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A unified mixed finite element approximations of the Stokes–Darcy coupled problem

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ABSTRACT

In this paper we develop and analyze a unified approximation of the velocity–pressure pair for the Stokes–Darcy coupled problem in a plane domain. It is well known that, stable finite element approximations for the Stokes problem may not be appropriate for Darcy problem and for the coupling of fluid flow (modeled by the Stokes equations) with porous media flow (modeled by the Darcy equation), and therefore, different spaces are commonly used for the discretizations of the Darcy and the Stokes problems. In this work we proposed a modification of the Darcy problem which allows us to apply the classical Mini-element to the whole coupled Stokes–Darcy problem. The proposed method is probably one of the cheapest method for continuous approximation of the coupled system, has optimal accuracy with respect to solution regularity, and has simple and straightforward implementations. Numerical experiments are also presented, which confirm the excellent stability and accuracy of our method.

1. Introduction

The development of efficient numerical methods to approximate the solution to the Stokes problem [1–9] to the Darcy problem [3,10–12] and, in particular, to the coupling of the fluid flow (modeled by the Stokes equation) with porous media flow (modeled by the Darcy equation), with the appropriate Beavers–Joseph–Saffman interface conditions, has been increasing in the last years (see [13–17] and the references therein) due to its importance in hydrology, biofluid dynamics and indeed in many different problems involving filtration (see, for example, [18]).

It is well known that the discretization of the velocity and the pressure, for both Stokes and Darcy problems and the coupled of them, has to be made in a compatible way in order to avoid instabilities. Since, usually, stable elements for the free fluid flow cannot been successfully applied to the porous medium flow, most of the finite element formulations developed for the Stokes–Darcy coupled problem are based on appropriate combinations of stable elements for the Stokes equations with stable elements for the Darcy equations. There are a lot of papers considering different finite element spaces in each flow region (see, for example, [13,14] and the references therein). In contrast to this, other articles use the same finite element spaces in both regions by, in general, introducing some penalizing terms. For example, in [17], a unified finite element has been formulated by using the Crouzeix–Raviart nonconforming element for the approximation of the velocity and piecewise constant functions for the approximation of the pressure in both region and adding penalizing terms corresponding to the jumps over the edges of the piecewise velocities, while in [16] the authors propose the same

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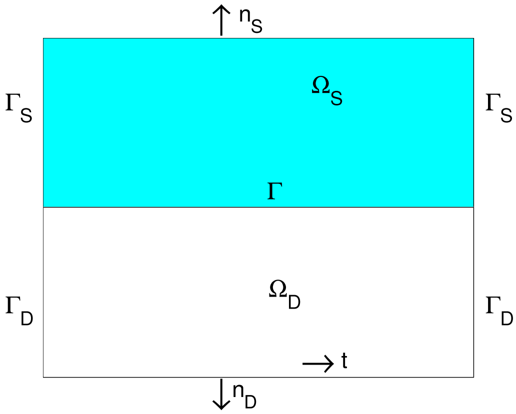

Fig. 1. Example of two-dimensional domain Ω.

nonconforming Crouzeix–Raviart element discretization on the entire domain which, is nonconforming in both the Stokes domain and Darcy domain and also a penalizing term is added. On the other hand, in [19] the authors propose an alternative formulation of the coupled problem which allows them use the classical Mini elements or Taylor–Hood elements. We are also focused on the development of a unified discretization (different to those consider in [19]) where the Stokes and the Darcy are approximated by using the same continuous finite element. Indeed, with this purpose in mind, we modified the mixed formulation (following the ideas given in [20]) in such a way that, the new problem has the same solution as the original and, independent of the mesh size, the stability condition for the new Stokes–Darcy problem reduces to the same as the Stokes problem.

The goal of this work is to apply the classical Mini-element to the modified coupled 2D Stokes–Darcy problem, which has simple and straightforward implementations. We prove that the formulation satisfies the discrete inf–sup conditions, obtaining as a result optimal accuracy with respect to solution regularity. Numerical experiments are also presented, which confirm the excellent stability and optimal performance of our method.

The rest of the paper is organized as follows. In Section 2 we state the classical Stokes–Darcy coupled problem. In Section 3 we present the modified coupled Stokes–Darcy problem. Section 4 is devoted to the finite element discretization and the error estimation. Finally, in Section 5, we present two numerical examples, in one the porous medium is entirely enclosed within the fluid region while in the other the two regions, fluid and porous, are only connected by the interface.

2. Problem statement

We consider an open, bounded and polygonal domain $\Omega \subset \mathbb{R}^2$ divided into two open subdomains with Lipschitz continuous boundaries Ω_S and Ω_D, where the indices S and D stand for fluid and porous, respectively. We assume that $\overline{\Omega} = \overline{\Omega}_S \cup \overline{\Omega}_D$, $\Omega_S \cap \Omega_D = \emptyset$ and $\overline{\Omega}_S \cap \overline{\Omega}_D = \Gamma$ so, Γ represents the interface between the fluid and the porous medium. The remaining parts of the boundaries are denoted by $\Gamma_S = \partial\Omega_S \setminus \Gamma$ and $\Gamma_D = \partial\Omega_D \setminus \Gamma$, as illustrated in Fig. 1.

We denote by $\boldsymbol{n}_S$ the unit outward normal direction on $\partial\Omega_S$ and by $\boldsymbol{n}_D$ the normal direction on $\partial\Omega_D$, oriented outward. On the interface Γ, we have $\boldsymbol{n}_S = -\boldsymbol{n}_D$.

The Stokes–Darcy coupled problem describes the motion of an incompressible viscous fluid occupying a region Ω_S which flows across the common interface into a porous medium living in another region Ω_D saturated with the same fluid. The mathematical model of this problem can be defined by two separate groups of equations and a set of coupling terms.

For any function $\mathbf{v}$ defined in Ω, taking into account that its restriction to Ω_S or to Ω_D could play a different mathematical role (especially their traces on Γ), we define $\mathbf{v}_S = \mathbf{v}|_{\Omega_S}$ and $\mathbf{v}_D = \mathbf{v}|_{\Omega_D}$.

In Ω_S, the fluid motion is governed by the Stokes equations for the velocity $\mathbf{u}_S$ and the pressure p_S:

$$\begin{cases} -\mu\Delta\mathbf{u}_S + \nabla p_S = \mathbf{f}_S, \text{ in } \Omega_S, \\ \operatorname{div}\mathbf{u}_S = 0, \text{ in } \Omega_S, \\ \mathbf{u}_S = \mathbf{0}, \text{ in } \Gamma_S, \end{cases} \tag{1}$$

where $\mathbf{f}_S \in (L^2(\Omega_S))^2$ represents the force per unit mass and $\mu > 0$ the viscosity.

In Ω_D, the porous media flow motion is governed by Darcy' law for the velocity $\mathbf{u}_D$ and the pressure p_D:

$$\begin{cases} \dfrac{\mu}{K}\mathbf{u}_D + \nabla p_D = \mathbf{f}_D, \text{ in } \Omega_D, \\ \operatorname{div}\mathbf{u}_D = g_D, \text{ in } \Omega_D, \\ \mathbf{u}_D \cdot \boldsymbol{n}_D = 0, \text{ in } \Gamma_D, \end{cases} \tag{2}$$

where $\mathbf{f}_D \in (L^2(\Omega_D))^2$ represents the force per unit mass, $g_D \in L^2(\Omega_D)$ a source and K denoting the permeability tensor reduced to a positive scalar in the isotropic case considered here.

In Γ, we consider the following boundary conditions (see, for example, [14]):

$$\begin{cases} \mathbf{u}_D \cdot \boldsymbol{n}_D + \mathbf{u}_S \cdot \boldsymbol{n}_S = 0, \\ p_S\, \boldsymbol{n}_S - \mu \nabla \mathbf{u}_S\, \boldsymbol{n}_S - p_D\, \boldsymbol{n}_S - \mu \dfrac{\alpha}{\sqrt{K}} (\mathbf{u}_S \cdot \boldsymbol{t})\boldsymbol{t} = 0, \end{cases} \tag{3}$$

where the first equation represents mass conservation and the second is due to the balance of normal forces and the Beavers–Joseph–Saffman condition, with $\nabla \mathbf{u} = \left(\frac{\partial u_i}{\partial x_j}\right)_{1\le i,j\le 2}$, α a parameter determined by experimental evidence and $\boldsymbol{t}$ the tangent vector on Γ (we recommend [21] for more details on the interface conditions).

We will denote with boldface the spaces consisting of vector value functions. The norms and seminorms in $\mathbf{H}^m(\mathcal{D})$, with m an integer, are denoted by $\|\cdot\|_{m,\mathcal{D}}$ and $|\cdot|_{m,\mathcal{D}}$ respectively and $(\cdot,\cdot)_{\mathcal{D}}$ denotes the inner product in $L^2(\mathcal{D})$ or $\mathbf{L}^2(\mathcal{D})$ for any subdomain $\mathcal{D} \subset \Omega$. The domain subscript is dropped for the case $\mathcal{D} = \Omega$. Let $\mathbf{H}(\mathrm{div}, \Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \ \mathrm{div}\, \mathbf{v} \in L^2(\Omega)\}$, $\mathbf{H}_0(\mathrm{div}, \Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \ \mathrm{div}\, \mathbf{v} \in L^2(\Omega), \mathbf{v} \cdot \boldsymbol{n}_D = 0 \text{ on } \Gamma_D\}$ and $L^2_0(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q = 0\}$.

We define the spaces

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega) : \mathbf{v}_S \in \mathbf{H}^1(\Omega_S), \mathbf{v} = \mathbf{0} \text{ on } \Gamma_S, \text{ and } \mathbf{v} \cdot \boldsymbol{n}_D = 0 \text{ on } \Gamma_D\}$$

and

$$Q = L^2_0(\Omega),$$

with the norms $\|\mathbf{v}\|_{\mathbf{V}} = (|\mathbf{v}|^2_{1,\Omega_S} + \|\mathbf{v}\|^2_{0,\Omega_D} + \|\,\mathrm{div}\,\mathbf{v}\|^2_{0,\Omega_D})^{\frac12} = (|\mathbf{v}|^2_{1,\Omega_S} + \|\mathbf{v}\|^2_{\mathbf{H}(\mathrm{div},\Omega_D)})^{\frac12}$ and $\|q\|_Q = \|q\|_0$ respectively.

Multiplying the first equation of (1) by a test function $\mathbf{v} \in \mathbf{V}$ and the second one by $q \in Q$, integrating by parts over Ω_S the terms involving $\Delta \mathbf{u}_S$ and ∇p_S, yield the variational form of Stokes equations:

$$\mu \int_{\Omega_S} \nabla \mathbf{u}_S : \nabla \mathbf{v} - \mu \int_\Gamma (\nabla \mathbf{u}_S\, \boldsymbol{n}_S) \cdot \mathbf{v}_S - \int_{\Omega_S} \mathrm{div}\, \mathbf{v}\, p_S + \int_\Gamma \mathbf{v}_S \cdot \boldsymbol{n}_S\, p_S = \int_{\Omega_S} \mathbf{f}_S \cdot \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{V},$$
$$\int_{\Omega_S} \mathrm{div}\, \mathbf{u}_S\, q = 0 \quad \forall q \in Q.$$

We apply a similar treatment to the Darcy equations by testing the first equation of (2) with a smooth function $\mathbf{v} \in \mathbf{V}$ and the second one by $q \in Q$, integrating by parts over Ω_D the terms involving ∇p_D, yield the variational form of Darcy equations:

$$\begin{aligned} \frac{\mu}{K} \int_{\Omega_D} \mathbf{u}_D \cdot \mathbf{v} - \int_{\Omega_D} p_D\, \mathrm{div}\, \mathbf{v} + \int_\Gamma \mathbf{v}_D \cdot \boldsymbol{n}_D\, p_D &= \int_{\Omega_D} \mathbf{f}_D \cdot \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{V}, \\ \int_{\Omega_D} \mathrm{div}\, \mathbf{u}_D\, q &= \int_{\Omega_D} g_D\, q \quad \forall q \in Q. \end{aligned} \tag{4}$$

Now, incorporating the boundary conditions (3) and taking into account that the vector valued functions in $\mathbf{V}$ have (weakly) continuous normal components on Γ (see Theorem 2.5 of [22]), the mixed variational formulation of the coupled problem (1)–(3) can be stated as follows [15,16]: Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ that satisfies

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = F(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}, \\ \qquad\qquad b(\mathbf{u}, q) = G(q) \quad \forall q \in Q, \end{cases} \tag{5}$$

where the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined on $\mathbf{V} \times \mathbf{V}$ and $\mathbf{V} \times Q$, respectively, as:

$$a(\mathbf{u}, \mathbf{v}) = \mu \int_{\Omega_S} \nabla \mathbf{u} : \nabla \mathbf{v} + \mu \frac{\alpha}{\sqrt{K}} \int_\Gamma (\mathbf{u}_S \cdot \boldsymbol{t})(\mathbf{v}_S \cdot \boldsymbol{t}) + \frac{\mu}{K} \int_{\Omega_D} \mathbf{u} \cdot \mathbf{v},$$

and

$$b(\mathbf{v}, q) = -\int_\Omega \mathrm{div}\, \mathbf{v}\, q.$$

By last, the linear forms F and G are defined as:

$$F(\mathbf{v}) = \int_{\Omega_D} \mathbf{f}_D\, \mathbf{v} + \int_{\Omega_S} \mathbf{f}_S\, \mathbf{v} \quad \text{and} \quad G(q) = -\int_{\Omega_D} g_D\, q.$$

It is easy to prove that a and b are continuous, b satisfies the continuous inf–sup condition and a is coercive on the null space of b (see, e.g., Lemma 3.3 of [15]). It is also clear that F and G are continuous and bounded. Then, using the classical theory of mixed methods (see, e.g., Theorem and Corollary 4.1 in Chapter I of [22]) it follows the well-posedness of the continuous formulation (5) and so the following theorem holds.

Theorem 2.1. *There exists a unique* $(\mathbf{u}, p) \in \mathbf{V} \times Q$ *solution to* (5). *In addition, there exists C, depending on the continuous inf–sup condition constant for b, the coercivity constant (on the null space of b) for a and the boundedness constants for a and b, such that*

$$\|\mathbf{u}\|_{\mathbf{V}} + \|p\|_Q \le C\{\|\mathbf{f}_S\|_{0,\Omega_S} + \|\mathbf{f}_D\|_{0,\Omega_D} + \|g_D\|_{0,\Omega_D}\}.$$

Remark 2.1. We observe that the mixed variational formulation of the coupled problem (5) is equivalent to the weak formulation (2.4) (and also (2.5)) of [15], with the particularity that, in our case, for any $\mathbf{v} \in \mathbf{H}$ we have that $\int_\Gamma (\mathbf{v}_S - \mathbf{v}_D) \cdot \boldsymbol{n}_S\, p_D = 0$.

It is well known that the discretization of the velocity and the pressure, for both Stokes and Darcy problems, and the coupled problem, has to be in a particular way to avoid instabilities (see, for example, [23]).

In fact, if we consider a Darcy problem in some domain $\mathcal{D}$ and finite element spaces $\mathbf{V}_h \subset \mathbf{H}_0(\mathrm{div}, \mathcal{D})$ and $Q_h \subset L^2_0(\mathcal{D})$, in order to approximate velocity and pressure, the following two conditions have to fulfilled:

(1) $\exists\, \alpha > 0 : \|\mathbf{v}_h\|^2_{0,\mathcal{D}} \ge \alpha \|\mathbf{v}_h\|^2_{\mathbf{H},\mathcal{D}}, \quad \forall \mathbf{v}_h$ such that: $(\mathrm{div}\, \mathbf{v}_h, q_h)_\mathcal{D} = 0, \quad \forall q_h \in Q_h$.
(2) The LBB conditions, i.e., there exists $\hat{\beta} > 0$ such that

$$\sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{V}_h} \frac{(\mathrm{div}\, \mathbf{v}_h, q_h)_\mathcal{D}}{\|\mathbf{v}_h\|_{\mathbf{H},\mathcal{D}}} \ge \hat{\beta} \|q_h\|_{0,\mathcal{D}} \qquad \forall q_h \in Q_h.$$

Therefore, the main difference to deal with the coupled problem is that, while in the Stokes problem the family of elements has only to satisfy the inf–sup conditions, the Darcy problem has to fulfill these two compatibility conditions. Indeed, for any function $\mathbf{v} \in \mathbf{H}^1(\mathcal{D})$ we have that $\|\mathbf{v}\|_{1,\mathcal{D}} \ge \|\mathbf{v}\|_{\mathbf{H},\mathcal{D}}$, it is clear that, if the family of finite elements satisfies the inf–sup condition related to the Stokes problem:

$$\sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{V}_h} \frac{(\mathrm{div}\, \mathbf{v}_h, q_h)_\mathcal{D}}{\|\mathbf{v}_h\|_{1,\mathcal{D}}} \ge \tilde{\beta} \|q_h\|_{0,\mathcal{D}} \quad \forall q_h \in Q_h,$$

also satisfy the LBB condition (2) but not necessary the condition (1), unless $\mathrm{div}(\mathbf{V}_h) = Q_h$. Thus, stable finite element approximations to the Stokes problem could not be appropriate for Darcy problem and, therefore, to the coupled problem under consideration.

3. A modified coupled Stokes–Darcy problem

In this section we introduce a modification to the Darcy equation, with the purpose in mind of the development of a unified discretization for the coupled problem, that is, the Stokes and Darcy parts be discretized using the same continuous finite element spaces.

The modification that we apply to the Darcy equation follows the idea given in [20] for linear elliptic equations. Indeed, we observe that taking the second equation of Darcy' problem (2) we can write, for any $\mathbf{v} \in \mathbf{V}$,

$$\int_{\Omega_D} (\mathrm{div}\, \mathbf{u}_D - g_D)\, \mathrm{div}\, \mathbf{v} = 0. \tag{6}$$

Then, by adding this equation to the first equation of the variational form given in (4), we get

$$\begin{aligned} \frac{\mu}{K} \int_{\Omega_D} \mathbf{u}_D \cdot \mathbf{v} + \int_{\Omega_D} \mathrm{div}\, \mathbf{u}_D\, \mathrm{div}\, \mathbf{v} - \int_{\Omega_D} p_D\, \mathrm{div}\, \mathbf{v} + \int_\Gamma \mathbf{v}_D \cdot \boldsymbol{n}_D\, p_D & \\ = \int_{\Omega_D} \mathbf{f}_D \cdot \mathbf{v} + \int_{\Omega_D} g_D\, \mathrm{div}\, \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{V} & \\ \int_{\Omega_D} \mathrm{div}\, \mathbf{u}_D\, q = \int_{\Omega_D} g_D\, q \quad \forall q \in Q. & \end{aligned} \tag{7}$$

From now on, we work with this modified variational form of Darcy equations.

In the same way that before, incorporating the boundary conditions (3) and remembering that, since $\mathbf{v} \in \mathbf{V}$, it has (weakly) continuous normal components on Γ, the variational form of the modified Stokes–Darcy problem can be written as follows: Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ satisfying

$$\begin{cases} \tilde{a}(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = L(\mathbf{v}) & \forall \mathbf{v} \in \mathbf{V}, \\ \qquad\qquad b(\mathbf{u}, q) = G(q) & \forall q \in Q, \end{cases} \tag{8}$$

where the bilinear forms $\tilde{a}(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined on $\mathbf{V} \times \mathbf{V}$, $\mathbf{V} \times Q$, respectively, as:

$$\tilde{a}(\mathbf{u}, \mathbf{v}) = \mu \int_{\Omega_S} \nabla \mathbf{u} : \nabla \mathbf{v} + \frac{\mu}{K} \int_{\Omega_D} \mathbf{u} \cdot \mathbf{v} + \int_{\Omega_D} \mathrm{div}\, \mathbf{u}\, \mathrm{div}\, \mathbf{v} + \mu \frac{\alpha}{\sqrt{K}} \int_\Gamma (\mathbf{u}_S \cdot \boldsymbol{t})(\mathbf{v}_S \cdot \boldsymbol{t}),$$

and

$$b(\mathbf{v}, q) = -\int_{\Omega} \operatorname{div} \mathbf{v}\, q.$$

By last, the linear forms L and G are defined as:

$$L(\mathbf{v}) = \int_{\Omega_D} \mathbf{f}_D \mathbf{v} + \int_{\Omega_S} \mathbf{f}_S \mathbf{v} + \int_{\Omega_D} g_D \operatorname{div} \mathbf{v} \quad \text{and} \quad G(q) = -\int_{\Omega_D} g_D\, q.$$

This problem has a unique solution. Indeed, when we considered problem (5), we saw that b was continuous and satisfied the continuous inf–sup condition and G was continuous and bounded. It is clear that, with the modification that we introduced, the new bilinear form $\tilde{a}$ is coercive and continuous and L is continuous and bounded. Then, applying the classical theory of mixed methods it follows the well-posedness of the continuous formulation (8).

Theorem 3.1. *There exists a unique $(\mathbf{u}, p) \in \mathbf{V} \times Q$ solution to (8). In addition, there exists a positive constant $\tilde{C}$, depending on the continuous inf–sup condition constant for b, the coercivity constant for $\tilde{a}$ and the boundedness constants for $\tilde{a}$ and b, such that*

$$\|\mathbf{u}\|_{\mathbf{V}} + \|p\|_Q \le \tilde{C}\{\|\mathbf{f}_S\|_{0,\Omega_S} + \|\mathbf{f}_D\|_{0,\Omega_D} + \|g_D\|_{0,\Omega_D}\}.$$

4. Finite element approximation of the modified Stokes–Darcy problem

In this section, we use the well known MINI-element, which has been introduced in [24], in order to approach the velocity and the pressure in the whole domain. Taking into account the modification that we introduced in the section above, this element (which is probably the cheapest continuous one for the approximation of the Stokes equation) can be successfully applied to the Stokes–Darcy modified coupled problem.

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of triangulations of Ω such that any two triangles in $\mathcal{T}_h$ share at most a vertex or an edge and each element $T \in \mathcal{T}_h$ is in either Ω_S or Ω_D. Let $\mathcal{T}_h^S$ and $\mathcal{T}_h^D$ be the corresponding induced triangulations of Ω_S and Ω_D. For any $T \in \mathcal{T}_h$, we denote by h_T the diameter of T and ρ_T the diameter of the largest ball inscribed into T and $\eta_T = \frac{h_T}{\rho_T}$. We assume that the family of triangulations is regular, i.e., there exists $\eta > 0$ such that $\eta_T \le \eta$ for all $T \in \mathcal{T}_h$ and $h > 0$. We also assume that the triangulation $\mathcal{T}_h$ satisfies that: for $T \in \mathcal{T}_h$, we have that T and Γ share at most a vertex or an edge (in particular, T cannot have two edges in Γ).

Let $\mathbf{V}_h \subset \mathbf{V}$ and $Q_h \subset Q$ be finite element spaces. The weak formulation (8) leads to the following discrete problem: Find $(\mathbf{v}_h, p_h) \in \mathbf{V}_h \times Q_h$ that satisfies

$$\begin{cases} \tilde{a}(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = L(\mathbf{v}_h) & \forall\, \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q_h) = G(q_h) & \forall\, q_h \in Q_h. \end{cases} \tag{9}$$

The discretization is said to be uniformly stable if there exist constants $\delta, \gamma > 0$, independent of h, such that

$$\begin{aligned} \tilde{a}(\mathbf{v}_h, \mathbf{v}_h) &\ge \delta \|\mathbf{v}_h\|_{\mathbf{V}}^2 \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h, \\ \sup_{\mathbf{0} \ne \mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{\mathbf{V}}} &\ge \gamma \|q_h\|_Q \quad \forall\, q_h \in Q_h. \end{aligned} \tag{10}$$

From now on, we will denote by C a generic positive constant, not necessarily the same at each occurrence, which may depend on the mesh only through the parameter η.

For $T \in \mathcal{T}_h$, let b_T be the standard cubic bubble given by:

$$\mathrm{b}_T = \begin{cases} \delta_{1,T}\delta_{2,T}\delta_{3,T} & \text{in } T \\ 0 & \text{in } \Omega \setminus T, \end{cases}$$

where $\delta_{1,T}$, $\delta_{2,T}$ and $\delta_{3,T}$ denote the barycentric coordinates of $T \in \mathcal{T}_h$. It is easy to check that the bubble function satisfies:

$$\int_T \mathrm{b}_T = Ch_T^2 \;\text{ and }\; \|\mathrm{b}_T\|_{1,T} \le C. \tag{11}$$

For any subdomain $\mathcal{D} \subseteq \Omega$, $k \in \mathbb{N}$, we denote by $P_k(\mathcal{D}) = \{v \in C^0(\mathcal{D}) : v|_T \in \mathcal{P}_k(T)\, \forall\, T \in \mathcal{T}_h \cap \mathcal{D}\}$.

We introduce the following notation

$$\mathcal{E} = \{\text{all edges in } \mathcal{T}_h\}, \qquad \mathcal{N} = \{\text{all vertices in } \mathcal{T}_h\},$$

and we denote by N the number of vertices on $\mathcal{N}$.

Let A be a set, we define

$$\mathcal{E}_A = \{\ell \in \mathcal{E} : \ell \subset A\}.$$

We decompose

$$\mathcal{E} = \mathcal{E}_{\Omega_S} \cup \mathcal{E}_{\Omega_D} \cup \mathcal{E}_{\Gamma_S} \cup \mathcal{E}_{\Gamma_D} \cup \mathcal{E}_{\Gamma}.$$

For $n \in \mathcal{N}$ we denote

$$\omega_n = \bigcup \{T \mid T \in \mathcal{T}_h \text{ and } n \in T\}.$$

For any $T \in \mathcal{T}_h$ we define

$$\omega_T = \bigcup \{\omega_n \mid n \text{ is a vertex of T }\}.$$

For $\ell \in \mathcal{E}_\Gamma$, we denote by T_S and T_D the two triangles sharing ℓ, with $T_S \in \mathcal{T}_h^S$ and $T_D \in \mathcal{T}_h^D$, and by $\omega_\ell = T_S \cup T_D$. We enumerate the vertices of T_S and T_D so that the vertices of ℓ are numbered first, i.e., let e_1 and e_2 be the vertices of ℓ we denote by e_3^S and e_3^D the vertices in Ω_S and Ω_D respectively. Now, we consider the classical edge-bubble function b_ℓ defined by:

$$\mathrm{b}_\ell = \begin{cases} \delta_{e_1,T_i}\delta_{e_2,T_i} & \text{in } T_i,\ i = S \text{ or } D, \\ 0 & \text{in } \Omega \setminus \omega_\ell, \end{cases}$$

and we define the space associated to b_ℓ by

$$B_{3,\ell} := \{v_h \in C^0(\Omega) : v_h|_{\omega_\ell} = \mathrm{b}_\ell \psi_{\omega_\ell},\ \forall\, \ell \in \mathcal{E}_\Gamma,\ \text{with } \psi_{\omega_\ell} \in C^0(\omega_\ell),\ \psi_{\omega_\ell}|_{T_i} \in P_1(T_i)$$
$$\text{and } \psi_{\omega_\ell}(e_3^i) = 0,\ i = S \text{ or } D\}.$$

On the other hand, the space associated to b_T is given by

$$B_3 := \{v_h \in C^0(\Omega) : v_h|_T = c_T \mathrm{b}_T,\ c_T \in \mathbb{R},\ \forall\, T \in \mathcal{T}_h\}.$$

The finite element spaces for velocities and pressures are

$$\begin{aligned} \mathbf{V}_h := \{&\mathbf{v}_h \in (C^0(\Omega_S))^2, \mathbf{v}_h \in (C^0(\Omega_D))^2 : \mathbf{v}_h|_T \in (P_1(T) \oplus B_3)^2 \quad \forall\, T \in \mathcal{T}_h : \mathcal{E}_T \cap \mathcal{E}_\Gamma = \emptyset, \\ &\text{and} \quad \mathbf{v}_h|_T \in (P_1(T) \oplus B_3)^2 \oplus B_{3,\ell}|_T\, \boldsymbol{n}_\ell \quad \forall\, T \in \mathcal{T}_h : \mathcal{E}_T \cap \mathcal{E}_\Gamma = \ell, \\ &\mathbf{v}_h = 0 \text{ on } \Gamma_S, \mathbf{v}_h \cdot \boldsymbol{n}_D = 0 \text{ on } \Gamma_D \text{ and } \mathbf{v}_h^D \cdot \boldsymbol{n}_D + \mathbf{v}_h^S \cdot \boldsymbol{n}_S = 0 \text{ on } \Gamma\} \end{aligned}$$

and

$$Q_h := \{q_h \in C^0(\Omega_S), q_h \in C^0(\Omega_D) : q_h|_T \in P_1(T)\ \forall\ T \in \mathcal{T}_h\} \cap L_0^2(\Omega).$$

We observe that, bases for the space $B_{3,\ell}$ can be easily obtained, for example, as follows: Let $\hat{T}$ be the classical reference triangle, i.e. the triangle of vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. For each triangle $T \subset \omega_\ell$, we denote by e_1, e_2 and e_3 the vertices of T, such that e_1 and e_2 are the vertices of ℓ and e_3 is the vertex of T that is not on Γ. If we denote by (x_j, y_j), $1 \le j \le 3$, the coordinates of the vertices e_j of T, then the affine transformation from $\hat{T}$ onto the triangle of vertices e_1, e_2 and e_3 can be defined as:

$$F(\hat{x}, \hat{y}) = \big(x_3 + (x_1 - x_3)\hat{x} + (x_2 - x_3)\hat{y}, y_3 + (y_1 - y_3)\hat{x} + (y_2 - y_3)\hat{y}\big),$$

we observe that F maps the edge $\overline{(1,0)(0,1)}$ into ℓ. In $\hat{T}$ we consider the Lagrange bases $\hat{\beta}_1$ and $\hat{\beta}_2$ such that: $\hat{\beta}_1(\frac{1}{4}, \frac{3}{4}) = 1$, $\hat{\beta}_1(\frac{3}{4}, \frac{1}{4}) = 0$ and $\hat{\beta}_1(0, 0) = 0$, and $\hat{\beta}_2(\frac{1}{4}, \frac{3}{4}) = 0$, $\hat{\beta}_2(\frac{3}{4}, \frac{1}{4}) = 1$ and $\hat{\beta}_2(0, 0) = 0$. Therefore, the corresponding bases functions in T are $\beta_{T,i} = \hat{\beta}_i \circ F^{-1}, \quad i = 1, 2$.

Then, we define the cubic bubbles $v_{\ell,1}$ and $v_{\ell,2}$ such that, $v_{\ell,i}|_T = \delta_{e_1,T}\delta_{e_2,T}\beta_{T,i}$, $i = 1, 2$ (see Fig. 2).

We remark that the velocity space $\mathbf{V}_h$ consists of all functions of the form

$$\mathbf{v} = \mathbf{v}^0 + \sum_{T \in \mathcal{T}_h} \mathbf{c}_T\, b_T + \sum_{\ell \in \mathcal{E}_\Gamma} (\alpha_{\ell,1} v_{\ell,1} \boldsymbol{n}_\ell + \alpha_{\ell,2} v_{\ell,2} \boldsymbol{n}_\ell),$$

where $\mathbf{v}^0$ is a piecewise linear vector field which are continuous on Ω_D and Ω_S, b_T is a bubble function on the triangle T, $\mathbf{c}_T$ is a constant vector, $v_{\ell,1}$ and $v_{\ell,2}$ are the bubble functions defined above with support on ω_ℓ, and $\alpha_{\ell,i}$, $i = 1, 2$ are constants.

The corresponding pressure space Q_h consists of continuous piecewise linear functions on Ω_D and Ω_S.

We notice that although the added bubble functions are continuous on $\overline{\Omega}$, since the piecewise linear functions are not supposed continuous on $\overline{\Omega}$, all functions in $\mathbf{V}_h$ and Q_h are allowed to be discontinuous across Γ.

We use the well known mixed finite element theory (see, e.g., Lemma 1.1 of [22]) to get the existence and uniqueness of a finite element solution of the discrete problem (9) for these spaces. Indeed, in order to prove the discrete inf–sup (10), we seek for an operator $\boldsymbol{\Pi}_h : \mathbf{H}_0^1(\Omega) \longrightarrow \mathbf{V}_h$ such that

$$\begin{cases} b(\mathbf{v} - \boldsymbol{\Pi}_h \mathbf{v}, q_h) = 0 \quad \forall\, \mathbf{v} \in \mathbf{H}_0^1(\Omega) \quad \forall\, q_h \in Q_h \\ \qquad \|\boldsymbol{\Pi}_h \mathbf{v}\|_{\mathbf{V}} \le C \|\mathbf{v}\|_1 \end{cases}$$

where $\|\mathbf{v}\|_1 = (\|v_1\|_1^2 + \|v_2\|_1^2)^{\frac{1}{2}}$.

To define the operator $\boldsymbol{\Pi}_h$ we use the Clèment's interpolator.

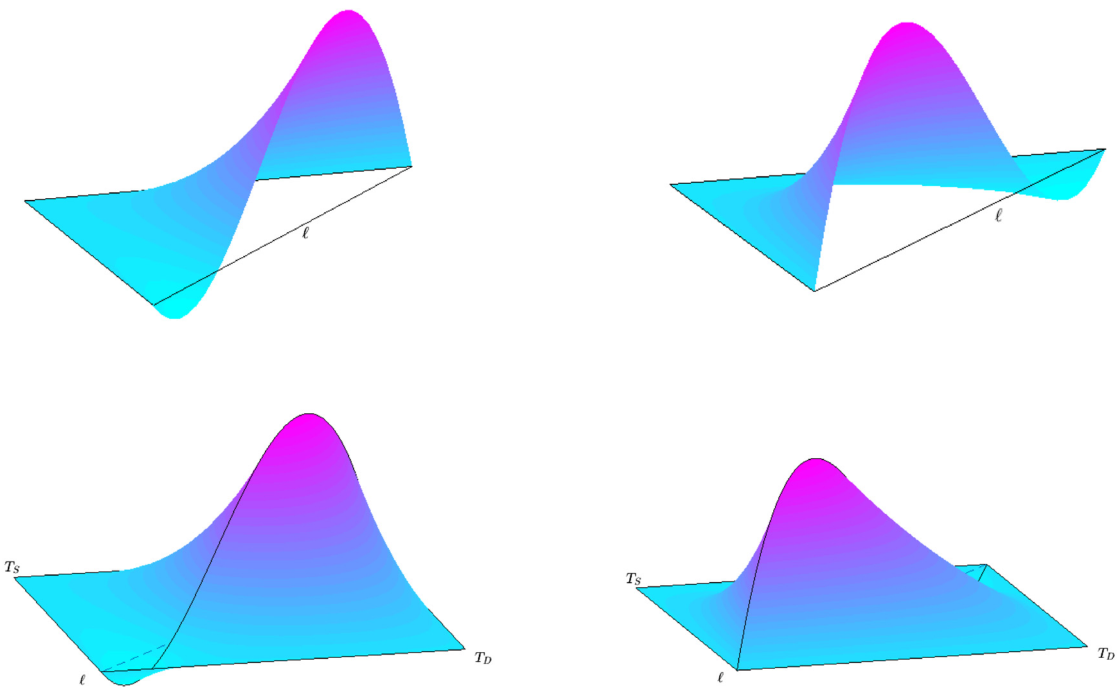

Fig. 2. The bubble functions $v_{\ell,1}|_T$ and $v_{\ell,2}|_T$ (top) and $v_{\ell,1}$ and $v_{\ell,2}$ on ω_ℓ (bottom).

For any $n \in \mathcal{N}$ and $v \in L^2(\omega_n)$, we can define $\mathcal{P}_{\omega_n} : L^2(\omega_n) \to P_0(\omega_n)$ the orthogonal projection of v on $P_0(\omega_n)$ with respect to the internal product in $L^2(\omega_n)$ that fulfills

$$\int_{\omega_n} v\, p_0 = \int_{\omega_n} \mathcal{P}_{\omega_n}(v)\, p_0 \qquad \forall\, p_0 \in P_0(\omega_n),$$

and therefore

$$\mathcal{P}_{\omega_n}(v) = \frac{1}{|\omega_n|} \int_{\omega_n} v.$$

Let $\{\phi_i\}_{i\in\{1,\cdots,N\}}$ be the Lagrange basis of $\mathcal{T}_h$, i.e., given a node n_i, $\phi_i(n_i) = 1$ and it is zero at the rest of the nodes of the mesh $\mathcal{T}_h$.

For any $\mathbf{v} = (v_1, v_2) \in \mathbf{L}^2(\Omega)$ and $n \in \mathcal{N}$ we can define $\mathbf{P}_{\omega_n}(\mathbf{v}) = (\mathcal{P}_{\omega_n}(v_1), \mathcal{P}_{\omega_n}(v_2))$. Then, let us consider a Clèment's interpolator as:

$$\mathcal{I}\mathbf{v}(x) = \sum_{i=1}^{N} \phi_i(x) \mathbf{P}_{\omega_{n_i}}(\mathbf{v}).$$

In order to construct the global operator $\mathbf{\Pi}_h$, we first impose a condition on each vertex $n \in \mathcal{N}$ according to its location in the domain.

$$\mathbf{\Pi}_h \mathbf{v}(n) = \begin{cases} \mathcal{I}\mathbf{v}(n) = \mathbf{P}_{\omega_n}(\mathbf{v}) & \text{if } n \in \Omega_S,\ n \in \Omega_D \text{ or } n \in \Gamma^\circ \\ \mathbf{0} & \text{other case,} \end{cases}$$

where Γ° denotes, as usual, the interior of Γ.

We observe that $\mathbf{\Pi}_h \mathbf{v} = ((\mathbf{\Pi}_h \mathbf{v})_1, (\mathbf{\Pi}_h \mathbf{v})_2)$ so, in order to simplify notation, we call $\mathbf{\Pi}_{h,j} \mathbf{v} = (\mathbf{\Pi}_h \mathbf{v})_j$ for $1 \le j \le 2$.

Now, for each $\ell \in \mathcal{E}_\Gamma$, we have two degrees of freedom more on ℓ and so we can impose that

$$\int_\ell \mathbf{\Pi}_h \mathbf{v} \cdot \boldsymbol{n}_\ell\, \gamma = \int_\ell \mathbf{v}^D \cdot \boldsymbol{n}_\ell\, \gamma, \quad \forall\, \gamma \in P_1(\ell),$$

where $\boldsymbol{n}_\ell$ stands for the unit normal vector on ℓ oriented outward Ω_D.

The other condition, related to the bubble on each triangle $T \in \mathcal{T}_h$, that we consider to define the operator is:

$$\int_T \mathbf{\Pi}_{h,j} \mathbf{v} = \int_T v_j \quad j = 1, 2, \tag{12}$$

Now, we write a formula for the global operator on each $T \in \mathcal{T}_h$. Note that there are two cases to consider:
(a) $T \in \mathcal{T}_h : \mathcal{E}_T \cap \mathcal{E}_\Gamma = \emptyset$.
(b) $T \in \mathcal{T}_h : \mathcal{E}_T \cap \mathcal{E}_\Gamma = \ell$.

(a) For any triangle $T \in \mathcal{T}_h$ has no sides on Γ, for both coordinates of the operators we are using the space $P_1 \oplus B_3$. We denote by n_i, $1 \le i \le 3$, its vertices and by β_i the Lagrange basis of T, i.e., $\beta_i(n_i) = 1$ and it is zero at the rest of the nodes of T. For each $j = 1, 2$, we observe that the operator restricted to T has the form:

$$\mathbf{\Pi}_{h,j}\mathbf{v}|_T = \sum_{i=1}^{3} \alpha_i^j \, \beta_i|_T + \gamma^j \, \mathrm{b}_T,$$

where

$$\alpha_i^j = \begin{cases} \mathcal{P}_{\omega_{n_i}}(v_j) & \text{if } n_i \in \Omega_S, n_i \in \Omega_D \text{ or } n_i \in \Gamma^\circ, \\ 0 & \text{if } n_i \in \overline{\Gamma}_S \text{ or } n_i \in \Gamma_D, \end{cases}$$

and the constant γ^j is obtaining by using (12), i.e.,

$$\sum_{i=1}^{3} \alpha_i^j \int_T \beta_i + \gamma^j \int_T \mathrm{b}_T = \int_T v_j, \qquad j = 1, 2,$$

and so,

$$\gamma^j = \frac{\int_T v_j - \sum_{i=1}^{3} \alpha_i^j \int_T \beta_i}{\int_T \mathrm{b}_T}, \qquad j = 1, 2.$$

Then,

$$\mathbf{\Pi}_{h,j}\mathbf{v}|_T = \sum_{i=1}^{3} \alpha_i^j \beta_i|_T + \left(\frac{\int_T v_j - \sum_{i=1}^{3} \alpha_i^j \int_T \beta_i}{\int_T \mathrm{b}_T} \right) \mathrm{b}_T.$$

We modify the projector (see, e.g., Chapter II section 6 of [25]) in order to consider the different conditions imposed over the vertices when we define the operator

$$\tilde{\mathcal{P}}_{\omega_{n_i}}(v_j) = \begin{cases} \mathcal{P}_{\omega_{n_i}}(v_j) & \text{if } n_i \in \Omega_S, n_i \in \Omega_D \text{ or } n_i \in \Gamma^\circ, \\ 0 & \text{if } n_i \in \overline{\Gamma}_S \text{ or } n_i \in \Gamma_D, \end{cases}$$

and therefore $\tilde{\mathbf{P}}_{\omega_{n_i}}(\mathbf{v}) = (\tilde{\mathcal{P}}_{\omega_{n_i}}(v_1), \tilde{\mathcal{P}}_{\omega_{n_i}}(v_2))$.
Next, we use the following modified interpolator

$$\tilde{\mathcal{I}}\mathbf{v}(x) = \sum_{i=1}^{N} \phi_i(x) \tilde{\mathbf{P}}_{\omega_{n_i}}(\mathbf{v}).$$

Using the previously defined we can rewrite the operator as

$$\mathbf{\Pi}_{h,j}\mathbf{v}(x)|_T = \tilde{\mathcal{I}}_j\mathbf{v}(x)|_T + \left(\frac{\int_T (v_j - \tilde{\mathcal{I}}_j\mathbf{v}(x)|_T)}{\int_T \mathrm{b}_T} \right) \mathrm{b}_T(x). \tag{13}$$

(b) Now, let $T \in \mathcal{T}_h$ be a triangle with a side on Γ, i.e., $\mathcal{E}_T \cap \mathcal{E}_\Gamma = \ell$. For each $j = 1, 2$, we observe that the operator restricted to T has the form:

$$\mathbf{\Pi}_{h,j}\mathbf{v}|_T = \sum_{i=1}^{3} \alpha_i^j \, \beta_i|_T + \gamma^j \, \mathrm{b}_T + \mathrm{b}_\ell \psi_{\omega_\ell}|_T n_{\ell,j},$$

where

$$\alpha_i^j = \begin{cases} \mathcal{P}_{\omega_{n_i}}(v_j) & \text{if } n_i \in \Omega_S, n_i \in \Omega_D \text{ or } n_i \in \Gamma^\circ, \\ 0 & \text{if } n_i \in \overline{\Gamma}_S \text{ or } n_i \in \Gamma_D. \end{cases}$$

We define, as above, the corresponding operator

$$\tilde{\mathcal{P}}_{\omega_{n_i}}(v_j) = \begin{cases} \mathcal{P}_{\omega_{n_i}}(v_j) & \text{if } n_i \in \Omega_S, n_i \in \Omega_D \text{ or } n_i \in \Gamma^\circ, \\ 0 & \text{if } n_i \in \overline{\Gamma}_S \text{ or } n_i \in \Gamma_D, \end{cases}$$

and the interpolator

$$\tilde{\mathcal{I}}\mathbf{v}(x) = \sum_{i=1}^{N} \phi_i(x)\tilde{\mathbf{P}}_{\omega_{n_i}}(\mathbf{v}),$$

with $\tilde{\mathbf{P}}_{\omega_{n_i}}(\mathbf{v}) = (\tilde{\mathcal{P}}_{\omega_{n_i}}(v_1), \tilde{\mathcal{P}}_{\omega_{n_i}}(v_2))$.

Now, in order to determinate γ^j and $v_\ell = b_\ell \psi_{\omega_\ell}$, we first observe that in this case, we have two degrees of freedom more on ℓ and we can impose that

$$\int_\ell \mathbf{\Pi}_h\mathbf{v} \cdot \boldsymbol{n}_\ell \, \gamma = \int_\ell \mathbf{v}^D \cdot \boldsymbol{n}_\ell \, \gamma, \quad \forall \, \gamma \in P_1(\ell), \tag{14}$$

where $\boldsymbol{n}_\ell$ stands for the unit normal vector on ℓ oriented outward Ω_D. Then, γ^j is obtained such that (12) holds.

Therefore, for each $j = 1, 2$, we observe that the operator restricted to T has the form:

$$\mathbf{\Pi}_{h,j}\mathbf{v}(x)|_T = \tilde{\mathcal{I}}_j\mathbf{v}(x)|_T + \left(\frac{\int_T (v_j - \tilde{\mathcal{I}}_j\mathbf{v}) - \int_T v_\ell n_{\ell,j}}{\int_T \mathrm{b}_T}\right)\mathrm{b}_T(x) + v_\ell(x)|_T n_{\ell,j}. \tag{15}$$

where, in view of condition (14), v_ℓ is such that

$$\int_\ell v_\ell \, \gamma = \int_\ell (\mathbf{v}^D - \tilde{\mathcal{I}}\mathbf{v}) \cdot \boldsymbol{n}_\ell \, \gamma, \quad \forall \, \gamma \in P_1(\ell), \tag{16}$$

with, for $T \subset \omega_\ell$, $v_\ell|_T = \delta_{e_1,T}\, \delta_{e_2,T}\, \psi_{\omega_\ell}|_T$. It is easy to prove that v_ℓ, i.e, ψ_{ω_ℓ} there exists and is unique. First, we note that the number of conditions defining ψ_{ω_ℓ} (two because $\gamma \in P_1(\ell)$) equals to the degrees of freedom of ψ_{ω_ℓ}. In order to show the existence of ψ_{ω_ℓ}, it is enough to prove uniqueness. So, take $\varphi_\ell = \psi_{\omega_\ell}|_\ell$, we have that $\varphi_\ell \in P_1(\ell)$ and $\int_\ell \delta_{e_1,T}\, \delta_{e_2,T}\, \varphi_\ell\, \gamma = 0 \quad \forall \gamma \in P_1(\ell)$, in particular we can take $\gamma = \varphi_\ell$ and then, $\int_\ell \delta_{e_1,T}\, \delta_{e_2,T}\, \varphi_\ell^2 = 0$. Therefore, $\varphi_\ell = 0$ and so, ψ_{ω_ℓ} is zero in ℓ. Then, since ψ_{ω_ℓ} is also zero in the vertices of ω_ℓ does not lie on ℓ, we conclude that $\psi_{\omega_\ell} \equiv 0$ as we wanted to see.

Now, we have to verify that the operator $\mathbf{\Pi}_h$ satisfies

$$b(\mathbf{v} - \mathbf{\Pi}_h\mathbf{v}, q_h) = 0, \quad \forall \, \mathbf{v} \in \mathbf{H}_0^1(\Omega), \quad \forall \, q_h \in Q_h.$$

Considering that

$$b(\mathbf{v}, q_h) = -\int_{\Omega_D} \operatorname{div} \mathbf{v} \, q_h - \int_{\Omega_S} \operatorname{div} \mathbf{v} \, q_h,$$

we have that

$$b(\mathbf{v} - \mathbf{\Pi}_h\mathbf{v}, q_h) = -\int_{\Omega_D} \operatorname{div}(\mathbf{v} - \mathbf{\Pi}_h\mathbf{v})\, q_h - \int_{\Omega_S} \operatorname{div}(\mathbf{v} - \mathbf{\Pi}_h\mathbf{v})\, q_h.$$

Now, adding on all triangles in both domains, integrating by parts on each triangle we get

$$\begin{aligned} b(\mathbf{v} - \mathbf{\Pi}_h\mathbf{v}, q_h) &= -\sum_{T \subset \Omega_D} \int_T \operatorname{div}(\mathbf{v} - \mathbf{\Pi}_h\mathbf{v})\, q_h - \sum_{T \subset \Omega_S} \int_T \operatorname{div}(\mathbf{v} - \mathbf{\Pi}_h\mathbf{v})\, q_h \\ &= \sum_{T \subset \Omega_D} \left(\int_T (\mathbf{v} - \mathbf{\Pi}_h\mathbf{v})\, \nabla q_h - \int_{\partial T} q_h(\mathbf{v} - \mathbf{\Pi}_h\mathbf{v}) \cdot \boldsymbol{n}_D \right) + \\ &\quad \sum_{T \subset \Omega_S} \left(\int_T (\mathbf{v} - \mathbf{\Pi}_h\mathbf{v})\, \nabla q_h - \int_{\partial T} q_h(\mathbf{v} - \mathbf{\Pi}_h\mathbf{v}) \cdot \boldsymbol{n}_S \right). \end{aligned}$$

For any $\ell \in \mathcal{E}_{\Omega_S} \cup \mathcal{E}_{\Omega_D}$ we choose a unit normal vector $\mathbf{n}_\ell$ and denote the two triangles sharing this edge T_{in} and T_{out}, with $\boldsymbol{n}_\ell$ pointing outwards T_{out}. We define

$$[\![\mathbf{v} \cdot \boldsymbol{n}_\ell]\!]_\ell := \left(\mathbf{v}|_{T_{\text{out}}}\right) \cdot \boldsymbol{n}_\ell - \left(\mathbf{v}|_{T_{\text{in}}}\right) \cdot \boldsymbol{n}_\ell,$$

which corresponds to the jump of the normal component of $\mathbf{v}$ across the edge ℓ. Notice that this value is independent of the chosen direction of the normal vector $\boldsymbol{n}_\ell$.

Rewriting the integrals on the borders of the triangles, we obtain

$$\begin{aligned} b(\mathbf{v} - \mathbf{\Pi}_h\mathbf{v}, q_h) &= \sum_{T \subset \Omega_D} \int_T (\mathbf{v} - \mathbf{\Pi}_h\mathbf{v})\, \nabla q_h + \sum_{T \subset \Omega_S} \int_T (\mathbf{v} - \mathbf{\Pi}_h\mathbf{v})\, \nabla q_h \\ &\quad - \frac{1}{2} \sum_{T \subset \Omega_D} \sum_{\ell \in \mathcal{E}_T \cap \Omega_D} \int_\ell [\![(\mathbf{v} - \mathbf{\Pi}_h\mathbf{v}) \cdot \mathbf{n}_\ell]\!]_\ell \, q_h - \sum_{\ell \in \mathcal{E}_{\Gamma_D}} \int_\ell (\mathbf{v} - \mathbf{\Pi}_h\mathbf{v}) \cdot \boldsymbol{n}_D \, q_h \end{aligned}$$

$$- \sum_{\ell\in\mathcal{E}_\Gamma}[\int_\ell(\mathbf{v}^D-\mathbf{\Pi}_h\mathbf{v})\cdot\,\boldsymbol{n}_D\,q_{D,h}+\int_\ell(\mathbf{v}^S-\mathbf{\Pi}_h\mathbf{v})\cdot\,\boldsymbol{n}_S\,q_{S,h}]$$
$$-\frac{1}{2}\sum_{T\subset\Omega_S}\sum_{\ell\in\mathcal{E}_T\cap\Omega_S}\int_\ell[\![(\mathbf{v}-\mathbf{\Pi}_h\mathbf{v})\cdot\mathbf{n}_\ell]\!]_\ell\;q_h-\sum_{\ell\in\mathcal{E}_{\Gamma_S}}\int_\ell(\mathbf{v}-\mathbf{\Pi}_h\mathbf{v})\cdot\boldsymbol{n}_S\,q_h$$
$$=I+II+III+IV+V+VI+VII.$$

We analyze the value of each of the previous terms:

(I–II) As $q_h|_T\in P_1$, their gradients are constant and by (12) we have that

$$\int_T(\mathbf{v}-\mathbf{\Pi}_h\mathbf{v})\,\nabla q_{S,h}=0\qquad\forall\;T\subset\Omega_S,$$
$$\int_T(\mathbf{v}-\mathbf{\Pi}_h\mathbf{v})\,\nabla q_{D,h}=0\qquad\forall\;T\subset\Omega_D.$$

(IV–VII) If $\ell\in\mathcal{E}_{\Gamma_S}$, $\mathbf{v}=\mathbf{0}=\mathbf{\Pi}_h\mathbf{v}$ and we have $\int_\ell(\mathbf{v}-\mathbf{\Pi}_h\mathbf{v})\cdot\boldsymbol{n}_S\,q_h=0$. On the other hand, if $\ell\in\mathcal{E}_{\Gamma_D}$, $\mathbf{v}=\mathbf{0}=\mathbf{\Pi}_h\mathbf{v}$ and so $\int_\ell(\mathbf{v}-\mathbf{\Pi}_h\mathbf{v})\cdot\boldsymbol{n}_D\,q_h=0$.

(III–VI) For the continuity of the normal component of $\mathbf{v}$ and $\mathbf{\Pi}_h\mathbf{v}$ we have that $\int_\ell[\![(\mathbf{v}-\mathbf{\Pi}_h\mathbf{v})\cdot\mathbf{n}_\ell]\!]_\ell\,q_h=0$, for any $\ell\in\mathcal{E}_{\Omega_S}\cup\mathcal{E}_{\Omega_D}$.

(V) If $\ell\in\mathcal{E}_\Gamma$, as $\mathbf{v}\in\mathbf{H}^1_0(\Omega)$ we have that $\int_\ell(\mathbf{v}^S-\mathbf{\Pi}_h\mathbf{v})\cdot\,\boldsymbol{n}_S\,q_{S,h}=\int_\ell(\mathbf{\Pi}_h\mathbf{v}-\mathbf{v}^D)\cdot\,\boldsymbol{n}_D\,q_{S,h}$. Thus, to prove that $\int_\ell(\mathbf{v}^D-\mathbf{\Pi}_h\mathbf{v})\cdot\,\boldsymbol{n}_D\,q_{D,h}+\int_\ell(\mathbf{v}^S-\mathbf{\Pi}_h\mathbf{v})\cdot\,\boldsymbol{n}_S\,q_{S,h}=0$, it is enough to see:

$$\int_\ell\mathbf{\Pi}_h\mathbf{v}\cdot\,\boldsymbol{n}_D\,\delta=\int_\ell\mathbf{v}^D\cdot\,\boldsymbol{n}_D\,\delta\quad\forall\,\delta\in P_1(\ell),$$

which holds from the property (14). Thus, we can ensure that the term V vanish.

As a consequence we can conclude that $b(\mathbf{v}-\mathbf{\Pi}_h\mathbf{v},q_h)=0\quad\forall\,\mathbf{v}\in\mathbf{H}^1_0(\Omega)\quad\forall\,q_h\in Q_h$.
Now, we need to prove that there exists a constant $C>0$, independent of h, such that

$$\|\mathbf{\Pi}_h\mathbf{v}\|_{\mathbf{V}}=(|\mathbf{\Pi}_h\mathbf{v}|^2_{1,\Omega_S}+\|\mathbf{\Pi}_h\mathbf{v}\|^2_{\mathbf{H}(\mathrm{div},\Omega_D)})^{\frac{1}{2}}\le C\|\mathbf{v}\|_1.$$

First we analyze $|\mathbf{\Pi}_h\mathbf{v}|_{1,\Omega_S}$.

$$|\mathbf{\Pi}_h\mathbf{v}|^2_{1,\Omega_S}=|\mathbf{\Pi}_{h,1}\mathbf{v}|^2_{1,\Omega_S}+|\mathbf{\Pi}_{h,2}\mathbf{v}|^2_{1,\Omega_S}.$$

Now, we can calculate the seminorm of the operators

$$|\mathbf{\Pi}_{h,j}\mathbf{v}|^2_{1,\Omega_S}=\sum_{T\subset\Omega_S:\mathcal{E}_T\cap\mathcal{E}_\Gamma=\emptyset}|\mathbf{\Pi}_{h,j}\mathbf{v}|^2_{1,T}+\sum_{T\subset\Omega_S:\mathcal{E}_T\cap\mathcal{E}_\Gamma\neq\emptyset}|\mathbf{\Pi}_{h,j}\mathbf{v}|^2_{1,T},\qquad j=1,2.$$

First, we analyze the term $I=\sum_{T\subset\Omega_S:\mathcal{E}_T\cap\mathcal{E}_\Gamma=\emptyset}|\mathbf{\Pi}_{h,j}\mathbf{v}|^2_{1,T}$. From (13) we have that

$$I\le C\left(\sum_{T\subset\Omega_S:\mathcal{E}_T\cap\mathcal{E}_\Gamma=\emptyset}|\tilde{\mathcal{I}}_j\mathbf{v}|^2_{1,T}+\sum_{T\subset\Omega_S:\mathcal{E}_T\cap\mathcal{E}_\Gamma=\emptyset}\frac{\left|\int_T(v_j-\tilde{\mathcal{I}}_j\mathbf{v})\right|^2}{(\int_T\mathrm{b}_T)^2}|\mathrm{b}_T|^2_{1,T}\right).$$

Applying (11) we obtain

$$I\le C\left(\sum_{T\subset\Omega_S:\mathcal{E}_T\cap\mathcal{E}_\Gamma=\emptyset}|\tilde{\mathcal{I}}_j\mathbf{v}(x)|^2_{1,T}+\sum_{T\subset\Omega_S:\mathcal{E}_T\cap\mathcal{E}_\Gamma=\emptyset}\frac{1}{h_T^4}\left|\int_T(v_j-\tilde{\mathcal{I}}_j\mathbf{v}(x))\right|^2\right).\tag{17}$$

Now, for the second term we have that

$$\left|\int_T(v_j-\tilde{\mathcal{I}}_j\mathbf{v}(x))\right|\le\int_T|v_j-\tilde{\mathcal{I}}_j\mathbf{v}(x)|\le|T|^{\frac{1}{2}}\|v_j-\tilde{\mathcal{I}}_j\mathbf{v}(x)\|_{0,T}.$$

Using $|T|^{\frac{1}{2}}\sim h_T$ and considering the approximation property given in the page 84 of [25] we obtain

$$\left|\int_T(v_j-\tilde{\mathcal{I}}_j\mathbf{v}(x))\right|\le Ch_T^2\|v_j\|_{1,\omega_T}.$$

For the first term in (17) we observe that, fixed i, $1\le i\le 3$, since the gradient of $\tilde{\mathcal{P}}_{\omega_{n_i}}(v_j)$ is zero, for $j=1,2$, applying an inverse estimate (see, e.g., Lemma 3.1 of [26]) and the previous approximation property we get

$$|\tilde{\mathcal{I}}_j\mathbf{v}(x)|_{1,T}=|\tilde{\mathcal{I}}_j\mathbf{v}(x)-\tilde{\mathcal{P}}_{\omega_{n_i}}(v_j)|_{1,T}\le C\frac{1}{h_T}\|\tilde{\mathcal{I}}_j\mathbf{v}(x)-\tilde{\mathcal{P}}_{\omega_{n_i}}(v_j)\|_{0,T}$$

$$\leq C\frac{1}{h_T}\left(\|\tilde{\mathcal{I}}_j\mathbf{v}(x)-v_j\|_{0,T}+\|v_j-\tilde{\mathcal{P}}_{\omega_{n_i}}(v_j)\|_{0,T}\right)$$
$$\leq C\|v_j\|_{1,\omega_T}+C\frac{1}{h_T}\|v_j-\tilde{\mathcal{P}}_{\omega_{n_i}}(v_j)\|_{0,T}.$$

Considering the approximation property given in the page 85 of [25] we get that

$$\|v_j-\tilde{\mathcal{P}}_{\omega_{n_i}}(v_j)\|_{0,T}\leq Ch_{\omega_{n_i}}|v_j|_{1,\omega_{n_i}},$$

then

$$|\tilde{\mathcal{I}}_j\mathbf{v}(x)|_{1,T}\leq C\|v_j\|_{1,\omega_T}+C\frac{h_{\omega_{n_i}}}{h_T}|v_j|_{1,\omega_{n_i}}.$$

Since $h_{\omega_{n_i}}\leq Ch_T$ (see, e.g., Lemma 1 of [27])

$$|\tilde{\mathcal{I}}_j\mathbf{v}(x)|_{1,T}\leq C\|v_j\|_{1,\omega_T}.$$

Thus, we conclude that, as the number of triangles in a neighborhood ω_{n_i} is bounded by a uniform constant,

$$I\leq C\sum_{T\subset\Omega_S:\mathcal{E}_T\cap\mathcal{E}_\Gamma=\emptyset}\|v_j\|^2_{1,\omega_T}\leq C\|v_j\|_1^2\leq C\|\mathbf{v}\|_1^2.$$

Now, we analyze the term $II=\sum_{T\in\Omega_S:\mathcal{E}_T\cap\mathcal{E}_\Gamma\neq\emptyset}|\mathbf{\Pi}_{h,j}\mathbf{v}|^2_{1,T}$, i.e., the case in which T has only one side on the interface that we denoted by ℓ.

From (15) we have that

$$II\leq C\Bigg(\sum_{T\subset\Omega_S:\mathcal{E}_T\cap\mathcal{E}_\Gamma\neq\emptyset}|\tilde{\mathcal{I}}_j\mathbf{v}|^2_{1,T}+\sum_{T\subset\Omega_S:\mathcal{E}_T\cap\mathcal{E}_\Gamma\neq\emptyset}\frac{\left|\int_T(v_j-\tilde{\mathcal{I}}_j\mathbf{v})\right|^2+\left|\int_T v_\ell n_{\ell,j}\right|^2}{(\int_T \mathrm{b}_T)^2}|\mathrm{b}_T|^2_{1,T}$$
$$+\sum_{T\subset\Omega_S:\mathcal{E}_T\cap\mathcal{E}_\Gamma\neq\emptyset}|v_\ell n_{\ell,j}|^2_{1,T}\Bigg). \quad (18)$$

We observe that $v_\ell|_T=\delta_{e_1,T}\delta_{e_2,T}\psi_{\omega_\ell}|_T$ and ψ_{ω_ℓ} can be obtained by solving the non singular system (16).

More precisely, since for each $T\subset\omega_\ell$, the function $\psi_{\omega_\ell}|_T$ can be written as $\psi_{\omega_\ell}|_T=\alpha_{\ell,1}\beta_{T,1}+\alpha_{\ell,2}\beta_{T,2}$ (with $\beta_{T,j}, j=1,2$, the Lagrange basis defined above), if we denote by $\beta_{\omega_\ell,j}$ the continuous functions defined in ω_ℓ such that $\beta_{\omega_\ell,j}|_T=\beta_{T,j}$, $j=1,2$, an easy calculation shows that

$$|\alpha_{\ell,j}|\leq\frac{C}{|\ell|}\max_{j=1,2}\left|\int_\ell(\mathbf{v}-\tilde{\mathcal{I}}\mathbf{v}(x))\cdot\boldsymbol{n}_\ell\,\beta_{\omega_\ell,j}\right|$$

and thus,

$$\int_T(v_\ell n_{\ell,j})^2=\int_T(\delta_{e_1,T}\,\delta_{e_2,T}\,\psi_{\omega_\ell}n_{\ell,j})^2=\int_T\left((\alpha_{\ell,1}\delta_{e_1,T}\,\delta_{e_2,T}\,\beta_{T,1}(x)+\alpha_{\ell,2}\delta_{e_1,T}\,\delta_{e_2,T}\,\beta_{T,2}(x))n_{\ell,j}\right)^2$$
$$\leq C\max_{i=1,2}|\alpha_{\ell,i}|^2\|\delta_{e_1,T}\,\delta_{e_2,T}\|^2_{0,T}$$
$$\leq C\frac{1}{|\ell|}\|\mathbf{v}-\tilde{\mathcal{I}}\mathbf{v}(x)\|^2_{0,\ell}\|\delta_{e_1,T}\,\delta_{e_2,T}\|^2_{0,T}$$
$$\leq C\frac{|T|}{|\ell|}\|\mathbf{v}-\tilde{\mathcal{I}}\mathbf{v}(x)\|^2_{0,\ell}\leq Ch_T^2\|\mathbf{v}\|^2_{1,\omega_T}$$

where we use that $\int_T\delta^{n_1}_{e_1,T}\delta^{n_2}_{e_2,T}dx=\frac{n_1!n_2!2!}{(n_1+n_2+2)!}|T|$ and $\|\mathbf{v}-\tilde{\mathcal{I}}\mathbf{v}(x)\|_{0,\ell}\leq C|\ell|^{\frac{1}{2}}\|\mathbf{v}\|_{1,\omega_T}$ (see, for example, [25,28]).

Hence, $|\int_T v_\ell n_{\ell,j}|\leq|T|^{\frac{1}{2}}\|v_\ell n_{\ell,j}\|_{0,T}\leq Ch_T^2\|\mathbf{v}\|_{1,\omega_T}$. Moreover, by using a classical inverse inequality, we get

$$|v_\ell n_{\ell,j}|_{1,T}\leq C\frac{1}{h_T}\|v_\ell n_{\ell,j}\|_{0,T}\leq C\|\mathbf{v}\|_{1,\omega_T}.$$

Therefore, by using these estimations in the expression (18) of the operator together with the fact that $|\tilde{\mathcal{I}}_j\mathbf{v}|_{1,T}\leq C\|v_j\|_{1,\omega_T}$ and $\left|\int_T(v_j-\tilde{\mathcal{I}}_j\mathbf{v})\right|\leq Ch_T^2\|v_j\|_{1,\omega_T}$ as we proved above, we can conclude that

$$|\mathbf{\Pi}_h\mathbf{v}|_{1,\Omega_S}\leq C\|\mathbf{v}\|_1.$$

Finally, we want to estimate $\|\mathbf{\Pi}_h\mathbf{v}\|_{\mathbf{H}(\mathrm{div},\Omega_D)}$.

Since $\|\mathbf{\Pi}_h\mathbf{v}\|_{\mathbf{H}(\mathrm{div},\Omega_D)}\leq\|\mathbf{\Pi}_h\mathbf{v}\|_{1,\Omega_D}$, with the same analysis to the previous one for $|\mathbf{\Pi}_{h,j}\mathbf{v}|^2_{1,\Omega_S}$, we can conclude that

$$\|\mathbf{\Pi}_h\mathbf{v}\|_{1,\Omega_D}\leq C\|\mathbf{v}\|_1.$$

Then, we have the following results.

Lemma 4.1. *The operator $\boldsymbol{\Pi}_h\mathbf{v}$ is bounded: There exists a positive constant C such that*

$$\|\boldsymbol{\Pi}_h\mathbf{v}\|_{\mathbf{V}} \leq C\|\mathbf{v}\|_1.$$

Lemma 4.2. *Discrete Inf-Sup Condition: There exists a positive constant β such that*

$$\sup_{\mathbf{0}\neq\mathbf{v}_h\in\mathbf{V}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{\mathbf{V}}} \geq \beta\|q_h\|_Q \quad \forall\, q_h \in Q_h.$$

Proof. Let $q_h \in Q_h$, it is a well known result (see, e.g., Corollary 2.4 in Chapter I of [22]) that there exists a vector valued function $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ and a constant $C_1 > 0$, independent of q_h such that $\nabla\cdot\mathbf{v} = -q_h$ and $\|\mathbf{v}\|_1 \leq C_1\|q_h\|_0$.

By the definition of $\boldsymbol{\Pi}_h\mathbf{v}$, we have $b(\mathbf{v} - \boldsymbol{\Pi}_h\mathbf{v}, q_h) = 0$. Then, Lemma 4.1 implies

$$b(\boldsymbol{\Pi}_h\mathbf{v}, q_h) = b(\mathbf{v}, q_h) = \|q_h\|_0^2 \geq \frac{1}{C_1}\|\mathbf{v}\|_1\|q_h\|_0 \geq \frac{1}{C\,C_1}\|\boldsymbol{\Pi}_h\mathbf{v}\|_{\mathbf{V}}\|q_h\|_0,$$

which completes the proof with $\beta = (C\,C_1)^{-1}$. □

Considering that the bilinear form $\tilde{a}$ is coercive and continuous, b is continuous and satisfies the discrete inf–sup condition, together with the abstract theory of mixed problems [6], immediately implies the following theorem.

Theorem 4.1. *There exists a unique solution $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ to the problem (9).*

Theorem 4.2. *Let $(\mathbf{u}, p) \in \mathbf{V} \times Q$ be the solution of the weak formulation (8) of the coupled problem. Let $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ be the solution of the discrete problem (9). Let the finite element spaces be chosen as in Section 4. Then, there exists a constant C such that:*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}} + \|p - p_h\|_Q \leq C\{\inf_{\mathbf{v}_h\in\mathbf{V}_h}\|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{V}} + \inf_{q_h\in Q_h}\|p - q_h\|_Q\}.$$

Corollary 4.1. *Let $(\mathbf{u}, p)$ be the solution of the coupled problems (1)–(2) together with the interface conditions (3) such that $\mathbf{u} \in \mathbf{V}$ and $p \in Q$ are smooth enough, that the norms on the right hand side of (19) are finite for some $r_1, r_2 \in (0, 1]$. Then, the discrete solution $(\mathbf{u}_h, p_h)$ of problem (9) satisfies the error estimation*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}} + \|p - p_h\|_Q \leq C\{h^{r_1}\|\mathbf{u}\|_{1+r_1,\Omega_S} + h^{r_2}\|\mathbf{u}\|_{1+r_2,\Omega_D} + h(|p|_{1,\Omega_S} + |p|_{1,\Omega_D})\}. \tag{19}$$

5. Numerical experiments

In this section we present some test cases to show the good performance of our method. We defined the individual errors by,

$$e_0(p_S) = \|p_S - p_{S,h}\|_{0,\Omega_S} \qquad e_0(p_D) = \|p_D - p_{D,h}\|_{0,\Omega_D}$$

$$e_0(\mathbf{v}_S) = \|\mathbf{v}_S - \mathbf{v}_{S,h}\|_{0,\Omega_S} \qquad e_0(\mathbf{v}_D) = \|\mathbf{v}_D - \mathbf{v}_{D,h}\|_{0,\Omega_D}$$

$$e_0(\operatorname{div}\mathbf{v}_S) = \|\operatorname{div}(\mathbf{v}_S - \mathbf{v}_{S,h})\|_{0,\Omega_S} \qquad e_0(\operatorname{div}\mathbf{v}_D) = \|\operatorname{div}(\mathbf{v}_D - \mathbf{v}_{D,h})\|_{0,\Omega_D}$$

$$e_1(\mathbf{v}_S) = |\mathbf{v}_S - \mathbf{v}_{S,h}|_{1,\Omega_S} \qquad e_1(\mathbf{v}_D) = |\mathbf{v}_D - \mathbf{v}_{D,h}|_{1,\Omega_D}$$

and the rates of convergence given by,

$$r_i(\square) = \frac{log(\frac{e_i(\square)}{e_i'(\square)})}{log(\frac{h}{h'})} \qquad \square \in \{\mathbf{v}_S, \mathbf{v}_D, \operatorname{div}\mathbf{v}_S, \operatorname{div}\mathbf{v}_D, p_S, p_D\} \text{ and } i = 0, 1$$

where h and h' denote two consecutive mesh-sizes with errors e_i and e_i'. Using the previous definition of r_i, we present for the first example, in Tables 1 and 2, the convergence history for a set of shape regular triangulations of the domain and, in Tables 3 and 4, the corresponding for the second one. For simplicity, all the parameters such as K, α and μ are set to 1. We mention that, since is difficult to construct examples satisfying the entire coupled Stokes–Darcy problem (1)–(3) (in particular, the homogeneous interface conditions (3)), the numerical experiments could include nonhomogeneous terms for the interface conditions and therefore conduce to modify (only) the right-hand side in (8).

We also comment that, in practice, mass conservation and Neumann condition have to be imposed in a weak way. Indeed, when we are assembling the system matrix we must add equations that ensures the normal continuity of the velocity and the boundary condition, i.e., $\int_\Gamma(\mathbf{v}_h^D\cdot\boldsymbol{n}_D + \mathbf{v}_h^S\cdot\boldsymbol{n}_S)\gamma = 0$ and $\int_{\Gamma_D}\mathbf{v}_h^D\cdot\boldsymbol{n}_D\,\gamma = 0,\ \forall\,\gamma \in \{C^0(\Gamma) : \gamma|_\ell \in P_1(\ell)\}$.

Table 1
Mesh-sizes, errors, and rates of convergence (Example 1).

h	$e_0(\mathbf{v}_S)$	$r_0(\mathbf{v}_S)$	$e_0(\mathbf{v}_D)$	$r_0(\mathbf{v}_D)$	$e_0(p_S)$	$r_0(p_S)$	$e_0(p_D)$	$r_0(p_D)$
0.0625	0.00007	2.1274	0.0112	2.3051	0.0069	1.8767	0.0093	1.9248
0.0313	0.00002	2.0078	0.0028	2.0194	0.0020	1.7874	0.0024	1.9266
0.0156	0.000004	2.0020	0.0008	1.7002	0.0006	1.6937	0.0007	1.8263

Table 2
Mesh-sizes, errors, and rates of convergence (Example 1).

h	$e_0(\text{div}\,\mathbf{v}_S)$	$r_0(\text{div}\,\mathbf{v}_S)$	$e_0(\text{div}\,\mathbf{v}_D)$	$r_0(\text{div}\,\mathbf{v}_D)$	$e_1(\mathbf{v}_S)$	$r_1(\mathbf{v}_S)$	$e_1(\mathbf{v}_D)$	$r_1(\mathbf{v}_D)$
0.0625	0.0123	1.0084	0.0224	1.4040	0.0188	1.0109	0.4832	1.2828
0.0313	0.0061	1.0023	0.0109	1.0416	0.0094	1.0026	0.2078	1.2177
0.0156	0.0031	1.0009	0.0055	0.9781	0.0047	1.0008	0.1009	1.0415

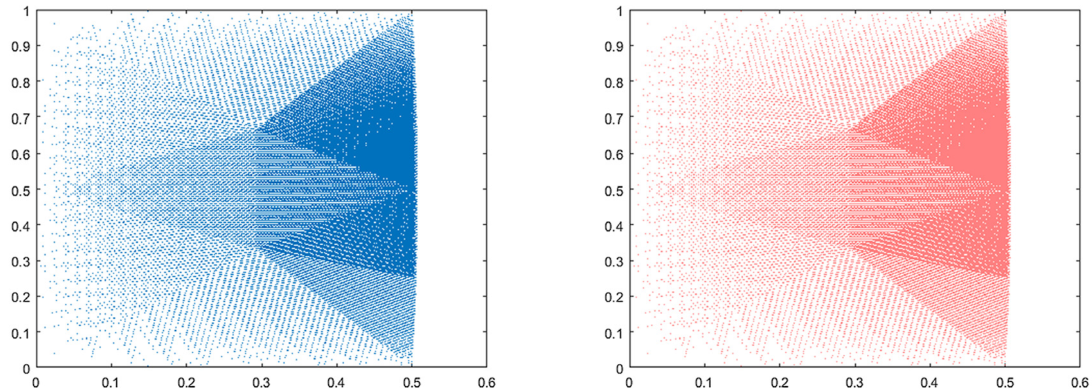

Fig. 3. $\mathbf{v}_S$ and $\mathbf{v}_{S,h}$ (Example 1).

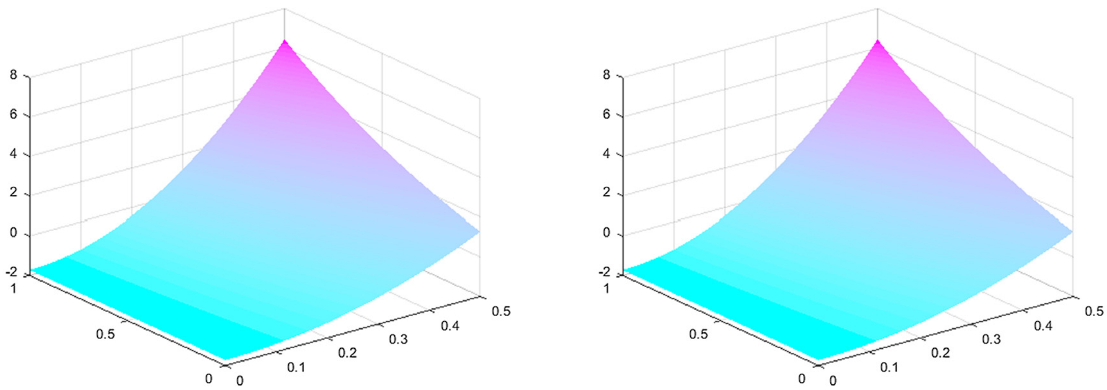

Fig. 4. p_S and $p_{S,h}$ (Example 1).

In the first example we consider the regions $\Omega_S = (0, \frac{1}{2}) \times (0, 1)$ and $\Omega_D = (\frac{1}{2}, 1) \times (0, 1)$. The interface, Γ, is the line $x = \frac{1}{2}$. We select the right-hand terms $\mathbf{f}_S, g_S =: \text{div}\,\mathbf{u}_S, \mathbf{f}_D, g_D$ and the boundary conditions according to the analytical solution given by

$$\mathbf{u}_S(x, y) = \begin{pmatrix} xy(1-y) \\ x^2(1-y)sin(y) \end{pmatrix} \qquad \mathbf{u}_D(x, y) = \begin{pmatrix} 2xy(1-y)(1-x) \\ xy^2(1-y) \end{pmatrix}$$

$$p_S(x, y) = 12x^2e^y \qquad p_D(x, y) = 16xy^3 - e - 2.$$

In this first example it is satisfied that $\mathbf{u}_D \cdot \boldsymbol{n_D} = 0$ in Γ_D and $\mathbf{u}_D \cdot \boldsymbol{n}_D + \mathbf{u}_S \cdot \boldsymbol{n}_S = 0$ in Γ.

In Figs. 3 and 5 we show the approximate and exact values of the velocities and in Figs. 4 and 6 of the pressures. It is clear from these figures that the finite element spaces used provide very accurate approximations to the unknowns.

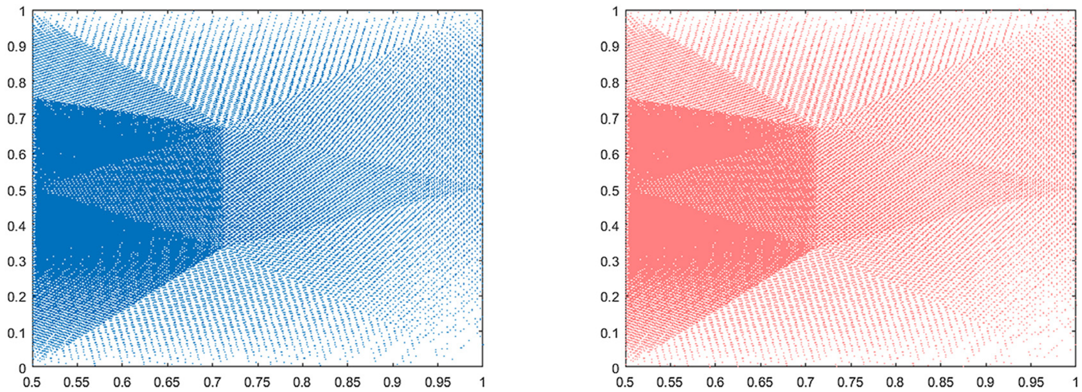

Fig. 5. $\mathbf{v}_D$ and $\mathbf{v}_{D,h}$ (Example 1).

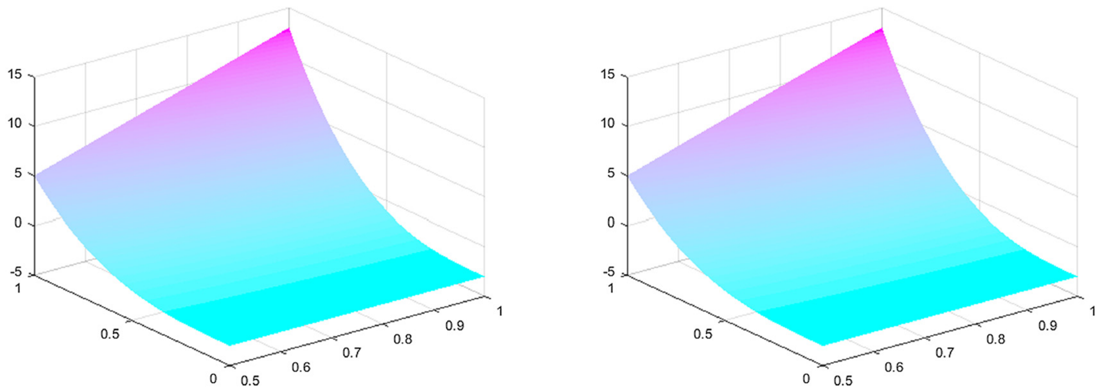

Fig. 6. p_D and $p_{D,h}$ (Example 1).

Table 3
Mesh-sizes, errors, and rates of convergence (Example 2).

h	$e_0(\mathbf{v}_S)$	$r_0(\mathbf{v}_S)$	$e_0(\mathbf{v}_D)$	$r_0(\mathbf{v}_D)$	$e_0(p_S)$	$r_0(p_S)$	$e_0(p_D)$	$r_0(p_D)$
0.0884	0.0046	2.0636	0.0491	0.8073	0.1937	1.4637	0.0049	1.2839
0.0442	0.0011	2.0480	0.0267	0.8809	0.0683	1.5030	0.0020	1.3002
0.0221	0.0003	2.0216	0.0140	0.9324	0.0241	1.5046	0.0008	1.3226

The purpose of this second example, which matches with Example 1 in [14], is to confirm the good performance of our mixed finite element scheme in comparison with other stable elements. Let $\Omega_D = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})$ and $\Omega_S = (-1, 1) \times (-1, 1) \setminus \Omega_D$ be a porous medium completely surrounded by a fluid. The particularity of this example is that there is no Γ_D because the boundary of Ω_D represents the interface, Γ. We set the appropriate forcing term $\mathbf{f}_S$ and the source g_D, such that the following solution to the Stokes–Darcy coupled problem, with $\mathbf{f}_D = \mathbf{0}$, is exact

$$\mathbf{u}_S(x, y) = \begin{pmatrix} -4(x^2-1)^2(y^2-1)y \\ 4(x^2-1)(y^2-1)^2x \end{pmatrix}$$

$$p_S(x, y) = -sin(x)e^y \qquad p_D(x, y) = -sin(x)e^y.$$

Figs. 7 and 8 show, respectively, the approximate and exact velocities and the approximate and exact values of the pressure for the Stokes region, while Figs. 9 and 10 display the corresponding for the Darcy region. Tables 3 and 4, which can be compared with Table 2 in [14], show that optimal rate of convergence can be also reached with our method.

We also observe that, in the two examples under consideration, the rate of convergence provided by Corollary 4.1 is attained by all the unknowns.

We emphasize that the numerical results confirm the good performance of the mixed finite element scheme with Mini element for the Stokes–Darcy coupled problem. We end this paper by mentioning that, the ideas used here for numerical approximation of the coupled problem, could be successfully applied (with perhaps eventual technical difficulties) to another family of elements that are known to be stable for the Stokes problem and it will be subject of future work.

Table 4
Mesh-sizes, errors, and rates of convergence (Example 2).

h	$e_0(\text{div}\,\mathbf{v}_S)$	$r_0(\text{div}\,\mathbf{v}_S)$	$e_0(\text{div}\,\mathbf{v}_D)$	$r_0(\text{div}\,\mathbf{v}_D)$	$e_1(\mathbf{v}_S)$	$r_1(\mathbf{v}_S)$	$e_1(\mathbf{v}_D)$	$r_1(\mathbf{v}_D)$
0.0884	0.2905	0.9892	0.0437	1.2140	0.9110	0.9935	0.2160	0.6935
0.0442	0.1449	1.0034	0.0189	1.2056	0.4558	0.9990	0.1478	0.5473
0.0221	0.0723	1.0037	0.0086	1.1382	0.2278	1.0001	0.1051	0.4925

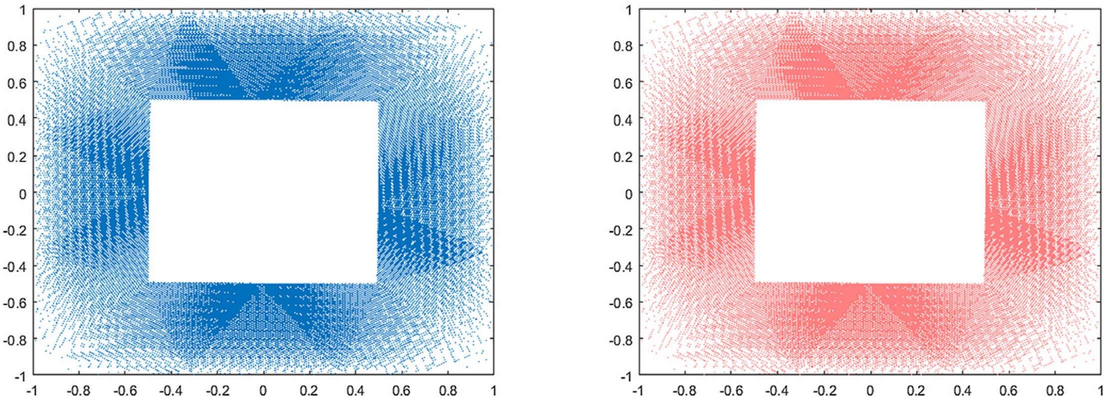

Fig. 7. $\mathbf{v}_S$ and $\mathbf{v}_{S,h}$ (Example 2).

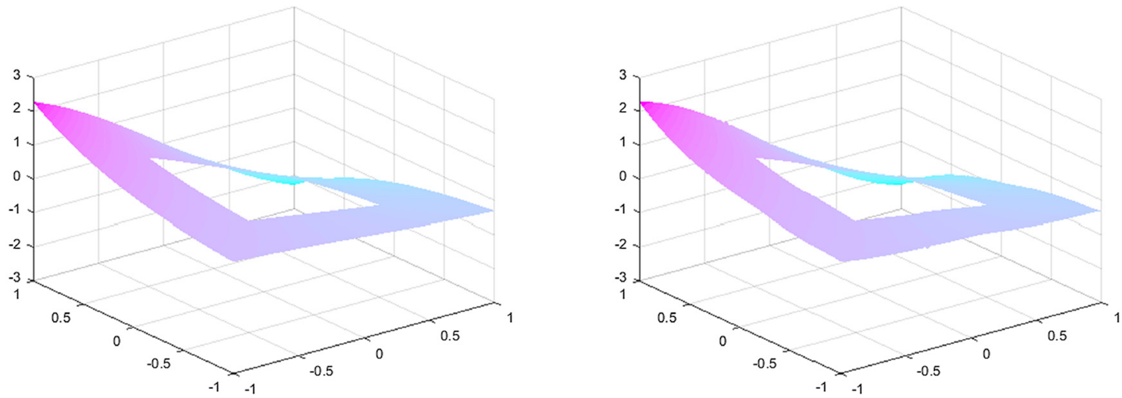

Fig. 8. p_S and $p_{S,h}$ (Example 2).

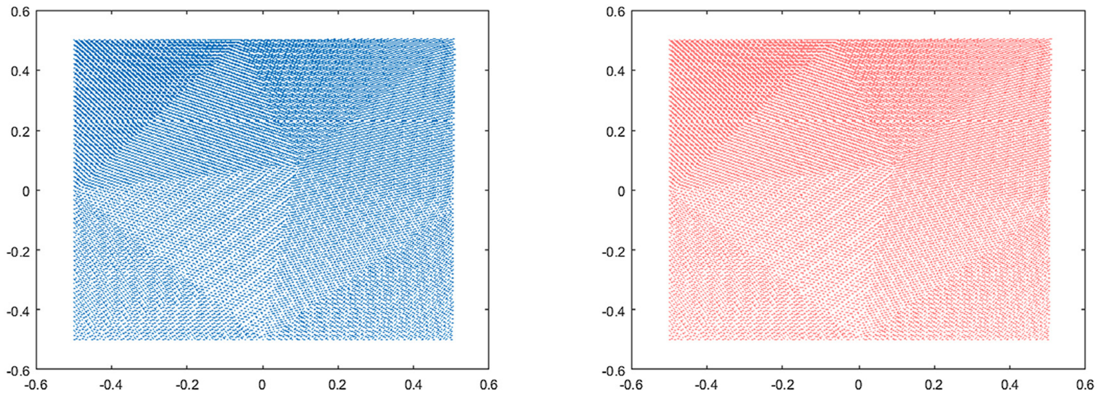

Fig. 9. $\mathbf{v}_D$ and $\mathbf{v}_{D,h}$ (Example 2).

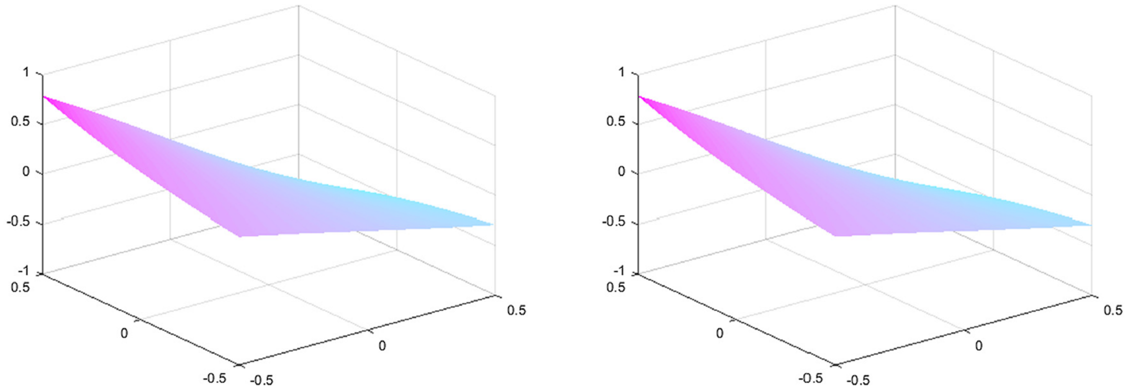

Fig. 10. p_D and $p_{D,h}$ (Example 2).

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