



Synchronization in lattice-embedded scale-free networks

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ABSTRACT

In this work, we study the effects of embedding a system of non-linear phase oscillators in a two-dimensional scale-free lattice. In order to analyze the effects of the embedding, we consider two different topologies. On the one hand, we consider a scale-free complex network where no constraint on the length of the links is taken into account. On the other hand, we use a method recently introduced for embedding scale-free networks in regular Euclidean lattices. In this case, the embedding is driven by a natural constraint of minimization of the total length of the links in the system. We analyze and compare the synchronization properties of a system of non-linear Kuramoto phase oscillators, when interactions between the oscillators take place in these networks. First, we analyze the behavior of the Kuramoto order parameter and show that the onset of synchronization is lower for non-constrained lattices. Then, we consider the behavior of the mean frequency of the oscillators as a function of the natural frequency for the two different networks and also for different values of the scale-free exponent. We show that, in contrast to non-embedded lattices that present a mean-field-like behavior characterized by the presence of a single cluster of synchronized oscillators, in embedded lattices the presence of a diversity of synchronized clusters at different mean frequencies can be observed. Finally, by considering the behavior of the mean frequency as a function of the degree, we study the role of hubs in the synchronization properties of the system.

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1. Introduction

The term *synchronicity* refers to the fact, widely observed in nature, that two or more components of a given system can eventually operate in unison. Ranging from molecular to astronomical scales, the universe has plenty of beautiful and surprising instances of synchronized behavior. In particular, the proper existence of life is supported by the ability of living organism to coordinate their rhythms with the natural cycles of the earthly environments.

As always occurs with emergent phenomena, the ultimate explanation of synchronicity resides in the capacity of the components of a given system to interact, even very weakly, and by means of these interactions to impose to all of them a cooperative and correlated regulatory behavior that gives rise to the appearance of global cyclic rhythms. What makes synchronicity especially astonishing is its ubiquity, and this is due to the large variety of interactions able to impose an overall organized behavior on the individual components of a system.

After the pioneering phenomenological observations of the prominent Dutch scientist Christiaan Huygens in the 17th century, physicists have tried to understand the microscopic origin of synchronization. However, it was only after the contributions of Wiener [1] and Winfree [2] in the 20th century that it was possible to lay down the mathematical principles that rule the modeling of synchronicity. In the middle of the 1970s, and following the ideas introduced by Winfree, Kuramoto

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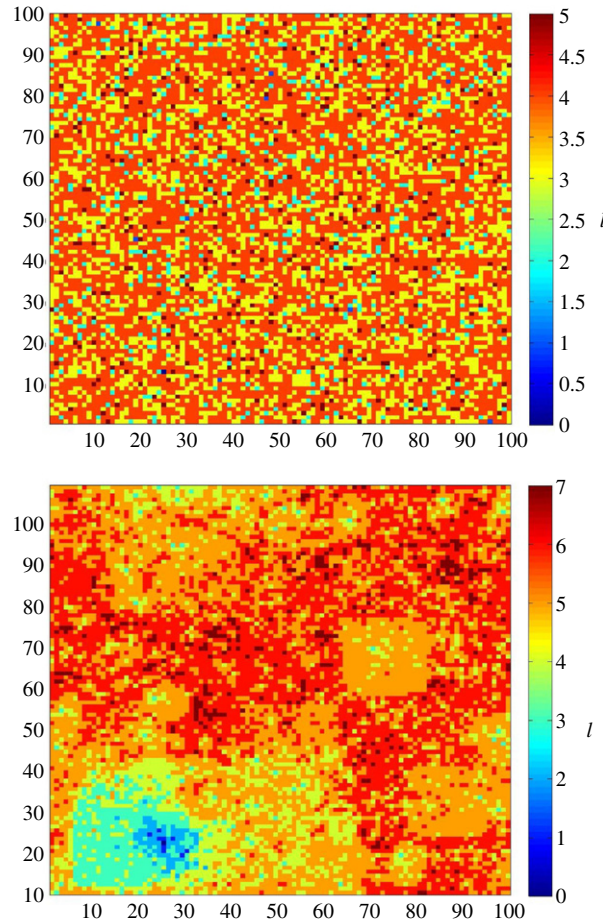


Fig. 1. Chemical distance l from the central site in non-embedded (top) and embedded (bottom) networks with $\alpha = 2.5$, $m = 4$, $M = 400$, and $A = 10$.

was able to develop for the first time a statistical mechanics approach to this problem [3]. Despite the simplicity of his model, he succeeded in explaining how the competition between the proper frequencies of the components and a *tiny* global interaction can give rise to a phase transition between a disordered phase, characterized by individual uncorrelated rhythms, and a synchronized phase, in which all the components oscillate in unison. Since then, the so-called Kuramoto model has attracted the attention of an increasing community of scientists, including not only physicists, but also biologists, chemists, and engineers, among many others [4–9].

As in any critical phenomenon, the appearance of a synchronized behavior via a phase transition takes place in a few universal ways, depending on some microscopic details of the systems, such as the nature and range of the interactions, the number of degrees of freedom of each component, and the topology of the network of connections. In this sense, a large corpus of papers and books have analyzed in detail the role of these ingredients in the macroscopic behavior of the whole system. And among the many aspects already analyzed, we are here concerned only with the role of the *topology* and the *geography* of connections. In this context, it is worth mentioning that one can find a great variety of topologies in the literature of the last decade, ranging from the mean-field fully connected version (originally analyzed by Kuramoto) to the nearest-neighbor low-dimensional networks usually used to model more realistic situations [4]. In between, a strong effort has been made to understand the phenomena of synchronization in complex natural and artificial networks by studying the Kuramoto model in small-world and/or scale-free structures [6,5]. Of particular interest for the present work is the paper by McGraw and Menzinger [10], who analyzed the effects of manipulating the clustering coefficient of scale-free networks without changing the degree distribution, and concluded that clustering promotes the synchronization of the most connected nodes.

In the study of complex networks, the notion of *geographical distance* is usually ignored, perhaps due to the cumbersome mathematical and numerical complications that its consideration requires. Instead, most of the complex lattices analyzed are actually mean-field-like networks, in the sense that the random wiring and/or rewiring mechanisms involved do not take into account any notion of distance between units. Nevertheless, our world has plenty of examples of complex lattices, both natural and artificial, in which the distance between nodes plays a fundamental role, as for instance brains and social networks, just to mention only two of them. Furthermore, many of these networks are actually embedded in

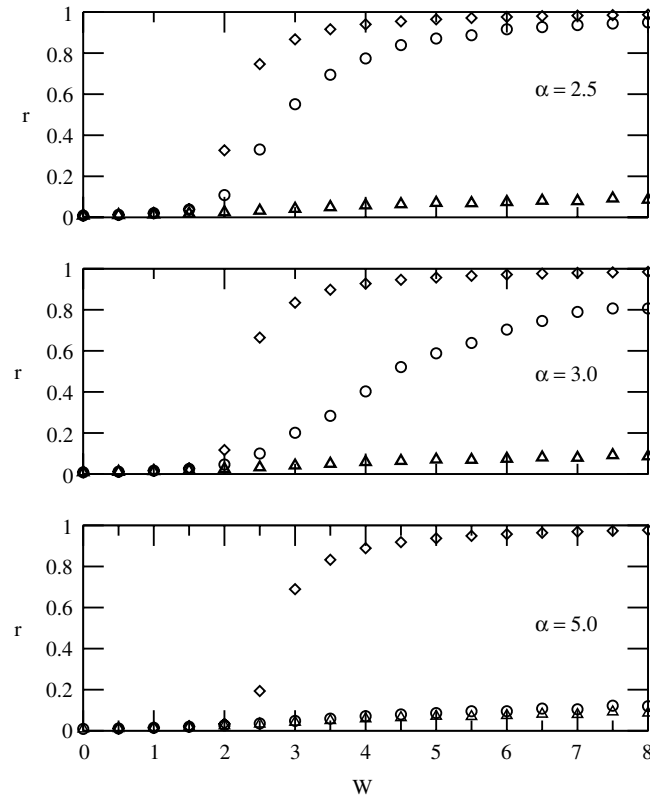


Fig. 2. Stationary value of the order parameter r versus coupling constant W for systems in non-embedded (diamonds) and embedded (circles) scale-free networks. We also plot the case of a nearest-neighbor regular square lattice (triangles), which for $\alpha = 5.0$ almost coincides with the embedded lattice. Note the strong dependency of the order parameter on the distribution exponent α for all embedded networks.

low-dimensional Euclidean spaces and also exhibit a power-law degree distribution, as occurs, for instance, in the neo-cortex of mammalian brains [11]. In particular, highly synchronized oscillations in certain areas of the visual cortex of cats have been reported a few years ago [12,13]. Inspired by this kind of phenomenon, Niebur et al. [14] investigated the dynamics of two-dimensional arrays of Kuramoto phase oscillators linked through different connection schemes. They observed that, for overall equal interaction strength, sparse coupling of the oscillators leads to a more rapid and robust phase locking than nearest-neighbor or locally dense connection topologies. It is worth stressing that precisely this kind of connection scheme captures some elements of cortical neuroanatomy.

Although so far not much importance has been given to this feature of real-world networks, in recent year physicists have begun to study the effect of topography on the behavior of networks. For example, Ref. [15] studied the collective synchronization behavior of oscillators on one-dimensional and two-dimensional small-world networks, in which the coupling strength depends on the geographical distance between nodes. They found a strong dependency of the synchronizability on the geographical distance of randomly added shortcuts, highlighting the important role that geographical effects play in network synchronization.

In this work, we report the results of a numerical study on the synchronization of a set of non-linear Kuramoto phase oscillators in a two-dimensional embedded scale-free network. In particular, we use a method proposed by Rozenfeld et al. [16,17] to embed networks in a low-dimensional Euclidean space. Each oscillator has its proper random natural frequency that characterizes its individual behavior, and is also attractively linked to a certain number of neighbor units, in such a way that these interactions tend to synchronize their movements. The neighborhood is chosen to guarantee the small-world and scale-free character of the total lattice.

We find that the embedding process hinders the global synchronization of the system. Nevertheless, we will see that, by controlling the strength of the coupling between oscillators, our system can display the formation of patches of locally synchronized units, proper of low-dimensional systems. Also, for sufficiently large couplings, the systems reach a fully synchronized state, as those observed in mean-field-like models.

2. The model

In order to analyze the role of geography in the overall behavior of the system, we define a complex structure of connectivities by linking the units through scale-free two-dimensional symmetric networks, assigning to each site a

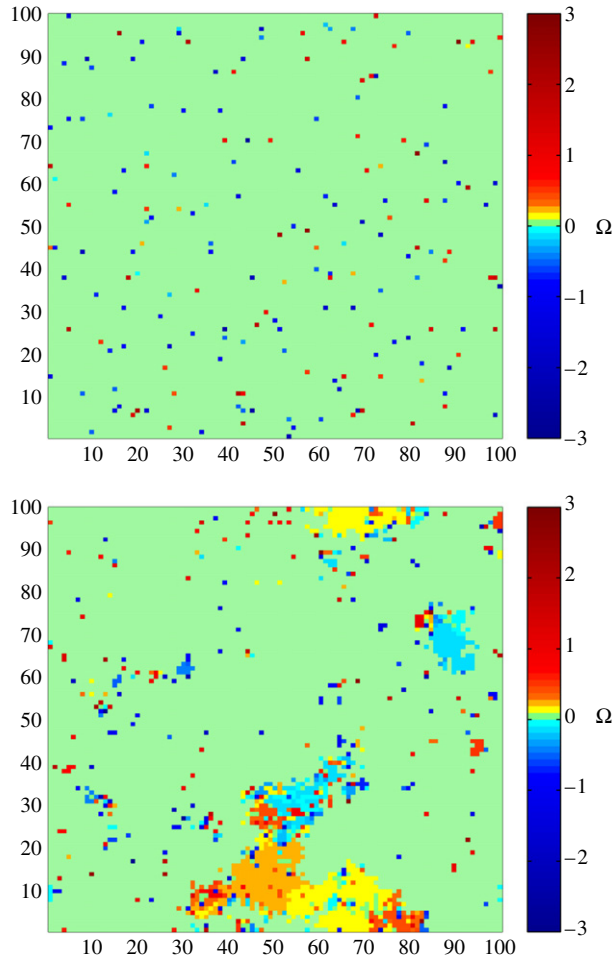


Fig. 3. Average frequency Ω for a non-embedded (top) and an embedded (bottom) network, both with $\alpha = 2.5$, $W = 3.0$, $m = 4$, $M = 400$, and $A = 10$.

neighborhood according to the algorithm introduced by Rozenfeld et al. [16,17]. The connection scheme is constructed as follows. First, we define a two-dimensional square lattice with periodic boundary conditions of size $N = L \times L$. To each oscillator i we assign a degree k_i taken at random from a power-law degree distribution $P(k) = Ck^{(-\alpha)}$, where C is the normalization constant, α is the power-law degree exponent, and the degree is bounded between m and M ($m \leq k \leq M$).

Next, we select at random a site i and connect it to its closest neighbors until its connectivity k_i is realized, or there are no more available sites to be connected to. Also, connections are forbidden between sites separated by a distance r larger than a certain upper bound $r_i^{\max} = A k_i^{1/2}$, where A is a constant. Finally, this procedure is repeated for all the sites of the network. By varying the value of A one can obtain a unique complex lattice or a self-similar structure of small complex patches [16,17]. We verified the power-law character of the resulting degree distributions which actually have an effective exponent α_e slightly smaller than the pre-assigned value α . In this way it is possible to obtain a network with a power-law degree distribution which, although it privileges links with nearest neighbours in Euclidean space, can eventually connect distant units. In order to determine the role of geography on the synchronization properties of the model, we also consider non-embedded networks, that is, randomly connected networks with the same power-law degree distribution and no length constraints.

Once we have constructed the network of interactions, we assign to each node a non-linear phase oscillator θ_i ($0 \leq \theta < 2\pi$). The dynamics of the whole system is ruled by the following set of coupled Kuramoto [3] differential equations:

$$\dot{\theta}_i = \omega_i + W \sum_{j \in V_i} \frac{1}{k_i} \sin(\theta_j - \theta_i), \quad i = 1 \dots N. \quad (1)$$

Here, ω_i is the natural frequency of oscillator i , W is the coupling constant, V_i is the neighborhood of site i , and k_i is its degree.

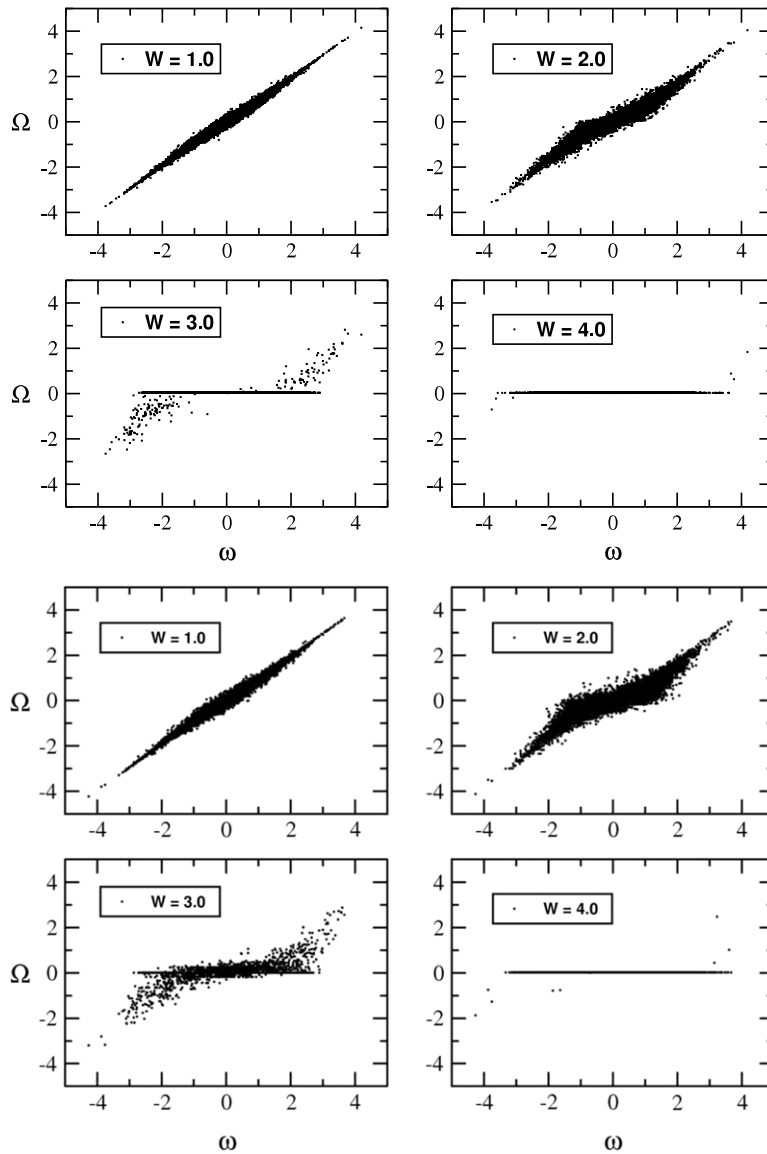


Fig. 4. Average frequency Ω as a function of natural frequency ω for non-embedded (four top plots) and embedded (four bottom plots) scale-free networks with power-law exponent $\alpha = 2.5$, $m = 4$, $M = 400$, $A = 10$, and four different coupling constants W .

3. Numerical simulation results

We consider systems with $N = L \times L$ nodes, where $L = 100$. Initially, the phases θ_i are chosen at random in the interval $0 \leq \theta \leq 2\pi$. The natural frequencies ω_i are also chosen at random from a Gaussian distribution with unit variance and zero mean, $g(\omega_i) = \exp(-\omega_i^2/2)/\sqrt{2\pi}$.

In this work we analyze three different cases:

- $\alpha = 2.5$ with $m = 4$ and $M = 400$;
- $\alpha = 3.0$ with $m = 4$ and $M = 100$;
- $\alpha = 5.0$ with $m = 4$ and $M = 40$.

In order to avoid the formation of self-similar connected structures inside our lattices, we use a unique value of $A = 10$ [16,17]. The choice of $m = 4$ guarantees that, for large values of the scale-free degree distribution exponent α , the network approximates a regular short-range square lattice.

Before beginning the study of synchronization itself, we analyze the effect of embedding in the spatial properties of the network. Starting from the central site of the network, we measure the chemical shell l that consists of all the sites that can be reached from the central site by a minimal number of l connecting links. In Fig. 1, we plot successive chemical shells for

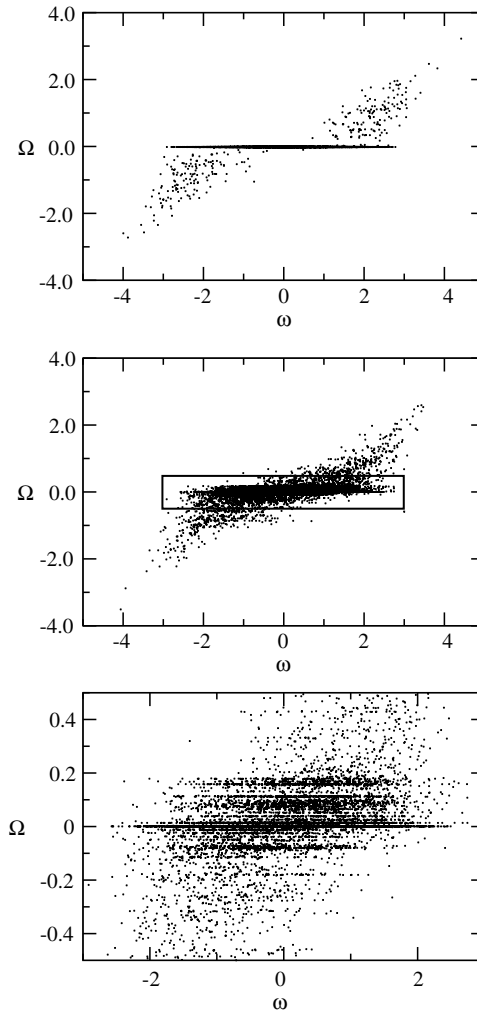


Fig. 5. Average frequency Ω as a function of natural frequency ω for non-embedded (top) and embedded (middle) scale-free networks with coupling constant $W = 3.0$, $\alpha = 3.0$, $m = 4$, $M = 100$, and $A = 10$. In the bottom figure, we present an enlargement of the embedded case, where the existence of clustered structures is clearly apparent.

both non-embedded (top) and embedded scale-free (bottom) lattices, with $\alpha = 2.5$. These figures clearly show the lack of spatial structure in the non-embedded lattice and the presence of spatial correlations introduced by the embedding.

As a first step in the analysis of the synchronization properties, we consider the Kuramoto order parameter,

$$r(t) = \left\| \frac{1}{N} \sum_{j=1}^N e^{i\theta_j(t)} \right\|, \quad (2)$$

where $0 \leq r(t) \leq 1$ measures the coherence of the oscillators. To calculate the time evolution of the system, we integrate Eq. (1) using Euler's method with a time step $\delta t = 0.02$.

In Fig. 2, we plot r , the stationary value of $r(t)$, as a function of the coupling constant W for three different values of the exponent α , namely, $\alpha = 2.5$ (top), $\alpha = 3.0$ (middle), and $\alpha = 5.0$ (bottom). The curves correspond to average values obtained over 50 realizations with different random initial phases and natural frequencies. In each figure, we plot both the case of non-embedded (diamonds) and the case of embedded (circles) lattices. For $\alpha = 5.0$, we have also included the results obtained with a nearest-neighbor regular square lattice of the same size (triangles). We verify that, independently of the value of α , the embedding process hinders the synchronization of the system. Nevertheless, both for $\alpha = 2.5$ and 3.0 , we observe that, for sufficiently large couplings, the complex two-dimensional networks order in a synchronized state. For $\alpha = 5.0$, we see that, as expected, the embedded lattice behaves almost as the nearest-neighbor model, and does not reach a synchronized state.

In order to obtain detailed information on the presence (or absence) of synchronized clusters in the system, we consider the average oscillation frequency Ω_i of each oscillator in a time interval of length T . The average is simply obtained by

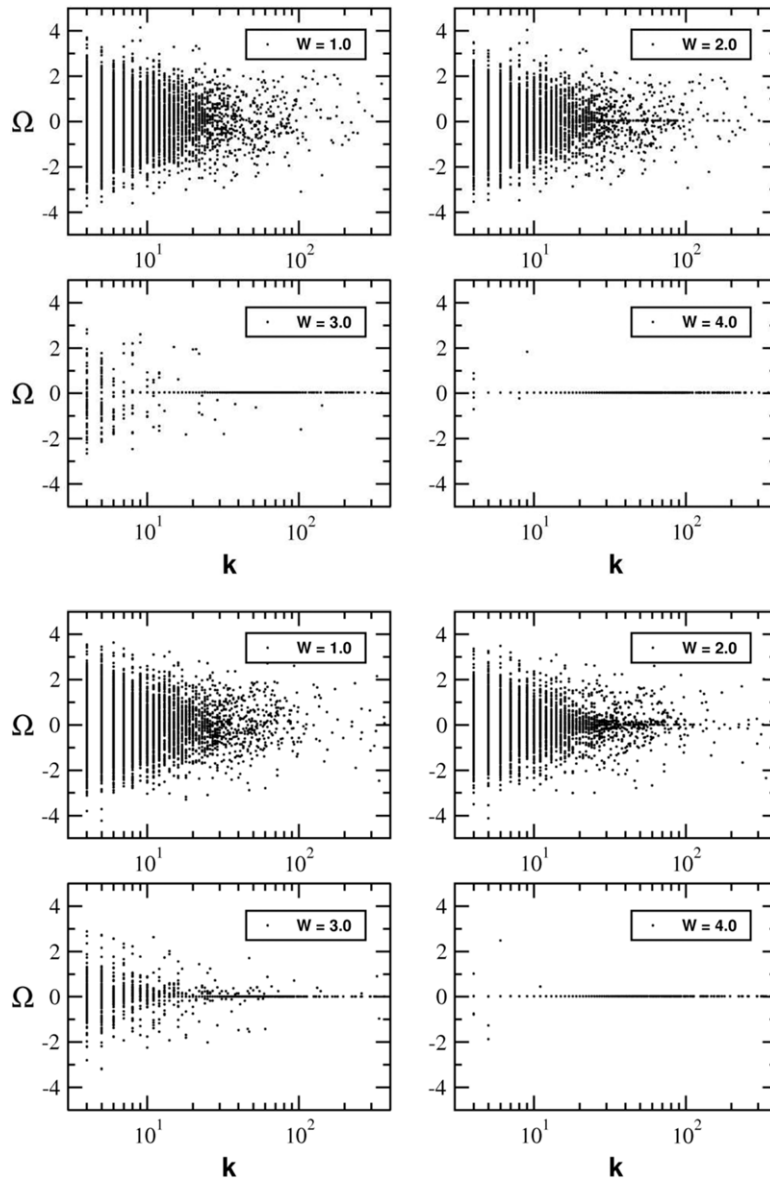


Fig. 6. Average frequency Ω as a function of degree k for a non-embedded (four top plots) and an embedded (four bottom plots) scale-free network with power-law exponent $\alpha = 2.5$, $m = 4$, $M = 400$, $A = 10$, and four different coupling constants.

calculating

$$\Omega_i = \frac{1}{T} \int_t^{t+T} \dot{\theta}_i(t') dt', \quad (3)$$

after a certain transient.

In Fig. 3, we present a comparison of the behavior of Ω between non-embedded (top) and embedded (bottom) networks when $\alpha = 2.5$ and $W = 3.0$. One can clearly see the presence of clusters of nodes with the same average frequency in the embedded networks, while no structure is present in the non-embedded networks. By comparing this figure with Fig. 1, one concludes that the geographical embedding introduces correlations that play a fundamental role in the synchronization pattern.

In order to advance further, we plot in Fig. 4 the average frequency Ω of each node as a function of its natural frequency ω for a non-embedded (top) and an embedded (bottom) scale-free network with power-law exponent $\alpha = 2.5$, and four different coupling constants, $W = 1.0, 2.0, 3.0$, and 4.0 . In these plots, the presence of clusters of synchronization is revealed by horizontal lines, which indicate groups of oscillators that have changed their natural frequency to the same average

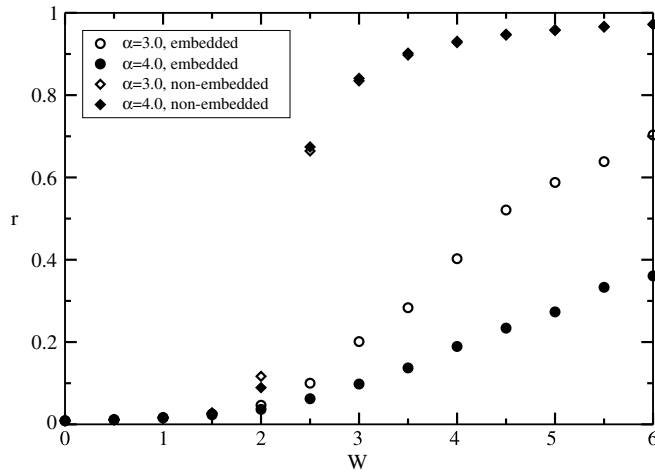


Fig. 7. Stationary value of the order parameter $|r|$ as a function of the coupling constant W for networks with $\langle K \rangle = 6.8$ with $N = 100 \times 100$ sites. The network parameters are $\alpha = 3.0$ with $m = 4$ and $M = 100$ (open symbols) and $\alpha = 4.0$ with $m = 5.0$ and $M = 200$ (full symbols). The results correspond to both embedded (circles) and non-embedded (diamond) structures.

frequency Ω . In both cases, we observe that, for low coupling constants, such as $W = 1.0$, the oscillators present only a slight deviation from their natural frequencies. On the other hand, for large coupling constants, such as $W = 4.0$, almost all the oscillators are synchronized and present the same average frequency. Also, in both cases, the last nodes to become synchronized are those near the extremes of the natural frequency distribution.

For intermediate couplings, the synchronization in non-embedded networks strongly resembles the mean-field model in which a single cluster of synchronization is formed, and grows as the coupling is increased. On the other hand, for embedded networks, there is a dispersion around a mean synchronization value, and only for sufficiently large values of the coupling constant is a single cluster observed. In order to analyze this effect in detail, we present in Fig. 5 a comparison of the behavior of Ω versus ω for a non-embedded (top) and an embedded (middle and bottom) scale-free network with exponent $\alpha = 3.0$ and coupling constant $W = 3.0$. As in the previous case ($\alpha = 2.5$), the non-embedded network presents a single cluster of synchronization (top) and the embedded network (middle) presents a dispersion around a mean value. As expected from the behavior of r observed in Fig. 2, the dispersion in the frequencies on non-synchronized oscillators grows with increasing α values. The bottom figure presents in detail the behavior of Ω around the mean, revealing a characteristic that clearly departs from the mean-field model. In fact, a number of clusters synchronized at different frequencies can be clearly observed. These clusters correspond to oscillators that are geographically apart and have synchronized locally (see Fig. 3). Note also that the synchronization frequencies of these clusters are very close, so, as the coupling constant is increased, the clusters easily synchronize between themselves. And eventually, for sufficiently large couplings, a single cluster emerges, as was observed in Fig. 4.

Next, we analyze the effect of embedding in the role of hubs. Fig. 6 shows the average frequency Ω versus degree k for non-embedded and embedded scale-free networks with power-law exponent $\alpha = 2.5$, and four different coupling constants. Note that the degree is plotted on a logarithmic scale, due to the broadness of its distribution. In both cases, we observe the same qualitative behavior, in which the nodes with the largest degree are the first to become synchronized, and those with a lower degree become synchronized as the coupling increases. This behavior indicates that the role of hubs in the synchronization of the system does not present a strong dependency with the embedding process we have chosen [10].

Finally, in Fig. 7, we compare the synchronization of embedded and non-embedded networks that share the same connectivity. We do this in order to analyze the effect of the embedding on the synchronization properties of the system. In particular, we present the stationary value of the order parameter $|r|$ as a function of the coupling constant W for networks with $\langle K \rangle = 6.8$ with $N = 100 \times 100$ sites. The network parameters are $\alpha = 3.0$ with $m = 4$ and $M = 100$ (open symbols) and $\alpha = 4.0$ with $m = 5.0$ and $M = 200$ (full symbols). For non-embedded networks (diamonds), the curves overlap for the different network parameters, and we do not observe any significant differences. On the other hand, for embedded networks (circles), a strong dependency on the network parameters is observed. The results show that the degree of synchronization depends not only on the mean connectivity and the topology of the network but also on its geography. In fact, we see that networks with the same degree distribution and the same connectivity present a different behavior when constructed on a mean-field-like structure or embedded in a low-dimensional lattice. The main point here is to realize that embedded lattices favor the formation of constrained geographical clusters. Summarizing, complex embedded structures allow the emergence of localized cluster of synchronized oscillators, which cannot be observed in mean-field structures. On the other hand, as we had already observed in Fig. 2, embedded lattices also improve the synchronization ability when compared with regular short-range lattices.

4. Conclusions

In this work, we have analyzed the effects of embedding in a two-dimensional square lattice on the synchronization properties of a system of non-linear Kuramoto phase oscillators. By comparing our results with those obtained on nearest-neighbor square lattices and mean-field scale-free networks, we were able to discern how the presence of a local complex structure in the interaction matrix affects both the order parameter and the spatial correlations of the model. As expected, the low dimensionality of the embedding lattice hinders global synchronization. Nevertheless, it allows for the formation of local synchronized structures, which are absent in the non-embedded model. It is worth stressing that, in both embedded and non-embedded networks, the hubs play a fundamental role. In fact, nodes with a higher degree synchronize first, and as the coupling constant is increased the nodes with a smaller degree become synchronized.

Summarizing, in this work we have presented a Kuramoto model in a geographically embedded network. The model includes many relevant features observed in real systems, such as physical, biological, technological, or social systems. In these systems, low dimensionality, power-law degree distribution, a small-world character, and some degree of locality are fundamental ingredients. Our results highlight the role of embedded structures not only in the appearance of order in low-dimensional lattices but also in the emergence of patches of locally synchronized units.

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