}$ contains those systems $F'_1, \ldots, F'_k$ of degrees $\alpha_{P_1}, \ldots, \alpha_{P_k}$ respectively having a common zero in X_{P+Q}, or equivalently systems $F_1, \ldots, F_k$ having a common zero in X_P. For all these families, the rightmost nontrivial map in $Kos(S', F'_1, \ldots, F'_k)_{\alpha_\delta+\alpha_Q+\alpha_P}$ cannot be onto for all the explained above. This concludes with the proof of Theorem 1.7 for an algebraically closed field K.

For the general case, as each of the vector spaces in $Kos(S'_{\overline{K}}, F'_1, \ldots, F'_k)_{\alpha_\delta+\alpha_Q+\alpha_P}$ has a monomial basis, which is also a monomial basis of the corresponding space in $Kos(S'_K, F'_1, \ldots, F'_k)_{\alpha_\delta+\alpha_Q+\alpha_P}$, and the matrices of the linear maps in both complexes (w.r.t. these bases) have their entries in K (in fact they are the same matrices), the claim follows straightforwardly for $Kos(S'_K, F'_1, \ldots, F'_k)_{\alpha_\delta+\alpha_Q+\alpha_P}$ as it holds for the first complex. □

Proof of Theorem 1.9. The hypothesis on the coordinates of δ implies that there is a one to one correspondence via (2.6) between the monomials in $S'_{\alpha_\delta+\alpha_Q+\alpha_P}$ and the elements of $(P+Q+\delta)\cap\mathbb{Z}^n$. The same applies to the monomials in $S'_{\alpha_\delta+\alpha_Q+\alpha_P-\alpha_{P_i}}$ and the elements of $(P_1+\ldots+P_{i-1}+P_{i+1}+\ldots+P_k+Q+\delta)\cap\mathbb{Z}^n,\ 1\le i\le k$.

The claim is now a restatement of (1) $\iff$ (3) in Theorem 1.7 after dehomogenizing the rightmost nontrivial map of $Kos(S', F'_1, \ldots, F'_k)_{\alpha_\delta+\alpha_P+\alpha_Q}$ to the torus via (2.3). □

4. Proof of Theorem 1.11

As $\pi(\Omega_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}})$ has positive codimension in its codomain (see (2.14)), then $F_1^g,\ldots,F_{n+1}^g$ do not have common zeroes in the toric variety X_{P+Q} over the algebraic closure of $\mathbb{K}$. From Theorem 1.7 we deduce straightforwardly that $Kos(S'_{\mathbb{K}},F_1^g,\ldots,F_{n+1}^g)_{\alpha_\delta+\alpha_Q+\alpha_P}$ is exact as a complex of $\mathbb{K}$-vector spaces, which is the first part of the statement.

The last part of the claim also follows straightforwardly from Theorem 1.7 because if $\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}\neq 1$, then we have that

$$\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}(f_1,\ldots,f_{n+1})\neq 0 \iff V_{X_{P+Q}}(F'_1,\ldots,F'_{n+1})=\emptyset$$

thanks to Proposition 2.8.

The remaining parts of the statement are proven in Propositions 4.4 and 4.5 below. We start by comparing the Koszul complex of the theorem with the complex $Kos(S'_{\mathbb{K}},F_1^g,\ldots,F_n^g)_{\alpha_\delta+\alpha_Q+\alpha_P}$, where we have removed F_{n+1}^g, and use the results proven in [BT22] for these kind of complexes.

Bernstein Theorem [Ber75] states that $V_{X_{P+Q}}(F_1^g,\ldots,F_n^g)$ is finite, lies in $\mathbb{T}^n_{\mathbb{K}}$, and its cardinality is equal to $MV(P_1,\ldots,P_n)$.

As the degree $\alpha_\delta+\alpha_Q+\alpha_P$ satisfies the hypothesis of Theorem 4.3 in [BT22] (thanks to Propositions 2.4 and 2.7 with $k=n$), it turns out that the following complex ((4.4) in [BT22]) is exact:

$$\begin{aligned}0\to (S'_{\mathbb{K}})_{\alpha_\delta+\alpha_Q+\alpha_{P_{n+1}}}\overset{\tilde\Psi_n}{\to}\cdots\to\bigoplus_{i=1}^{n}(S'_{\mathbb{K}})_{\alpha_\delta+\alpha_Q+\alpha_P-\alpha_{P_i}}\overset{\tilde\Psi_1}{\to}(S'_{\mathbb{K}})_{\alpha_\delta+\alpha_Q+\alpha_P}&\\ \overset{\tilde\Psi_0}{\to}\left(S'_{\mathbb{K}}/I\right)_{\alpha_\delta+\alpha_Q+\alpha_P}\to 0,&\end{aligned}\tag{4.1}$$

with $I:=\langle F_1^g,\ldots,F_n^g\rangle\subset S'_{\mathbb{K}}$. Note that this complex coincides with the complex $Kos(S'_{\mathbb{K}},F_1^g,\ldots,F_n^g)_{\alpha_\delta+\alpha_Q+\alpha_P}$ except for the last term.

From Theorem 4.3 in [BT22], we also get that

$$\dim\left(S'_{\mathbb{K}}/I\right)_{\alpha_\delta+\alpha_Q+\alpha_P}=MV(P_1,\ldots,P_n).\tag{4.2}$$

because of Bernstein Theorem. Note that if $MV(P_1,\ldots,P_n)=0$, (4.1) implies that $Kos(S'_{\mathbb{K}},F_1^g,\ldots,F_n^g)_{\alpha_\delta+\alpha_Q+\alpha_P}$ is exact.

We want to relate (4.1) with the exact complex $Kos(S'_{\mathbb{K}},F_1^g,\ldots,F_n^g,F_{n+1}^g)_{\alpha_\delta+\alpha_Q+\alpha_P}$, which is

$$0\to (S'_{\mathbb{K}})_{\alpha_\delta+\alpha_Q}\overset{\Psi_{n+1}}{\to}\cdots\to\bigoplus_{i=1}^{n+1}(S'_{\mathbb{K}})_{\alpha_\delta+\alpha_Q+\alpha_P-\alpha_{P_i}}\overset{\Psi_1}{\to}(S'_{\mathbb{K}})_{\alpha_\delta+\alpha_Q+\alpha_P}\to 0.\tag{4.3}$$

In (4.1), all the $\mathbb{K}$-vector spaces have standard monomial bases, except the last one. Clearly one can choose a (class of) monomial basis in this quotient, which we also assume that has been fixed. Let $\widetilde{\mathrm{Det}}$ be the determinant of (4.1) with respect to the monomial bases of all the nontrivial vector spaces, and Det the determinant of (4.3) with respect to the monomial bases of its nontrivial vector spaces. Following Proposition 2.10, let $\tilde M_i$, $0\le i\le n$, be a square submatrix of the one given by $\tilde\Psi_i$ such that $\det(\tilde M_i)\neq 0$ for all i, so that

$$\widetilde{\mathrm{Det}}=\pm\prod_{i=0}^{n}\det(\tilde M_i)^{(-1)^i}.$$

Note that obviously $\tilde M_i$ does not depend on the coefficients of F_{n+1}^g for all $i=0,\ldots,n$.

Lemma 4.1. *There exists a square matrix M_1 of maximal rank of the form* $\begin{pmatrix} \tilde{M}_1 & A \\ * & B \end{pmatrix}$, *such that it is a maximal square submatrix of the matrix of Ψ_1, with* $\begin{pmatrix} A \\ B \end{pmatrix}$ *having $MV(P_1,\ldots,P_n)$ columns containing each of them coefficients of F^g_{n+1}.*

Proof. From (4.1) and (4.2), we deduce that the image of $\tilde{\Psi}_1$ has corank $MV(P_1,\ldots,P_n)$ in $(S'_{\mathbb{K}})_{\alpha_\delta+\alpha_Q+\alpha_P}$. As (4.3) is exact, we deduce that one can complete the linearly independent set of columns of the submatrix of Ψ_1 indexed by the columns of $\tilde{M}_1$ (which has maximal rank), with $MV(P_1,\ldots,P_n)$ columns coming from the multiplication by F^g_{n+1}. This concludes with the proof. $\square$

Proposition 4.2. *Given M_1 as above, for all $i=2,\ldots,n$, after sorting properly the monomial bases, there exists a submatrix of maximal rank of Ψ_i of the form*

$$M_i = \begin{pmatrix} \tilde{M}_i & * \\ \mathbf{0} & L_i \end{pmatrix},$$

with L_i not depending on the coefficients of F^g_{n+1}, and there is another matrix M_{n+1} -submatrix of maximal rank of Ψ_{n+1} not depending on the coefficients of F^g_{n+1}- such that

$$\mathrm{Det} = \pm \prod_{i=1}^{n+1} \det(M_i)^{(-1)^{i+1}}. \tag{4.4}$$

Proof. For $i=2,\ldots,n$, we can decompose Ψ_i as follows

$$\begin{array}{c} \left(\bigoplus_{1\le j_1<\ldots<j_i\le n} (S'_{\mathbb{K}})_{\alpha_\delta+\alpha_Q+\alpha_P-\sum_{s=1}^{i}\alpha_{P_{j_s}}}\right) \oplus \left(\bigoplus_{1\le j_1<\ldots<j_{i-1}\le n} (S'_{\mathbb{K}})_{\alpha_\delta+\alpha_Q+\alpha_P-\alpha_{P_{n+1}}-\sum_{s=1}^{i-1}\alpha_{P_{j_s}}}\right) \\ \downarrow \\ \left(\bigoplus_{1\le j_1<\ldots<j_{i-1}\le n} (S'_{\mathbb{K}})_{\alpha_\delta+\alpha_Q+\alpha_P-\sum_{s=1}^{i-1}\alpha_{P_{j_s}}}\right) \oplus \left(\bigoplus_{1\le j_1<\ldots<j_{i-2}\le n} (S'_{\mathbb{K}})_{\alpha_\delta+\alpha_Q+\alpha_P-\alpha_{P_{n+1}}-\sum_{s=1}^{i-2}\alpha_{P_{j_s}}}\right) \end{array} \tag{4.5}$$

(when $i=2$, the last factor has only one nontrivial term $(S'_{\mathbb{K}})_{\alpha_\delta+\alpha_Q+\alpha_P-\alpha_{P_{n+1}}}$).

Let $B_i = \tilde{B}_i \cup B'_i$, for $i=1,\ldots,n$, be the decomposition of the chosen basis of $\bigoplus_{1\le j_1<\ldots<j_i\le n+1} (S'_{\mathbb{K}})_{\alpha_\delta+\alpha_Q+\alpha_P-\sum_{s=1}^{i}\alpha_{P_{j_s}}}$, where $\tilde{B}_i$ is a basis of $\bigoplus_{1\le j_1<\ldots<j_i\le n} (S'_{\mathbb{K}})_{\alpha_\delta+\alpha_Q+\alpha_P-\sum_{s=1}^{i}\alpha_{P_{j_s}}}$ and B'_i is a basis of $\bigoplus_{1\le j_1<\ldots<j_{i-1}\le n} (S'_{\mathbb{K}})_{\alpha_\delta+\alpha_Q+\alpha_P-\alpha_{P_{n+1}}-\sum_{s=1}^{i-1}\alpha_{P_{j_s}}}$.

With this order of bases, we have that the matrix of Ψ_i with respect to the bases B_i and B_{i-1} can be decomposed as

$$|\Psi_i|_{B_i,B_{i-1}} = \begin{pmatrix} |\tilde{\Psi}_i|_{\tilde{B}_i,\tilde{B}_{i-1}} & A_i \\ \mathbf{0} & N_i \end{pmatrix},$$

with N_i not depending on the coefficients of F^g_{n+1}.

Denote with $\tilde{I}_i \subseteq \tilde{B}_i$ the set such that $\tilde{M}_i = |\tilde{\Psi}_i|_{\tilde{B}_i\setminus\tilde{I}_i,\tilde{I}_{i-1}}$. Here and in the rest of the proof, for a matrix M with columns and rows indexed by sets $\mathcal{C}$ and $\mathcal{R}$ respectively, given $\mathcal{C}_0\subseteq\mathcal{C}$ and $\mathcal{R}_0\subseteq\mathcal{R}$, we will write $M_{\mathcal{C}_0,\mathcal{R}_0}$ for the submatrix of M consisting of the columns indexed by $\mathcal{C}_0$ and the rows indexed by $\mathcal{R}_0$.

Assume that, for every $j < i$, a subset $I_j \subseteq B_j$ of the form $I_j = \tilde{I}_j \cup I'_j$ with $I'_j \subseteq B'_j$ has been chosen such that $M_j = |\Psi_j|_{B_j \setminus I_j, I_{j-1}}$ is a submatrix of maximal rank of $|\Psi_j|_{B_j, B_{j-1}}$ as in the statement of the Proposition.

We have that $\tilde{M}_i$ is a submatrix of $|\tilde{\Psi}_i|_{\tilde{B}_i, \tilde{I}_{i-1}}$ of maximal rank equal to $\#\tilde{I}_{i-1} = \dim(\ker(\tilde{\Psi}_{i-1})) = \dim(\mathrm{Im}(\tilde{\Psi}_i))$. In order to obtain the matrix M_i, we consider $|\Psi_i|_{B_i, I_{i-1}} = \begin{pmatrix} |\tilde{\Psi}_i|_{\tilde{B}_i, \tilde{I}_{i-1}} & (A_i)_{B'_i, \tilde{I}_{i-1}} \\ \mathbf{0} & (N_i)_{B'_i, I'_{i-1}} \end{pmatrix}$. This matrix has full row rank equal to $\#I_{i-1} = \dim(\ker(\Psi_{i-1})) = \dim(\mathrm{Im}(\Psi_i))$; then, we can choose $\#I_{i-1}$ linearly independent columns. By the construction of $\tilde{M}_i$, from the columns indexed by $\tilde{B}_i$, those corresponding to $\tilde{B}_i \setminus \tilde{I}_i$ are a maximal linearly independent subset. We extend it to a basis of the column space of $|\Psi_i|_{B_i, I_{i-1}}$ by adding some columns indexed by B'_i. Let $I'_i \subseteq B'_i$ be the index set of the columns we do not add and $I_i := \tilde{I}_i \cup I'_i$. We have that $M_i := |\Psi_i|_{B_i \setminus I_i, I_{i-1}}$ is a submatrix of $|\Psi_i|_{B_i, B_{i-1}}$ of maximal rank, and it is of the form

$$M_i = \begin{pmatrix} |\tilde{\Psi}_i|_{\tilde{B}_i \setminus \tilde{I}_i, \tilde{I}_{i-1}} & (A_i)_{B'_i \setminus I'_i, \tilde{I}_{i-1}} \\ \mathbf{0} & (N_i)_{B'_i \setminus I'_i, I'_{i-1}} \end{pmatrix} = \begin{pmatrix} \tilde{M}_i & * \\ \mathbf{0} & L_i \end{pmatrix}$$

The claim now follows straightforwardly since L_i is a submatrix of N_i, which does not depend on the coefficients of F^g_{n+1}.

To finish the proof, note that having chosen $M_1, \ldots, M_n$ univocally determines $M_{n+1} = |\Psi_{n+1}|_{B_{n+1}, I_n}$, since Ψ_{n+1} is injective. As $\tilde{M}_n = |\tilde{\Psi}_n|_{\tilde{B}_n, \tilde{I}_{n-1}}$ (because $\tilde{\Psi}_n$ is also injective), it turns out that $\tilde{I}_n = \emptyset$ and so, the columns of M_n are indexed by a set $B_n \setminus I_n$ with $I_n \subseteq B'_n$. Then, the rows of $|\Psi_{n+1}|_{B_{n+1}, I_n}$ do not depend on the coefficients of F^g_{n+1}. This concludes with the proof of the Proposition. □

Corollary 4.3. *For all* $i = 1, \ldots, n+1$, *the degree of* Det *with respect to the coefficients of* F^g_i *is equal to* $MV(P_1, \ldots, \check{P}_i, \ldots, P_{n+1})$.

Proof. From Lemma 4.1 and Proposition 4.2 we deduce the claim for $i = n+1$. By permuting the input polynomials, one can get the same statement for any $i = 1, \ldots, n$. □

Proposition 4.4. *Up to an invertible element in* $\mathbb{Q}$, Det *is equal to the* gcd *in* $\mathbb{Q}[c_{i,b}]$ *of all the maximal minors of* Ψ_1.

Proof. Let $R := \mathbb{Q}[c_{i,b}]$, and consider $Kos(S'_R, F^g_1, \ldots, F^g_{n+1})_{\alpha_\delta + \alpha_Q + \alpha_P}$, which is a complex of R-modules. As this complex becomes $Kos(S'_{\mathbb{K}}, F^g_1, \ldots, F^g_{n+1})_{\alpha_\delta + \alpha_Q + \alpha_P}$ after tensoring it with $K(R) = \mathbb{K}$, which is exact, we deduce then that it is generically exact as defined in (2.16). We will use Theorem 2.12 to prove our claim.

To do so, let $p \in R$ be any irreducible polynomial. Then, p depends on some coefficient $c_{i_0,b}$. Assume w.l.o.g. that $i_0 = n+1$. From Proposition 4.2, all the matrices M_j are invertible in the local ring $R_{\langle p \rangle}, j \geq 2$, as these minors cannot be a multiple of p because they do not depend on the coefficients of F^g_{n+1}. We deduce then straightforwardly that $Kos(S'_{R_{\langle p \rangle}}, F^g_1, \ldots, F^g_{n+1})_{\alpha_\delta + \alpha_Q + \alpha_P}$ is exact except at the rightmost nontrivial map, which implies then that the j-th cohomology of the complex $H^j\big(Kos(S'_R, F^g_1, \ldots, F^g_{n+1})_{\alpha_\delta + \alpha_Q + \alpha_P}\big)$ is zero when localized in $\langle p \rangle$ for all $j > 0$, and for all irreducible $p \in R$. The claim now follows from Theorem 2.12. □

Proposition 4.5. *The* gcd *in* $\mathbb{Z}[c_{i,b}]$ *of all the maximal minors of* Ψ_1 *is equal to* $\pm \operatorname{Res}_{\mathcal{A}_1, \ldots, \mathcal{A}_{n+1}}$.

Proof. From Proposition 4.4, we have that Det $\in \mathbb{Q}[c_{i,b}]$. Consider its factorization in irreducibles in this ring: Det $= \prod_{j=1}^{N} p_j^{e_j}$. If p_j does not coincide up to a nonzero rational factor with $\mathrm{Elim}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$, then the set $\{p_j = 0\} \cap \{\mathrm{Elim}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}} \neq 0\}$ is not empty in any algebraically closed field of characteristic zero K. Pick a system of polynomials $F_1, \ldots, F_{n+1} \in S_K$ with coefficients in this set. From Theorem 1.7, the specialized complex $Kos(S'_K, F'_1, \ldots, F'_{n+1})_{\alpha_\delta+\alpha_Q+\alpha_P}$ is exact. But this implies that the map Ψ_1 is onto, so one of the maximal minors of it must be nonzero. As p_j is a common factor of all these minors, this gives a contradiction.

So, we have that Det $= \lambda\, \mathrm{Elim}^e_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$ for a suitable $e \in \mathbb{N}$, and $\lambda \in \mathbb{Q}^\times$. To compute e we use Corollary 4.3 and Proposition 2.8(1). We then have that Det $= \lambda\, \mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$. On the other hand, all the maximal minors of Ψ_1 belong to $\mathbb{Z}[c_{i,b}]$, and we have just shown that there are no irreducible elements in $\mathbb{Q}[c_{i,b}]$ dividing it apart from $\mathrm{Elim}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$. So, the gcd of all these maximal minors is equal to $z \in \mathbb{Z}^\times$ times $\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$. To show that $z = \pm 1$, it is enough to exhibit a minor which is not zero modulo p for any prime integer $p \in \mathbb{Z}$. These can be found from the Canny-Emiris type matrices produced in [DJS23]. For instance, in Proposition 4.13 in loc. cit it is shown that there exists a nonzero maximal minor of Ψ_1 whose initial term with respect to some monomial order equals a monomial, i.e. this term does not vanish modulo p for any prime $p \in \mathbb{Z}$. This completes the proof. □

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