

New derivation for the equations of motion for particles in electromagnetism

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We present equations of motion for charged particles using balanced equations, and without introducing explicitly divergent quantities. This derivation contains as particular cases some well known equations of motion, as the Lorentz-Dirac equations. An study of our main equations in terms of order of the interaction with the external field conduces us to the Landau-Lifshitz equations. We find that the analysis in second order show a special behavior. We give an explicit presentation up to third order of our main equations, and expressions for the calculation of general orders.

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I. INTRODUCTION

The main question we would like to study in this article is how far can the notion of a particle, as a point like object be extended, in the realm of classical electromagnetism. Of course in the textbooks one encounters extensive discussions of test particles; which are normally used for the very definition of the electromagnetic fields; namely, the electromagnetic fields are those that enter into the Lorentz force acting on test particles. However, the physical system becomes more complicated when one considers finite charged particles. On the one hand, such a particle radiates, and therefore the equations of motion should reflect the loss of energy momentum of the particle. On the other hand, a finite point like source involves fields that have a divergent behavior as one approaches the particle; which in turn would imply a divergent contribution to the total stress energy-momentum tensor of the system.

This problem has been tackled in the past using different techniques, as, for example, the idea of studying “spheres” in the limit when they become smaller and smaller. In fact this guided the early works of Lorentz, Abraham, and Dirac [1]. In this article we present a discussion of the problem in which we avoid dealing with infinite terms. Our final result is a generalization of the equations found in the past, which contain previous works as particular choices.

In the next section we present the basic notation used below. Section III is devoted to the presentation of the balance equations, which is our main tool in this work. The deduced equations of motion are a generalization of other similar equations found in the literature. We also present in this section a generalized notion of total momentum for the charged particle. Notably the new equations of motion involve new degrees of freedom, that are discussed below. A particular choice of the new degrees of freedom conduces to the celebrated Lorentz-Dirac equations of motion. In Sec. IV we study the nature of the solutions to the general problem. Those solutions present difficulties in the physical interpretation, which motivate us to study the equations of motion in terms of orders of the

strength of the interaction with the external fields, which is presented in Sec. V. In the last section, we summarize our results. An appendix is added where some properties of the solutions are presented.

II. DEALING WITH PARTICLES IN ELECTROMAGNETISM**A. Basic equations of a test charged particle**

Let us start by quickly reviewing the main equations of electromagnetism. The dynamics of the electromagnetic fields is governed by Maxwell equations,

$$\nabla_a F^{ab} = \kappa J^b, \quad (1)$$

and

$$\nabla_{[a} F_{bc]} = 0. \quad (2)$$

While the dynamic of *test particles* is determined by the Lorentz force f^a given by

$$f^a = q F^{ab} v_b, \quad (3)$$

where one is considering a test particle of charge q and four-velocity v^a .

The immediate question is what is the force that acts on a particle? (not test particle).

The nature of the problem is that a (nontest) particle radiates if it is accelerated; and therefore the use of the Lorentz force will imply an imbalance of energy and momentum. Several approaches have appeared to answer this question, and Lorentz, Abraham, and Dirac [1] have made important contributions to this goal, which have ended in what is widely known as the Lorentz-Dirac equations of motion (see below). We here present these well-known equations of motion as a special case of our equations by demanding balance of energy and momentum at an asymptotic sphere defined by the intersection of the future null cone of the particle with future null infinity.

B. Notation associated with a timelike curve

Let us start with a massive point particle with mass m_A whose trajectory, in a flat space-time (M, η_{ab}) , is given by the timelike curve C , which in a Cartesian coordinate system x^a reads

$$x^a = z^a(\tau), \quad (4)$$

with τ meaning the proper time of the particle along C .

The associated four-velocity of this particle is

$$v^a = \frac{dz^a}{d\tau}, \quad (5)$$

and the signature of the flat metric is chosen such that $v_a v^a = 1$. (Note that we are using units in which $c = 1$, so that v^a has no units.) Now, for each point $Q = z(\tau)$ of C , we draw a future null cone $\mathcal{C}_Q$ with vertex in Q . If we call x_P^a the Minkowskian coordinates of an arbitrary point on the cone $\mathcal{C}_Q$, then we can define the retarded radial distance on the null cone by

$$r = v_a(x_P^a - z^a(\tau)). \quad (6)$$

A null vector l^a is defined by

$$l^a = \frac{x_P^a - z^a(\tau)}{r}. \quad (7)$$

And since l^a is null, one has

$$l_a l^a = 0. \quad (8)$$

Then one can see that

$$v_a l^a = 1. \quad (9)$$

One can introduce null polar coordinates associated to the timelike curve C in the following way. Let u be the null coordinate which is constant on the future null cones defined for points Q of C and such that $u = \tau(Q)$. Let $(\zeta, \bar{\zeta})$ be stereographic coordinates of the sphere of directions at the point Q . Then, the relation between Minkowskian coordinates x^a and null polar coordinates $(u, r, \zeta, \bar{\zeta})$ is given by

$$x^a = z^a(u) + r l^a(u, \zeta, \bar{\zeta}). \quad (10)$$

Let us note that by defining $\hat{l}^a$ to be the null vector that corresponds to a stationary motion; which implies that $\hat{l}^a = \hat{l}^a(\zeta, \bar{\zeta})$, one can see that l^a and $\hat{l}^a$ will be proportional. In particular, defining

$$V(u, \zeta, \bar{\zeta}) = v^a(u) \hat{l}_a(\zeta, \bar{\zeta}), \quad (11)$$

one can see that

$$l_a(u, \zeta, \bar{\zeta}) = \frac{1}{V(u, \zeta, \bar{\zeta})} \hat{l}_a(\zeta, \bar{\zeta}). \quad (12)$$

C. Electromagnetic fields of a nontest charged particle

The retarded electromagnetic field of a particle with charge e , can be given in terms of the potential and/or the electromagnetic field itself.

The retarded potential is

$$A^a(x) = \frac{e v^a(u)}{r}. \quad (13)$$

The corresponding electromagnetic field is [2]

$$\begin{aligned} F_{ab} &= 2e \left(\frac{1}{rV} \hat{l}_{[a} \dot{V}_{b]} + \frac{1}{r^2 V} \left(1 - \frac{r \dot{V}}{V} \right) \hat{l}_{[a} V_{b]} \right) \\ &= 2e \left(\frac{1}{r} \left[l_{[a} \dot{V}_{b]} - \frac{\dot{V}}{V} l_{[a} V_{b]} \right] + \frac{1}{r^2} l_{[a} V_{b]} \right), \end{aligned} \quad (14)$$

where a dot means derivative with respect to u and we have chosen Gaussian units, so that $\kappa = 4\pi$.

It is convenient to also have at hand the spinor components of the electromagnetic field, which are given by

$$\phi_0 = 0, \quad (15)$$

$$\phi_1 = -\frac{e}{2r^2}, \quad (16)$$

and

$$\phi_2 = -\frac{eV}{r} \bar{\delta} \left(\frac{\dot{V}}{V} \right). \quad (17)$$

The symbol $\bar{\delta}$ denotes the edth operator [3] of the unit sphere, and $\bar{\delta}$ is its complex conjugate.

III. BALANCE EQUATIONS FOR A NONTEST CHARGED PARTICLE

A. The conservation law for the total energy-momentum tensor

The main tool to derive the balanced equations of motion for charged particles is the conservation law of the total energy-momentum tensor, namely

$$\nabla \cdot T = 0. \quad (18)$$

Let $K_{\underline{c}}^b$ be four translational Killing vectors; where we are using the numeric index $\underline{c} = 0, 1, 2, 3$ and the abstract index b . Then from the conservation law (18) one also obtains that

$$\nabla_a (T^a_b K_{\underline{c}}^b) = 0. \quad (19)$$

Then, let $\mathcal{V}$ be the four volume which has as boundaries the two spacelike hypersurfaces, Σ' at its future boundary and Σ at its past boundary, as depicted in Fig. 1. Also, each spacelike hypersurface has the same two dimensional boundary S .

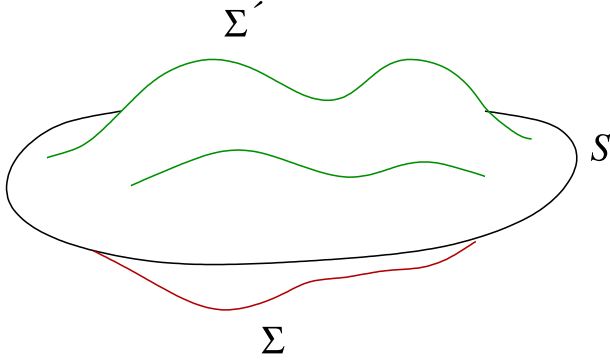


FIG. 1 (color online). Two spacelike hypersurfaces, Σ' and Σ , are the boundary of the internal four volume $\mathcal{V}$. In turn, each of these spacelike hypersurfaces has the same two dimensional boundary S .

Given a three form D , Stoke's theorem tells us that

$$\int_{\mathcal{V}} dD = \int_{\Sigma'} D - \int_{\Sigma} D. \quad (20)$$

In our case we take, for each Killing vector $K_{\underline{c}}^b$, the three form $D_{\underline{c}abc} = T^d_e K_{\underline{c}}^e \epsilon_{abcd}$, so that $dD_{\underline{c}abcd} = k \nabla_f (T^f_e K_{\underline{c}}^e) \epsilon_{abcd}$, where k is a constant.

Therefore our main equation is

$$0 = \int_{\mathcal{V}} dD_{\underline{c}} = \int_{\Sigma'} D_{\underline{c}} - \int_{\Sigma} D_{\underline{c}}, \quad (21)$$

since due to the conservation equation (19) the left-hand side vanishes. It is because of this reason that actually the total momentum is determined by S , and not by the particular hypersurface with boundary S .

This equation is intimately related to the equations of motion, since Eqs. (21) are telling us that the difference of the total momentum calculated at Σ' and Σ vanishes. We will apply these equations to the case of a particle, even if one has indications that each term in the difference could be ill defined; since it could contain infinite terms, however its difference is finite.

Let S be a sphere at future null infinity defined as the asymptotic sphere of the future null cone of a point $Q(\tau)$, and let Σ be the future null cone of this point. Let $\tilde{S}$ be the corresponding asymptotic sphere for the point $Q(\tau + d\tau)$ on the curve C , and let $\tilde{\Sigma}$ be the future null cone of this point. Let us call Σ^+ the hypersurface at future null infinity with boundaries S and $\tilde{S}$. Then in the last equation we can identify $\Sigma' = \tilde{\Sigma} \cup \Sigma^+$. See Fig. 2 for a graphical representation of the hypersurfaces.

Therefore, assuming the same integrand, one has the relation

$$\int_{\tilde{\Sigma}} - \int_{\Sigma} = - \int_{\Sigma^+}, \quad (22)$$

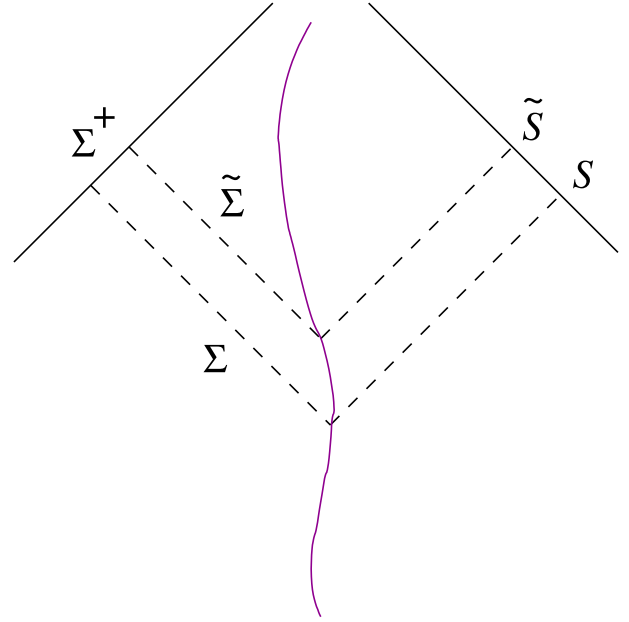


FIG. 2 (color online). Two hypersurfaces reaching future null infinity.

which says that the momentum at $\tilde{S}$ is the momentum at S , minus the flux through Σ^+ .

Let us differentiate between a charged particle, which we will call the system of interest A , and the rest of the world, which we will call system B . Then, the total energy-momentum tensor can be decompose as the following sum of terms:

$$T = T_{(m)A} + T_{(m)B} + T_{(EM)A} + T_{(EM)B} + T_{(EM)A.B}, \quad (23)$$

where we have distinguished: the mechanical term (m) of particle A , the mechanical term of system B , the electromagnetic term (EM) of particle A , the electromagnetic term of system B , the electromagnetic term of the products of fields of particle A times fields of system B .

Considering the vector $T_b^a K_{\underline{c}}^b$; whose divergence is zero, one can apply Eq. (22). Each term of (23) contributes on the left-hand side with a corresponding dP term, i.e. a difference of momentum at both null hypersurfaces. The right-hand side can be expressed as

$$- \int_{\Sigma^+} = - \int_S d\tau, \quad (24)$$

where S is the intersection of the null cone Σ with future null infinity, and we are considering an infinitesimal $d\tau$.

Therefore, one has

$$\begin{aligned} & dP_{(m)A} + dP_{(m)B} + dP_{(EM)A} + dP_{(EM)B} \\ & + dP_{(EM)A.B} + dP_{(EM)B.A} \\ & = - \int_S T_{ab} K^b n^a \frac{dS^2}{V^2} d\tau, \end{aligned} \quad (25)$$

where K^b is a translational Killing vector, n^a is the null vector normal to future null infinity satisfying $n^a l_a = 1$, and dS^2 is the surface element of the unit sphere. We have also indicated separately the difference in momentum of system A due to the existence of system B ($dP_{(\text{EM})A,B}$), and the difference in momentum of system B due to the existence of system A ($dP_{(\text{EM})B,A}$).

For the sake of simplicity in the discussion, let us consider first the case in which system B consists of a smooth distribution of matter and charges with no radiation at future null infinity. Furthermore, let us assume that system B can be represented by a Lagrangian formulation, including the actions of system A on B . Therefore, under variations of the B fields, the Lagrangian will induce the equations of motion

$$\frac{dP_{(\text{m})B}}{d\tau} + \frac{dP_{(\text{EM})B}}{d\tau} + \frac{dP_{(\text{EM})B,A}}{d\tau} = 0; \quad (26)$$

in other words, the terms $dP_{(\text{m})B} + dP_{(\text{EM})B} + dP_{(\text{EM})B,A}$ are balanced in Eq. (25), which implies

$$\frac{dP_{(\text{m})A}}{d\tau} + \frac{dP_{(\text{EM})A}}{d\tau} = -\frac{dP_{(\text{EM})A,B}}{d\tau} - \int_S T_{ab} K^b n^a \frac{dS^2}{V^2}. \quad (27)$$

Then one can think that the term $dP_{(\text{EM})A}$ could contain infinite contributions at the particle world line C , since the self fields will be taken into account. However, we know that the right-hand side of the equation is finite; therefore, the sum of both terms on the left is finite.

To calculate the flux term it is convenient to note that

$$\frac{1}{4\pi} \int l^a \frac{dS^2}{V^2} = v^a, \quad (28)$$

$$\frac{1}{4\pi} \int l^a l^b \frac{dS^2}{V^2} = \frac{4}{3} v^a v^b - \frac{1}{3} \eta^{ab} \quad (29)$$

and

$$\frac{1}{4\pi} \int l^a l^b l^c \frac{dS^2}{V^2} = 2v^a v^b v^c - \frac{1}{3} (\eta^{ab} v^c + \eta^{bc} v^a + \eta^{ca} v^b), \quad (30)$$

which are a generalization of the relations that appear in [4].

The radiative part of T_{ab} is given by

$$t_{ab} = \frac{1}{2\pi} \phi_2 \bar{\phi}_2 l_a l_b, \quad (31)$$

so that the integration on S is more precisely

$$\begin{aligned} \int_S \frac{1}{2\pi} \phi_2 \bar{\phi}_2 l_\epsilon \frac{dS^2}{V^2} &= \frac{e^2}{2\pi} \int_S V^2 \bar{\delta} \left(\frac{\dot{V}}{V} \right) \bar{\delta} \left(\frac{\dot{V}}{V} \right) l_\epsilon \frac{dS^2}{V^2} \\ &= \frac{e^2}{2\pi} \int_S \bar{\delta}_V \left(\frac{\dot{V}}{V} \right) \bar{\delta}_V \left(\frac{\dot{V}}{V} \right) l_\epsilon \frac{dS^2}{V^2}, \end{aligned} \quad (32)$$

where $\bar{\delta}_V$ is the edth [3] operator for the metric with surface element $\frac{dS^2}{V^2}$.

Now, let us note that for any two vectors A^a and B^b , and defining the quantities $A = A^a l_a$ and $B = B^a l_a$, one has

$$A^a B_a = AB + A \bar{\delta}_V \bar{\delta}_V B + B \bar{\delta}_V \bar{\delta}_V A - \bar{\delta}_V A \bar{\delta}_V B - \bar{\delta}_V A \bar{\delta}_V B, \quad (33)$$

where we are assuming the Minkowski metric in the contraction of vectors.

Therefore, noting that $\frac{\dot{V}}{V} = \dot{v}^a l_a$, one deduces that

$$\begin{aligned} \bar{\delta}_V \left(\frac{\dot{V}}{V} \right) \bar{\delta}_V \left(\frac{\dot{V}}{V} \right) &= \frac{1}{2} \left(\frac{\dot{V}}{V} \right)^2 + \frac{\dot{V}}{V} \bar{\delta}_V \bar{\delta}_V \frac{\dot{V}}{V} - \frac{1}{2} \dot{v}^a \dot{v}_a \\ &= -\frac{1}{2} \left(\frac{\dot{V}}{V} \right)^2 - \frac{1}{2} \dot{v}^a \dot{v}_a \\ &= -\frac{1}{2} (\dot{v}^a l_a \dot{v}^b l_b + \dot{v}^a \dot{v}_a), \end{aligned} \quad (34)$$

where we have used that

$$\bar{\delta}_V \bar{\delta}_V \frac{\dot{V}}{V} = -\frac{\dot{V}}{V}. \quad (35)$$

Then, Eq. (32) becomes

$$\begin{aligned} \int_S \frac{1}{2\pi} \phi_2 \bar{\phi}_2 l_\epsilon \frac{dS^2}{V^2} &= \frac{e^2}{2\pi} \int_S \bar{\delta}_V \left(\frac{\dot{V}}{V} \right) \bar{\delta}_V \left(\frac{\dot{V}}{V} \right) l_\epsilon \frac{dS^2}{V^2} \\ &= -\frac{e^2}{4\pi} \int_S (\dot{v}^a l_a \dot{v}^b l_b + \dot{v}^a \dot{v}_a) l_\epsilon \frac{dS^2}{V^2} \\ &= -e^2 \left[\left(-\frac{1}{3} \dot{v}^a \dot{v}^b \eta_{ab} v_\epsilon + \dot{v}^a \dot{v}_a v_\epsilon \right) \right] \\ &= -\frac{2}{3} e^2 \dot{v}^a \dot{v}_a v_\epsilon. \end{aligned} \quad (36)$$

Finally, the flux term contributes with

$$-\int_S T_{ab} K_\epsilon^b n^a \frac{dS^2}{V^2} = \frac{2}{3} e^2 \dot{v}^a \dot{v}_a v_\epsilon. \quad (37)$$

The other term on the right-hand side of (27) includes all the contributions due to forces that system B exerts on particle A ; for the case of electromagnetic interactions one has

$$-\frac{dP_{(\text{EM})A,B\epsilon}}{d\tau} = -eF(B)_{a\epsilon} v^a = eF(B)_{\epsilon a} v^a, \quad (38)$$

which is the standard Lorentz force. In order to elucidate the role of units, we clarify that we are using units of time for τ (the proper time), so that a dot, or proper time derivative, implies an extra 1/second in any expression.

The other terms on the left-hand side of (27), not only include the mechanical term proportional to the acceleration $\dot{v}_\epsilon$, but they also include other terms coming from the contributions due to the self-fields of the particle. The main idea in our presentation is not to treat both terms separately but as a unity, namely, the variation of the total momentum.

In the customary treatment in which the terms are treated separately, one has to deal with infinities and complicated arguments for their cancellation. We show here that the variation of the total momentum does not include difficulties.

The variation of the total momentum does not have arbitrary vectorial dependence, since to begin with, the mechanical contribution only depends on the four-velocity, and the energy-momentum tensor depends on the electromagnetic tensor, and this depends on the velocity and acceleration. Therefore, the time derivative of the total momentum can only depend at most on $(v^c, \dot{v}^c, \ddot{v}^c)$; in other words, it cannot depend on higher derivatives like $(\ddot{v}^c)$.

The coefficients of this vectors must be finite, since the right-hand side is finite. Therefore, although at first sight one could expect divergent behavior for the expressions of the self-fields, one must understand that there are cancellation of infinite like terms between the mechanical and the field momenta, that as a result provide with the finite coefficients.

In other words, we do not attempt to decompose the variation of total momentum into a mechanical and electromagnetic part, but consider an expression for the variation of the total momentum, which we know is finite, and of the form

$$\dot{P}_c = m'v_c + m\dot{v}_c + \alpha\ddot{v}_c, \quad (39)$$

where m' , m and α are understood as functions of τ .

From Eq. (27) we conclude then that the equations of motion can be written as

$$m'v_c + m\dot{v}_c + \alpha\ddot{v}_c = eF(B)_{ca}v^a + \frac{2}{3}e^2\dot{v}^a\dot{v}_av_c. \quad (40)$$

Let us also note that the total momentum must be expressed as

$$P_c = Mv_c + \alpha\dot{v}_c, \quad (41)$$

with $M = M(\tau)$. This expression for the total momentum is a generalization of expression (4.4) found in Ref. [5] through renormalization procedures.

By taking the time derivative of the last equation and comparing it with Eq. (39), we get

$$\dot{M} = m', \quad (42)$$

$$M + \dot{\alpha} = m. \quad (43)$$

Let us note that there are at least two notions of mass. One has the parameters m , which we will call *inertial mass*. Also one has the parameter M , that we will call *rest mass*, in order to differentiate them.

Therefore the final form of the equations of motion is

$$M\dot{v}_c + (M + \dot{\alpha})v_c + \alpha\ddot{v}_c = eF(B)_{ca}v^a + \frac{2}{3}e^2\dot{v}^a\dot{v}_av_c. \quad (44)$$

This is our main result, the most general equations of motion for charged particles in the framework we have

presented, in which it is required the balance of radiated momentum. Let us note that in our approach we have avoided treating explicitly the elimination of infinite renormalizations or similar techniques.

We recognize new degrees of freedom present in this form of the equations of motion, namely the scalar $\alpha(\tau)$, and also $M(\tau)$. The latter has been considered already by other authors (see below).

Now, if we contract the expression Eq. (44) with the vector v^c , and using the notation $-a^2 \equiv \dot{v}^a\dot{v}_a$, we obtain

$$\dot{M} + \alpha a^2 = -\frac{2}{3}e^2 a^2. \quad (45)$$

Since this is one equation for two unknown, one must prescribe $\alpha(\tau)$ or $M(\tau)$, or a relation among them, as we will do below.

It is interesting to note that if we replace $\dot{M}$ in the last equation into (44) one obtains

$$(M + \dot{\alpha})\dot{v}_c = eF(B)_{ca}v^a - \alpha(\ddot{v}_c - a^2v_c), \quad (46)$$

which only sets the dynamics for 3 degrees of freedom since it is orthogonal to the four-velocity v . Therefore, one has the option to consider the original equations (44), or the two independent equations (46) and (45).

In presenting Eqs. (44)–(46) we have given preference to the rest mass M ; alternatively, if one gives preference to the inertial mass m , one can reexpress them as

$$(\dot{m} - \ddot{\alpha})v_c + m\dot{v}_c + \alpha\ddot{v}_c = eF(B)_{ca}v^a + \frac{2}{3}e^2\dot{v}^a\dot{v}_av_c, \quad (47)$$

$$\dot{m} - \ddot{\alpha} + \alpha a^2 = -\frac{2}{3}e^2 a^2, \quad (48)$$

and

$$m\dot{v}_c = eF(B)_{ca}v^a - \alpha(\ddot{v}_c - a^2v_c). \quad (49)$$

In summary, we have arrived at the general equations of motion for a charged particle, in which two new degrees of freedom appear, and also the order of the equations for v has changed, from the Lorentz equations. So, the set of basic variables, can be considered (v^a, M, α) or (v^a, m, α) , where the velocity vector is in turn expressed in terms of the position of the particle. Since v is always assumed to have unit modulus, one has four equations for 5 degrees of freedom. Therefore, one has the liberty to choose a relation among the two new degrees of freedom. We will discuss below how different choices conduce us to known cases and also to new physically interesting ones.

It is probably worthwhile to emphasize that we have arrived at these general equations of motion by balancing the retarded radiation field of the charged particle when the rest of the system does not radiate. This is a very strong assumption, on the rest of the electromagnetic system, and, in particular, would exclude the interaction of the particle with perfect conductors, or a dispersive permeable medium, for example. Our approach is a technique valuable

to calculate the “corrections” to the Lorentz force that take into account the retarded radiation fields; however, after one has calculated such corrections, one would expect these to be valid in a general situation.

B. Historical choices on the general equations of motion

1. The Lorentz-Dirac equations

We have seen before that there are two natural notions of mass that appear in our treatment. Therefore, it is interesting to consider the particular cases in which each of them is required to be a constant of the motion; since, in particular, real elementary particles as the electron seem to have a constant mass.

The case $\dot{m} = 0$ will be considered below; we will consider here the case $\dot{M} = 0$.

This particular choice can be understood as follows. From (41), we see that $M = m - \dot{\alpha}$ can be interpreted as the rest mass of the electron (which has contributions of the electromagnetic field generated by the particle). Then if we want to describe particles whose total rest mass remains constant, we must require that $\dot{M} = \dot{m} - \ddot{\alpha} = 0$, then from Eq. (45), we obtain that

$$\alpha = -\frac{2}{3}e^2. \quad (50)$$

Furthermore, one can also deduce from Eq. (43) that the two notions of mass coincide, and therefore one also has $\dot{m} = 0$.

Therefore, one arrives at the following equations of motion for the charge given by

$$m\dot{v}^a = eF(B)^a_b v^b + \frac{2}{3}e^2(\ddot{v}^a + \dot{v}^b \dot{v}_b v^a), \quad (51)$$

which are the well-known Lorentz-Dirac equations of motion for the electron.

It is probably worthwhile to recall that these equations have several problems, as it has been investigated in the past by several authors. First of all they are third order differential equations for the position of the charged particle, which is contrary to the generally accepted idea of mechanics for particle. Second, they have the so-called problem of the runaway solutions. Several of the implied problems were discussed by Dirac and we will not review them here.

2. The Bonnor equations

Another approach to the two mass question is to require that the two notions coincide. Then, let us consider the choice $M = m$. From this one immediately obtains that $\dot{\alpha} = 0$, and we recognize that the choice $\alpha = -\frac{2}{3}e^2$ coincides with the previous case; but instead we study here the general $\dot{m} \neq 0$ case, which must satisfy

$$\dot{m} = -(\frac{2}{3}e^2 + \alpha)a^2, \quad (52)$$

where as noted before, α must be a constant.

In particular if we set $\alpha = 0$, one arrives at the equations of motion with varying mass, namely

$$m\dot{v}_c = eF(B)_{ca}v^a, \quad (53)$$

with

$$\dot{m} = -\frac{2}{3}e^2a^2. \quad (54)$$

This equation was studied by [6]. Unfortunately it has the unphysical consequence that the mass of the particle could vanish in a finite time.

IV. STUDY OF THE GENERAL EQUATIONS OF MOTION FOR CHARGED PARTICLES

A. Behavior of the general solution for two particular cases

We study here general properties of the solutions that can be deduced prior to the use of the extra liberty for the choice of a condition among the two new degrees of freedom.

a. The case $a^2 = 0$. Let us consider first the case in which $a^2 = 0$ for all times. In this case, all the terms in Eqs. (44) collapse to zero, and the particle moves along a geodesic. One can observe that the terms involving $\dot{v}$ and $\ddot{v}$ are zero; then the only two remaining terms containing the external fields and the first one proportional to v are orthogonal, so that they must vanish independently. In particular, one has $F(B) = 0$ and $\dot{M} = 0$.

b. The case $F(B) = 0$. The other case that it is important to be considered is the case in which the external fields $F(B)$ are zero, and study the general behavior of a^2 for large values of the time variable in order to see the nature of the runaway solutions problem in this setting. So now we invert the logic and think what is the behavior of a^2 in terms of the behavior of the new degrees of freedom M and α . We observe then that by contracting the equations of motion (44) with $\dot{v}$, one obtains

$$(M + \dot{\alpha})a^2 + \frac{\alpha}{2}\dot{a}^2 = 0. \quad (55)$$

This constitutes a simple first order differential equation for a^2 , whose behavior is determined by the sign of the coefficients. In this setting a runaway behavior implies that the coefficients have opposite signs. Since one can choose one relation for the two extra degrees of freedom, we would like to explore possible conditions for them. If one were interested in an asymptotic nonincreasing solution for a^2 , one would probably consider $(M + \dot{\alpha}) > 0$ and $\alpha > 0$. It is interesting to note that if one chooses as initial data for α and $\dot{\alpha}$ a tiny positive value for them, then Eq. (55) would imply an initial fast exponential decay of a^2 . This, will in turn imply, that initially $M \sim m$, and from (45) that $\dot{M}$ would become very small in a short time. This choice of very small initial conditions of the α degree of freedom avoid the problem of the runaway solutions, for the case in which $F(B) = 0$ for all times. Instead, the discussion

becomes more complicated if one thinks in the situation in which the external field is turned on and off. When the external field is turned on, one can arrange so that $\dot{a}^2 > 0$, which will imply from (55) that α rapidly adopts a negative value (provided one has managed to maintain $m = M + \dot{\alpha} > 0$). If later the external field is turned off again, then the particle will enter the region of vanishing field with the wrong initial condition and one would probably be in the presence of a runaway solution. So there is no universal choice of the initial conditions of the α degree of freedom that would exclude the runaway solutions. Also, the behavior $\alpha > 0$ would imply, from (45), a condition $\dot{M} < 0$, which could have the problem that M would vanish in a finite time, if a^2 were bounded from below by a nonzero value. From this analysis we can not exclude the possibility of a solution where both α and a^2 go asymptotically to zero, for large values of the time coordinate, but with a nonzero M value. The other possibility is to have $\alpha < 0$, but would conduce us to the runaway solution problem (provided one has managed to maintain $m = M + \dot{\alpha} > 0$). Therefore, we conclude that the runaway solution problem is generic in this dynamic too. The case $(M + \dot{\alpha}) < 0$ is ruled out since this would imply that the inertial mass is negative. The value $\alpha = 0$ was not considered due to the fact that it would collapse to the previously studied case of $a^2 = 0$ (since we are examining the case $F(B) = 0$).

B. Behavior of the solution with $\dot{m} = 0$

Let us recall that m can be understood as the inertial mass. The reason that condition $\dot{m} = 0$ is worth studying comes also from the following considerations. The time derivative of the *rest mass*¹ of the particle is given by $\dot{M} = \dot{m} - \ddot{\alpha}$. Therefore, the condition $\dot{m} = 0$ means that the inertial mass is constant, and the variation in the rest mass is only due to the electromagnetic fields contribution to the mass. Note that the Lorentz-Dirac equations are contained in this family, as it is case $F(B) = 0$ considered in the last subsection.

Treating the equations of motion as exact to any order leads one to possible divergent behavior for the new degrees of freedom as we will see next.

In this case, Eq. (45) can be completely expressed in terms of α , which must satisfy

$$\ddot{\alpha} - a^2 \alpha = \frac{2}{3} e^2 a^2. \quad (56)$$

It is convenient to define $\beta = \alpha + \frac{2}{3} e^2$, since then Eq. (56), can be written as

$$\ddot{\beta} - a^2 \beta = 0. \quad (57)$$

In Appendix there is a brief discussion of the properties of solutions of Eq. (57) in terms of the global properties of

¹Recall that M is the factor of v in the expression for the total momentum.

$a^2(\tau)$. There, a list of theorems is presented where non-divergent properties for a^2 are assumed for large values of τ . These results suggest that one could in principle be able to choose initial conditions for β to select decaying solutions. That is, one in general would find independent solutions for β ; one that would grow with time and the other that would decay to zero. It is somehow remarkable that one could select a solution for β so that α tends asymptotically to the Lorentz-Dirac value. However, we emphasize that in order to find the decreasing solution one must know the whole future history of the motion of the particle; therefore, this is a nonlocal analysis of the motion, which, in particular, is not based just on initial conditions.

Let us note that defining the vector $f^a = eF(B)^a_b v^b$, Eqs. (49) become

$$m\dot{v}^a = f^a - \alpha(\dot{v}^a - a^2 v^a), \quad (58)$$

which with (56) constitute two independent equations. Equation (58) has the same form as the Lorentz-Dirac equations, and it would agree with it if one would take the solution $\alpha = -\frac{2}{3} e^2$, as mentioned above.

There is a general concern with the dynamical equations, which involves the choice of two constants for the initial conditions for Eq. (56). Then in turn one would have the set of solutions of the equations of motion (44) affected by the arbitrary initial choice for the α degree of freedom.

V. DYNAMICS IN TERMS OF ORDERS OF THE STRENGTH OF THE INTERACTION WITH EXTERNAL FIELDS

The difficulties found in the study of the general solutions, of the last section, suggest that the original equations of motion must be understood in terms of an analysis in orders of the interaction with the external fields, which we present next.

A. Equation of motion up to second order in the external fields

It is important to remark that Eqs. (44) are exact equations in the classical framework for charged particles. It is generally believed that a classical description of particles must break down when one considers real elementary particles at microscopic levels. Presumably, one would be able to describe the behavior of the new degrees of freedom from quantum electrodynamics. All this suggests that Eqs. (44) should be understood in terms of orders of the interaction with the external field $F(B)$.

Therefore next, we present a study of our main equations (44) in first orders of the strength of the interaction with the external field, but still considering the physical condition $\dot{m} = 0$. This condition allows to interpret $M(\tau)$ in terms of $\alpha(\tau)$.

Note that contracting f^a with Eq. (44), one obtains

$$mf^a v_a + \alpha f^a \dot{v}_a = -\mathbf{f}^2, \quad (59)$$

with $\mathbf{f}^2 \equiv -f_a f^a$. At this point it is convenient to recall that the equations of motion for a test particle with negligible mass and charge μ and q , respectively, is

$$\mu \dot{v}_c = qF(B)_{ca} v^a, \quad (60)$$

from which one deduces that the behavior of $\alpha(\tau)$ is completely due to the interaction of the charge e with the external field F_{ab} . If one considers the correction to the equations of motion as arising from the strength of different terms appearing in Eq. (44), one is tempted to consider terms of order $\mathcal{O}(\mathbf{f}^p)$ and also to consider order in $\mathcal{O}(e^q)$, since the e^2 appears as an independent factor in the radiation term.

Let us note that from Eqs. (60) one can deduce that $\dot{v} = \mathcal{O}(\mathbf{f})$. Then, since we know that $\alpha = \mathcal{O}(\mathbf{f}^0)$, one would have, from Eq. (59) that $\dot{v} = \mathcal{O}(\mathbf{f}) = \mathcal{O}(\mathbf{f})$. Then, since α is expected to have some nontrivial $\mathcal{O}(e^q)$ order, the second term on the left-hand side of (59) is of higher order than the rest, as can also be deduced from the fact that

$$\dot{v} = \frac{1}{m} f + \mathcal{O}(\mathbf{f}^+), \quad (61)$$

where $\mathcal{O}(\mathbf{f}^+)$ means higher order than $(\mathbf{f})$, as for example $\mathcal{O}(e^2 \mathbf{f})$, and for simplicity we have omitted the vectorial index.

Now we will consider Eqs. (58) up to order $\mathcal{O}(\mathbf{f}^2)$. We therefore study Eq. (57) up to order $\mathcal{O}(\mathbf{f}^2)$. For this purpose we will use $a = \frac{f}{m}$ and suggest a β of the form

$$\beta(\tau) = A_0(\tau) + A_1(\tau)\mathbf{f} + A_2(\tau)\mathbf{f}^2 + \mathcal{O}(\mathbf{f}^3). \quad (62)$$

Then equating order like terms in Eq. (57), one has

$$\ddot{A}_0 = 0, \quad (63)$$

$$(A_1 \ddot{\mathbf{f}}) = 0, \quad (64)$$

and

$$(A_2 \ddot{\mathbf{f}}^2) = A_0 \frac{\mathbf{f}^2}{m^2}. \quad (65)$$

The solution of Eq. (64) implies two constants of integration that must be of $\mathcal{O}(\mathbf{f})$, but for the general physical situation of a nonstationary $\mathbf{f}$ we find no universal way to assign a constant. Therefore, we chose those two constants to be zero; in other words, we set $A_1 \mathbf{f} = 0$.

Instead, in the solution of (63) one finds $A_0 = A_{00} + A_{01}\tau$, where A_{00} and A_{01} are constants.

Replacing the solution of these terms in Eqs. (58) and keeping terms up to $\mathcal{O}(\mathbf{f}^2)$, one obtains

$$m\dot{v}^a = f^a + \left(\frac{2}{3}e^2 - A_0\right)\left(\frac{1}{m}\dot{f}^a - \frac{\mathbf{f}^2}{m^2}v^a\right). \quad (66)$$

It is hard to give a physical meaning to a dynamical system that depends on the two parameters involved in A_0 . Even if one chooses $A_{01} = 0$ to avoid the time dependence, one is still left with a constant that probably should be determined from another theoretical framework like quantum electrodynamics. From the classical point of view one could consider studying the cases in which A_0 is taken to be a constant proportional to e^2 . A peculiar case would be to consider $A_0 = \frac{2}{3}e^2$, since this choice would just cancel the other terms proportional to e^2 , and therefore conduce to the Lorentz force for the charged particle. This would be unacceptable in our approach to the equations of motion that takes into account the change in momentum due to the radiation emitted by the particle. However, considering a background quantum nature of particles, one could think in a value of the form

$$A_0 = \lambda \frac{h}{c^2}, \quad (67)$$

where h is Planck constant, c the velocity of light, and λ a number without units that should be determined from quantum electrodynamics. Then, since we are considering second order expressions, in the equations of motion (66), in place of $\dot{f}_c$ one must use

$$\begin{aligned} \dot{f}_{(2)c} &= (eF(\dot{B})_{cb}v^b) = e(F(\dot{B})_{cb})v^b + eF(B)_{cb}\dot{v}^b \\ &= e(F(\dot{B})_{cb})v^b + \frac{e}{m}F(B)_{cb}f^b. \end{aligned} \quad (68)$$

In a complete classical framework, one would not have any physical argument for a nonzero A_0 , and therefore one would be forced to take $A_0 = 0$, leading to the equations of motion

$$m\dot{v}^a = f^a + \frac{2}{3}e^2 \frac{1}{m}\dot{f}_{(2)}^a - \frac{2}{3}e^2 \frac{\mathbf{f}^2}{m^2}v^a, \quad (69)$$

which coincide with the Landau-Lifshitz [7] equations of motion.

B. Study of the equations of motion up to third order in the interaction with external fields

Now we would like to study Eqs. (44), or equivalently (49) up to third order.

Let us note that by having obtained the second order equations of motion in the previous section; one has a second order acceleration $\dot{v}_{(2)}$, namely the one that satisfies Eqs. (69). Then, third order in turn is defined to satisfy the equations of motion

$$m\dot{v}_{(3)}^b = f^b - \alpha_{(2)}(\ddot{v}_{(2)}^b - a_{(2)}^2 v^b), \quad (70)$$

where $\ddot{v}_{(2)}^b$ is the τ derivative of

$$\dot{v}_{(2)}^b = \frac{1}{m}f^b + \left(\frac{2}{3}e^2 - A_0\right)\frac{1}{m^2}\dot{f}_{(2)}^b - \left(\frac{2}{3}e^2 - A_0\right)\frac{\mathbf{f}^2}{m^3}v^b, \quad (71)$$

and one should consider terms up to order 3.

In this way one can see that this procedure can be generalized to any higher order where $\dot{v}_{(n)}^b$ satisfies the differential equations

$$m\dot{v}_{(n)}^b = f^b - \alpha_{(n-1)}(\ddot{v}_{(n-1)}^b - a_{(n-1)}^2 v^b), \quad (72)$$

where $\dot{v}_{(n-1)}^b$ is the time derivative of

$$\dot{v}_{(n-1)}^b = \frac{1}{m}f^b - \alpha_{(n-2)}(\ddot{v}_{(n-2)}^b - a_{(n-2)}^2 v^b), \quad (73)$$

and one should only consider terms up to order n .

Then, coming back to the third order calculation, let us consider β of the form

$$\beta(\tau) = A_0(\tau) + A_1(\tau)\mathbf{f} + A_2(\tau)\mathbf{f}^2 + A_3(\tau)\mathbf{f}^3 + \mathcal{O}(\mathbf{f}^4). \quad (74)$$

Then equating order like terms in Eq. (57) one obtains Eqs. (63) and (64), while instead of (65) one now has

$$(A_2\ddot{\mathbf{f}}^2) = A_0a_{(2)}^2, \quad (75)$$

and also

$$(A_3\ddot{\mathbf{f}}^3) = A_0a_{(3)}^2, \quad (76)$$

where $a_{(2)}^2$ and $a_{(3)}^2$ are the order $\mathcal{O}(\mathbf{f}^2)$ and $\mathcal{O}(\mathbf{f}^3)$, respectively, of a^2 .

If $A_0 \neq 0$, the main difficulty in this case is the fact that in the product βa^2 of Eq. (57) one needs the explicit integral of (75), which will involve a couple of integration constants [or order $\mathcal{O}(\mathbf{f}^2)$] associated to the choice of initial time for the integration. Since the integration constants must be of order $\mathcal{O}(\mathbf{f}^2)$, they cannot be associated just with the charge “ e ” or to universal constants; therefore, it would be very difficult to give physical meaning to a physical dynamical system that depends on the arbitrary choice of initial conditions [even if one manage to choose them of order $\mathcal{O}(\mathbf{f}^2)$].

All this seems to indicate that the only physically sensible choice is to take $A_0 = 0$, since in this case one would have to solve the homogeneous problem, with the natural choice for all arbitrary constants to be zero. In other words, β must be zero and we are conduced to the equations

$$m\dot{v}_{(3)}^b = f^b + \frac{2}{3}e^2(\ddot{v}_{(2)}^b - a_{(2)}^2 v^b), \quad (77)$$

where

$$\dot{v}_{(2)}^b = \frac{1}{m}f^b + \frac{2}{3}e^2\frac{1}{m^2}\dot{f}_{(2)}^b - \frac{2}{3}e^2\frac{\mathbf{f}^2}{m^3}v^b, \quad (78)$$

in which we have replaced $\alpha_{(2)} = -\frac{2}{3}e^2$ and one should consider terms up to order 3.

This difficulty will appear in higher orders too, from which we conclude that $\alpha_{(n \geq 2)} = -\frac{2}{3}e^2$, and that the second order case treated previously is sort of peculiar, since it is the only order that allows for a physically permissible nonzero β .

VI. FINAL COMMENTS

The equations of motion (44) are general equations for charged particles that are derived from the condition of balance of variation of total momentum with the radiated momentum. In this derivation, we have avoided dealing explicitly with infinite contributions to the momentum.

We have shown how different versions of equations of motion for charged particles can be obtained from our general equations, in particular, the celebrated Lorentz-Dirac equations.

Studying the properties of solutions to the general equations of motion we have found the possibility to choose initial conditions for the new degrees of freedom, which select decaying modes to the Lorentz-Dirac value. However, this choice is possible only if one knows the whole history of the world line of the particle. This is reminiscent of the notion of a horizon in general relativity, which depends on the whole history of the space-time. Even if one manage to choose these preferable initial conditions, one is still left with the problem of runaway solutions, or equivalently of pre-accelerations. If one were forced to a theoretical framework based purely on classical considerations, this would be a sort of dead-end. Nonetheless, if one uses the known physical information that the ultimate nature of real particles is of a quantum kind, and therefore classical theoretical frameworks should be understood as approximate models of the real world, one would think that one should not demand a real physical interpretation to the exact classical equations. Instead one is tempted to consider the corrections to the Lorentz force as also being related to terms that probably should be calculated from quantum electrodynamics. If so, then the original equations should be understood in terms of orders of the strength of the interactions with the external fields. In particular, there is a natural limit to the strength of the interactions that must be considered in order to avoid the quantum creation of pairs of particles.

In the study in terms of orders of the strength of interaction with the external fields we have found at second order the equations of motion (69). These equations have also been supported in the derivation of Ref. [8] based on a different setting, namely the study of first effects for particles with “small” charge and mass. Instead, here we have considered finite charge and mass particles but study the exact equations of motion in terms of orders of the strength of the interaction with the external fields. Then, it is no surprise that both approaches agree at first orders.

It is important to remark however that the original Landau-Lifshitz equations are of second order and

coincide with our equations (69). Our second order equations (66) could account for first order quantum corrections to the classical equations.

Instead, the equations of motion (72) and (73) presented here, are a generalization valid up to any desired higher order.

Summarizing, although we have found more general equations of motion for charged particles, we have shown that our set contains the main cases studied in the past. We also give arguments that indicate that one should not take the exact general equations (44) as the physically relevant ones. Instead, to our understanding the only physically reasonable treatment of the equations of motion (44) is through the notion of finite orders in terms of $\mathcal{O}(\mathbf{f})$, so that we conclude that the equations of motion applicable to classical particles, but with finite charge, are (69), in second order, or Eqs. (72) and (73) in higher orders.

The Appendix presents the properties of the solutions to Eq. (57).

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APPENDIX A: PROPERTIES OF β FUNCTION

Here we would like to recall some of the properties of the solutions to Eq. (57).

Th. A.1² Let $a^2(\tau) > 0$ be continuous in $(-\infty, \infty)$. Then the equation,

$$\ddot{\beta} - a^2\beta = 0, \quad (57)$$

has one and only one solution $\beta_1(\tau)$ passing through $(0,1)$, which is positive and strictly decreasing for all τ and one and only one solution $\beta_2(\tau)$ through $(0,1)$, which is positive and strictly increasing for all τ . Furthermore,

$$a\beta_1 \in L_2(0, \infty), \quad \dot{\beta}_1 \in L_2(0, \infty).$$

The notation for $L_2(0, \infty)$ comes from the Lebesgue spaces $L_p(b, c)$, which contain the set of integrable functions such that

$$\|f\|_p = \left[\int_b^c |f|^p d\tau \right]^{1/p} < \infty. \quad (A1)$$

Th. A.2 If $0 < a_{\inf}^2 < a^2 < a_{\sup}^2 < \infty$, then the solution β_1 satisfies

$$e^{-a_{\inf}\tau} \leq \beta(\tau) \leq e^{-a_{\sup}\tau} \quad (A2)$$

for $\tau > 0$.

²Page 359 in Ref. [9].

Th. A.3³ Let a^2 be positive and continuous in $[0, \infty)$ and posses continuous first and second order derivatives. Set

$$H = \frac{\frac{5}{16}(a^2)^2 - \frac{1}{4}a^2\ddot{a}^2}{a^5}, \quad (A3)$$

and suppose that

$$H \in L_1(0, \infty). \quad (A4)$$

Then there exist constants c_1 and c_2 such that

$$\beta_1(\tau) = c_1 \frac{-e^{\int_0^\tau a(\tau')d\tau'}}{\sqrt{a}} (1 + R_1(\tau)), \quad (A5)$$

$$\beta_2(\tau) = c_2 \frac{e^{\int_0^\tau a(\tau')d\tau'}}{\sqrt{a}} (1 + R_2(\tau)), \quad (A6)$$

where

$$|R_1(\tau)| \leq e^{\int_\tau^\infty |H(\tau')|d\tau'} - 1, \quad (A7)$$

and a positive constant c such that $|R_2(\tau)| \leq c|R_1(\tau)|$.

Th. A.4⁴ In the equation

$$\ddot{\beta} - a^2\beta = 0 \quad (57)$$

let a^2 be a continuous complex-valued function for large τ satisfying

$$\int_\tau^\infty \tau |a^2(\tau)| d\tau < \infty, \quad (A8)$$

or, more generally

$$A(\tau) \equiv \int_\tau^\infty a^2(\tau') d\tau' = \lim_{T \rightarrow \infty} \int_\tau^T a^2(\tau') d\tau' \text{ exists} \quad \text{and} \quad \int_\tau^\infty \sup_{\tau \leq r < \infty} |A(r)| d\tau < \infty.$$

Then, there is a pair of solutions β_0 and β_1 satisfying, as $\tau \rightarrow \infty$,

$$\beta_0(\tau) \sim 1, \quad \dot{\beta}_0(\tau) = o\left(\frac{1}{\tau}\right), \quad (A9)$$

$$\beta_1(\tau) \sim \tau, \quad \dot{\beta}_1(\tau) \sim 1. \quad (A10)$$

Conversely, if $a^2(\tau)$ is real valued and does not change signs and if (57) has a solution satisfying (A9) or (A10), then (A8) holds.

Th. A.5⁵ In the equation

$$\ddot{\beta} - (\lambda^2 + q(\tau))\beta = 0, \quad (A11)$$

let $\lambda > 0$ and $q(\tau)$ be a complex-valued continuous function for large τ satisfying

³Page 445 in Ref. [9].

⁴Page 380 in Ref. [10].

⁵Page 381 in Ref. [10].

$$\int^{\infty} |q(\tau)| d\tau < \infty, \quad (\text{A12})$$

or, more generally,

$$\int^{\infty} q(t) e^{-2\lambda t} dt = \lim_{T \rightarrow \infty} \int^T \text{exists} \quad \text{and} \quad (\text{A13})$$

$$\int^{\infty} e^{2\lambda \tau} \sup_{\tau \leq s < \infty} \left| \int_s^{\infty} q(r) e^{-2\lambda r} dr \right| d\tau < \infty.$$

Then, (A11) has solutions β_0, β_1 satisfying

$$\beta_0 \sim -\frac{\dot{\beta}_0}{\lambda} \sim e^{-\lambda \tau}, \quad \beta_1 \sim -\frac{\dot{\beta}_1}{\lambda} \sim e^{\lambda \tau}. \quad (\text{A14})$$

Conversely, if $q(\tau)$ is real valued and does not change signs and if (A11) has a solution β_0 or β_1 satisfying the corresponding conditions in (A14), then (A12) holds.

An example of a solution that does not satisfy any of the hypotheses of the theorems is the case $a^2 = \frac{2}{\tau}$. The two independent solutions for β are proportional to τ^2 and to $1/\tau$.

However, let us note that the case $a^2 = \frac{2}{\tau^2 + \delta^2} + \epsilon^2$, satisfies the hypothesis of the last theorem, which states that there is an exponentially decreasing solution of the form $\beta_0 \sim e^{-\epsilon \tau}$.

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