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2012 Class. Quantum Grav. 29 073001

(http://iopscience.iop.org/0264-9381/29/7/073001)

IOP PUBLISHING CLASSICAL AND QUANTUM GRAVITY

Class. Quantum Grav. **29** (2012) 073001 (45pp) doi:10.1088/0264-9381/29/7/073001

TOPICAL REVIEW

Geometric inequalities for axially symmetric black holes

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Received 16 November 2011, in final form 10 February 2012
Published 12 March 2012
Online at stacks.iop.org/CQG/29/073001

Abstract
A geometric inequality in general relativity relates quantities that have both a physical interpretation and a geometrical definition. It is well known that the parameters that characterize the Kerr–Newman black hole satisfy several important geometric inequalities. Remarkably enough, some of these inequalities also hold for dynamical black holes. This kind of inequalities play an important role in the characterization of the gravitational collapse; they are closely related with the cosmic censorship conjecture. Axially symmetric black holes are the natural candidates to study these inequalities because the quasi-local angular momentum is well defined for them. We review recent results in this subject and we also describe the main ideas behind the proofs. Finally, a list of relevant open problems is presented.

PACS numbers: 04.70.Bw, 04.20.Dw, 04.20.Ex, 02.40.Ky, 02.30.Jr, 02.30.Xx

Contents

1. Introduction

Geometric inequalities have an ancient history in the literature of mathematics. A classical example is the isoperimetric inequality for closed plane curves given by

$$L^2 \geqslant 4\pi A, \tag{1}$$

where A is the area enclosed by a curve C of length L, and where equality holds if and only if C is a circle (for a review on this subject see [94]). General relativity is a geometric theory, and hence, it is not surprising that geometric inequalities appear naturally in it. As we will see, many of these inequalities are similar in spirit to the isoperimetric inequality (1). However, general relativity as a physical theory provides an important extra ingredient. It is often the case that the quantities involved have a clear physical interpretation and the expected behavior of the gravitational and matter fields often suggest geometric inequalities, which can be highly non-trivial from the mathematical point of view. The interplay between geometry and physics gives to geometric inequalities in general relativity their distinguished character.

A prominent example is the positive mass theorem. The physics suggests that the mass of the spacetime (which is represented by a pure geometrical quantity [11, 18, 30]) should be positive and equal to zero if and only if the spacetime is flat. From the geometrical mass definition, without the physical picture, it would be very hard to conjecture this inequality. In fact the proof turn out to be very subtle [102, 103, 118].

A key assumption in the positive mass theorem is that the matter fields should satisfy an energy condition. This condition is expected to hold for all physically realistic matter. It is remarkable that such a simple condition encompasses a huge class of physical models and that it translates into a pure geometrical condition. These kinds of general properties that do not depend very much on the details of the model are not easy to find for astrophysical objects (like stars or galaxies), which usually have a very complicated structure. And hence it is difficult to obtain simple geometric inequalities among the parameters that characterize them.

In contrast, black holes represent a unique class of very simple macroscopic objects that play, in some sense, the role of 'elementary particles' in the theory. The black hole uniqueness theorem ensures that stationary black holes in electro-vacuum are characterized by three parameters, which can be taken to be the area A of the black hole, the angular momentum J and the charge q. The mass m is calculated in terms of these parameters by an explicit formula (cf equation (8)). It is well known that these parameters satisfy certain geometrical inequalities that restrict their range. These inequalities are direct consequences of the explicit formula (8). Among them, we note first the following:

$$m \geqslant \sqrt{\frac{A}{16\pi}}, \tag{2}$$

which will lead to the Penrose inequality for dynamical black holes. Also, we have the following two inequalities that will play a central role in this review:

$$m^2 \geqslant \frac{q^2 + \sqrt{q^4 + 4J^2}}{2}, \qquad A \geqslant 4\pi\sqrt{q^4 + 4J^2}. \tag{3}$$

The equality in (2) is achieved for the Schwarzschild black hole. The equality in both inequalities (3) is achieved for extreme black holes.

However, black holes are not stationary in general. Astrophysical phenomena, such as the formation of a black hole by gravitational collapse or a binary black hole collision, are highly dynamical. For such systems, the black hole cannot be characterized by few parameters as in the stationary case. In fact, even stationary but non-vacuum black holes have a complicated structure (e.g., black holes surrounded by a rotating ring of matter, see the numerical studies in [7]). Remarkably, inequalities (2)–(3) extend (under appropriate assumptions) to the fully dynamical regime. Moreover, inequalities (2)–(3) are deeply connected with properties of the global evolution of Einstein equations, in particular with the cosmic censorship conjecture. The main subject of this review is to present a series of recent results which are mainly concerned with the dynamical versions of inequalities (3).

To extend the validity of inequalities (2)–(3) to non-stationary black holes the first difficulty is how to define the physical parameters involved, most notably the angular momentum J of a dynamical black hole. To define quasi-local quantities is in general a difficult problem (see the review [108]). However, for axially symmetric black holes, the angular momentum (via Komar's formula) is well defined and it is conserved in vacuum. Essentially, for this reason inequalities (3) have been mostly studied for axially symmetric black holes. An exception are inequalities which involve only the electric charge, since the charge is well defined as a quasi-local quantity without any symmetry assumption.

The plan of the review is as follows. In section 2 we describe the heuristic physical arguments that support these inequalities and connect them with global properties of a gravitational collapse. In section 3 we present an overview of the main results concerning these inequalities that have been recently obtained. We also describe the main ideas behind the proofs. Two important geometrical quantities involved in (3), mass and angular momentum, have distinguished properties in axial symmetry (in contrast to the electric charge). These properties play a fundamental role. We describe in some detail the angular momentum in section 4 and the mass in section 5. Finally in section 6 we present the relevant open problem in this area.

2. The physical picture

The most important example of a geometric inequality for dynamical black holes is the Penrose inequality. In a seminal article [95], Penrose proposed a physical argument that connects global properties of the gravitational collapse with geometric inequalities under the initial conditions. For a recent review about this inequality, see [88] and references therein. Since it will play an important role in what follows, let us review the Penrose argument.

We will assume that the following statements hold in a gravitational collapse.

(i) Gravitational collapse results in a black hole (weak cosmic censorship).
(ii) The spacetime settles down to a stationary final state. We will further assume that at some finite time all the matter have fallen into the black hole, and hence, the exterior region is electro-vacuum.

Conjectures (i) and (ii) constitute the standard picture of the gravitational collapse. Relevant examples, where this picture is confirmed (and where the role of angular momentum is analyzed), are the collapse of neutron stars studied numerically in [16, 58].

Before going into the Penrose argument, let us analyze the final stationary state postulated in (ii). The black hole uniqueness theorem implies that the final state is given by the Kerr–Newman black hole (we emphasize however that many important aspects of the black holes uniqueness still remain open, see [34] for a recent review on this problem). Let us denote by m_0, A_0, J_0 and q_0, respectively, the mass, area, angular momentum and charge of the remainder Kerr–Newman black hole. In order to describe a black hole, the parameters of the Kerr–Newman family of solutions of Einstein equations should satisfy the remarkable inequality

$$d \geqslant 0, \tag{4}$$

where we have defined

$$d = m_0^2 - q_0^2 - \frac{J_0^2}{m_0^2}. \tag{5}$$

This inequality is equivalent to

$$m_0^2 \geqslant \frac{q_0^2 + \sqrt{q_0^4 + 4J_0^2}}{2}. \tag{6}$$

From Newtonian considerations (take $q = 0$ for simplicity), we can interpret this inequality as follows (see [109]): in a collapse, the gravitational attraction ($\approx m_0^2/r^2$) at the horizon ($r \approx m_0$) dominates over the centrifugal repulsive forces ($\approx J_0^2/m_0 r^3$). It is important to recall that the Kerr–Newman solution is well defined for any choice of the parameters, but it only represents a black hole if inequality (6) is satisfied.

The Kerr–Newman black hole is called extreme if the equality in (6) is satisfied:

$$m_0^2 = \frac{q_0^2 + \sqrt{q_0^4 + 4J_0^2}}{2}. \tag{7}$$

The area of the black hole horizon is given by the important formula

$$A_0 = 4\pi \left(2m_0^2 - q_0^2 + 2m_0\sqrt{d}\right). \tag{8}$$

Note that this expression has meaning only if inequality (6) holds.

Penrose argument runs as follows. Let us consider a gravitational collapse. Take a Cauchy surface S in the spacetime such that the collapse has already occurred. This is shown in figure 1. Let Σ denotes the intersection of the event horizon with the Cauchy surface S and let A be its area. Let (m, q, J) be the total mass, charge and angular momentum at spacelike infinity, respectively. These quantities can be computed from the initial surface S. By the black hole area theorem, we have that the area of the black hole increase with time and hence

$$A_0 \geqslant A. \tag{9}$$

Since gravitational waves carry positive energy, the total mass of the spacetime should be bigger than the final mass of the black hole

$$m \geqslant m_0. \tag{10}$$

The difference $m - m_0$ is the total amount of gravitational radiation emitted by the system.

The area A_0 of the remainder black hole is given by equation (8) in terms of the final parameters (J_0, q_0, m_0). It is a monotonically increasing function of m_0 (for fixed q_0 and J_0), namely the derivative $\partial A_0/\partial m_0$ is positive (we explicitly calculate this derivative below). Then, using this monotonicity and inequalities (9) and (10), we obtain

$$A \leqslant A_0 \leqslant 4\pi \left(2m^2 - q_0^2 + 2m\left(m^2 - q_0^2 - \frac{J_0^2}{m^2}\right)^{1/2}\right), \tag{11}$$

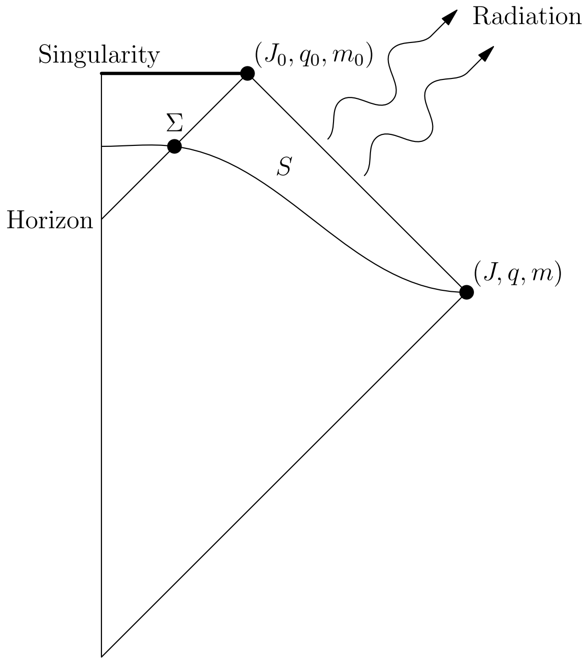

Figure 1. Schematic representation of a gravitational collapse.

where the important point is that on the right-hand side appears the total mass m (instead of m_0), which can be calculated on the Cauchy surface S. The parameters q_0 and J_0 are not known *a priori* but since they appear with negative sign we have

$$A \leqslant A_0 \leqslant 4\pi \left(2m^2 - q_0^2 + 2m\left(m^2 - q_0^2 - \frac{J_0^2}{m^2}\right)^{1/2}\right) \leqslant 16\pi m^2. \tag{12}$$

The following still remains an important point: how to estimate the area A of Σ in terms of geometrical quantities that can be locally computed under the initial conditions. Recall that in order to know the location of the event horizon the whole spacetime is needed. Assume that the surface S contains a future trapped two-surface Σ_0 (the trapped condition is a local property). By a general result on black hole spacetimes, we have that the surface Σ_0 should be contained in Σ. But that does not necessarily means that the area of Σ_0 is smaller than the area of Σ. Consider all surfaces $\tilde{\Sigma}$ enclosing Σ_0. Denote by $A_{\min}(\Sigma_0)$ the infimum of the areas of all such surfaces. Then, we clearly have that $A(\Sigma) \geqslant A_{\min}(\Sigma_0)$. The advantage of this construction is that $A_{\min}(\Sigma_0)$ is a quantity that can be computed from the Cauchy surface S. Using this inequality and inequalities (11) and (12), we finally obtain the Penrose inequality

$$m \geqslant \sqrt{\frac{A_{\min}(\Sigma_0)}{16\pi}}. \tag{13}$$

For further discussion, we refer to [88] and references therein.

The Penrose argument is remarkable because it ends up in an inequality that can be written purely in terms of the initial conditions. On the other hand, the proof of such inequality gives indirect evidences of the validity of conjectures (i) and (ii).

Can we include the parameters q and J in inequality (13) to obtain a stronger version of it? The problem is, of course, how to relate the final state parameters (q_0, J_0) with the initial state ones (q, J).

If the matter fields are not charged, then the charge is conserved:

$$q = q_0. \tag{14}$$

And hence in that case we have the following version of the Penrose inequality with charge:

$$A_{\min}(\Sigma_0) \leqslant A \leqslant 4\pi \left(2m^2 - q^2 + 2m(m^2 - q^2)^{1/2}\right). \tag{15}$$

The case of angular momentum is more complicated. Angular momentum is in general non-conserved. There exists no simple relation between the total angular momentum J of the initial conditions and the angular momentum J_0 of the final black hole. For example, a system can have $J = 0$ initially, but collapse to a black hole with the final angular momentum $J_0 \neq 0$. We can imagine that under the initial conditions there are two parts with opposite angular momenta, one of them falls in to the black hole and the other escapes to infinity.

Axially symmetric vacuum spacetimes constitute a remarkable exception because the angular momentum is conserved. In that case, we have

$$J = J_0. \tag{16}$$

We discuss this conservation law in detail in section 5.1. The physical interpretation of (16) is that axially symmetric gravitational waves do not carry angular momentum.

For non-vacuum axially symmetric spacetimes, the angular momentum is no longer conserved. Matter can transfer angular momentum even in axial symmetry. This is also true for the electromagnetic field. However, in the electro-vacuum case, a remarkable effect occurs. First, the sum of the gravitational and the electromagnetic angular momenta is conserved. Second, at spacelike infinity, only the gravitational angular momentum is nonzero. In section 4, we prove these two facts. Hence, if J and J_0 denote now the total angular momenta, then the conservation (16) still holds for axially symmetric electro-vacuum spacetimes and we obtain the full Penrose inequality valid for axially symmetric electro-vacuum initial conditions:

$$A_{\min}(\Sigma_0) \leqslant A \leqslant 4\pi \left(2m^2 - q^2 + 2m\left(m^2 - q^2 - \frac{J^2}{m^2}\right)^{1/2}\right). \tag{17}$$

We emphasize that in this inequality the total angular momentum J can be computed at any closed surface that surround the black hole using formula (122). When this surface is at infinity, the angular momentum is given by the gravitational angular momentum (i.e. the Komar integral of the axial Killing field).

Inequality (17) implies the bound

$$m^2 \geqslant \frac{q^2 + \sqrt{q^4 + 4J^2}}{2}. \tag{18}$$

Of course, inequality (18) can be deduced directly with the same argument without using the area theorem. The first place where this conjecture was formulated is in [54] (see also [73]).

Inequality (18) can be viewed as a simplified version of the Penrose inequality. The major difference is that the area of the horizon does not appears in (18). Only charges, which are essentially topological, appear on the right-hand side of this inequality.

Inequality (18) is a global inequality for two reasons. First, it involves the total mass m of the spacetime. Second, it assumes global restrictions on the initial data: axial symmetry and electro-vacuum. We will discuss these assumptions in more detail in section 3.

The area A, the angular momentum J in axial symmetry and the charge q are quasi-local quantities (in particular, the right-hand side of (18) is purely quasi-local). Namely, they carry information on a bounded region of the spacetime. In contrast with a local quantity, like a tensor field, which depends on a point of the spacetime, or a global quantity (like the total mass), which depends on the whole initial conditions. A natural question is whether dynamical black holes satisfy purely quasi-local inequalities. The relevance of this kind of inequalities is that they provide a much finer control on the dynamics of black holes than the global versions.

It is well known that the energy of the gravitational field cannot be represented by a local quantity. The best one can hope is to obtain a quasi-local expression. These are the so-called quasi-local mass definitions (see [108] and reference therein). Consider formula (8) for the horizon area for the Kerr–Newman black holes. From this expression, we can write the mass in terms of the other parameters as follows:

$$m_{\rm bh} = \sqrt{\frac{A}{16\pi} + \frac{q^2}{2} + \frac{\pi\,(q^4 + 4J^2)}{A}}. \tag{19}$$

In this equation, we have dropped the subindex 0 on the right-hand side and also denoted the mass on the left-hand side by $m_{\rm bh}$ to emphasize that this expression can in principle be defined for any black hole (i.e. not necessarily stationary). With this interpretation, this expression is known as the Christodoulou [28] mass of the black hole.

For a dynamical black hole, expression (19) is in principle just a definition. Does formula (19) represent the quasi-local mass of a non-stationary black hole? Let us analyze its physical behavior. We discuss first the relation of this quasi-local mass with the total mass of the spacetime. For only one black hole, we expect the following inequality to be true:

$$m \geqslant m_{\rm bh}. \tag{20}$$

This inequality implies the Penrose inequality (17) but it is stronger (see the discussion in [88]). However, it is important to emphasize that for the case of many black holes, this inequality does not hold. In fact, it is possible to find counter examples, if we take the area as additive [117] or the quasi-local masses as additives [51]. This is expected, since the interaction energy of the black holes need to be taken into account (see the discussion in [51]). For only one black hole, inequality (20) in axial symmetry is an open problem in this general form; we will further discuss it in section 6.

We discuss now the purely quasi-local properties of (19). Formula (19) trivially satisfies inequality (18). This is, of course, just because the Kerr black hole satisfies this bound. Hence, if we accept (19) as the correct formula for the quasi-local mass of an axially symmetric black hole, then (19) provides the, rather trivial, quasi-local version of (18).

Consider the evolution of $m_{\rm bh}$. By the area theorem, we know that the horizon area will increase. If we assume axial symmetry and electro-vacuum, then the total angular momentum (gravitational plus electromagnetic) will be conserved at the quasi-local level. On physical grounds, one would expect that in this situation the quasi-local mass of the black hole should increase with the area, since there is no mechanism at the classical level to extract mass from the black hole. In effect, the only way to extract mass from a black hole is by extracting angular momentum through a Penrose process. But the angular momentum transfer is forbidden in electro-vacuum axial symmetry. Then, one would expect that both the area A and the quasi-local mass $m_{\rm bh}$ should monotonically increase with time.

Let us take a time derivative of $m_{\rm bh}$ (denoted by a dot). To analyze this, it is illustrative to write down the complete differential, namely the first law of thermodynamics

$$\delta m_{\rm bh} = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J + \Phi_H \delta q, \tag{21}$$

where

$$\kappa = \frac{1}{4m_{\rm bh}}\left(1 - \left(\frac{4\pi}{A}\right)^2 (q^4 + 4J^2)\right), \quad \Omega_H = \frac{4\pi J}{A m_{\rm bh}}, \quad \Phi_H = \frac{4\pi\,(m_{\rm bh} + \sqrt{d})q}{A}. \tag{22}$$

Here, $m_{\rm bh}$ is given by (19) and d (defined in equation (5)) is written in terms of A and J and q as

$$d = \frac{1}{m_{\rm bh}^2}\left(\frac{A}{16\pi}\right)^2\left(1-(q^4+4J^2)\left(\frac{4\pi}{A}\right)^2\right)^2. \tag{23}$$

In equation (21), we have followed the standard notation for the formulation of the first law; we emphasize, however, that in our context this equation is a trivial consequence of (19).

Under our assumptions, from formula (19), we obtain

$$\dot{m}_{bh} = \frac{\kappa}{8\pi}\dot{A}, \tag{24}$$

where we have the angular momentum J and the charge q conserved. Since, by the area theorem, we have

$$\dot{A} \geqslant 0, \tag{25}$$

the time derivative of $m_{\rm bh}$ will be positive (and hence the mass $m_{\rm bh}$ will increase with the area) if and only if $\kappa \geqslant 0$:

$$4\pi\sqrt{q^4+4J^2} \leqslant A. \tag{26}$$

Then, it is natural to conjecture that (26) should be satisfied for any black hole in axial symmetry. If the horizon violates (26), then in the evolution the area will increase but the mass $m_{\rm bh}$ will decrease. This will indicate that the quantity $m_{\rm bh}$ does not have the desired physical meaning. Also, a rigidity statement is expected. Namely, the equality in (26) is reached only by the extreme Kerr black hole given by the formula

$$A = 4\pi(\sqrt{q^4+4J^2}). \tag{27}$$

The final picture is that the size of the black hole is bounded from below by the charge and angular momentum, and the minimal size is realized by the extreme Kerr–Newman black hole. This inequality provides a remarkable quasi-local measure of how far a dynamical black hole is from the extreme case, namely an 'extremality criterion' in the spirit of [24], although restricted only to axial symmetry. In [43], it has been conjectured that, within axial symmetry, to prove the stability of a nearly extreme black hole is perhaps simpler than for a Schwarzschild black hole. It is possible that this quasi-local extremality criterion will have relevant applications in this context. Note also that inequality (26) allows one to define, at least formally, the positive surface gravity density (or temperature) of a dynamical black hole by formula (22) (see [15, 14] for a related discussion of the first law in dynamical horizons).

If inequality (26) is true, then we have a non-trivial monotonic quantity (in addition to the black hole area) $m_{\rm bh}$ in electro-vacuum

$$\dot{m}_{bh} \geqslant 0. \tag{28}$$

It is important to emphasize that the physical arguments presented above in support of (26) are certainly weaker in comparison with the ones behind the Penrose inequalities (13), (15) and (17). A counter example of any of these inequalities will prove that the standard picture of the gravitational collapse is wrong. On the other hand, a counter example of (26) will just prove that the quasi-local mass (19) is not appropriate to describe the evolution of a non-stationary black hole. One can imagine other expressions for quasi-local mass may be more involved in axial symmetry. In contrast, reversing the argument, a proof of (26) will certainly suggest that the mass (19) has physical meaning for non-stationary black holes as a natural quasi-local mass (at least in axial symmetry). Also, inequality (26) provide a non-trivial control of the size of a black hole valid at any time.

Finally, it is important to explore the physical scope of validity of these geometrical inequalities. Are they valid for other macroscopic objects? The Penrose inequality (17) is clearly not valid for an arbitrary region in the spacetime. Namely, consider an arbitrary 2-surface of area A, which is not necessarily a black hole boundary. We can make A arbitrarily large keeping the mass (total or quasi-local) small (e.g., take a region in Minkowski).

For inequalities (18) and (26), the situation is less obvious. To have an intuitive idea of the order of magnitude involved, it is important to include the relevant constants in these inequalities. Let G be the gravitational constant and c the speed of light. Then, these inequalities are written as follows:

$$m^2 \geqslant \frac{1}{G} \frac{q^2 + \sqrt{q^4 + 4J^2c^2}}{2} \tag{29}$$

and

$$4\pi \frac{G}{c^4} \sqrt{q^4 + 4J^2c^2} \leqslant A. \tag{30}$$

For readers' convenience, we include the explicit values of the constants (in centimeters, grams and seconds):

$$G = 6.67 \times 10^{-8}\,\mathrm{cm}^3\,\mathrm{g}^{-1}\,\mathrm{s}^{-2}, \quad c = 3 \times 10^{10}\,\mathrm{cm}\,\mathrm{s}^{-1}. \tag{31}$$

The values of the fundamental physical constants used in this section were taken from [90].

Let us analyze first the global inequality (29). It is useful to split it into cases with zero charge and zero angular momentum, respectively:

$$\sqrt{G} \geqslant \frac{|q|}{m} \tag{32}$$

and

$$\frac{G}{c} \geqslant \frac{|J|}{m^2}. \tag{33}$$

For inequality (32), consider an electron and a proton; for these particles, the quotient on the right-hand side is given by

$$\frac{|q_e|}{m_e} = 0.53 \times 10^{18}\,\mathrm{g}^{-1/2}\,\mathrm{cm}^{3/2}\,\mathrm{s}^{-1}, \tag{34}$$

$$\frac{|q_e|}{m_p} = 2.87 \times 10^{14}\,\mathrm{g}^{-1/2}\,\mathrm{cm}^{3/2}\,\mathrm{s}^{-1}. \tag{35}$$

Since in these units we have $\sqrt{G} \approx 10^{-4}$, we see that these particles grossly violate the global inequality (32). And hence ordinary charged matter would also violate it (a similar discussion has been presented in [62] and [73]).

For the angular momentum case (33), we can also consider an elementary particle. In that case, the angular momentum of a particle with spin s (recall that $s = 1/2$ for the electron and the proton) is given by

$$J = \sqrt{s(s+1)}\hbar, \quad \hbar = 1.05 \times 10^{-27}\,\mathrm{cm}^2\,\mathrm{s}^{-1}\,\mathrm{g}, \tag{36}$$

where $\hbar$ is the Planck constant. For example, for the electron, we have

$$\frac{|J|}{m_e^2} = 1.12 \times 10^{27}\,\mathrm{cm}^2\,\mathrm{s}^{-1}\,\mathrm{g}^{-1}. \tag{37}$$

Since

$$\frac{G}{c} = 2.22 \times 10^{-18}\,\mathrm{cm}^2\,\mathrm{s}^{-1}\,\mathrm{g}^{-1}, \tag{38}$$

inequality (33) is also violated by several order of magnitude for elementary particles. Instead of an elementary particle, we can consider an ordinary rotating object. It is clear that there exists an ordinary object for which $J/m^2 \approx 1$ (say a rigid sphere of mass 1 g, radius 1 cm and angular velocity $1\,\mathrm{s}^{-1}$), and hence, inequality (33) is also violated for ordinary rotating bodies.

We conclude that ordinary matter does not satisfy in general the global inequality (29). This inequality should be interpreted as a property of electro-vacuum gravitational fields on complete regular initial conditions where both the charge and the angular momentum are 'produced by the topology' and not by matter sources (unless, of course, they are inside a black hole horizon). By 'produced by the topology' we mean the following. The angular momentum and the electric charge are defined as integrals over closed two-dimensional surfaces. In electro-vacuum, these integrals are conserved (we discuss this in detail in section 4), and hence, they are zero if the topology of the initial conditions is trivial (i.e. $\mathbb{R}^3$). In order to have non-trivial charges in electro-vacuum, the initial conditions should have some 'holes'. This non-trivial topology signals the presence of a black hole.

For the quasi-local inequality (30), it is also convenient to distinguish between the cases with zero charge and zero angular momentum, respectively:

$$A \geqslant \frac{G}{c^4} 4\pi q^2 \tag{39}$$

and

$$A \geqslant 8\pi \frac{G}{c^3} |J|. \tag{40}$$

Since the charge is discrete, in unit of q_e, it make sense to calculate the following characteristic radius:

$$r_0 = \frac{q_e G^{1/2}}{c^2} = 1.38 \times 10^{-34}\,\mathrm{cm}. \tag{41}$$

We see that r_0 is one order of magnitude less than the Planck length l_p given by

$$l_p = \left(\frac{G\hbar}{c^3}\right)^{1/2} = 1.6 \times 10^{-33}\,\mathrm{cm}. \tag{42}$$

If we assume that the particle or the macroscopic object has spherical shape, we can define the area radius r by $A = 4\pi r^2$. Then, inequality (39) for a particle of charge q_e has the form

$$r \geqslant r_0. \tag{43}$$

The proton charge radius is $r_p = 0.87 \times 10^{-12}$ cm according to [96] and $r_p = 0.84 \times 10^{-12}$ cm according to the recent calculation presented in [90]. Hence, the proton satisfies inequality (43). This inequality is also consistent with the upper bound for the electron radius 10^{-20} cm measured in [52].

For the case of angular momentum, using relation (36), we can compute the following quotient for an elementary particle:

$$r_0 = \left(2\frac{G}{c^3}|J|\right)^{1/2} = \sqrt{2}(s(s+1))^{1/4} l_p. \tag{44}$$

We see that for a particle of spin s of order 1 the minimal radius is of the order of the Planck length l_p, and hence, it is also satisfied for elementary particles.

More relevant is the case of an ordinary rotating body. The angular momentum J of a rigid body is given by

$$J = I\omega, \tag{45}$$

where I is the moment of inertia and ω is the angular velocity. Consider an ellipsoid of revolution with semi-axes a and b, rotating along the b axis. The moment of inertia along the axis of rotation is given by

$$I = \frac{2}{5}ma^2, \tag{46}$$

where m is the mass of the ellipsoid, which is assumed to have constant density. The area of the ellipsoid satisfies the following elementary inequality:

$$A \geqslant 2\pi a^2. \tag{47}$$

The equality in (47) is achieved in the limit $b \to 0$, namely, the bound is sharp.

Inserting equations (45) and (46) into inequality (40), and using the bound (47) and the fact that it is sharp, we obtain that inequality (40) is satisfied by the ellipsoid if and only if the following inequality holds:

$$\frac{5}{8}\frac{c^3}{G} \geqslant m\omega. \tag{48}$$

It is interesting to note that in inequality (48) neither the area nor the radii of the body appear. The inequality relates only the mass and the angular velocity of the body. Also, the body is not assumed to be nearly spherical and the parameters for the ellipsoid are arbitrary.

The value on the left-hand side of this inequality is

$$\frac{5}{8}\frac{c^3}{G} = 2.53 \times 10^{38}\ \mathrm{s}^{-1}\ \mathrm{g}. \tag{49}$$

For the sun, we have the following values:

$$m_{\mathrm{sun}} = 1.989 \times 10^{33}\ \mathrm{g}, \qquad \omega = 2.90 \times 10^{-6}\ \mathrm{rad\,s}^{-1}, \tag{50}$$

and hence,

$$m\omega = 5.77 \times 10^{27}\ \mathrm{s}^{-1}\mathrm{g}. \tag{51}$$

We see that the inequality is satisfied for the sun. In order to violate (48), a body should be very massive and highly spinning, a natural candidate for that is a neutron star. For the fastest spinning neutron star found to date (see [70]), we have

$$\omega \approx 4.5 \times 10^3\ \mathrm{rad\,s}^{-1}. \tag{52}$$

Assuming that the neutron stars has about three solar masses (which appears to be a reasonable upper bound for the mass, see [85]), we obtain

$$m\omega \approx 2.7 \times 10^{37}\ \mathrm{s}^{-1}\mathrm{g}. \tag{53}$$

Inequality (48) is still satisfied; however, the value (53) is remarkably close to the upper limit (49).

The example of the ellipsoid shows the elementary relation between shape and angular momentum in classical mechanics, which is valid even for non-spherical bodies. There is no such relation between shape and the electric charge of a body. In fact, we will present some counter examples for charged highly prolated objects that violate inequality (39). However, remarkably enough, for charged 'round surfaces' (we will define this concept later on), an inequality between area and charge can be proved (see theorem 3.5). On the other hand, the example of the ellipsoid suggests that the scope of validity of the inequality between area and angular momentum (40) (or the related inequality (48)) for axial symmetric bodies is much larger.

3. Results and main ideas

In this section, we present the main results concerning inequalities (18) and (26) that have been recently proved in the literature. We also discuss the general strategy of their proofs.

3.1. Global inequality

The proof of the inequality between total mass and charge (namely, setting $J = 0$ in (18)), which is valid without any symmetry assumptions, has been known for some time. The first proof was provided in [62] and [60] using spinorial arguments similar to the Witten proof of the positive mass theorem [118]. See also [73]. A related inequality was proved in [91] with similar techniques. In [19], the proof was generalized to include low-differentiable metrics. For this inequality, an interesting rigidity result is expected: the equality holds if and only if the initial data are embed into the Majumdar–Papapetrou static spacetime (see [65] for a discussion of these spacetimes in relations with black holes). The rigidity statement has also been established, but with supplementary hypotheses, in [62] and [37]. Very recently a new proof was provided in [81], which removes all remaining hypotheses. Also, in this review, a new approach is presented. The strategy is to combine the Jang equation method with the spinorial proof of the positive mass theorem.

The inclusion of angular momentum in axial symmetry (which is the main subject of this review) involves complete different techniques. In particular, no spinorial proof of these inequalities are available so far (see, however, [119] where a related inequality is proved using spinors). The first proof of the global inequality (18) (with no electric charge) was provided in a series of articles [41, 40, 39], which end up in the global proof given in [44].

In [31] and [33], the result was generalized and the proof was simplified. In [35] and [38], the charge was included. As a sample of the most general result currently available, we present the following theorem proved in [35] and [38].

Theorem 3.1. *Consider an axially symmetric, electro-vacuum, asymptotically flat and maximal initial data set with two asymptotic ends. Let m, J and q denote the total mass, angular momentum and charge, respectively, at one of the ends. Then, the following inequality holds:*

$$m^2 \geqslant \frac{q^2 + \sqrt{q^4 + 4J^2}}{2}. \tag{54}$$

For the precise definition, fall-off conditions and assumptions on the electro-vacuum initial data, we refer to [35] and [38]. For simplicity, in section 5.1 we discuss in detail only the pure vacuum case.

Recall that for asymptotically flat initial data, the total mass m, the total charge q and the total angular momentum J (without any symmetry assumption) are well defined as integrals over two-spheres at infinity for a given asymptotic end. That is, all the quantities involved in (54) are well defined for generic asymptotically flat data, which are not necessarily axially symmetric. However, the inequality does not hold without the symmetry assumption. General families of counter examples have been constructed in [74] for pure vacuum and complete manifolds.

Under the hypothesis of this theorem (namely, electro-vacuum and axial symmetry), both the angular momentum and the electric charge are defined as conserved quasi-local integrals (we discuss this in detail in section 4). In particular, if the topology of the manifold is trivial (i.e. $\mathbb{R}^3$), then these quantities are zero, and hence theorem 3.1 reduces to the positive mass theorem. In order to have nonzero charge or angular momentum, we need to allow non-trivial topologies, e.g., manifolds with two asymptotic ends as is the case in theorem 3.1 (see

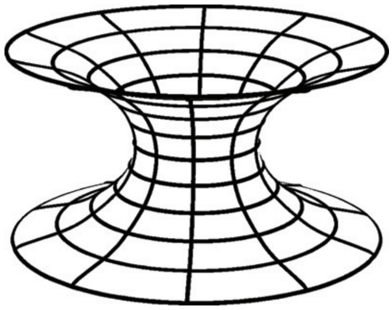

Figure 2. Initial data with two asymptotically flat ends.

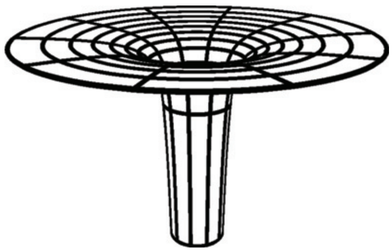

Figure 3. The cylindrical end on extreme Kerr black hole initial data.

figure 2). An important initial data set that satisfies the hypothesis of the theorem is provided by a slice t = constant in the non-extreme Kerr–Newman black hole in the standard Boyer–Lindquist coordinates.

This theorem has three main limitations: (i) the initial data are assumed to be maximal, (ii) there is no rigidity statement and (iii) the data are assumed to have only two asymptotic ends. Let us discuss these points in more detail. The maximal condition plays a crucial role in the proof since it ensure a positive-definite scalar curvature. A relevant open problem is how to remove this condition; we will discuss it in more detail in section 6.

Extreme Kerr–Newman initial data, which reach the equality in (54), is not asymptotically flat at both ends. The data have a cylindrical end and an asymptotically flat end (see figure 3). Hence, these data are excluded in the hypothesis of theorem 3.1. In order to include the equality case, we need to enlarge the class of data. An example is given by the following theorem proved in [44], which includes the rigidity statement.

Theorem 3.2. *Consider a vacuum Brill initial data set that satisfies condition 2.5 in [44]. Then, the inequality*

$$m \geqslant \sqrt{|J|} \tag{55}$$

holds. Moreover, the equality in (55) holds if and only if the data are a slice of the extreme Kerr spacetime.

The precise definition of the Brill class of data can be seen in [44]. The main advantage of these kinds of data is that they encompass both class of asymptotics: cylindrical and asymptotic flatness. We discuss this in section 5.1. The condition 2.5 (see [44] for details) mentioned in this theorem implies that the initial data have non-trivial angular momentum only at one end; however, multiple extra ends with zero angular momentum are allowed. This condition involves also other restrictions that are technical. In a very recent work [105], these technical conditions have been removed and also an interesting new approach to the variational problem is presented.

Inequality (54) (which in particular implies (55)) is expected to hold for manifolds with an arbitrary number of asymptotic ends; this generalization is probably to the most important open problem regarding this kind of inequalities (we discuss this in detail in section 6). There exist, however, a very interesting partial result [33]. In order to describe it, we need to introduce the mass functional $\mathcal{M}$; this functional is defined in section 5. It plays a major role in all the proofs, as it is explained in section 3.3. This functional represents a lower bound for the mass. Moreover, the global minimum of this functional (under appropriate boundary conditions that preserve the angular momentum) is achieved by an harmonic map with prescribed singularities. As we will see in section 3.3, this is the main strategy in the proofs of all the previous theorems that are valid for two asymptotic ends. Remarkably enough in [33], the existence and uniqueness of this singular harmonic map have been proved also for manifolds with an arbitrary number of asymptotic ends. In this review, the following theorem is proved.

Theorem 3.3. *Consider an axially symmetric, vacuum asymptotically flat and maximal initial data with N asymptotic ends. Denote by m_i and J_i ($i = 1, \ldots N$), respectively, the mass and angular momentum of the end i. Take an arbitrary end (say 1); then, the mass at this end satisfies the inequality*

$$m_1 \geqslant \mathcal{M}(J_2, \ldots, J_N), \tag{56}$$

where $\mathcal{M}(J_2, \ldots, J_N)$ denotes the numerical value of the mass functional $\mathcal{M}$ evaluated at the corresponding harmonic map.

This theorem reduces the proof of the inequality with multiples ends to compute the value of the mass functional on the corresponding harmonic map and verify the inequality

$$\mathcal{M}(J_2, \ldots, J_N) \geqslant \sqrt{|J_1|}. \tag{57}$$

We further discuss this theorem in section 3.3.1.

Strong numerical evidence for the fact that inequality (57) holds for three asymptotic ends has been provided in [48]. The numerical methods used in that article are related with the harmonic map structure of the equations. We will describe them in section 5.2.

All the previous results assume complete manifolds without inner boundaries. The inclusion of inner boundary is important to prove the Penrose inequality with angular momentum (17). Boundary conditions in relation with the mass functional $\mathcal{M}$ were studied in [61] in order to prove a version of the Penrose inequality in axial symmetry. In [36], inner boundaries were also included and an interesting new lower bound for the mass is obtained, which depends only on the inner boundary. Finally, we mention that in [80], numerical evidence for the validity of the Penrose inequality (17) has been presented.

3.2. Quasi-local inequalities

Quasi-local inequalities between area and charge have been proved in [59] for stable minimal surfaces on time symmetric initial data. The following theorem proved in [47] generalizes this result for generic dynamical black holes.

Consider Einstein equations with the cosmological constant Λ:

$$G_{\mu\nu} = 8\pi\left(T^{\mathrm{EM}}_{\mu\nu} + T_{\mu\nu}\right) - \Lambda g_{\mu\nu}, \tag{58}$$

where $T^{\mathrm{EM}}_{\mu\nu}$ is the electromagnetic energy–momentum tensor defined in terms of the electromagnetic field $F_{\mu\nu}$ by (85). The electric charge of an arbitrary closed, oriented, two-surface $\mathcal{S}$ embedded in the spacetime is defined by (86).

Theorem 3.4. *Given a closed marginally trapped surface $\mathcal{S}$ satisfying the spacetime stably outermost condition in a spacetime that satisfies Einstein equations (58) with the non-negative cosmological constant Λ and such that the non-electromagnetic matter fields $T_{\mu\nu}$ fulfill the dominant energy condition, the following inequality holds:*

$$A \geqslant 4\pi q^2, \tag{59}$$

where A and q are the area and the charge of $\mathcal{S}$.

For the definition of marginally trapped surfaces (which is standard) and the stably condition, see [2, 3, 79, 47] (see also [66, 98]). This theorem is a completely quasi-local result that applies to general dynamical black holes without any symmetry assumption. It is also important to emphasize that the matter is not assumed to be uncharged, namely it is allowed that $\nabla_\mu F^{\mu\nu} \neq 0$ (which is equivalent to $\nabla^\mu T^{\mathrm{EM}}_{\mu\nu} \neq 0$). The only condition imposed in the non-electromagnetic matter field stress–energy tensor $T_{\mu\nu}$ is that it satisfies the dominant energy condition. In this theorem, it is also possible to include the magnetic charge and Yang–Mills charges (see [47] and [77]).

In [107], an interesting generalization of theorem 3.4 is presented in which the stability requirement is removed at the expense of introducing the principal eigenvalue of the stability operator and also the cosmological constant (with arbitrary sign) is added.

At the end of section 2, we have observed that a variant of inequality (59) is expected to hold for ordinary macroscopic charged objects (which are not necessarily black holes) at least if they are 'round enough'. In fact there exists an interesting and highly non-trivial counter example to (59) for macroscopic objects. This counter example was constructed by Bonnor in [22] (see also the discussion in [47]) and it can be summarized as follows: for any given positive number k, there exist static, isolated, non-singular bodies, satisfying the energy conditions, whose surface area A satisfies $A \leqslant kq^2$. The body is a highly prolated spheroid of electrically counterpoised dust. From the physical point of view, we are saying that for an ordinary charged object (in contrast to a black hole), we need to control another parameter (the roundness) in order to obtain an inequality between area and charge. Remarkably enough, it is possible to encode this intuition in the geometrical concept of isoperimetric surface: we say that a surface $\mathcal{S}$ is isoperimetric (or 'round') if among all surfaces that enclose the same volume as $\mathcal{S}$ does, $\mathcal{S}$ has the least area. Then, based on the results proved in [29], the following theorem was obtained in [47] for isoperimetric surfaces.

Theorem 3.5. *Consider an electro-vacuum, maximal initial data, with a non-negative cosmological constant. Assume that $\mathcal{S}$ is a stable isoperimetric sphere. Then,*

$$A \geqslant \frac{4\pi}{3} q^2, \tag{60}$$

where q is the electric charge of $\mathcal{S}$.

Note that inequality (60) has a different coefficient to (59). We recall that the notion of stable isoperimetric surface is very similar to the case of a stable minimal surface: the differential operator is identical, the only difference is that the allowed test functions should integrate to zero on the surface and this is precisely the condition that the deformations preserve the volume (see [17]).

It is also possible to prove interesting variants of theorem 3.4 that are valid for generic surfaces (i.e. not necessarily trapped or minimal) but in order to obtain these results global assumptions on the initial data should be made (i.e. in contrast to theorem 3.4, these are not a purely quasi-local results): the two-surfaces are embedded on initial conditions that are complete, maximal and asymptotically flat. The non-electromagnetic matter fields are assumed to be non-charged and they should satisfy the dominant energy condition on the whole initial data (see theorem 2.2 in [47]).

As in the case of the global inequality, quasi-local inequalities with angular momentum involve different techniques in comparison with the pure charged case. Their study started very recently.

The quasi-local inequality with angular momentum and charge (26) was first conjectured to hold in stationary spacetimes in [10]. In that article, the extreme limit of this inequality was analyzed and also numerical evidence for validity in the stationary case was presented (using the numerical method and code developed in [7]). In a series of articles [68, 69], inequality (26) was proved for stationary black holes. See also [6].

It is important to emphasize that the stationary non-vacuum case is highly non-trivial. The physical situation is, e.g., a black hole surrounded by a ring of matter (which do not touch the black hole). To illustrate the complexity of this case, we mention that the Komar mass (which is only defined in the stationary case) can be negative for these black holes (for the Kerr–Newman black hole is always positive), see [8, 9]. It is interesting to note that for this class of stationary spacetimes, there exists a remarkable relation of the form $(8\pi J)^2 + (4\pi q^2)^2 = A^+A^-$, where A^+ and A^- denote the areas of event and Cauchy horizon, respectively. This result has been proved in the following series of articles [5, 67, 4].

In the dynamical regime, the inequality was conjectured to hold in [45] based on the heuristic argument mentioned in section 2. In that article, the main relevant techniques for its proof were also introduced, namely the mass functional on the surface and its connections with the area (we discuss this in section 5.3). A global proof (but with technical restrictions) was obtained in [1, 55]. The first general and pure quasi-local result was proven in [50], where the relevant role of the stability condition for minimal surfaces was pointed out.

Theorem 3.6. *Consider axisymmetric, vacuum and maximal initial data, with a non-negative cosmological constant. Assume that the initial data contain an orientable closed stable minimal axially symmetric surface $\mathcal{S}$. Then,*

$$A \geqslant 8\pi |J|, \tag{61}$$

where A is the area and J is the angular momentum of $\mathcal{S}$. Moreover, if the equality in (61) holds, then $\Lambda = 0$ and the local geometry of the surface $\mathcal{S}$ is an extreme Kerr throat sphere.

The extreme throat sphere geometry, with the angular momentum J, was defined in [45] (see also [1] and [50]). This surface captures the local geometry near the horizon of an extreme Kerr black hole and it is defined as follows. The sphere is embedded in initial data with an intrinsic metric given by

$$\gamma_0 = 4J^2\, e^{-\sigma_0}\, d\theta^2 + e^{\sigma_0} \sin^2\theta\, d\phi^2, \tag{62}$$

where σ_0 is the same as given in (201). Moreover, the sphere must be totally geodesic, the twist potential evaluated at the surface must be given by ω_0 defined in (201) and the components of the second fundamental

$$K_{ij}\xi^i = K_{ij}n^jn^i = K_{ij}\eta^j\eta^i = 0 \tag{63}$$

must vanish at the surface. Here, K_{ij} denotes the second fundamental form of the initial data, n^i is the unit normal vector to the surface and η^i is the axial Killing field. Note that the functions σ_0 and ω_0, which characterize the intrinsic and extrinsic geometries of the surface, respectively, depend only on the angular momentum parameter J. The geometry of an axially symmetric initial data set is described in detail in section 5.1. In particular, the twist potential is determined by the second fundamental form K_{ij}, using equation (175). The adapted coordinates system used in (62) is defined is section 4.1.

This theorem has two main restrictions: the first one is the maximal condition and the second is vacuum. Remarkably enough, it is possible not only to avoid both restrictions but also to provide a pure spacetime proof (i.e. no mention of a three-dimensional hypersurface) of this inequality, in which axisymmetry is only imposed on $\mathcal{S}$. This generalization is proved in [79].

Theorem 3.7. *Given an axisymmetric closed marginally trapped surface $\mathcal{S}$ satisfying the (axisymmetry-compatible) spacetime stability outermost condition, in a spacetime with non-negative cosmological constant and fulfilling the dominant energy condition, the following inequality holds:*

$$A \geqslant 8\pi |J|, \tag{64}$$

where A and J are, respectively, the area and (Komar) angular momentum of $\mathcal{S}$. If equality holds, then $\mathcal{S}$ is a section of a non-expanding horizon with the geometry of extreme Kerr throat sphere.

The concept of non-expanding horizon is explained in [79]; it essentially means that the shear vanished at the surface.

It is important to note that the angular momentum that appears in (64) is the gravitational one (i.e. the Komar integral). The matter fields also have angular momentum and it can be transferred to the black hole; however, inequality (64) remains true even in that case. In fact, this inequality is non-trivial for the Kerr–Newman black hole; we discuss this in detail in section 4.

In [99], it has been pointed out that there exists a connection between the global inequalities described in section 3.1 and the quasi-local inequalities. This is obtained by linking the relevant mass functionals $\mathcal{M}$ and $\mathcal{M}^{\mathcal{S}}$ (we discuss these mass functionals in section 3.3).

Inequality (64) has played an important role in the proofs of the non-existence of stationary two black holes configurations (see [93, 32]).

Finally, we mention that there exists a very interesting generalization of (64) to black holes in higher dimensions [72].

3.3. Main ideas

In this section, we present the main ideas behind the proofs of the global inequality (54) and the quasi-local inequality (64). This section should be considered as a guide in which the technicalities are avoided. In sections 4 and 5, we discuss in detail the main relevant properties of the angular momentum and mass in axial symmetry, which constitute the essential part of the proofs.

3.3.1. Global inequalities. The starting point in the proof of inequality (54) (we consider the case $q = 0$ for simplicity) is the formula for the total mass m given by equation (184). This formula represents the total mass as a positive-definite integral over a maximal (here the maximal condition plays a crucial role) initial surface. This integral representation is a generalization of the Brill mass formula discovered in [26] (in section 5, we further discuss this formula and provide the relevant references). The formula holds in a particular coordinate system that is called isothermal (see lemma 5.1).

The integrand in equation (184) has two kinds of terms: dynamical and stationary. The dynamical terms vanish for a stationary solution like Kerr. The stationary part leads to the relevant mass functional $\mathcal{M}$ defined by (185). This functional provides an obvious lower bound to the total mass:

$$m \geqslant \mathcal{M}(\sigma, \omega). \tag{65}$$

The mass functional $\mathcal{M}$ depends on two functions σ and ω. The function σ is essentially the norm of the axial Killing vector (see equation (179)). The function ω is the twist potential of the Killing vector (see section 4). These two functions can be freely prescribed on the initial data. This is an important and far from obvious property since the constraint equations (162) and (163) should be satisfied. This fact allows one to formulate a variational principle for the functional $\mathcal{M}$. The other important ingredient for this variational principle is the behavior of the angular momentum. The angular momentum is prescribed by the value of ω at the axis (see section 4). Then, if the variations of ω vanish at the axis the angular momentum will be preserved. With these two ingredients, it is hence possible to reduce the proof of inequality (54) to a pure variational problem for the functional $\mathcal{M}(\sigma, \omega)$.

The second step of the proof is to solve this variational problem. This is the most difficult part and also the most interesting since it reveals the geometric properties of the mass functional $\mathcal{M}$. Let us discuss the general strategy of the proof.

Let (σ_0, ω_0) be the corresponding functions obtained from the extreme Kerr initial data with the angular momentum J; then, we have that

$$\mathcal{M}(\sigma_0, \omega_0) = \sqrt{|J|}. \tag{66}$$

The heuristic discussed in section 2 and equation (67) suggest that the following inequality holds:

$$\mathcal{M}(\sigma, \omega) \geqslant \sqrt{|J|}, \tag{67}$$

for all (σ, ω), such that ω has the same value at the axis as the function ω_0. Moreover, the equality in (67) is reached if and only if $\sigma = \sigma_0$ and $\omega = \omega_0$. This is precisely the variational problem. Note that the variational problem is formulated purely in terms of the functional $\mathcal{M}(\sigma, \omega)$, without any reference to the constraint equations (162) and (163).

The first evidence that this variational problem will have the expected solution is that the Euler–Lagrange equations of the functional $\mathcal{M}$ are equivalent to the stationary axially symmetric Einstein equations; in particular, extreme Kerr satisfies these equation [41]. The second evidence (which is harder to prove) is that the second variation of $\mathcal{M}$ is positive definite evaluated at extreme Kerr [40]. To prove this positivity property, it is crucial to make contact with the harmonic map theory (in this case, through the Carter identity). With these ingredients, it is possible to show that extreme Kerr is a local minimum of the mass, and hence, inequality (67) is proved in an appropriately defined neighborhood of extreme Kerr. This local proof was given in [40].

To have a global proof of (67) (i.e. without any smallness assumption), more subtle properties of the mass functional are required. A crucial step is to realize that the mass functional $\mathcal{M}$ is essentially the renormalized energy of a harmonic map into the hyperbolic plane [44] (we discuss this in section 5.2). This kind of harmonic map has been extensively studied in the literature. The problem here is that the map is singular at the axis, and hence, the standard techniques do not apply directly. To use the harmonic map theory, we need to handle these singularities, and this is the main technical difficulty. In [44], the proof of this variational result was done using estimates, inspired by the work of Weinstein [111–116] (see also [86]), which relies on particular properties of this functional (inversion symmetry). In subsequent works [31, 33, 35, 38], the proof was simplified and improved using general results on harmonic maps (more precisely the existence result [71]). In these proofs, the connection with the harmonic maps theory is more transparent and the problem of the singularities is clearly isolated.

Finally, let us mention the following important point regarding initial data with multiple ends. The multiple ends appear in the variational problem (67) as singular points of the functions (σ, ω). Remarkably, even in that case it is possible to solve completely the variational problem. This is precisely the content of theorem 3.3 (proved in [33]). The only missing piece in order to prove inequality (54) in that case is the following. Theorem 3.3 ensures the existence of a global minimum but the value of $\mathcal{M}$ at the global minimum is unknown. In the case of two asymptotic ends, we know that extreme Kerr is the global minimum, and hence we can explicitly compute the value (66).

3.3.2. Quasi-local inequalities. The global inequality (54) applies to a complete three-dimensional manifold. In contrast, the quasi-local inequality (64) applies to a closed two-surface. In principle, it is *a priori* not clear at all that there is a relation between these two kinds of inequalities. The two physical heuristic arguments presented in section 2 in support for them are very different. In particular, it is far from obvious that the mass functional $\mathcal{M}$ can play a role for the quasi-local inequalities. Remarkably enough, it turns out that a suitable adapted mass functional $\mathcal{M}^{\mathcal{S}}$ over a two-surface plays a very similar role to $\mathcal{M}$. The motivation for the definition of $\mathcal{M}^{\mathcal{S}}$ given by (204) is discussed in section 5.3. The new mass functional $\mathcal{M}^{\mathcal{S}}$ and its connection with the area represent the kernel of the proof of (64). Let us discuss this.

On the extreme Kerr initial data, there exists an important canonical two-surface, namely the intersection of the Cauchy surface with the horizon. On a surface given by $t =$ constant in the Boyer–Lindquist coordinates, this two-surface is located at infinity on the cylindrical end (see figure 3). Let us call $(\bar{\sigma}_0, \bar{\omega}_0)$ the value of the functions (σ_0, ω_0) (defined in section 3.3.1) on this two-surface. The functions $(\bar{\sigma}_0, \bar{\omega}_0)$ will play for $\mathcal{M}^{\mathcal{S}}$ a similar role to the functions (σ_0, ω_0) for $\mathcal{M}$. Namely, first, they satisfy the Euler–Lagrange equations of $\mathcal{M}^{\mathcal{S}}$. Second, the second variation of $\mathcal{M}^{\mathcal{S}}$ is positive definite evaluated at $(\bar{\sigma}_0, \bar{\omega}_0)$ (see [45]). There is also a connection with the harmonic maps energy (we discuss this in section 5.3). Using similar kinds of arguments as described in section 3.3.1, it is possible to prove the following inequality:

$$2|J| \leqslant \mathrm{e}^{(\mathcal{M}^{\mathcal{S}}(\sigma,\omega)-8)/8} \tag{68}$$

for all functions (σ, ω), such that ω has the same values at the poles of the two-surface as ω_0, and with equality if and only if we have $\sigma = \bar{\sigma}_0$ and $\omega = \omega_0$. A local version of this inequality was first proved in [45]. The global version (68) was proved in [1]. The rigidity statement was proved in [50]. We emphasize that inequality (68) (in complete analogy to inequality (67)) is a property of the functional $\mathcal{M}^{\mathcal{S}}$, and the geometry of the two-surface $\mathcal{S}$ does not intervene at all.

Inequality (68) is interesting but, in the light of the discussion presented in 3.3.1, it is somehow expected. What is crucial and completely unexpected is the relation between $\mathcal{M}^{\mathcal{S}}$ and the area of the surface $\mathcal{S}$. This relation was founded locally in [45] and globally in [1]. By local, in this context, we mean that the relation holds for surfaces in an appropriately defined neighborhood of an extreme Kerr throat geometry. On the other hand, global means that the relation holds for general surfaces. In [50], the important connection with the stability condition was proved for minimal surfaces and in [79] for marginally trapped surfaces. The final inequality essentially reads as follows. For a two-surface $\mathcal{S}$, if (i) it is minimal [50] (or marginally trapped [79]) and (ii) it is stable, then the following inequality holds:

$$A \geqslant 4\pi \mathrm{e}^{(\mathcal{M}^{\mathcal{S}}-8)/8}. \tag{69}$$

The minimal and marginally trapped conditions are requirements on the extrinsic curvature of $\mathcal{S}$ (namely, on the trace of the second fundamental form). The stability condition is a requirement for the derivatives of the second fundamental form. In the spacetime version presented in [79]), inequality (69) is a consequence of a flux inequality (see lemma 1 in that reference), where the geometric and physical meaning of each term is apparent.

4. Angular momentum in axial symmetry

Axial symmetry plays a major role in the inequalities that include angular momentum presented in the previous sections for two main reasons: for the global inequality (54) it is the conservation of angular momentum implied by axial symmetry which is relevant and for the quasi-local inequality (61) it is the very definition of quasi-local angular momentum (only possible in axial symmetry) which is important. These two properties are closely related for vacuum spacetimes, since the Komar integral provides both the conservation law and the definition of quasi-local angular momentum. For non-vacuum spacetimes in axial symmetry, the Komar integral still provides a meaningful expression for the gravitational angular momentum but it is no longer conserved. Nevertheless, as we already mentioned in section 2, in the electro-vacuum case, the sum of the gravitational and electromagnetic angular momenta is conserved, and hence, in that case, the proof of the global inequality (54) is possible. In the general non-vacuum case in axial symmetry (that is, with general matter sources which are not electromagnetic), no simple and universal relation between mass and angular momentum is expected. For example, in [58], models of neutron stars in axial symmetry have been numerically constructed such that they violate inequality (54). Nevertheless, remarkably, the quasi-local inequality (61) for black holes is still valid in the general non-vacuum axially symmetric case.

In this section, we summarize relevant results concerning angular momentum in axial symmetry. Although these results are not new, they are not easy to find in the literature, notably, the conservation of angular momentum in the electro-vacuum case and the relation between Komar integrals and potentials in the presence of matter fields. The conservation of angular momentum in the electro-vacuum, together with the relevant Komar integral for the electromagnetic angular momentum, was discovered in [106], in a much more general setting. We present a simpler derivation of this result in the language of differential forms.

Let us begin with some general remarks about conserved quantities in General Relativity (see also [78] for a related discussion and [108] for the general problem of how to define quasi-local angular momentum without symmetries).

Let M be a four-dimensional manifold with the metric $g_{\mu\nu}$ (with signature $(-+++)$) and the Levi-Civita connection ∇_μ. On this curved background, let us consider an arbitrary energy–momentum tensor $T_{\mu\nu}$, which satisfies the conservation equation

$$\nabla_\mu T^{\mu\nu} = 0. \tag{70}$$

It is well known that if the spacetime admit a Killing vector field η^μ

$$\nabla_{(\mu}\eta_{\nu)} = 0, \tag{71}$$

then the vector

$$K_\mu = T_{\mu\nu}\eta^\nu \tag{72}$$

is divergence free

$$\nabla_\mu K^\mu = 0. \tag{73}$$

This equation provides an integral conservation law via the Gauss theorem. For some of the computations in this section, it is convenient to use differential forms instead of tensors. We will denote them in boldface. Let $\boldsymbol{K}$ be the 1-form defined by (72). Equation (73) is equivalent to

$$d^*\boldsymbol{K} = 0, \tag{74}$$

where d is the exterior derivative and the dual of p-form is defined with respect to the volume element $\epsilon_{\mu\nu\lambda\gamma}$ of the metric $g_{\mu\nu}$ by the standard formula

$$^*\alpha_{\mu_1\cdots\mu_{4-p}} = \frac{1}{p!}\alpha^{\nu_1\cdots\nu_p}\epsilon_{\nu_1\cdots\nu_p\mu_1\cdots\mu_{4-p}}. \tag{75}$$

Let Ω denotes a four-dimensional, orientable, region in M and let $\partial\Omega$ be its three-dimensional boundary. Then, using (74) and the Stokes theorem, we obtain

$$0 = \int_\Omega d^*\boldsymbol{K} = \int_{\partial\Omega} {}^*\boldsymbol{K}. \tag{76}$$

Note that in this equation, the region Ω is arbitrary and the boundary $\partial\Omega$ can have many disconnected components.

Consider a spacelike three-surface S. The conserved quantity corresponding to the Killing vector η^μ is defined with respect to S by

$$K(S) = \int_S {}^*\boldsymbol{K}. \tag{77}$$

The interpretation of equation (76) in relation to the quantity $K(S)$ is the following. Let Ω to be a timelike cylinder, such that its boundary $\partial\Omega$ is formed by the bottom and the top spacelike surfaces S_1 and S_2 and the timelike piece $\mathcal{C}$. Then, we have (taking the corresponding orientation)

$$0 = \int_{\partial\Omega} {}^*\boldsymbol{K} = K(S_1) - K(S_2) + \int_{\mathcal{C}} {}^*\boldsymbol{K}. \tag{78}$$

The integral over the timelike surface $\mathcal{C}$ is the flux of $K(S)$. Equation (78) is interpreted as the conservation law of the quantity $K(S)$. The region Ω can also be chosen to have a null boundary $\mathcal{C}$, equation (78) remains identical and the interpretation is similar.

If the Killing vector η^μ is also a symmetry of the tensor $T_{\mu\nu}$ (we have not assumed that so far), namely

$$\pounds_\eta T_{\mu\nu} = 0, \tag{79}$$

where $\pounds$ denote Lie derivatives, then the following vector is also divergence free:

$$\hat{K}_\mu = 8\pi\left(T_{\mu\nu}\eta^\nu - \tfrac{1}{2}T\eta_\mu\right), \tag{80}$$

where T is the trace of $T_{\mu\nu}$. In fact there is a whole family of divergence-free tensors since $T\eta_\mu$ is divergence free. Hence, the previous discussion applies to this vector as well.

Note that the conserved quantities $K(S)$ are naturally defined as integrals over spacelike three-surfaces. In flat spacetime, it is possible to convert these integrals into a boundary integral over two-surfaces. This gives the quasi-local representation of conserved quantities (see the discussion in the introduction of [108]). In a curved background, this is in general not possible. However, as we will see, this is possible for the particular case of the electromagnetic field.

Before analyzing the angular momentum, it is important to study the electric charge. The electric charge is of course relevant in our discussion since it appears in the inequalities discussed in the previous sections. But more importantly, even if we want to analyze these inequalities in pure vacuum, the electric charge represents the simpler ‘conserved charge’ on a curved spacetime. Its definition and properties serve as model for all the other conserved quantities, like the angular momentum.

The Maxwell equations on $(M, g_{\mu\nu})$ are given by

$$\nabla^{\mu} F_{\mu\nu} = -4\pi j_{\nu}, \tag{81}$$

$$\nabla_{[\mu} F_{\nu\alpha]} = 0. \tag{82}$$

In terms of forms, they are written as

$$d^{*}\mathbf{F} = 4\pi^{*}\mathbf{j}, \tag{83}$$

$$\mathrm{d}\mathbf{F} = 0. \tag{84}$$

The energy–momentum tensor of the electromagnetic field is given by

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\lambda}F_{\nu}{}^{\lambda} - \frac{1}{4}g_{\mu\nu}F_{\lambda\gamma}F^{\lambda\gamma}\right). \tag{85}$$

Let $\mathcal{S}$ be a closed orientable two-surface embedded in M (in the following, all two-surfaces will be assumed to be closed and orientable). The electric charge q of $\mathcal{S}$ is defined by

$$q(\mathcal{S}) = \frac{1}{4\pi}\int_{\mathcal{S}} {}^{*}\mathbf{F}. \tag{86}$$

Let S be a three-surface with boundary $\mathcal{S}$; then, using the Stokes theorem and Maxwell equation (83), we obtain

$$q(\mathcal{S}) = \int_{S} {}^{*}\mathbf{j}. \tag{87}$$

This equation is interpreted as follows. From equation (83), we deduce the conservation law for the current $\mathbf{j}$ analogous to (74):

$$d^{*}\mathbf{j} = 0. \tag{88}$$

And hence taking the same region Ω and using Stokes theorem we obtain the analog expression as (78) for the current $\mathbf{j}$

$$0 = \int_{S_1} {}^{*}\mathbf{j} - \int_{S_2} {}^{*}\mathbf{j} + \int_{\mathcal{C}} {}^{*}\mathbf{j}. \tag{89}$$

Using (87), we finally obtain

$$q(\mathcal{S}_1) - q(\mathcal{S}_2) = \int_{\mathcal{C}} {}^{*}\mathbf{j}. \tag{90}$$

This is the conservation law for the electric charge. Note that on the left-hand side of (90), we have integrals over two-surfaces, in contrast with (78) where integrals over three-surfaces appear. This is because we have an extra equation (i.e. the Maxwell equation (83)) that allows us to write the integral (87) in the form (86).

When $\mathbf{j} = 0$, the charge has the same value, namely

$$q(\mathcal{S}_1) = q(\mathcal{S}_2), \tag{91}$$

and we say that the charge is strictly conserved.

We turn now to angular momentum for axially symmetric spacetimes. We begin with the definition of axial symmetry.

Definition 4.1. *The spacetime $(M, g_{\mu\nu})$ is said to be axially symmetric if its group of isometries has a subgroup isomorphic to $SO(2)$.*

We will denote by η^μ the Killing field generator of the axial symmetry. The orbits of η^μ are either points or circles. The set of point orbits Γ is called the axis of symmetry. Assuming that Γ is a surface, it can be proved that η^μ is spacelike in a neighborhood of Γ (see [89]). We will further assume that the Killing vector is always spacelike outside Γ. Note that if this condition is not satisfied, then the spacetime will have closed causal curved; in particular, it cannot be globally hyperbolic.

The form η_μ will be denoted by $\boldsymbol{\eta}$, and the square of its norm by η:

$$\eta = \eta^\mu \eta_\mu = |\boldsymbol{\eta}|^2. \tag{92}$$

We have used the notation η^μ to denote the Killing vector field and η to denote the square of its norm to be consistent with the literature. However, in this section, to avoid confusion between η^μ and its square norm η, we will denote the vector field η^μ by $\bar{\eta}$ in equations involving differential forms in the index-free notation.

Consider now Einstein equations on an axially symmetric spacetime

$$G_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{93}$$

Note that the Killing equation (71) implies that $T_{\mu\nu}$ satisfies (79).

The Komar integral (it is also appropriate to call it the Komar charge) of the Killing field is defined over a two-dimensional surface $\mathcal{S}$ as follows:

$$J(\mathcal{S}) = \frac{1}{16\pi}\int_{\mathcal{S}} \epsilon_{\mu\nu\lambda\gamma}\nabla^\lambda \eta^\gamma = \frac{1}{16\pi}\int_{\mathcal{S}} {}^*\mathrm{d}\boldsymbol{\eta}. \tag{94}$$

Hence, as we discussed above, via the Stokes theorem, we obtain

$$J(\mathcal{S}) = \frac{1}{16\pi}\int_S d^*\mathrm{d}\boldsymbol{\eta}, \tag{95}$$

where S is a three-dimensional surface with boundary $\mathcal{S}$. It is a classical result [82] (see also [110]) that the integrand in (95) can be computed in terms of the Ricci tensor

$$\nabla_{[\mu}(\epsilon_{\nu\alpha]\beta\gamma}\nabla^\beta \eta^\gamma) = \tfrac{2}{3}R^\mu{}_\nu \eta^\nu \epsilon_{\mu\alpha\beta\gamma}. \tag{96}$$

In terms of forms, this equation is written as

$$d^*\,\mathrm{d}\boldsymbol{\eta} = 2^*\boldsymbol{K}, \tag{97}$$

where we have defined the 1-form $\boldsymbol{K}$ by

$$\boldsymbol{K} \equiv K_\mu = R_{\mu\nu}\eta^\nu. \tag{98}$$

Using Einstein equations (93), the form $\boldsymbol{K}$ can be written in terms of the energy–momentum tensor

$$K_\mu = 8\pi\left(T_{\mu\nu}\eta^\nu - \tfrac{1}{2}T\eta_\mu\right). \tag{99}$$

Note that this expression is identical to (80). Then, repeating the same argument, we obtain the conservation law for angular momentum in axial symmetry, which is the exact analog to the charge conservation (90):

$$J(\mathcal{S}_1) - J(\mathcal{S}_2) = \frac{1}{8\pi}\int_C {}^*\mathbf{K}. \tag{100}$$

The right-hand side of this equation represents the change in the angular momentum of the gravitational field, which is produced on the left-hand side, namely the angular momentum of the matter fields. Note that the angular momenta of the matter fields are written as integrals over three-dimensional surfaces. In particular, in vacuum, we have the strict conservation of angular momentum:

$$J(\mathcal{S}_1) = J(\mathcal{S}_2). \tag{101}$$

We consider now the case where the energy–momentum tensor in Einstein equations (93) is given purely by the electromagnetic field (85). For simplicity, we consider the case with no currents $j = 0$, which is the relevant one since only in that case we obtain conserved quantities. In that case we have that (85) satisfies equation (70) and the source-free Maxwell equations are given by

$$d^*\boldsymbol{F} = 0, \tag{102}$$

$$\mathrm{d}\boldsymbol{F} = 0. \tag{103}$$

We also assume that the Maxwell fields are axially symmetric:

$$\pounds_\eta \boldsymbol{F} = 0. \tag{104}$$

Consider the 1-forms defined by

$$\boldsymbol{\alpha} = \boldsymbol{F}(\bar{\eta}), \quad \boldsymbol{\beta} = {}^*\boldsymbol{F}(\bar{\eta}), \tag{105}$$

where we have used the standard notation $\boldsymbol{F}(\bar{\eta}) = F_{\mu\nu}\eta^\mu$ to denote contractions of forms with vector fields. Using the general expression for the action of the Lie derivative on forms

$$\pounds_{\bar{\eta}}\boldsymbol{\omega} = d[\boldsymbol{\omega}(\bar{\eta})] + (d\boldsymbol{\omega})(\bar{\eta}), \tag{106}$$

the Maxwell equations (102)–(103) and condition (104), we obtain

$$\mathrm{d}\boldsymbol{\alpha} = 0, \quad \mathrm{d}\boldsymbol{\beta} = 0. \tag{107}$$

It follows that locally there exist functions χ and ψ, such that

$$\boldsymbol{\alpha} = \mathrm{d}\chi, \quad \boldsymbol{\beta} = \mathrm{d}\psi. \tag{108}$$

The form $\boldsymbol{K}$ defined by (99) has the following expression for the electromagnetic field:

$$\boldsymbol{K} = 2\left(\boldsymbol{F}(\alpha) - \tfrac{1}{4}\boldsymbol{\eta}|F|^2\right). \tag{109}$$

We have that (see [116])

$${}^*(\boldsymbol{\eta} \wedge (\boldsymbol{F}(\alpha)) = \boldsymbol{\alpha} \wedge \boldsymbol{\beta}. \tag{110}$$

Using (108), we obtain

$$d(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = 0, \tag{111}$$

and hence, there exists locally a 1-form $\boldsymbol{\gamma}$, such that

$$\mathrm{d}\boldsymbol{\gamma} = \boldsymbol{\alpha} \wedge \boldsymbol{\beta}, \tag{112}$$

where γ is given by

$$\boldsymbol{\gamma} = \tfrac{1}{2}(\chi\, \mathrm{d}\psi - \psi\, \mathrm{d}\chi). \tag{113}$$

Note that

$$\boldsymbol{\gamma}(\bar{\eta}) = 0. \tag{114}$$

From equation (110), we deduce

$${}^*(\boldsymbol{\eta} \wedge \boldsymbol{K}) = 2\boldsymbol{\alpha} \wedge \boldsymbol{\beta}. \tag{115}$$

We use the following identity that is valid for arbitrary 1-forms:

$${}^*(\boldsymbol{\eta} \wedge \boldsymbol{K}) \wedge \boldsymbol{\eta} = \eta^*\boldsymbol{K} - {}^*\boldsymbol{\eta}(\boldsymbol{K}(\bar{\eta})). \tag{116}$$

Using (112), we finally obtain our main formula

$${}^*\boldsymbol{K} = 2d(\boldsymbol{\gamma} \wedge \hat{\boldsymbol{\eta}}) + 2\boldsymbol{\gamma} \wedge \mathrm{d}\hat{\boldsymbol{\eta}} + {}^*\hat{\boldsymbol{\eta}}(\boldsymbol{K} \cdot \boldsymbol{\eta}), \tag{117}$$

where we have defined

$$\hat{\boldsymbol{\eta}} = \frac{\boldsymbol{\eta}}{\eta}. \tag{118}$$

It is important to note that

$$d\hat{\boldsymbol{\eta}}(\bar{\eta}) = 0. \tag{119}$$

Using equation (117), we integrate ${}^*\boldsymbol{K}$ over a three-surface S tangential to η^μ, with boundary $\mathcal{S}$. Using that η^μ is tangential to S, it follows that the restriction of the 3-form ${}^*\hat{\boldsymbol{\eta}}$ to S is zero. For the second term in (117), we use equations (114) and (119) to obtain the same conclusion. Hence, we have

$$\int_S {}^*\boldsymbol{K} = \int_S 2d(\boldsymbol{\gamma} \wedge \hat{\boldsymbol{\eta}}) = 2\int_{\mathcal{S}} \boldsymbol{\gamma} \wedge \hat{\boldsymbol{\eta}}, \tag{120}$$

where in the last equality we have used the Stokes theorem.

We summarize the previous calculation in the following lemma, which is a re-writing of the results that have been obtained in [106].

Lemma 4.2. *Consider an axially symmetric spacetime for which the Einstein–Maxwell equations (93), (85), (102) and (103) are satisfied. Let S be an orientable three-surface, tangent to the axial Killing field η^μ, with boundary (possible disconnected) $\mathcal{S}$. Then, we have*

$$J(\mathcal{S}) = \frac{1}{4\pi}\int_{\mathcal{S}} \boldsymbol{\gamma} \wedge \hat{\boldsymbol{\eta}}, \tag{121}$$

where $J(\mathcal{S})$ is the Komar integral given by (94), $\boldsymbol{\gamma}$ is defined in terms of the electromagnetic field by (113) and $\hat{\boldsymbol{\eta}}$ is given by (118).

We can define a 'total angular momentum' that is conserved in electro-vacuum:

$$J_T(\mathcal{S}) = \frac{1}{16\pi}\int_{\mathcal{S}} {}^*d\boldsymbol{\eta} - 4\,\boldsymbol{\gamma} \wedge \hat{\boldsymbol{\eta}}. \tag{122}$$

We note that since the surface $\mathcal{S}$ is tangent to η^μ and all the fields are axially symmetric, the surface integrals are in fact line integrals on the quotient manifold $M \setminus SO(2)$.

Formula (122) was studied in [13, 12] for rotating isolated horizons. This formula has also been recently studied at null infinity in connection with the center of mass and general definition of angular momentum for asymptotically flat (at null infinity) spacetimes, see [83].

A very important example where lemma 4.2 applies is the Kerr–Newman black hole. Consider the Kerr–Newman black hole with parameters (m, a, q). The total angular momentum is given by $J_T = am$. This is equal to the Komar integral evaluated at infinity, since the electromagnetic field decays and does not contribute at infinity. However, at the horizon, the Komar angular momentum is not J_T. The angular momentum at the horizon has the decomposition (122).

For the Kerr–Newman black hole, the Komar angular momentum at the horizon J is given by (see, e.g., [97, p 222]):

$$J = a\frac{r_+^2 + a^2}{2r_+}\left(1 + \frac{q^2}{2a^2}\left(1 - \frac{r_+^2 + a^2}{ar_+}\right)\arctan\left(\frac{a}{r_+}\right)\right), \tag{123}$$

where r_+ is the horizon radius:

$$r_+ = m + (m^2 - a^2 - q^2)^{1/2}. \tag{124}$$

The area of the horizon is given by

$$A = 4\pi(r_+^2 + a^2). \tag{125}$$

Equation (125) is of course identical to equation (8). As we have already mentioned, it is well known that the area satisfies the inequality

$$A \geqslant 4\pi\sqrt{q^4 + 4J_T^2}. \tag{126}$$

This in particular implies

$$A \geqslant 8\pi|J_T|. \tag{127}$$

But this inequality relates the total angular momentum J_T. It is *a priori* not obvious if the following inequality holds:

$$A \geqslant 8\pi|J|, \tag{128}$$

where J is given by (123), namely the Komar integral at the horizon. Note that this is precisely the inequality proved in theorem 3.6 and that the Kerr–Newman black hole satisfies all the hypotheses of that theorem.

Let us check explicitly that indeed (128) is satisfied. Expression (123) is remarkably complicated and to better analyze it let us rewrite it in the following form. Using (125), we have

$$J = \frac{A}{8\pi}\epsilon, \tag{129}$$

where

$$\epsilon = \frac{a}{r_+}\left(1 + \frac{q^2}{2a^2}\left(1 - \frac{r_+^2 + a^2}{ar_+}\right)\arctan\left(\frac{a}{r_+}\right)\right), \tag{130}$$

Instead of using a it is convenient to use $x = a/r_+$ as a free parameter. In terms of x, the function ϵ is written as

$$\epsilon = x\left(1 + \frac{q^2}{2r_+^2}f(x)\right), \tag{131}$$

where

$$f(x) = \frac{1}{x^2}\left(1 - \left(\frac{1}{x} + x\right)\arctan(x)\right). \tag{132}$$

We take as free parameters (q, r_+, x). Note that $-1 \leqslant x \leqslant 1$, and hence we have

$$|\epsilon| \leqslant \left|1 + \frac{q^2}{2r_+^2}f(x)\right|. \tag{133}$$

Fix (q, r_+). It can be explicitly checked that the function $f(x)$ is non-positive and has a unique global minimum at $x = 0$, where $f(0) = -2/3$. Hence, it follows that

$$|\epsilon| \leqslant 1. \tag{134}$$

4.1. Potentials

The potentials for the axial Killing vector play an important role in the mass functional described in section 5.

It is instructive to analyze first the electric charge and its potential in axial symmetry. Assume first that the Maxwell equations are source free. Then, we have found the potentials χ and ψ defined by equation (108). In particular, the potential ψ determines the electric charge over an axially symmetric two-surface $\mathcal{S}$. In order to see that, it is convenient to consider a tetrad $(l^\mu, k^\mu, \xi^\mu, \eta^\mu)$ and coordinate system (θ, ϕ) adapted to an axially symmetric two-surface defined as follows (see figure 4). For simplicity, we will assume that $\mathcal{S}$ has the topology of a two-sphere. Let us consider the null vectors ℓ^μ and k^ν spanning the normal plane to $\mathcal{S}$

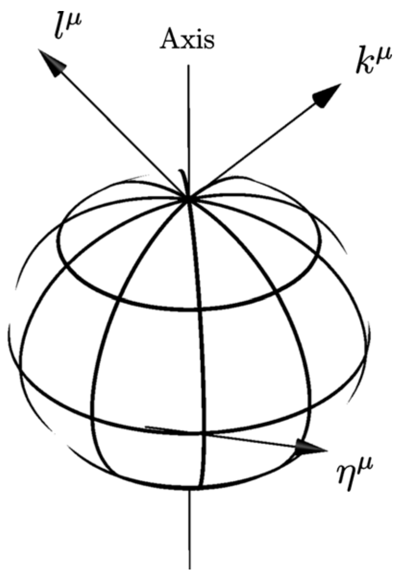

Figure 4. Adapted tetrad for an axially symmetric two-sphere.

and normalized as $\ell^\mu k_\mu = -1$, leaving a (boost) rescaling freedom $\ell'^\mu = f\ell^\mu$, $k'^\mu = f^{-1}k^\mu$. By assumption, η^μ is tangent to $\mathcal{S}$, it has on the surface closed integral curves and it vanishes exactly at two points, which are the intersection of the axis Γ with $\mathcal{S}$. We normalize the vector η^μ so that its integral curves have an affine length of 2π. Let us choose a coordinate ϕ on $\mathcal{S}$ such that $\eta^\mu = \partial/\partial\phi$. The other vector of the tetrad, which is tangent to $\mathcal{S}$ and orthogonal to η^μ, will be denoted by ξ^μ and assume that it has unit norm. We define the coordinate θ, such that ξ^μ is proportional to $\partial/\partial\theta$ and such that $\theta = \pi, 0$ are the poles of $\mathcal{S}$.

The induced metric and the volume element on $\mathcal{S}$ (written as spacetime projectors) are given by $q_{\mu\nu} = g_{\mu\nu} + \ell_\mu k_\nu + \ell_\nu k_\mu$ and $\epsilon_{\mu\nu} = 2^{-1}\epsilon_{\lambda\gamma\mu\nu}\ell^\lambda k^\gamma$, respectively. The area measure on $\mathcal{S}$ is denoted by ds. Since the surface is axially symmetric, we have $\mathcal{L}_\eta q_{\mu\nu} = 0$.

Using this tetrad, the charge over an axially symmetric surface $\mathcal{S}$ is written as follows:

$$q(\mathcal{S}) = \frac{1}{2\pi}\int_{\mathcal{S}} \eta^{-1/2}\xi^\mu\beta_\mu \,\mathrm{d}s, \tag{135}$$

where β_μ the same as defined in (105). In the source-free case, we can use the potential ψ defined by (108) to obtain

$$q(\mathcal{S}) = \frac{1}{2\pi}\int_{\mathcal{S}} \eta^{-1/2}\xi^\mu\nabla_\mu\psi \,\mathrm{d}s = \int_0^\pi \partial_\theta\psi \,\mathrm{d}\theta. = \psi(\pi) - \psi(0). \tag{136}$$

That is, the charge is given by the difference of the value of the potential ψ at the poles of the surface $\mathcal{S}$.

We have seen that the potential ψ is only defined in the source-free case as a scalar function in the spacetime. However, on the surface $\mathcal{S}$, it is always possible to define a potential $\bar{\psi}$ by the equation

$$\xi^\mu\psi_\mu = \xi^\mu\nabla_\mu\bar{\psi} \tag{137}$$

on the surface, since this equation involves only a derivative with respect to θ. The potential $\bar{\psi}$ is only defined on the surface. By definition, in the source-free case, where the potential

ψ also exists, the two functions are equal up to a constant on the surface. This constant is irrelevant since it does not affect the charge.

Consider now the potential for the angular momentum. The twist vector of η^μ is defined by

$$\omega_\mu = \epsilon_{\mu\nu\lambda\gamma}\eta^\nu\nabla^\lambda\eta^\gamma. \tag{138}$$

It is a well-known result that the vacuum equations $R_{\mu\nu} = 0$ imply that

$$\mathrm{d}\boldsymbol{\omega} = 0, \tag{139}$$

where $\boldsymbol{\omega}$ is the 1-form defined by (138). Hence, there exists a potential ω, such that

$$\boldsymbol{\omega} = \mathrm{d}\omega. \tag{140}$$

The function ω is the twist potential of the Killing vector η^μ; it contains all the information of the angular momentum, as we will see. In the non-vacuum case, the twist potential is not defined. In the following, we will not assume vacuum.

In terms of the adapted tetrad, we have that the Komar expression is given by

$$J = \frac{1}{8\pi}\int_{\mathcal{S}} \nabla^\mu\eta^\nu l_\mu k_\nu \,\mathrm{d}s. \tag{141}$$

We use the relation

$$\nabla_\mu\eta_\nu = \tfrac{1}{2}\eta^{-1}\epsilon_{\mu\nu\lambda\gamma}\eta^\lambda\omega^\gamma + \eta^{-1}\eta_{[\nu}\nabla_{\mu]}\eta \tag{142}$$

to write the Komar integral in the following form:

$$J = \frac{1}{16\pi}\int_{\mathcal{S}} \eta^{-1}\epsilon_{\mu\nu\lambda\gamma}\eta^\lambda\omega^\gamma\ell^\mu k^\nu \,\mathrm{d}s. \tag{143}$$

It is clear that the vector defined by

$$\xi_\gamma = \eta^{-1/2}\epsilon_{\mu\nu\lambda\gamma}\eta^\lambda\ell^\mu k^\nu \tag{144}$$

is orthogonal to l^μ, k^μ, η^μ and it has unit norm and hence is the other member of the tetrad. Then, we have

$$J = \frac{1}{16\pi}\int_{\mathcal{S}} \eta^{-1/2}\xi^\mu\omega_\mu \,\mathrm{d}s. \tag{145}$$

We emphasize that this expression is valid only for axially symmetric surfaces.

If we assume vacuum, then we can use the twist potential defined by (140) to obtain

$$J = \frac{1}{8}\int_0^\pi \partial_\theta\omega \,\mathrm{d}\theta = \frac{1}{8}\left(\omega(\pi) - \omega(0)\right). \tag{146}$$

Equations (145) and (146) are the analogs to equations (135) and (136) for the charge.

We consider now the non-vacuum case. We define the following vector (a part of the extrinsic curvature of $\mathcal{S}$ with the role of a connection on its normal cotangent bundle, see, e.g., the discussion in [63, 23]):

$$\Omega^{(\ell)}_\mu = -k^\gamma q^\lambda{}_\mu\nabla_\lambda\ell_\gamma. \tag{147}$$

Since $\mathcal{S}$ is axially symmetric, the tetrad can be chosen such that

$$\pounds_\eta l^\mu = \pounds_\eta k^\mu = 0. \tag{148}$$

In particular, this implies that

$$l^\mu\nabla_\mu\eta^\nu - \eta^\mu\nabla_\mu l^\nu = 0. \tag{149}$$

Using this equation, we obtain

$$\eta^\mu\Omega^{(\ell)}_\mu = -k^\mu\eta^\nu\nabla_\nu\ell_\mu = -k^\mu\ell^\nu\nabla_\nu\eta_\mu. \tag{150}$$

From this equation, we obtain the following equivalent expression for the Komar angular momentum (see, e.g., [78]):

$$J = \frac{1}{8\pi}\int_{\mathcal{S}} \eta^{\mu}\Omega^{(\ell)}_{\mu}\,\mathrm{d}s. \tag{151}$$

Remarkably, as we will see in the following, the vector $\Omega^{(\ell)}_{\mu}$ determines even in the non-vacuum case a potential on the surface (we will denote it by $\bar{\omega}$), which coincides with the twist potential ω defined above in the vacuum case. We emphasize that the potential $\bar{\omega}$ will only be defined on the two-surface $\mathcal{S}$, in contrast to the twist potential ω that (in the vacuum case) is defined by equations (139) and (140) on a region of the spacetime.

By construction, the vector $\Omega^{(\ell)}_{\mu}$ is tangent to the surface $\mathcal{S}$. Since we have assumed that $\mathcal{S}$ has the $\mathbb{S}^2$ topology, there exists functions $\hat{\omega}$ and λ such that the vector has the following decomposition on $\mathcal{S}$ in terms of a divergence-free and an exact form:

$$\Omega^{(\ell)}_{A} = T_A + \nabla_A\lambda, \qquad T_A = \epsilon_{AB}\nabla^B\hat{\omega}. \tag{152}$$

The functions $\hat{\omega}$ and λ are fixed up to a constant. In equation (152), we have used the capital indices (which run from 1 to 2) to emphasize that this is an intrinsic equation on the two-surface $\mathcal{S}$.

By assumption, both $\Omega^{(\ell)}_{A}$ and the intrinsic metric on $\mathcal{S}$ are axially symmetric; then, it follows that the functions $\hat{\omega}$ and λ are also axially symmetric (i.e. they depend only on θ). In particular, it follows that

$$\xi^A T_A = 0, \qquad \eta^A\nabla_A\lambda = 0, \tag{153}$$

where ξ^A is the vector tangent to $\mathcal{S}$ previously defined. Since the norm η is also an axially symmetric function, equation (153) implies that

$$T^A D_A\eta = 0. \tag{154}$$

Equation (154) ensures the integrability conditions for the existence of a function $\bar{\omega}$ such that

$$T_A = \frac{1}{2\eta}\epsilon_{AB}\nabla^B\bar{\omega}. \tag{155}$$

Note that equation (155) is valid only in axial symmetry. Collecting these results, we obtain the decomposition

$$\Omega^{(\ell)}_{A} = \frac{1}{2\eta}\epsilon_{AB}\nabla^B\bar{\omega} + \nabla_A\lambda. \tag{156}$$

In particular, using (153), we obtain

$$\eta^A\Omega^{(\ell)}_{A} = \frac{1}{2\eta}\epsilon_{AB}\eta^A\nabla^B\bar{\omega}. \tag{157}$$

We write equation (157) with spacetime indices and we use the following representation for the tetrad vector ξ^{μ}:

$$\xi_{\nu} = \eta^{-1/2}\epsilon_{\mu\nu}\eta^{\mu}, \tag{158}$$

to finally obtain

$$\eta^{\mu}\Omega^{(\ell)}_{\mu} = \tfrac{1}{2}\eta^{-1/2}\xi^{\mu}\nabla_{\mu}\bar{\omega}. \tag{159}$$

Using equation (151) and integrating, we finally obtain

$$J = \frac{1}{8}\int_0^{\pi}\partial_{\theta}\bar{\omega}\,\mathrm{d}\theta = \frac{1}{8}(\bar{\omega}(\pi) - \bar{\omega}(0)). \tag{160}$$

The relevance of this construction is that the function $\bar{\omega}$, which is only defined at the surface $\mathcal{S}$, plays the role of a potential that can be defined in the non-vacuum case [79]. To see the relation between $\bar{\omega}$ and ω, note that we have

$$\xi^\mu \omega_\mu = \xi^\mu \nabla_\mu \bar{\omega}. \tag{161}$$

This equation is valid always (i.e. non-vacuum) and it gives the function $\bar{\omega}$ in terms of the twist vector ω_μ. In the non-vacuum case, the twist potential ω is not defined but the function $\bar{\omega}$ is always well defined. In fact, we can take (161) as definition of $\bar{\omega}$, since the right-hand side is only a derivative with respect to θ. On the other hand, in the vacuum case, equation (161) implies that ω and $\bar{\omega}$ differs by a constant that is irrelevant since it does not contribute to the angular momentum.

5. Mass in axial symmetry

The main goal of this section is to present the mass formula for axially symmetric data (184) and the mass functional for two-surfaces (204). We also discuss their geometrical properties in connection with harmonic maps.

5.1. Axially symmetric initial data

The global geometrical inequality (54) is studied on an axially symmetric, asymptotically flat initial data set. In order to present the mass formula, we first review the basic definitions and properties of this kind of initial data. For simplicity, we concentrate only on the pure vacuum case (see [35, 38] for the electro-vacuum case).

An *initial data set* for the Einstein vacuum equations is given by a triplet (S, h_{ij}, K_{ij}), where S is a connected three-dimensional manifold, h_{ij} is a Riemannian metric and K_{ij} is a symmetric tensor field on S. The fields are assumed to satisfy the vacuum constraint equations

$$D_j K^{ij} - D^i K = 0, \tag{162}$$

$$R - K_{ij} K^{ij} + K^2 = 0, \tag{163}$$

where D and R are the Levi-Civita connection and the scalar curvature associated with h_{ij}, respectively, and $K = K_{ij} h^{ij}$. In these equations, the indices are moved with the metric h_{ij} and its inverse h^{ij}.

The initial data are called *maximal* if

$$K = 0. \tag{164}$$

This condition is crucial because it implies via the Hamiltonian constraint (163) that the scalar curvature R is non-negative. In fact on the right-hand side of equation (163), it is possible to add a non-negative matter density and all the arguments behind the proof of theorem 3.1 will also apply since they rely on lower bounds for R.

The manifold S is called *Euclidean at infinity*, if there exists a compact subset $\mathcal{K}$ of S, such that $S \setminus \mathcal{K}$ is the disjoint union of a finite number of open sets U_k, and each U_k is diffeomorphic to the exterior of a ball in $\mathbb{R}^3$. Each open set U_k is called an *end* of S. Consider one end U and the canonical coordinates x^i in $\mathbb{R}^3$, which contains the exterior of the ball to which U is diffeomorphic to. Set $r = \left(\sum (x^i)^2\right)^{1/2}$. An initial data set is called *asymptotically flat* if S is Euclidean at infinity, the metric h_{ij} tends to the Euclidean metric and K_{ij} tends to zero as $r \to \infty$ in an appropriate way. These fall-off conditions (see [18, 30] for the optimal fall-off rates) imply the existence of the total mass m (or ADM mass [11]) defined at each end U by

$$m = \frac{1}{16\pi} \lim_{r\to\infty} \int_{\mathcal{S}_r} (\partial_j h_{ij} - \partial_i h_{jj}) s^i \, \mathrm{d}s, \tag{165}$$

where ∂ denotes partial derivatives with respect to x^i, $\mathcal{S}_r$ is the Euclidean sphere $r =$ constant in U, s^i is its exterior unit normal and ds is the surface element with respect to the Euclidean metric.

The angular momentum is also defined as a surface integral at infinity (supplementary fall-off conditions must be imposed, see, e.g., the review articles [108, 78] and reference therein). Let β^i be an infinitesimal generator for rotations with respect to the flat metric associated with the end U; then, the angular momentum J in the direction of β^i is given by

$$J = \frac{1}{8\pi} \lim_{r\to\infty} \int_{\mathcal{S}_r} (K_{ij} - K h_{ij}) \beta^i s^j \, \mathrm{d}s. \tag{166}$$

The above discussion applies to general asymptotically flat initial data. We have seen that at any end the total mass m and the total angular momentum J are the well-defined quantities. We consider now axially symmetric initial data. As in the spacetime definition 4.1, we say that the Riemannian manifold (S, h_{ij}) is axially symmetric if its group of isometries has a subgroup isomorphic to $SO(2)$. We will denote by η^i the Killing field generator of the axial symmetry and by Γ the axis. The initial data set is called axially symmetric if in addition η^i is also a symmetry of K_{ij}:

$$\pounds_\eta K_{ij} = 0. \tag{167}$$

On an axially symmetric spacetime, there exist axially symmetric initial conditions, and conversely, axially symmetric initial data evolve into an axially symmetric spacetime. However, on an axially symmetric spacetime, it is also possible to take initial conditions that are not axially symmetric in the sense defined above. These conditions will of course also evolve into an axially symmetric spacetime, but the Killing vector is 'hidden' on them (these kinds of initial data were studied in [20]). We will not consider such data here since on an axially symmetric spacetime it is always possible to choose initial conditions that are explicitly axially symmetric in the sense defined above.

For axially symmetric data, we have the Komar integral discussed in section 4. This integral can be calculated in terms on the initial data if we choose $\mathcal{S} \subset S$. There is a very simple relation expression for the Komar integral on the initial data:

$$J = \frac{1}{8\pi} \int_{\mathcal{S}} K_{ij} \eta^i s^j \, \mathrm{d}s. \tag{168}$$

The equivalence between (168) and the original definition (94) can be seen as follows. Consider the tetrad adapted to $\mathcal{S}$ defined in section 4.1. Assume that $\mathcal{S} \subset S$. By assumption, the axial Killing vector is tangent to the three-dimensional surface S. Denote by n^μ the timelike unit normal to the three-surface S, and let s^μ be the spacelike unit normal to the two-surface $\mathcal{S}$, which lies on S. These vectors can be written in terms of ℓ^μ and k^μ as follows:

$$n^\mu = \tfrac{1}{\sqrt{2}}(\ell^\mu + k^\mu), \quad s^\mu = \tfrac{1}{\sqrt{2}}(\ell^\mu - k^\mu). \tag{169}$$

Using these expression, we can write the integrand in (94) as

$$J(\mathcal{S}) = \frac{1}{8\pi} \int_{\mathcal{S}} n^\lambda s^\gamma \nabla_\lambda \eta_\gamma. \tag{170}$$

The second fundamental form can be written (as spacetime tensor) in terms of the unit normal n^μ as

$$K_{\mu\nu} = -h^\lambda_\mu \nabla_\lambda n_\nu. \tag{171}$$

Using $\eta^\mu n_\mu = 0$ and s^μ is tangent to S, from (171), we obtain

$$K_{\mu\nu} s^\mu \eta^\nu = n^\nu s^\lambda \nabla_\lambda \eta_\nu. \tag{172}$$

From (172) and (170), we obtain (168).

Comparing expression (168) with (166), we see that if we choose in (166) the vector $\beta^i = \eta^i$ (i.e. the generator of axial rotations), then these two expressions are equivalent since $\eta^i s_i = 0$ for an sphere at infinity.

It is possible to calculate the potential ω for the spacetime Killing field defined in section 4 in terms of K_{ij} as follows. Define the vector K_i by

$$K_i = \epsilon_{ijk} S^j \eta^k, \quad S_i = K_{ij}\eta^j, \tag{173}$$

where ϵ_{ijk} is the volume element with respect to the metric h_{ij}. Then, as a consequence of equations (162) and (167), we have that

$$D_{[i}K_{j]} = 0. \tag{174}$$

Then, the potential ω is defined by

$$K_i = -\tfrac{1}{2} D_i \omega. \tag{175}$$

It can be checked that this is the same potential as defined in the previous section (see [42]) evaluated on the initial surface. The importance of expression (175) is that it allows one to calculate ω in terms of the initial conditions.

The functions ω and η have the information of the stationary part of the initial conditions. The dynamical part is characterized by the functions η' and ω' defined as

$$\eta' = -2K_{ij}\eta^i\eta^j, \quad \omega' = \epsilon_{ijk}\eta^i D^j \eta^k. \tag{176}$$

The notation comes from the fact that they are related with the time derivative of these functions, namely if n^μ is the timelike normal of the initial data, we have (see [42])

$$\eta' = n^\mu \nabla_\mu \eta, \quad \omega' = n^\mu \nabla_\mu \omega. \tag{177}$$

The whole initial data can be constructed in terms of $(\eta, \omega; \eta', \omega')$. These functions are the initial data for the wave map equations that characterize the evolution in axial symmetry. This is clearly seen in the quotient representation (see [49]). For our present purpose, the relevant part of the initial conditions is contained in (η, ω).

So far, we have discussed local implications of axial symmetry. Suppose that S is simply connected and asymptotically flat (with possible multiple ends). It can be proved (see [31]) that in such a case the analysis essentially reduces to consider a manifold of the form $S = \mathbb{R}^3 \setminus \sum_{k=0}^N i_k$, where i_k are the points in $\mathbb{R}^3$. These points represent the extra asymptotic ends of S. Moreover, in [31], it has been proved that on S there exists the following global coordinate system that will be essential in what follows.

Lemma 5.1 (Isothermal coordinates)**.** *Consider an axially symmetric, asymptotically flat (with possible multiple ends) Riemannian manifold (S, h_{ij}), where S is assumed to be simply connected. Then, there exists a global coordinate system (ρ, z, φ), such that the metric has the following form:*

$$h = \mathrm{e}^{(\sigma+2q)}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^2 e^\sigma (\mathrm{d}\varphi + A_\rho\, \mathrm{d}\rho + A_z\, \mathrm{d}z)^2, \tag{178}$$

where the functions σ, q, A_ρ, A_z do not depend on φ. In these coordinates, the axial Killing vector is given by $\eta = \partial/\partial\varphi$ and the square of its norm is given by

$$\eta = \rho^2\, \mathrm{e}^\sigma. \tag{179}$$

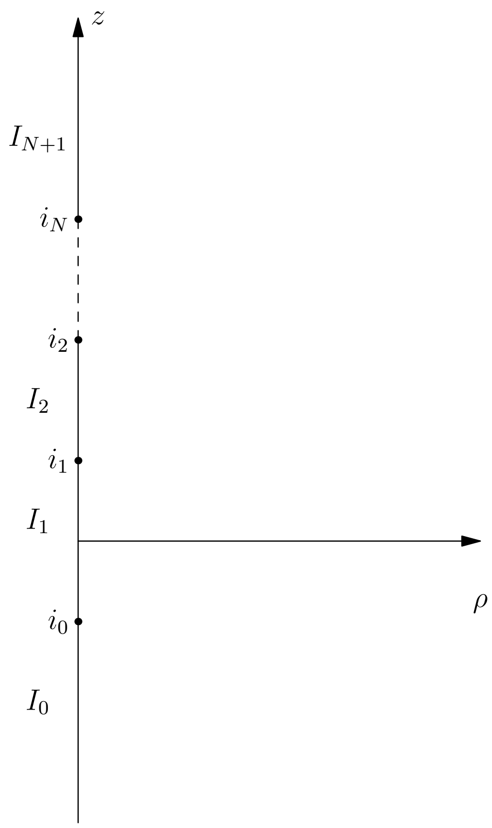

Figure 5. Axially symmetric data with N asymptotic ends.

Using this coordinate system, the end points i_k are located at the axis $\rho = 0$ of $\mathbb{R}^3$. Define the intervals I_k, $1 \leqslant k \leqslant N-1$, to be the open sets in the axis between i_k and i_{k-1}; we also define I_0 and I_N as $z < i_0$ and $z > i_N$, respectively (see figure 5). The manifold S is Euclidean at infinity with $N+1$ ends. In effect, for each i_k take a small open ball B_k of radius $r_{(k)}$ and centered at i_k, where $r_{(k)}$ is small enough, such that B_k does not contain any other $i_{k'}$ with $k' \neq k$. Let $\bar{B}_R$ be a closed ball, with a large radius R, such that $\bar{B}_R$ contains all points i_k in its interior. The compact set $\mathcal{K}$ is given by $\mathcal{K} = \bar{B}_R \setminus \sum_{k=1}^{N} B_k$ and the open sets U_k are given by $B_k \setminus i_k$, for $1 \leqslant k \leqslant N$, and U_0 is given by $\mathbb{R}^3 \setminus \bar{B}_R$. Our choice of coordinate makes an artificial distinction between the end U_0 (which represents $r \to \infty$) and the other ones. This is convenient for our purpose because we want to work always at one fixed end. We emphasize that the isothermal coordinates (ρ, z, φ) (and hence the corresponding spherical radius $r = \sqrt{\rho^2 + z^2}$) are globally defined on S. In what follows, we will always use these coordinates.

The smoothness of the initial data on the axis implies that the potential ω is constant on each interval I_k. These constants are directly related to the angular momentum of the end points i_k. In effect, the angular momentum of an end i_k is defined to be the Komar integral with respect to a surface $\mathcal{S}_k$ that encloses only that point. Using the formula (146), we obtain

$$J_k \equiv J(\mathcal{S}_k) = \tfrac{1}{8}(\omega|_{I_k} - \omega|_{I_{k-1}}). \qquad (180)$$

Let $\mathcal{S}_0$ be a surface that enclose all the end points i_k; then, the total angular momentum of the end $r \to \infty$ is given by

$$J_0 \equiv J(\mathcal{S}_0) = \tfrac{1}{8}(\omega|_{I_0} - \omega|_{I_N}), \tag{181}$$

which is equivalent to

$$J_0 = \sum_{k=1}^{N} J_k. \tag{182}$$

The mass (165) is defined as a boundary integral at infinity. For axially symmetric initial data, there is an equivalent representation of the mass as a positive-definite integral over the three-dimensional slice. This formula was discovered by Brill [26] and allowed him to prove the first version of the positive mass for pure vacuum axially symmetric gravitational waves. This formula was generalized to include metrics with non-hypersurface orthogonal Killing vectors (i.e. with nonzero A_ρ and A_z in the representation (178)) in [61] and for non-trivial topologies in [41]. In particular, this includes the topology of the Kerr initial data. See also [42] and [49] for related discussion on this mass formula.

Take isothermal coordinates and assume that the initial data are maximal (i.e. condition (164) holds). Then, the total mass (165) is given by the following positive-definite integral:

$$m = \frac{1}{32\pi}\int_{\mathbb{R}^3}\left[|\partial\sigma|^2 + \frac{e^{(\sigma+2q)}\omega'^2}{\eta^2} + 2e^{(\sigma+2q)}K^{ij}K_{ij}\right] dv, \tag{183}$$

where $|\partial\sigma|^2 = \sigma_{,\rho}^2 + \sigma_{,z}^2$ and dv is the flat volume element in $\mathbb{R}^3$, namely $dv = \rho\, dz\, d\rho\, d\varphi$. Since the functions are axially symmetric, the integration in φ is trivial; we keep, however, this notation because it will be important in section 5.2.

The remarkable fact of integral (183) is that it applies also to the case of multiples ends; the integral is taking over all the extra ends i_k since it cover the whole $\mathbb{R}^3$.

The integrand in (183) can be further decomposed as follows (see [49] for details):

$$m = \frac{1}{32\pi}\int_{\mathbb{R}^3}\left[\frac{e^{(\sigma+2q)}}{\eta^2}(\eta'^2 + \omega'^2) + F^2 + |\partial\sigma|^2 + \frac{|\partial\omega|^2}{\eta^2}\right] dv. \tag{184}$$

Here, the term F^2 is essentially the square of the second fundamental form of the two-surface that is obtained by the quotient of the manifold S by the symmetry $SO(2)$ (this two-surface is a half-plane $\rho \geqslant 0$ minus the end points i_k). The important point in this representation is that it clearly separates the dynamical and the stationary terms in the mass formula. In fact, this formula can be seen as the geometrical conserved energy of the wave map that appear in the evolution equations (see [49]).

The first three terms in the integrand of (184) correspond to the dynamical part for the mass. They vanished for stationary solutions. The last two terms correspond to the stationary part. They give the total mass for Kerr. Explicitly, the stationary part is given by

$$\mathcal{M}(\sigma,\omega) = \frac{1}{32\pi}\int_{\mathbb{R}^3}(|\partial\sigma|^2 + \rho^{-4}e^{-2\sigma}|\partial\omega|^2)\, dv. \tag{185}$$

And hence we obtain the important bound

$$m \geqslant \mathcal{M}(\sigma,\omega). \tag{186}$$

We have presented the mass formula but we have not discussed the asymptotic conditions that ensures that this integral is well defined. We discuss them now, trying to focus only in the essential aspects; for the technical details, we refer to [44] and [31]. All the important features can be seen already in the term $|\partial\sigma|^2$ in (185). We will concentrate on this term only in what follows.

It is instructive to see how the function σ behaves for the Kerr black hole initial data with mass m. These initial data have two asymptotic ends, namely we have only one extra end point i_1. In the limit $r \to \infty$ (i.e. the end U_0), we have

$$\sigma = \frac{2m}{r} + O(r^{-2}). \tag{187}$$

At the end i_1, we have

$$\sigma = -4\log(r) + O(1) \tag{188}$$

for the non-extreme case, and

$$\sigma = -2\log(r) + O(1) \tag{189}$$

for the extreme case. We see that the fall-off behavior at the end i_1 is different for the non-extreme and extreme cases; this reflects of course the difference between an asymptotically flat end and a cylindrical end (see figures 2 and 3, respectively).

To include this kind of fall-off behavior, let us consider the following conditions. In the limit $r \to \infty$, we impose

$$\sigma = o(r^{-1/2}), \quad \partial\sigma = o(r^{-3/2}). \tag{190}$$

At the end point i_k, we impose

$$\sigma = o\big(r_{(k)}^{-1/2}\big), \quad \partial\sigma = o\big(r_{(k)}^{-3/2}\big). \tag{191}$$

Despite the formal similarity of (191) and (190), they are of a very different nature. Conditions (190) are essentially the standard asymptotic flat conditions on the metric; they include in particular the behavior (187) of the Kerr initial data. On the other hand, conditions (191) on the extra ends i_k are much more general than asymptotic flatness. Note that asymptotic flatness at the ends i_k in these coordinates essentially implies a behavior like (190). But conditions (188) includes also (189) (in fact also any logarithmic behavior with any coefficient).

The motivation for the fall-off conditions (191) and (190) is that they ensure that the integral of $|\partial\sigma|^2$ is finite both at infinity and at the end points i_k. And the advantage of condition (191) is that it encompasses both kind of behavior: asymptotically flat and cylindrical. This is the main motivation of the Brill data definition that appears in theorem 3.2 (see [44] for details).

5.2. Harmonic maps

The crucial property of the mass functional defined in (185) is its relation to the energy of harmonic maps from $\mathbb{R}^3$ to the hyperbolic plane $\mathbb{H}^2$: they differ by a boundary term. To see this relation, consider first the mass functional $\mathcal{M}$ defined on a bounded region Ω, such that Ω does not intersect the axis Γ (given by $\rho = 0$):

$$\mathcal{M}_\Omega = \frac{1}{32\pi}\int_\Omega \left(|\partial\sigma|^2 + \rho^{-4}\,\mathrm{e}^{-2\sigma}|\partial\omega|^2\right)\mathrm{d}v. \tag{192}$$

We consider this integral for general functions σ and ω that are not necessarily axially symmetric. The function $\log\rho$ is harmonic outside the axis:

$$\Delta\log\rho = 0, \quad \text{in} \quad \mathbb{R}^3 \setminus \Gamma, \tag{193}$$

where Δ is the flat Laplacian in $\mathbb{R}^3$. Using equation (193), we obtain the following identity:

$$\mathcal{M}_\Omega = \mathcal{M}'_\Omega - \oint_{\partial\Omega} 4\frac{\partial\log\rho}{\partial s}(\log\rho + \sigma)\,\mathrm{d}s, \tag{194}$$

where the derivative is taken in the normal direction to the boundary $\partial\Omega$ and $\mathcal{M}'_\Omega$ is given by

$$\mathcal{M}'_\Omega = \frac{1}{32\pi}\int_\Omega \left(\frac{|\partial\eta|^2 + |\partial\omega|^2}{\eta^2}\right) \mathrm{d}v. \tag{195}$$

Recall that η is defined by (179).

The functional $\mathcal{M}'_\Omega$ defines an energy for maps $(\eta, \omega) : \mathbb{R}^3 \to \mathbb{H}^2$, where $\mathbb{H}^2$ denotes the hyperbolic plane $\{(\eta, \omega) : \eta > 0\}$, equipped with the negative constant curvature metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}\eta^2 + \mathrm{d}\omega^2}{\eta^2}. \tag{196}$$

The Euler–Lagrange equations for the energy $\mathcal{M}'_\Omega$ are given by

$$\Delta \log \eta = -\frac{|\partial\omega|^2}{\eta^2}, \tag{197}$$

$$\Delta\omega = 2\frac{\partial\omega\partial\eta}{\eta}. \tag{198}$$

The solutions of (197)–(198), i.e. the critical points of $\mathcal{M}'_\Omega$, are called harmonic maps from $\mathbb{R}^3 \to \mathbb{H}^2$. Since $\mathcal{M}_\Omega$ and $\mathcal{M}'_\Omega$ differ only by a boundary term, they have the same Euler–Lagrange equations.

Harmonic maps have been intensively studied; in particular, the Dirichlet problem for target manifolds with negative curvature has been solved [64, 101, 100] and [71]. The last article is particularly relevant here. However, these results do not directly apply in our case because the equations are singular at the axis. One of the main technical complications in the proofs of theorems 3.1 and 3.2 is precisely how to handle the singular behavior at the axis.

We present in what follows the main strategy in the proof of inequality (67) (we follow the approach presented in [33] and [38]). The core of the proof is the use of a theorem due to Hildebrandt *et al* [71] for harmonic maps. In that work, it is shown that if the domain for the map is compact, connected and with non-void boundary, and the target manifold has negative sectional curvature, then minimizers of the harmonic energy with the Dirichlet boundary conditions exist, are smooth and satisfy the associated Euler–Lagrange equations. That is, harmonic maps are the minimizers of the harmonic energy for the given Dirichlet boundary conditions. Also, the solutions of the Dirichlet boundary value problem are unique when the target manifold has negative sectional curvature. Therefore, we want to use the relation between $\mathcal{M}$ and the harmonic energy $\mathcal{M}'$ in order to prove that minimizers of $\mathcal{M}'$ are also minimizers of $\mathcal{M}$.

There are two main difficulties in doing so. First, the harmonic energy $\mathcal{M}'$ is not defined for the functions that we are considering if the domain of integration includes the axis. Second, we are not dealing with a Dirichlet problem. To overcome these difficulties, an appropriate compact domain is chosen which does not contain the singularities. Then, a partition function is used to interpolate between extreme Kerr initial data outside this domain and general data inside, also constructing an auxiliary intermediate region. This solves two difficulties in the sense that now the Dirichlet problem on a compact region can be considered, and the harmonic energy is well defined for this domain of integration. This allows us to show that the mass functional for Kerr data is less than or equal to the mass functional for the auxiliary interpolating data. The final step is to show that as we increase the compact domain to cover all $\mathbb{R}^3$, the auxiliary data converge to the mass functional for the original general data. This is the subtle and technical part of the proof.

Finally, we mention that it is possible to construct a heat flow for equations (197)–(198). Note that the first existence result for harmonic maps used a heat flow [53]. In our present setting, an appropriate heat flow that incorporates the singular boundary conditions is constructed as follows. Consider functions (σ, ω) that depend on an extra parameter t. Then, we define the following flow:

$$\dot{\sigma} = \Delta\sigma + \frac{e^{-2\sigma}|\partial\omega|^2}{\rho^4}, \tag{199}$$

$$\dot{\omega} = \Delta\omega - 2\frac{\partial\omega\partial\eta}{\eta}, \tag{200}$$

where the dot denotes partial derivative with respect to t. Equations (199)–(200) represent the gradient flow of the energy (185). The important property of the flow is that the energy $\mathcal{M}$ is monotonic under appropriate boundary conditions. This flow has been used in [48] as an efficient method for numerically computing both the solution and the value of the energy $\mathcal{M}$ at a stationary solution.

5.3. Two-surfaces and the mass functional

In this section, we want to define a mass functional over a two-surface $\mathcal{S}$. This functional plays a major role in the proofs of the quasi-local inequalities. Let us motivate first the definition.

Consider the mass functional (185) defined over $\mathbb{R}^3$. Assume that on $\mathbb{R}^3$ we have a foliation of two-surfaces. For example, take the spherical coordinates (r, θ, φ) and the two-surface $r =$ constant. We can split the integral (185) into an integral over the two surface and a radial integral. However, the integral over the two-surface alone will not have any intrinsic meaning since, of course, its integrand depends on r. To obtain an intrinsic expression, we need to somehow avoid the r dependence taking some kind of limit. For extreme Kerr initial data, this limit is provided naturally on the cylindrical end. The functions σ and ω have a well-defined and non-trivial limit there (in contrast with the asymptotic flat end where they tend to zero). Also, all the radial derivatives go to zero at the cylindrical end. This surface define an extreme Kerr throat surface (see the discussion in [46] and [45] for more details) and it is characterized by only one parameter: the angular momentum J. Explicitly, the functions in that limit are given by

$$\sigma_0 = \ln(4|J|) - \ln(1 + \cos^2\theta), \quad \omega_0 = -\frac{8J\cos\theta}{1 + \cos^2\theta}. \tag{201}$$

And the area of this two-surface is given by

$$A_0 = 8\pi|J|. \tag{202}$$

Consider the following functional over a two-sphere:

$$\mathcal{M}^{\mathcal{S}} = \int_0^\pi \left(|\partial_\theta\sigma|^2 + 4\sigma + \frac{|\partial_\theta\omega|^2}{\eta^2}\right)\sin\theta\, d\theta. \tag{203}$$

We can write this integral as an integral over the unit sphere $\mathbb{S}^2$ in the following way:

$$\mathcal{M}^{\mathcal{S}} = \frac{1}{2\pi}\int_{S^2}\left(|D\sigma|^2 + 4\sigma + \frac{|D\omega|^2}{\eta^2}\right) ds_0, \tag{204}$$

where $ds_0 = \sin\theta\, d\theta\, d\phi$ is the area element of the standard metric in $\mathbb{S}^2$ and D is the covariant derivative with respect to this metric. In complete analogy with the discussion in section 5.2, we consider this integral for the general functions $\sigma(\theta, \varphi)$ and $\omega(\theta, \varphi)$ on $\mathbb{S}^2$, which are not necessarily axially symmetric.

The Euler–Lagrange equations of this functional are given by

$$\Delta_0\sigma - 2 = -\frac{|\partial_\theta\omega|^2}{\tilde{\eta}^2}, \tag{205}$$

$$\Delta_0\omega = 2\frac{\partial_\theta\omega\partial_\theta\tilde{\eta}}{\tilde{\eta}}, \tag{206}$$

where

$$\tilde{\eta} = \sin^2\theta\, \mathrm{e}^{\sigma}. \tag{207}$$

The functional (204) is relevant because the extreme Kerr throat surface (201) is a solution of the Euler–Lagrange equations. Equations (205)–(206) can also be deduced from the stationary axially symmetric equations (197)–(198) taking the limit to the cylindrical end (given by $r \to 0$ in these coordinates) and using the fall-off behavior of the functions at the cylindrical end:

$$\sigma = -2\log(r) + \tilde{\sigma}(\theta) + O(r^{-1}), \quad \omega = \tilde{\omega}(\theta) + O(r^{-1}). \tag{208}$$

Note that 2 that appears in the second term on the left-hand side of (205) arises from the characteristic $-2\log(r)$ fall-off behavior at the cylindrical end (we have already discussed this property in equation (189)). Also, this 2 produces the linear term 4σ in the mass functional (204).

The value of the functional (204) at the extreme Kerr throat sphere is given by

$$\mathcal{M}^{\mathcal{S}} = 8(\ln(2|J|) + 1). \tag{209}$$

The connection between the mass functional (204) and the energy of harmonic maps between $\mathbb{S}^2$ and $\mathbb{H}^2$ is very similar to the one described in the previous section for the mass functional $\mathcal{M}$. Namely, consider the functional

$$\mathcal{M}'^{\mathcal{S}}_{\Omega} = \frac{1}{2\pi}\int_{\Omega}\frac{|D\eta|^2 + |D\omega|^2}{\eta^2}\,\mathrm{d}s_0 \tag{210}$$

defined on some domain $\Omega \subset \mathbb{S}^2$, such that Ω does not include the poles. Integrating by parts and using the identity

$$\Delta_0(\log(\sin\theta)) = -1, \tag{211}$$

where Δ_0 is the Laplacian on $\mathbb{S}^2$, we obtain the following relation between $\mathcal{M}$ and $\mathcal{M}'$:

$$\mathcal{M}'^{\mathcal{S}}_{\Omega} = \mathcal{M}^{\mathcal{S}}_{\Omega} + 4\int_{\Omega}\log\sin\theta\,\mathrm{d}s_0 + \oint_{\partial\Omega}(4\sigma + \log\sin\theta)\frac{\partial\log\sin\theta}{\partial s}\,\mathrm{d}l, \tag{212}$$

where s denotes the exterior normal to Ω, $\mathrm{d}l$ is the line element on the boundary $\partial\Omega$ and we have used the obvious notation $\mathcal{M}^{\mathcal{S}}_{\Omega}$ to denote the mass functional (204) defined over the domain Ω. The difference between $\mathcal{M}^{\mathcal{S}}$ and $\mathcal{M}'^{\mathcal{S}}$ are the boundary integral plus the second term that is just a numerical constant. Note that if we integrate over $\mathbb{S}^2$, this constant term is finite:

$$\int_{\Omega}\log\sin\theta\,\mathrm{d}s_0 = 2\log 2 - 2. \tag{213}$$

The boundary terms however diverges at the poles.

In an analogous way as it was described in the previous section, the functional $\mathcal{M}'^{\mathcal{S}}$ defines an energy for maps $(\eta, \omega) : \mathbb{S}^2 \to \mathbb{H}^2$, where $\mathbb{H}^2$ denotes the hyperbolic plane $\{(\tilde{\eta}, \omega) : \tilde{\eta} > 0\}$, equipped with the negative constant curvature metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}\tilde{\eta}^2 + \mathrm{d}\omega^2}{\tilde{\eta}^2}. \tag{214}$$

The Euler–Lagrange equations for the energy $\mathcal{M}'$ are called harmonic maps from $S^2 \to \mathbb{H}^2$. Since $\mathcal{M}^{\mathcal{S}}$ and $\mathcal{M}'^{\mathcal{S}}$ differ only by a constant and boundary terms, they have the same Euler–Lagrange equations.

The variational problem for the mass functional on the two-surface is very similar to the one for the mass functional on $\mathbb{R}^3$ described in section 5.2 (see [1] for the details).

6. Open problems

In this final section, we present the main open problems regarding these geometrical inequalities. In the light of the recent results of [72], there now exists a very interesting open door to higher dimensions, but this lies out of the scope of this review, and hence, in this section, we restrict ourselves to four spacetime dimensions. Our aim is to present open problems that are relevant (and probably involve the discovery of new techniques) and at the same time appear feasible to solve.

We begin with the global inequality (54). The two main open problems are the following:

- remove the maximal condition,
- generalization for asymptotic flat manifolds with multiple ends.

The situation for the maximal condition in theorems 3.1 and 3.2 resembles the strategy of the proof of positive mass theorem by Schoen and Yau [102–104].[1] That proof was performed first for maximal initial data and then extended for general data. It is conceivable that similar techniques (i.e. the use of the Jang equation) can be used here also, but it is far from obvious how to extend these ideas to the present case.

The most relevant open problem regarding the global inequality (54) is its validity for manifolds with multiple asymptotic ends with non-trivial angular momentum. The physical heuristic argument presented in section 2 applies to that case, and hence there is little doubt that the inequality holds. In particular, as we already mentioned in section 3.1, for the case of three asymptotic ends, there is strong numerical evidence for the validity of the inequality [48].

Theorem 3.3 (proved in [33]) reduces the proof of the inequality to prove the bound (57) for the mass functional evaluated at the stationary solution. The case of three asymptotic ends (which, roughly speaking, is equivalent to saying that we have two black holes) is special for the following reason. There exist exact stationary axially symmetric solutions of the Einstein equations (the so-called double-Kerr-NUT solutions [84, 92]), which represent two Kerr-like black holes. These solutions contain singularities that prevent them qualifying as a genuine equilibrium states for binary black holes. In fact, one of the main parts in the strategy to prove that the uniqueness theorem holds for the binary case is to prove that these solutions are always singular [93]. However, even if these solutions are singular, they can be useful to prove the bound (57), because in order to qualify as a stationary point of the mass functional $\mathcal{M}$ all we need is that the functions (η, ω) are regular. It is conceivable that some of these exact solutions have this property (that means, of course, that other coefficients of the metric are singular), and hence, with the explicit expression for (η, ω) provided by them, it will possible to evaluate $\mathcal{M}$ and check the bound (57). In the articles [87, 27], the geometrical inequality (54) has been studied for these exact solutions. These results provide a guide for which subclasses of these solutions are potentially useful to prove the bound (57). Unfortunately, the solutions, although explicit, are very complicated and it is very difficult to compute the mass functional $\mathcal{M}$ for them[2].

To compute the value of $\mathcal{M}$ for this kind of exact solution would certainly be a very interesting result that not only will prove inequality (54) for the three-asymptotic-end case but also will hopefully provide some new interpretation of the double-Kerr-NUT solutions. However, this result will be confined to the three-asymptotic-end case and probably will not put light into the mechanism of the variational problem for the mass functional $\mathcal{M}$ with multiple

[1] I thank Marcus Khuri for pointing this out to me and for relevant discussion on this subject.

[2] I thank Piotr Chrusciel for relevant discussion on this point.

ends. The basic property which is expected to satisfy $\mathcal{M}$ is the following: if an extra black hole is added, with arbitrary angular momentum, then the value of $\mathcal{M}$ increases. This property is of course another way of saying that the force between the black holes is always attractive (in particular, it cannot be balanced by the spin–spin repulsive interaction). The variational problem for the mass functional $\mathcal{M}$ with multiple ends appears to have a remarkable structure. In particular, there is formal similarity between this problem and the kind of singular boundary problems for harmonic maps studied in [21].

We mention in section 2 that there is a clear physical connection between the global inequality (54) and the Penrose inequality with angular momentum in axial symmetry. Hence, it is appropriate to list here the Penrose inequality as a relevant open problem for axially symmetric black holes (for more detail on this problem, see the review [88]).

- Prove the Penrose inequality with angular momentum (see equation (17)).

However, it is important to emphasize that it not clear that the techniques used to prove theorems 3.1 and 3.2 will help to solve this problem. The reason is the following. The proof of the Penrose inequality involves an inner boundary, namely the black hole horizon. On the other hand, theorems 3.1 and 3.2 refer to complete manifolds without inner boundaries. As we mentioned in section 3.1, there are results in axial symmetry, which include inner boundaries (i.e. [61, 36]) and use similar techniques as in theorems 3.1 and 3.2 (namely, the representation of the mass as a positive-definite integral in axial symmetry, see section 5). However, the boundary for the Penrose inequality has a very important property: it should be an outer minimal surface (for simplicity, we discuss only the Riemannian case). This property is very difficult to incorporate in a standard boundary value problem. To see if the mass formula in axial symmetry is useful to prove the Penrose inequality with angular momentum, the natural first step is to prove the Riemannian Penrose inequality without angular momentum in axial symmetry using this mass formula. The results presented in [61, 36] contribute in this direction, but so far the problem remains open. Maybe there exists a combination of the global flows techniques developed in [75, 25] for the Riemannian Penrose inequality with the mass functional that incorporates the angular momentum in axial symmetry. For example, it is suggestive that the strategy for the proof of the charged Penrose inequality (15) given in [76] and [75] consists of first proving inequality (20) using a flow and a lower bound to the scalar curvature that resembles the mass functional. However, a generalization of this construction to include the angular momentum is far from obvious.

We turn now to the quasi-local inequalities. The three main problems are the following.

- Include the charge and the electromagnetic angular momentum in axial symmetry for inequality (64).
- Isoperimetric inequalities in axial symmetry with angular momentum. That is, a version for theorem 3.5 (in axial symmetry) with angular momentum instead of charge.
- A generalization of inequality (64) without axial symmetry.

The inclusion of charge in inequality (64) is important, of course, since charge is the other relevant parameter that characterized the Kerr–Newman black hole. But it is also relevant for another reason. The angular momentum that appears in this inequality is the Komar gravitational angular momentum. In the generalization with charge, it is expected that the total angular momentum (i.e. gravitational plus electromagnetic) appears in this inequality. This is shown, including also the magnetic charge, in [56]. In addition, this is important in connection with the rigidity statement. The work on this problem is in progress [57].

We mention in section 2 that a version of inequality (64) for isoperimetric surfaces (instead of black hole horizons) could have interesting astrophysical applications, since apparently highly spinning neutron stars approach to the limit value predicted by this kind of inequality. Such a theorem will be analogous to theorem 3.5 for the charge. However, it is by no means clear that similar techniques to the one used in the proofs of theorem 3.6 can be applied to that case.

Finally, we mention the problem of finding versions of inequality (64) without any symmetry assumption. In contrast with the other open problems presented here, this is not a well-defined mathematical problem since there is no unique notion of quasi-local angular momentum in the general case. However, exploring the scope of the inequality in regions close to axial symmetry (in some appropriate sense) can perhaps provide such a notion. From the physical point of view, we do not see any reason why this inequality should only hold in axial symmetry.

Acknowledgments

The author thanks Robert Geroch, José Luis Jaramillo, Carlos Kozameh and Walter Simon for illuminating discussions during the preparation of this review. The author was supported by CONICET (Argentina). This work was supported in part by grant PIP 6354/05 of CONICET (Argentina), grant Secyt-UNC (Argentina) and the Partner Group grant of the Max Planck Institute for Gravitational Physics (Germany).

References

[1] Aceña A, Dain S and Clément M E G 2011 Horizon area–angular momentum inequality for a class of axially symmetric black holes *Class. Quantum Grav.* **28** 105014 (arXiv:1012.2413)

[2] Andersson L, Mars M and Simon W 2005 Local existence of dynamical and trapping horizons *Phys. Rev. Lett.* **95** 111102 (arXiv:gr-qc/0506013)

[3] Andersson L, Mars M and Simon W 2008 Stability of marginally outer trapped surfaces and existence of marginally outer trapped tubes *Adv. Theor. Math. Phys.* **12** 853–88

[4] Ansorg M and Hennig J 2008 The inner Cauchy horizon of axisymmetric and stationary black holes with surrounding matter *Class. Quantum Grav.* **25** 222001 (arXiv:0810.3998)

[5] Ansorg M and Hennig J 2009 The inner Cauchy horizon of axisymmetric and stationary black holes with surrounding matter in Einstein–Maxwell theory *Phys. Rev. Lett.* **102** 221102 (arXiv:0903.5405)

[6] Ansorg M, Hennig J and Cederbaum C 2011 Universal properties of distorted Kerr–Newman black holes *Gen. Rel. Grav.* **43** 1205–10 (arXiv:1005.3128)

[7] Ansorg M and Petroff D 2005 Black holes surrounded by uniformly rotating rings *Phys. Rev.* D **72** 024019 (arXiv:gr-qc/0505060)

[8] Ansorg M and Petroff D 2006 Negative Komar mass *Class. Quantum Grav.* **23** L81 (arXiv:gr-qc/0607091)

[9] Ansorg M and Petroff D 2007 Negative Komar masses in regular stationary spacetimes *Proc. MGXI Meeting* pp 1600–2 (arXiv:0708.3899)

[10] Ansorg M and Pfister H 2008 A universal constraint between charge and rotation rate for degenerate black holes surrounded by matter *Class. Quantum Grav.* **25** 035009 (arXiv:0708.4196)

[11] Arnowitt R, Deser S and Misner C W 1962 The dynamics of general relativity *Gravitation: An Introduction to Current Research* ed L Witten (New York: Wiley) pp 227–65 (arXiv:gr-qc/0405109)

[12] Ashtekar A *et al* 2000 Isolated horizons and their applications *Phys. Rev. Lett.* **85** 3564–7 (arXiv:gr-qc/0006006)

[13] Ashtekar A, Beetle C and Lewandowski J 2001 Mechanics of rotating isolated horizons *Phys. Rev.* D **64** 044016 (arXiv:gr-qc/0103026)

[14] Ashtekar A and Krishnan B 2002 Dynamical horizons: energy, angular momentum, fluxes and balance laws *Phys. Rev. Lett.* **89** 261101 (arXiv:gr-qc/0207080)

[15] Ashtekar A and Krishnan B 2003 Dynamical horizons and their properties *Phys. Rev.* D **68** 104030 (arXiv:gr-qc/0308033)

[16] Baiotti L *et al* 2005 Three-dimensional relativistic simulations of rotating neutron-star collapse to a Kerr black hole *Phys. Rev.* D **71** 024035 (arXiv:gr-qc/0403029)

[17] Barbosa J L, Do Carmo M and Eschenburg J 1988 Stability of hypersurfaces of constant mean curvature in Riemannian manifolds *Math. Z.* **197** 123–38

[18] Bartnik R 1986 The mass of an asymptotically flat manifold *Commun. Pure Appl. Math.* **39** 661–93

[19] Bartnik R A and Chruściel P T 2005 Boundary value problems for Dirac-type equations *J. Reine Angew. Math.* **579** 13–73 (arXiv:math.DG/0307278)

[20] Beig R and Chruściel P T 1997 Killing initial data *Class. Quantum Grav.* **14** A83–92

[21] Bethuel F, Brezis H and Hélein F 1994 *Ginzburg–Landau vortices (Progress in Nonlinear Differential Equations and Their Applications* vol 13*)* (Boston, MA: Birkhäuser)

[22] Bonnor W B 1998 A model of a spheroidal body *Class. Quantum Grav.* **15** 351

[23] Booth I and Fairhurst S 2007 Isolated, slowly evolving and dynamical trapping horizons: geometry and mechanics from surface deformations *Phys. Rev.* D **75** 084019 (arXiv:gr-qc/0610032)

[24] Booth I and Fairhurst S 2008 Extremality conditions for isolated and dynamical horizons *Phys. Rev.* D **77** 084005 (arXiv:0708.2209)

[25] Bray H L 2001 Proof of the Riemannian Penrose conjecture using the positive mass theorem *J. Differ. Geom.* **59** 177–267 (arXiv:math.DG/9911173)

[26] Brill D 1959 On the positive definite mass of the Bondi–Weber–Wheeler time-symmetric gravitational waves *Ann. Phys.* **7** 466–83

[27] Cabrera-Munguia I, Manko V and Ruiz E 2010 Remarks on the mass-angular momentum relations for two extreme Kerr sources in equilibrium *Phys. Rev.* D **82** 124042 (arXiv:1010.0697)

[28] Christodoulou D 1970 Reversible and irreversible transformations in black-hole physics *Phys. Rev. Lett.* **25** 1596–7

[29] Christodoulou D and Yau S-T 1988 Some remarks on the quasi-local mass *Mathematics and General Relativity (Santa Cruz, CA, 1986) (Contemporary Mathematics* vol 71*)* (Providence, RI: American Mathematical Society) pp 9–14

[30] Chruściel P 1986 Boundary conditions at spatial infinity from a Hamiltonian point of view *Topological Properties and Global Structure of Space-Time (Erice, 1985) (NATO Advanced Science Institutes Series B: Physics* vol 138*)* (New York: Plenum) pp 49–59

[31] Chrusciel P T 2008 Mass and angular-momentum inequalities for axisymmetric initial data sets: part I. Positivity of mass *Ann. Phys.* **323** 2566–90 (arXiv:0710.3680)

[32] Chrusciel P T, Eckstein M, Nguyen L and Szybka S J 2011 Existence of singularities in two-Kerr black holes arXiv:1111.1448

[33] Chruściel P T, Li Y and Weinstein G 2008 Mass and angular-momentum inequalities for axisymmetric initial data sets: part II. Angular momentum *Ann. Phys.* **323** 2591–613 (arXiv:0712.4064)

[34] Chrusciel P T and Costa J L 2008 On uniqueness of stationary vacuum black holes *Proc. Géométrie Diffrentielle, Physique Mathématique, Mathématiques et Société, Astérisque* vol 321 pp 195–265 (arXiv:0806.0016)

[35] Chrusciel P T and Costa J L 2009 Mass, angular momentum and charge inequalities for axisymmetric initial data *Class. Quantum Grav.* **26** 235013 (arXiv:0909.5625)

[36] Chrusciel P T and Nguyen L 2011 A lower bound for the mass of axisymmetric connected black hole data sets *Class. Quantum Grav.* **28** 125001 (arXiv:1102.1175)

[37] Chrusciel P T, Reall H S and Tod P 2006 On Israel–Wilson–Perjes black holes *Class. Quantum Grav.* **23** 2519–40 (arXiv:gr-qc/0512116)

[38] Costa J L 2010 Proof of a Dain inequality with charge *J. Phys. A Math. Theor.* **43** 285202 (arXiv:0912.0838)

[39] Dain S 2006 Angular momentum–mass inequality for axisymmetric black holes *Phys. Rev. Lett.* **96** 101101 (arXiv:gr-qc/0511101)

[40] Dain S 2006 Proof of the (local) angular momentum–mass inequality for axisymmetric black holes *Class. Quantum Grav.* **23** 6845–55 (arXiv:gr-qc/0511087)

[41] Dain S 2006 A variational principle for stationary, axisymmetric solutions of Einstein's equations *Class. Quantum Grav.* **23** 6857–71 (arXiv:gr-qc/0508061)

[42] Dain S 2008 Axisymmetric evolution of Einstein equations and mass conservation *Class. Quantum Grav.* **25** 145021 (arXiv:0804.2679)

[43] Dain S 2008 The inequality between mass and angular momentum for axially symmetric black holes *Int. J. Mod. Phys.* D **17** 519–23 (arXiv:0707.3118 [gr-qc])

[44] Dain S 2008 Proof of the angular momentum–mass inequality for axisymmetric black holes *J. Differ. Geom.* **79** 33–67 (arXiv:gr-qc/0606105)

[45] Dain S 2010 Extreme throat initial data set and horizon area–angular momentum inequality for axisymmetric black holes *Phys. Rev.* D **82** 104010 (arXiv:1008.0019)

[46] Dain S and Clément M E G 2011 Small deformations of extreme Kerr black hole initial data *Class. Quantum Grav.* **28** 075003 (arXiv:1001.0178)
[47] Dain S, Jaramillo J L and Reiris M 2012 Area–charge inequality for black holes *Class. Quantum Grav.* **29** 035013 (arXiv:1109.5602)
[48] Dain S and Ortiz O E 2009 Numerical evidences for the angular momentum–mass inequality for multiple axially symmetric black holes *Phys. Rev.* D **80** 024045 (arXiv:0905.0708)
[49] Dain S and Ortiz O E 2010 Well-posedness, linear perturbations and mass conservation for the axisymmetric Einstein equations *Phys. Rev.* D **81** 044040
[50] Dain S and Reiris M 2011 Area–angular-momentum inequality for axisymmetric black holes *Phys. Rev. Lett.* **107** 051101 (arXiv:1102.5215)
[51] Dain S, Weinstein G and Yamada S 2011 Counterexample to a Penrose inequality conjectured by Gibbons *Class. Quantum Grav.* **28** 085015 (arXiv:1012.4190)
[52] Dehmelt H 1988 A single atomic particle forever floating at rest in free space: new value for electron radius *Phys. Scr.* **1988** 102
[53] Eells J Jr and Sampson J H 1964 Harmonic mappings of Riemannian manifolds *Am. J. Math.* **86** 109–60
[54] Friedman J L and Mayer S 1982 Vacuum handles carrying angular momentum; electrovac handles carrying net charge *J. Math. Phys.* **23** 109–15
[55] Clément M E G 2011 Comment on 'Horizon area–angular momentum inequality for a class of axially symmetric black holes' arXiv:1102.3834
[56] Clement M E G and Jaramillo J L 2011 Black hole area–angular momentum–charge inequality in dynamical non-vacuum spacetimes arXiv:1111.6248
[57] Clement M E G, Jaramillo J L and Reiris M 2011 in preparation
[58] Giacomazzo B, Rezzolla L and Stergioulas N 2011 Collapse of differentially rotating neutron stars and cosmic censorship *Phys. Rev.* D **84** 024022 (arXiv:1105.0122)
[59] Gibbons G 1999 Some comments on gravitational entropy and the inverse mean curvature flow *Class. Quantum Grav.* **16** 1677–87 (arXiv:hep-th/9809167)
[60] Gibbons G W, Hawking S W, Horowitz G T and Perry M J 1983 Positive mass theorems for black holes *Commun. Math. Phys.* **88** 295–308
[61] Gibbons G W and Holzegel G 2006 The positive mass and isoperimetric inequalities for axisymmetric black holes in four and five dimensions *Class. Quantum Grav.* **23** 6459–78 (arXiv:gr-qc/0606116)
[62] Gibbons G W and Hull C M 1982 A Bogomolny bound for general relativity and solitons in $N = 2$ supergravity *Phys. Lett.* B **109** 190–4
[63] Gourgoulhon E 2005 A generalized Damour–Navier–Stokes equation applied to trapping horizons *Phys. Rev.* D **72** 104007 (arXiv:gr-qc/0508003)
[64] Hamilton R S 1975 *Harmonic Maps of Manifolds With Boundary (Lecture Notes in Mathematics* vol 471*)* (Berlin: Springer)
[65] Hartle J and Hawking S 1972 Solutions of the Einstein–Maxwell equations with many black holes *Commun. Math. Phys.* **26** 87–101
[66] Hayward S 1994 General laws of black hole dynamics *Phys. Rev.* D **49** 6467 (arXiv:gr-qc/9306006)
[67] Hennig J and Ansorg M 2009 The inner Cauchy horizon of axisymmetric and stationary black holes with surrounding matter in Einstein–Maxwell theory: study in terms of soliton methods *Ann. Henri Poincare* **10** 1075–95 (arXiv:0904.2071)
[68] Hennig J, Ansorg M and Cederbaum C 2008 A universal inequality between the angular momentum and horizon area for axisymmetric and stationary black holes with surrounding matter *Class. Quantum. Grav.* **25** 162002
[69] Hennig J, Cederbaum C and Ansorg M 2010 A universal inequality for axisymmetric and stationary black holes with surrounding matter in the Einstein–Maxwell theory *Commun. Math. Phys.* **293** 449–67 (arXiv:0812.2811)
[70] Hessels J W *et al* 2006 A radio pulsar spinning at 716 Hz *Science* **311** 1901–4 (arXiv:astro-ph/0601337)
[71] Hildebrandt S, Kaul H and Widman K-O 1977 An existence theorem for harmonic mappings of Riemannian manifolds *Acta Math.* **138** 1–16
[72] Hollands S 2011 Horizon area–angular momentum inequality in higher dimensional spacetimes arXiv:1110.5814
[73] Horowitz G T 1984 The positive energy theorem and its extensions *Asymptotic Behavior of Mass and Spacetime Geometry (Corvallis, OR, 1983) (Lecture Notes in Phys.* vol 202*)* ed F J Flaherty (Berlin: Springer) pp 1–21
[74] Huang L-H, Schoen R and Wang M-T 2011 Specifying angular momentum and center of mass for vacuum initial data sets *Commun. Math. Phys.* **306** 785–803 (arXiv:1008.4996)
[75] Huisken G and Ilmanen T 2001 The inverse mean curvature flow and the Riemannian Penrose inequality *J. Differ. Geom.* **59** 352–437

[76] Jang P S 1979 Note on cosmic censorship *Phys. Rev.* D **20** 834–837

[77] Jaramillo J 2012 Area inequalities for stable marginally trapped surfaces *Proc. VI International Meeting on Lorentzian Geometry, Granada 2011 (Springer Proceedings in Mathematics)* to appear arXiv:1201.2054

[78] Jaramillo J L and Gourgoulhon E 2011 Mass and angular momentum in general relativity *Mass and Motion in General Relativity (Fundamental Theories of Physics* vol 162*)* ed L Blanchet, A Spallicci and B Whiting (Berlin: Springer) pp 87–124 (arXiv:1001.5429)

[79] Jaramillo J L, Reiris M and Dain S 2011 Black hole area–angular momentum inequality in non-vacuum spacetimes arXiv:1106.3743

[80] Jaramillo J L, Vasset N and Ansorg M 2008 A numerical study of Penrose-like inequalities in a family of axially symmetric initial data *EAS Publ. Ser.* **30** 257–60 (arXiv:0712.1741)

[81] Khuri M and Weinstein G 2011 Rigidity in the positive mass theorem with charge to appear

[82] Komar A 1959 Covariant conservation laws in general relativity *Phys. Rev.* **119** 934–6

[83] Kozameh C N and Quiroga G 2012 Spin and center of mass in axially symmetric Einstein–Maxwell spacetimes arXiv:1202.5972

[84] Kramer D and Neugebauer G 1980 The superposition of two Kerr solutions *Phys. Lett.* A **75** 259–61

[85] Lattimer J and Prakash M 2004 The physics of neutron stars *Science* **304** 536–42 (arXiv:astro-ph/0405262)

[86] Li Y Y and Tian G 1992 Regularity of harmonic maps with prescribed singularities *Commun. Math. Phys.* **149** 1–30

[87] Manko V, Ruiz E and Sadovnikova M 2011 Stationary configurations of two extreme black holes obtainable from the Kinnersley–Chitre solution *Phys. Rev.* D **84** 064005 (arXiv:1105.2646)

[88] Mars M 2009 Present status of the Penrose inequality *Class. Quantum Grav.* **26** 193001 (arXiv:0906.5566)

[89] Mars M and Senovilla J M 1993 Axial symmetry and conformal Killing vectors *Class. Quantum Grav.* **10** 1633–47 (arXiv:gr-qc/0201045)

[90] Mohr P J, Taylor B N and Newell D B 2008 Codata recommended values of the fundamental physical constants: 2006 *Rev. Mod. Phys.* **80** 633–730

[91] Moreschi O M and Sparling G A J 1984 On the positive energy theorem involving mass and electromagnetic charges *Commun. Math. Phys.* **95** 113–20

[92] Neugebauer G 1980 A general integral of the axially symmetric stationary Einstein equations *J. Phys. A Math. Gen.* **13** L19

[93] Neugebauer G and Hennig J 2011 Stationary two-black-hole configurations: a non-existence proof arXiv:1105.5830

[94] Osserman R 1978 The isoperimetric inequality *Bull. Am. Math. Soc.* **84** 1182–238

[95] Penrose R 1973 Naked singularities *Ann. New York Acad. Sci.* **224** 125–34

[96] Pohl R *et al* 2010 The size of the proton *Nature* **466** 213–6

[97] Poisson E 2004 *A Relativist's Toolkit* (Cambridge: Cambridge University Press)

[98] Racz I 2008 A Simple proof of the recent generalisations of Hawking's black hole topology theorem *Class. Quantum Grav.* **25** 162001 (arXiv:0806.4373)

[99] Reiris M 2011 in preparation

[100] Schoen R and Uhlenbeck K 1982 A regularity theory for harmonic maps *J. Differ. Geom.* **17** 307–35

[101] Schoen R and Uhlenbeck K 1983 Boundary regularity and the Dirichlet problem for harmonic maps *J. Differ. Geom.* **18** 253–68

[102] Schoen R and Yau S T 1979 On the proof of the positive mass conjecture in general relativity *Commun. Math. Phys.* **65** 45–76

[103] Schoen R and Yau S T 1981 The energy and the linear momentum of space-times in general relativity *Commun. Math. Phys.* **79** 47–51

[104] Schoen R and Yau S T 1981 Proof of the positive mass theorem: part II *Commun. Math. Phys.* **79** 231–60

[105] Schoen R and Zhou X 2011 in preparation

[106] Simon W 1985 Gravitational field strength and generalized Komar integral *Gen. Rel. Grav.* **17** 439

[107] Simon W 2011 Comment on 'Area–charge inequality for black holes' arXiv:1109.6140

[108] Szabados L B 2004 Quasi-local energy–momentum and angular momentum in GR: A review article *Living Rev. Relativ.* **7** cited on 8 August 2005

[109] Wald R 1971 Final states of gravitational collapse *Phys. Rev. Lett.* **26** 1653–5

[110] Wald R M 1984 *General Relativity* (Chicago, IL: University of Chicago Press)

[111] Weinstein G 1990 On rotating black holes in equilibrium in general relativity *Commun. Pure Appl. Math.* **43** 903–48

[112] Weinstein G 1992 The stationary axisymmetric two-body problem in general relativity *Commun. Pure Appl. Math.* **45** 1183–203

[113] Weinstein G 1994 On the force between rotating co-axial black holes *Trans. Am. Math. Soc.* **343** 899–906

[114] Weinstein G 1995 On the Dirichlet problem for harmonic maps with prescribed singularities *Duke Math. J.* **77** 135–65

[115] Weinstein G 1996 Harmonic maps with prescribed singularities on unbounded domains *Am. J. Math.* **118** 689–700

[116] Weinstein G 1996 *N*-black hole stationary and axially symmetric solutions of the Einstein/Maxwell equations *Commun. Partial Differ. Eqns* **21** 1389–430

[117] Weinstein G and Yamada S 2005 On a Penrose inequality with charge *Commun. Math. Phys.* **257** 703–23

[118] Witten E 1981 A new proof of the positive energy theorem *Commun. Math. Phys.* **80** 381–402

[119] Zhang X 1999 Angular momentum and positive mass theorem *Commun. Math. Phys.* **206** 137–55