Logic and **L**ogical **P**hilosophy (2021)
DOI: 10.12775/LLP.2021.001
Published online: June 30, 2021

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Beyond Mixed Logics

Abstract. In order to define some interesting consequence relations, certain generalizations have been proposed in a many-valued semantic setting that have been useful for defining what have been called *pure*, *mixed* and *order-theoretic* consequence relations. But these generalizations are insufficient to capture some other interesting relations, like other intersective mixed relations (a relation that cannot be defined as a mixed relation, but only as the intersection of two mixed relations) or relations with a conjunctive (or, better, "universal") interpretation for multiple conclusions. We propose a broader framework to define these cases, and many others, and to set a common background that allows for a direct compared analysis. At the end of the work, we illustrate some of these comparisons.

Keywords: Many-valued logics; mixed consequence relations; mapping matrix; multiple conclusions interpretations

1. Introduction

A logic can be defined as a set of valid argument schemes over a given language, which in turn can be captured by a definition of logical consequence, a relation between propositions that serve as premises and propositions that serve as conclusions. Logical consequence has been widely understood as preserving some property from the premises to the conclusions, usually truth as in classical logic. To get a preservation relation, the definition of a logical consequence must have some structural constraints, such as *reflexivity*, *transitivity* or *monotonicity*.

However, reasons for relaxing each of these constraints have been raised, not just for getting different preservation relations, but for get-

Special Issue: Logics and Their Interpretations. Guest Editors: Henrique Antunes and Damian Szmuc

ting non-preservation relations as well; logical consequences that lack some of these constraints are called substructural logics. The aim of this work is to offer a general framework for a class of logics that includes substructural logics, allowing a common background for their relations and their interpretations. We will focus on a semantic setting, based on many-valued logics, where the notions of satisfaction and counterexamples play an important role. Within this setting, an extensive group of consequence relations have been studied, primarily *mixed* consequence relations.

Mixed consequence relations hold between premises that satisfy a standard for premises and conclusions that satisfy a standard for conclusions. A preservation relation could be viewed as imposing the same standard for premises and for conclusions; as in the case of a truth-preserving relation, it holds whenever the premises are (all) true, the conclusion will be also true, i.e. whenever the premises satisfy the *being-true*-standard, the conclusion will satisfy the *being-true*-standard. But these standards can differ, dropping the preservation-feature, as in the case of a relation that holds whenever the premises are (all) not false, the conclusion will be true, i.e. whenever the premises satisfy the *being-not-false*-standard, the conclusion will satisfy the *being-true*-standard.

Both above relations are the same relation when talking about a two-valued logic, but are not when talking about other many-valued logics. In this latter case, some of *reflexivity*, *transitivity* and *monotonicity* do not obtain, so these mixed relations correspond to substructural logics. There are, however, some other related logics that cannot be defined with mixed consequence relations, but can be defined as their intersection [3, p. 2199].

Emmanuel Chemla, Paul Égré and Benjamin Spector [3] have claimed that a subclass of mixed consequence relations and their intersections forms a "natural" class of consequence relations. This subclass is defined by some "respectable" properties (with a similar role to the one played by *reflexivity*, *transitivity* and *monotonicity* in Tarski's program [24]), proposed to identify within the three-valued logics some specific logical systems. These systems are, in turn, already "identified or introduced on various independent grounds, usually with different motivations in mind" [3, p. 2194]. Despite their success, these properties define a narrow class, leaving aside other consequence relations that are interesting on their own.

One case is caused by a restrictive criterion for premises and conclusions standards, based on some order between the elements of the standards [3, pp. 2197-2198]. Relaxing this constraint allows, in a three-valued setting, for a standard like *being-determinate* (as opposed to *being-indeterminate*, a standard that indeterminate sentences would satisfy).[1] According to Chemla et al. [3, n. 3], Sharvit [22] explores a consequence relation where the premises and conclusions standards are the *being-determinate*-standard, and its intersections with some other mixed relations. These consequence relations and their intersections with "respectable" relations are not "respectable" relations themselves, but can be interpreted in a useful way, along with other relations they leave out of consideration.

Another case concerns inferences with multiple conclusions, and the interpretation for this multiplicity. The usual way of interpreting a mixed consequence relation claims that it holds if whenever *every* premise satisfy the premises standard, then *some* of the conclusions will satisfy the conclusions standard. But this is not the only way of interpreting inferences in a multiple-conclusion system, because a relation could ask for *every* conclusion to satisfy the conclusions standard. Despite it not being the usual way it is understood, it has been explored by [5]. If this change becomes available, the considered mixed relations would have some "new" counterparts. If the two groups of mixed relations were allowed to interact, being intersected for instance, some other relations would be obtained.

Though we will not determine whether all of these relations are interesting on their own, we will offer a broader representation of notions of entailment that serves as a common ground for their analysis (and the analysis of their interactions). To do so, we will recover in Section 2 the "natural" entailment notions proposed by Chemla et al. [3] and, like them, analyse a three-valued case. We will not limit ourselves to the consequence relations they propose, but will also mention the examples above to compare them with.

In Section 3, we will motivate an interpretation for consequence relations, not based on what standard every or some propositions satisfy, but based on when the premises and the conclusions satisfy some con-

[1] This presupposes a value interpreted as *being-false*, a value interpreted as *being-true*, and a value interpreted as *being-indeterminate*. The third one could be interpreted in a distinct way, but what matters in the example is the exclusion of the third value over the other two values (*determined* values, persisting in the initial interpretation).

straints, i.e. based on what counts as a counterexample for a statement with respect to a consequence relation and when an inference could have a counterexample. Though it is not a new interpretation for consequence relations, it will allow us to introduce a semantic notion for relating sets of propositions with sets of semantic values. This notion will be the basis for the broader entailment notion, and we will introduce some operations over consequence relations.

Section 4 will show how to reformulate the previous generalizations as cases of the general one, through some translation results. In this section we will also show some results about the relation between the examples analysed in Section 2.

2. Consequence relation generalizations

Throughout this work, we will follow Chemla et al. and will consider a many-valued setting. We will focus on the relations between sets of premises and conclusions when they take certain combinations of semantic (truth) values, independently of the language involved.

We will leave aside their methodological proposal, regarding other aspects, allowing multiple conclusions (a set of conclusions instead of a single conclusion) and not giving special attention to a possible order of the semantic values. Though still valid, their results hold for a framework with single conclusions and some constraints on the order of the semantic values. The shift to a multiple conclusion framework and no constraints on the order is motivated by its generality.

Let's begin with the semantic values, which usually refer to alethic notions such as *truth* and *falsity*, but also *indeterminacy*, *nonsense* and others.[2] We could think of a three-valued semantic where the values 1, $\frac{1}{2}$ and 0 are treated as representing *truth*, *indeterminacy* and *falsity*, respectively. Another example of multi-valued semantics is an infinite-valued semantics, where every real number, for example, in the $[0, 1]$ interval represents a degree of *truth*, between a full degree of *truth* and its complete opposite, i.e. a full degree of *falsity*. For a set of semantic

[2] Typically, these last notions are characterized in (at least) three-valued semantics. For an interpretation for the third value as indeterminate, see, for example, [13]. For a reading of non-classical values as nonsensical, see, among others, [2], [11] and [23].

values $\mathcal{SV}$, each semantic value is associated with an intended interpretation. Our present work is focused only on the finite sets of values, but conceivably it could be extended to infinite sets.

To define propositions, Chemla et al. propose the notion of an index set, that can be viewed as valuations, though valuations are linked to specific languages. We will adopt the notion of valuation, as it is usually understood. For any language $\mathbf{L}$, its well formed formulae conform a set $FOR_{\mathbf{L}}$. A valuation v is a function from $FOR_{\mathbf{L}}$ to $\mathcal{SV}$.

In addition, they are interested in sets with more semantic structure, where the semantic values are a partially ordered set; moreover, there are always a greatest element and a least element. In the above examples, 1 is the greatest element and 0 is the least element. This structure is relevant for their results, but we will see that it plays no role in our broader framework.

Finally, given a language $\mathbf{L}$, a consequence relation X is a subset of $\mathscr{P}(FOR_{\mathbf{L}}) \times \mathscr{P}(FOR_{\mathbf{L}})$. For a set of propositions (or formulas) Γ as premises and a set of propositions (or formulas) Δ as conclusions, we write $\Gamma \models_X \Delta$ to state that the set Δ is a consequence of the set Γ. For reasons of convenience, we will not consider cases where Γ or Δ are empty sets; this would lead us to present unnecessarily complicated technicalities.[3] In order to reduce the notation for sets that does not include the empty set, take $\mathscr{P}^*(X)$ as $\mathscr{P}(X) - \varnothing$.

The addition of constraints would limit what counts as a consequence relation. There are many ways of doing so. Chemla et al. chose the most standard (semantic) way, which is to impose a relation between the semantic values that the premises and conclusions take. Actually, they present *three* ways of adding constraints in this sense. Two of them make use of the *designated value* notion. The designated values are a subset of the semantic values, and they are usually thought as the range

[3] Notice that following this path we have some loss: given a language and a consequence relation, for some formulae it is possible to show they can be derived as conclusions no matter what set of premises is considered, even an empty set of premises. We could express a similar fact stating that these formulae can be derived from an arbitrary set of premises, without mentioning the empty set. This equivalence will be relative to some specific logic, which should be monotonic, or at least monotonic regarding premises of inferences with an empty set of premises. Nevertheless, we are relying for the equivalence on the intuitive idea that a sentence is valid if and only if it follows from everything—to emulate sentential validities—, and that a sentence is antivalid if and only if everything follows from it—to represent sentential antivalities. (For more about antivalidities, see [21] and [1].)

of values that tell us that a sentence is assertible. This, again, can be thought as having more structure in the set of semantic values; the values are ordered, so it is usually demanded that if a value is designated, every higher value is also designated.[4] Chemla et al. exclude the empty set of values and the whole set of semantic values as designated values.

The designated values, under this interpretation, can be used for defining relations that preserve the assertion conditions from premises to conclusions, or for defining relations that make explicit independent conditions for the assertion of premises and conclusions. The former relations are called *pure consequence relations* and the latter, *mixed consequence relations*; the pure relations are a special case of the mixed relations. The set of designated values can be thought of as a standard the propositions satisfy (be they premises or conclusions). When talking about any subset of the semantic values, we will use the word "standard" instead of "designated values" for its neutrality regarding the order of the values.

Definition 2.1. Let $S \subseteq \mathcal{SV}$. X is a *pure* consequence relation:
$\Gamma \models_X \Delta$ iff for every valuation v, if every premise $\gamma \in \Gamma, v(\gamma) \in S$, then for some conclusion $\delta \in \Delta, v(\delta) \in S$.[5]

Definition 2.2. Let $S_a \subseteq \mathcal{SV}$ and $S_b \subseteq \mathcal{SV}$. X is a *mixed* consequence relation:
$\Gamma \models_X \Delta$ iff for every valuation v, if every premise $\gamma \in \Gamma, v(\gamma) \in S_a$, then for some conclusion $\delta \in \Delta, v(\delta) \in S_b$.

In the third way Chemla et al. add constraints to the definition of consequence relation. The interpretation of the values does not rest on the notion of designated values, but on the order between them. There is also the goal of preserving some characteristic: "the conclusion should not meet a worse standard than whatever standard is set for the premises" [3, p. 2193], where "worse" must be understood in relation to the order between values. These consequence relations are called *order-theoretic consequence relations*.

[4] In particular, this is how Chemla et al. understood the idea of designated value. But this is not mandatory; as we will soon see, [22] uses a logic based on a consequence relation that they called *dd*. And in [16], several logics are introduced, some of which are such that the set of designated values does not form a filter or an upset.

[5] We use subindexes in $\Gamma \models_X \Delta$ where X is the label of a consequence relation. We make extensive use of this symbolism in the rest of the article.

Definition 2.3. X is an *order-theoretic* consequence relation: $\Gamma \models_X \Delta$ iff for every valuation v, there is a premise $\gamma \in \Gamma$ and a conclusion $\delta \in \Delta$, such that $v(\gamma) \leq v(\delta)$.

The first two generalizations for defining consequence relations are easy to compare, given pure consequence relations are the special case of mixed consequence relations where S_a is equal to S_b. The third one was shown to be equivalent to the intersection of every pure consequence relation which standard is a set of designated values [3]. When we consider the mixed relations where the standards are taken to be a set of designated values and the order-theoretic relations, we obtain what Chemla et al. have called "respectable" relations. They can be shown to be related, but always taking one detour or another. This also happens with some other relations, as we will show below.

As an example, let's consider a three-valued setting where the set of semantic values $\mathcal{SV}$ is $\{0, \frac{1}{2}, 1\}$. Under some interpretations, this values are interpreted as *truth*, *indeterminacy* and *falsity*, and there is an intended alethic order: $0 < \frac{1}{2} < 1$. Given this order, there are only two possible sets of designated values: $\{1\}$ and $\{\frac{1}{2}, 1\}$. Below we give the complete list of standards (subsets of the semantic values) with a name, linked to a standards interpretation (we consider only the non-empty subsets of $\mathcal{SV}$):

$$
\begin{array}{ccccl}
\text{(falsity)} & \text{F} & = & \{0\} \\
\text{(indeterminacy)} & \text{I} & = & \{\frac{1}{2}\} \\
\text{(truth)} & \text{T} & = & \{1\} \\
\text{(non-truth)} & \text{NT} & = & \{0, \frac{1}{2}\} \\
\text{(non-indeterminacy)} & \text{NI} & = & \{0, 1\} \\
\text{(non-falsity)} & \text{NF} & = & \{\frac{1}{2}, 1\} \\
\text{(domain of values)} & \text{SV} & = & \{0, \frac{1}{2}, 1\}
\end{array}
$$

The names follow the intended interpretation, though there are many other notations for these sets of values. [18] work on a three-valued setting where the 1 is interpreted as *being strictly true* and $\frac{1}{2}$ as *being tolerantly true* (but not *strictly true*). From this, the set $\{1\}$ is interpreted for the notion of *strict truth* and the set $\{\frac{1}{2}, 1\}$ for the notion of *tolerant truth*, noted as $\mathcal{S}$ and $\mathcal{T}$, respectively. Another relevant subset is $\{0, 1\}$, called d for *definedness*. We will just retain the names of the consequence relations induced by this notation, but will use the above names for the sets, i.e. the standards.

If we look at sets of designated values, we obtain two pure consequence relations: one that preserves the truth of the premises, and one that preserves their non-falsity. The former is called the *ss*-relation and the latter the *tt*-relation, and are defined as below:

DEFINITION 2.4.

$$\Gamma \models_{ss} \Delta \text{ iff } \forall v : \text{if } \forall \gamma \in \Gamma(v(\gamma) \in \{1\}), \text{ then } \exists \delta \in \Delta(v(\delta) \in \{1\})$$

DEFINITION 2.5.

$$\Gamma \models_{tt} \Delta \text{ iff } \forall v : \text{if } \forall \gamma \in \Gamma(v(\gamma) \in \{\tfrac{1}{2}, 1\}), \text{ then } \exists \delta \in \Delta(v(\delta) \in \{\tfrac{1}{2}, 1\})$$

If we put no restrictions on the sets, we obtain five more relations. Among them, we will just use explicitly the *dd*-relation.

DEFINITION 2.6.

$$\Gamma \models_{dd} \Delta \text{ iff } \forall v : \text{if } \forall \gamma \in \Gamma(v(\gamma) \in \{0, 1\}), \text{ then } \exists \delta \in \Delta(v(\delta) \in \{0, 1\})$$

Regarding the mixed consequence relations, *ss* and *tt* are two of them (when given the same standard for premises as for conclusions),[6] but there are other two linked to sets of designated values: the relations *st* and *ts* (which are instances of what is called *p*-consequence by [9], and *q*-consequence by [14], respectively).

DEFINITION 2.7.

$$\Gamma \models_{st} \Delta \text{ iff } \forall v : \text{if } \forall \gamma \in \Gamma(v(\gamma) \in \{1\}), \text{ then } \exists \delta \in \Delta(v(\delta) \in \{\tfrac{1}{2}, 1\})$$

DEFINITION 2.8.

$$\Gamma \models_{ts} \Delta \text{ iff } \forall v : \text{if } \forall \gamma \in \Gamma(v(\gamma) \in \{\tfrac{1}{2}, 1\}), \text{ then } \exists \delta \in \Delta(v(\delta) \in \{1\})$$

Again, if we put no constraints on the standards, we obtain other forty-four consequence relations, some of them marginally studied. But there is only one order-theoretic consequence relation, defined as in Definition 2.3, sometimes called the $\leq$-relation; $\leq$ is equivalent to the intersection of *ss* and *tt*, and cannot be defined as a mixed consequence

[6] If we were to consider some specific languages, we could relate *ss* and *tt* to the known systems **K3** and **LP**, as presented by Stephen Kleene [12] and by Graham Priest [17]. **LP** is the logic that results from taking *tt* and a standard language with Strong Kleene valuations, and **K3** is the logic that results from taking *ss* and the same language.

relation, as many other relations that can only be obtained as the intersection of two other relations.

The relations *ss*, *tt*, *st*, *ts* and $\leq$ are useful for many topics, such as semantic paradoxes and the representation of the phenomenon of vagueness.[7] These are paradigmatic cases of the area related to mixed consequence relations. But the area faces other and more anomalous cases. Take for instance the case of Sharvit [22] that explores the intersection of *dd* and other relations (such as *st*); from the moment we restrict ourselves to work only with sets of designated values, it seems we lose some interesting relations.

Notwithstanding, even if we put no constraints on the above generalizations, they all read multiple conclusions in a specific way, but there are other ways of reading this multiplicity (though less canonical). They all interpret the entailment notions as stating that when all the premises have some property, then *some* of the conclusions satisfy a given condition, though not necessarily all conclusions do. These entailment notions read the multiplicity of premises in a universal way (they all share a property) and the multiplicity of conclusions in an existential way (one of them has a given property). They can be read under a different interpretation. [5] presents a universal reading both for premises and for conclusions. Something similar can be done with premises: they can be read in an existential way.[8]

The links between the above consequence relations are interesting on their own, but the comparisons are not made against a single common

[7] There are many places where these applications can be found. For example, an extensive presentation of **LP** and its application to semantic paradoxes, see [17]. Non-transitive approaches to logical consequence were discussed, previously, in many works, but the particular application of **ST** to this field is original of Cobreros, Egré, Ripley and van Rooij in [7]. In [6] they also propose it as a suitable solution to the problem of vagueness. Recently, Rohan French in [10] discussed it in connection with the paradoxes of self-reference. (As with **LP** and **K3**, **ST** and **TS** are the resulting logics of taking a standard language with Strong Kleene valuations and, for **ST**, the relation *st* and, for **TS**, the relation *ts*.)

[8] We prefer to call it an "existential" reading, because the disjunctive interpretation is attached to the notion of designated value, so if we read the conclusions as disjuncts, whenever one conclusion has a designated value, the disjunction will also have a designated value; but what about not-designated standards? If the standard (for conclusions) is $\{0\}$, the disjunction of the conclusions could miss, in a particular valuation, the falsity of one false conclusion in the presence of one true conclusion. The same happens, *mutatis mutandis*, with the "universal" reading as preferable over a conjunctive interpretation.

background. Often there are some interesting features and we ignore the rest of them. For example, if we focus on the combinations of standards for mixed relations, we will probably have no interest in more than one multiple conclusions interpretation; or if we focus on the multiple conclusions interpretations, we will probably restrict ourselves to a limited group of mixed relations. We lack a common background. In the following section we provide a broader framework to make up for this absence.

3. Mapping matrices and consequence relations

In this many-valued logic framework we can, in principle, establish if an inference is valid or not by listing every possible case for this inference, and checking if there is any counterexample for the inferences validity. This can be done because the definition of a consequence distinguishes between allowed (permitted) cases and not-allowed (prohibited) cases.[9]

Take for instance the consequence relation for classical logic. An inference is valid if and only if every case where the premises are all true, is also a case where some conclusion is also true. Or, in other words, an inference is valid, unless the premises are all true and the conclusions are all false. Leaving aside whether we are in a two- or three-valued setting, this consists in a condition for an inference being invalid, as opposed to being valid, by making explicit prohibited cases.

We can list the permitted and prohibited cases for the previous example. The consequence definition tells us that if every premise is true, then some conclusion is true. So, the cases where every premise is true and every conclusion is true, are permitted cases, but also the cases where every premise is true, some conclusion is true and some conclusion is false. Not so the cases where every premise is true and every conclusion is false. Every other case is a case where the antecedent of the definition is not satisfied (those cases where the premises are not all true), and it is a, trivially, permitted case.

Each consequence relation mentioned in the previous sections gets this permitted-prohibited distinction. The pure and mixed consequence

[9] Another way of saying this is that some valuations are permitted while others are prohibited. Though we might have made that choice, we prefer not to, as it is not that the valuations that correspond to prohibited cases are not genuine valuations or anything like that.

relations get it by selecting a characteristic for the set of premises and a characteristic for the set of conclusions. Every case in which the set of premises lacks the premise-characteristic, is a permitted case. Every case in which this set has the premise-characteristic, should be confronted with the conclusion-characteristic: if the set of conclusions has the conclusion-characteristic, it is a permitted case, but if not it is a prohibited case.

The order-theoretic relations get the distinction by characterizing the set of premises and the set of conclusions, and giving a comparison criterion between the characterizations. Every case in which the characterization of the set of premises and the characterization of the set of conclusions satisfy the comparison criterion is a permitted case. Every other case is a prohibited case.

The characterization of the sets (i.e. having some defined property) allows us to tell if, for an inference, every case is permitted or not. This is so because the definition of a consequence relation links every combination of characterizations either with a permitted case or with a prohibited case. A consequence relation can be seen as a constraint on the possible combinations of characterizations; it sets what is permitted and what is prohibited.

In order to give a general strategy, we propose to characterize the sets of premises and conclusions as a mapping between sets of propositions and sets of values. A set of propositions can be valued in different ways, depending on the set of semantic values considered. Each subset of the semantic values is linked to a different characterization of a set of propositions. The mapping can be seen as a triadic relation, such that a valuation maps a set of propositions over a set of values, whenever (i) for each value of the subset, some proposition takes this value, and (ii) every proposition takes a value of the subset. The mapped set of propositions could be either a set of premises or a set of conclusions, and the subset of values could be any standard. The following definition introduces some notation that eases the presentation of the following:

Definition 3.1. A valuation v maps a set of propositions Σ over a standard S, i.e. $\mathsf{map}(v, \Sigma, S)$, if and only if (i) and (ii):

$$
\begin{array}{ll}
\text{(i)} & \forall s \in S\ (\exists \sigma \in \Sigma\ (v(\sigma) = s)) \\
\text{(ii)} & \forall \sigma \in \Sigma\ (\exists s \in S\ (v(\sigma) = s))
\end{array}
$$

This is equivalent to saying that $\mathsf{map}(v, \Sigma, S)$ holds if and only if $v[\Sigma] = S$, a shorthand for $\{v(\sigma) \mid \sigma \in \Sigma\} = S$.

Notice that, given a valuation, a non-empty set of propositions Σ is always mapped with only one standard S:

FACT 3.1. Given a valuation v, a non-empty set of propositions Σ and the set of semantic values $\mathcal{SV}$, there is only one $S_i \subseteq \mathcal{SV}$ such that $\mathsf{map}(v, \Sigma, S_i)$.

Fact 3.1 is the result of taking any valuation function and noticing that, because it is a function, it meets the *uniqueness* property (that is, for a given input, there is only one defined output).

Thus, in the previous three-valued setting, we can formulate seven non-empty standards. Given Fact 3.1, a valuation maps an inference $\Gamma \models \Delta$ to a pair of standards $\langle S_a, S_b \rangle$. One valuation of the propositions of the inference can fall in any of the 49 combinations of mappings for the pair $\langle S_a, S_b \rangle$.

Having developed the notion of a *mapping*, we would now like to represent the mapping constraints of a consequence relation (the pairs of standards for which a valuation is permitted to map an inference) and the "valuation" mappings for an inference (the set of mappings for every possible valuation of the propositions involved in an inference). We define below the notion of a *mapping matrix* which allows us to enumerate sets of pairs of permitted combinations of mappings (for consequence relations) and mappings linked to sets of valuations (for inferences).[10]

DEFINITION 3.2. Let $\mathcal{SV}$ be a set of semantic values. Then $M \subseteq \mathscr{P}^*\mathcal{SV} \times \mathscr{P}^*\mathcal{SV}$ is a *mapping matrix*. M is represented as a matrix of dimensions $|\mathscr{P}^*\mathcal{SV}| \times |\mathscr{P}^*\mathcal{SV}|$, such that for every m_{ij} of the matrix, $m_{ij} = 1$ if and only if $\langle i, j \rangle \in M$ (otherwise, $m_{ij} = 0$).

A mapping matrix is then a relation between standards, such that its matricial representation links rows with premises-mappings and links columns with conclusions-mappings.

[10] We work with the usual mathematical notion for a *matrix*: "a matrix is defined as any rectangular array of elements from a field" [8, p. 193]. In our case, the elements of a matrix are from a Boolean algebra, defined over the set $\{0, 1\}$.

If a consequence relation allows every combination of mappings, we could represent this trivial relation with a matrix, as follows:[11]

$$M_1 = \begin{array}{c} \\ \textsc{f} \\ \textsc{i} \\ \textsc{t} \\ \textsc{nt} \\ \textsc{ni} \\ \textsc{nf} \\ \textsc{sv} \end{array} \begin{array}{c} \begin{array}{ccccccc} \textsc{f} & \textsc{i} & \textsc{t} & \textsc{nt} & \textsc{ni} & \textsc{nf} & \textsc{sv} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

As said above, a '1' represents that the combination of row (premises-mapping) and column (conclusions-mapping) is permitted. But we know that not all the combinations of mapping must be allowed, since some valuations serve as counterexamples for invalid inferences. We could represent a consequence relation that allows no combination (there would be no valid inferences for this relation) with the following matrix, where a '0' represents that the combination of row and column is prohibited:

$$M_0 = \begin{array}{c} \\ \textsc{f} \\ \textsc{i} \\ \textsc{t} \\ \textsc{nt} \\ \textsc{ni} \\ \textsc{nf} \\ \textsc{sv} \end{array} \begin{array}{c} \begin{array}{ccccccc} \textsc{f} & \textsc{i} & \textsc{t} & \textsc{nt} & \textsc{ni} & \textsc{nf} & \textsc{sv} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

One more concrete example will shed more light on the matter. Consider the *ts*-relation, as analysed before. Then, the permitted and pro-

[11] The order is arbitrary, but we choose the following convention: first, top to bottom and left to right, the mapping for standards with one element, then for standards with two elements, and finally the only standard with the three elements of the set of semantic values.

hibited combinations of mappings are represented as follows:

$$M_{ts} = \begin{array}{c} \\ \text{F} \\ \text{I} \\ \text{T} \\ \text{NT} \\ \text{NI} \\ \text{NF} \\ \text{SV} \end{array} \begin{array}{c} \begin{array}{ccccccc} \text{F} & \text{I} & \text{T} & \text{NT} & \text{NI} & \text{NF} & \text{SV} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

The relevant combinations are those that map (all) the premises to a subset of the non-false standard. So, if one of these combinations maps the conclusions to a set that shares a value with the true standard, then it is a permitted case, else it is a prohibited one. For every other case, it is a permitted case.

These matrices are representations of the consequence relations. Actually, a more precise description is to say that these matrices are representations of *just the permitted* groups of valuations or mappings for any pair of sets of propositions, for a given consequence relation.[12]

The matrix M_1 represents a consequence relation[13] for which an inference $\langle \Gamma, \Delta \rangle$ is valid if and only if, for every valuation v, there is a pair of standards $\langle S_a, S_b \rangle \in \mathscr{P}^*\mathcal{SV} \times \mathscr{P}^*\mathcal{SV}$, such that $\mathsf{map}(v, \Gamma, S_a)$ and $\mathsf{map}(v, \Delta, S_b)$. These will hold for every valuation v, given that v will always map a set of propositions to a subset of $\mathcal{SV}$ (see Fact 3.1), so it will also do it for the set of premises and the set of conclusions. Therefore every inference will be valid according to this consequence relation.

Something quite different happens with the matrix M_0, according to which an inference $\langle \Gamma, \Delta \rangle$ is valid if and only if, for every valuation v, there is no pair of standards $\langle S_a, S_b \rangle \in \mathscr{P}^*\mathcal{SV} \times \mathscr{P}^*\mathcal{SV}$, such that $\mathsf{map}(v, \Gamma, S_a)$ and $\mathsf{map}(v, \Delta, S_b)$. To put it in similar words to before,

[12] We could have given a representation for the prohibited mappings; this would exchange every '1' for a '0' and vice versa. It would not make any difference doing so, but instead of giving the notions we present in the following, it would give us their dual notions.

[13] It does for a three-valued setting. For a two-valued setting the matrix corresponding to a similar constraint for inferences, i.e. no constrains, is $M_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ because there are three non-empty standards that can interact with each other.

an inference $\langle\Gamma,\Delta\rangle$ is valid if and only if, for every valuation v, there is a pair of standards $\langle S_a, S_b\rangle \in \varnothing \times \varnothing$,[14] such that $\mathsf{map}(v,\Gamma,S_a)$ and $\mathsf{map}(v,\Delta,S_b)$. In this case, no valuation satisfies this clause, given that Fact 3.1 enters in direct contradiction with it. In consequence, there are no valid inferences for the consequence relation encoded by M_0.

For both M_1 and M_0 we have said that every valuation maps an inference to a pair of standards that belongs to some specific set of pairs (in the case of M_1 this set is $\mathscr{P}^*\mathcal{SV} \times \mathscr{P}^*\mathcal{SV}$, and in the case of M_0 it is $\varnothing \times \varnothing$). These pairs are encoded by the matrix, representing the permitted cases for an inference.

Thus, the matrix M_{ts} will also encode the pairs of standards $\langle S_a, S_b\rangle$ (one for premises, one for conclusions) that are allowed, but with a different set of pairs of standards. This is done by having a '1' in the corresponding rows and columns that would encode the permitted cases and a '0' in the corresponding rows and columns that would encode the prohibited cases. Every row in the matrix corresponds to a non-empty standard, as each column also does. So, for M_{ts}, the set of pairs of standards should have every $\langle S_a, S_b\rangle \in \mathscr{P}^*\mathcal{SV} \times \mathscr{P}^*\mathcal{SV}$ such that, either $S_a \nsubseteq \{\frac{1}{2}, 1\}$ (i.e. NF) or if $S_a \subseteq \{\frac{1}{2}, 1\}$, then $\varnothing \neq (S_b \cap \{1\})$ (i.e. S_b shares at least one element with T).

The mapping matrices are useful for characterizing a consequence relation, through the notion of *satisfaction*: for a given consequence relation X, a valuation v satisfies an inference $\Gamma \models_X \Delta$ if and only if exists a pair $\langle S_a,S_b\rangle$ such that $\mathsf{map}(v,\Gamma,S_a)$, $\mathsf{map}(v,\Delta,S_b)$, and $\langle S_a,S_b\rangle \in M_X$.

Before we give a condition for an inference to be valid according to a consequence relation, we introduce the notion of a *valuation matrix.* It will allows us to determine if every valuation satisfies a consequence relation definition.

DEFINITION 3.3. Let V be the set of valuations over a set of propositions. $M_V(\Gamma,\Delta)$ is the *valuation matrix* for the pair of sets of propositions $\langle\Gamma,\Delta\rangle$ if and only if $M_V(\Gamma,\Delta)$ is a mapping matrix and for every $\langle S_a,S_b\rangle$ in $\mathscr{P}^*\mathcal{SV} \times \mathscr{P}^*\mathcal{SV}$ and every $v \in V$, there is a v such that $\mathsf{map}(v,\Gamma,S_a)$ and $\mathsf{map}(v,\Delta,S_b)$ if and only if $\langle S_a,S_b\rangle$ is in $M_V(\Gamma,\Delta)$.

[14] Remember that we are working with non-empty standards S_a and S_b.

Consider, for instance, the inference going from the set $\Gamma = \{p, q\}$ to set $\Delta = \{p\}$. The valuation matrix $M_V(\Gamma, \Delta)$ is the following:

$$M_V(\Gamma,\Delta) = \begin{array}{c} \\ \text{F} \\ \text{I} \\ \text{T} \\ \text{NT} \\ \text{NI} \\ \text{NF} \\ \text{SV} \end{array} \begin{array}{c} \begin{array}{ccccccc} \text{F} & \text{I} & \text{T} & \text{NT} & \text{NI} & \text{NF} & \text{SV} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Notice that the columns corresponding to the standards with two or more values have only '0's; this is due to Δ having just one proposition, and therefore being impossible for it to be mapped to those standards. The same happens to the row associated to the standard with every value. Being p in both Γ and Δ, if a valuation assigns a specific value to every premise, it also assigns it to the conclusion. And so on for each of the nine possible valuations for the propositions in Γ and Δ.

Now that we know, given the set of valuations V, how to represent what standards are mapped to a given inference (a pair $\langle\Gamma, \Delta\rangle$), it would be useful to know if V respects the permitted cases determined by a consequence relation X. For doing so, we can compare the corresponding matrices, checking if the mappings represented in $M_V(\Gamma, \Delta)$ meet the mappings allowed by M_X.

Definition 3.4. Let $M_V(\Gamma, \Delta)$ be the valuation matrix of the inference from Γ to Δ, and let M_X be the mapping matrix for the consequence relation X. An inference $\Gamma \models_X \Delta$ is *valid* if and only if, for every $\langle S_a, S_b\rangle$ in $\mathscr{P}^*\mathcal{SV} \times \mathscr{P}^*\mathcal{SV}$, if $\langle S_a, S_b\rangle$ is in $M_V(\Gamma, \Delta)$, then it is in M_X.

This definition for logical validity is broader than those presented before.[15] We will analyse some applications of it in the next section. But we want to formulate first some operations between matrices that

[15] An anonymous referee asks if this should not be a result that must be proven, i.e. that this way of defining *validity* is equivalent to the conditions for an inference to be valid in a given consequence relation X. But this is not a result of the equivalence of two validity definitions. We are giving a way to define for any consequence relation X its corresponding validity conditions, by stating the relation that should obtain between the mapping matrix of X (that is M_X) and the valuation matrix of any

will help us in the analysis. The first one concerns the preceding definition; we propose an order for the matrices, for a twofold purpose: (i) to compare a valuation matrix for an inference and a consequence relation matrix, so we can know if the former valuations are represented also by the latter matrices; and (ii) to compare the permitted mappings of a consequence relation with the permitted mappings of another.

Definition 3.5. $M_x \leq M_y$ if and only if, for every $\langle S_a, S_b \rangle$ in $\mathscr{P}^*\mathcal{SV} \times \mathscr{P}^*\mathcal{SV}$, if $\langle S_a, S_b \rangle$ is in M_x, then it is in M_y, which is equivalent to say that $M_x \subseteq M_y$.

Hence, we can derive from Definition 3.4 the following:

Fact 3.2. Let $M_V(\Gamma, \Delta)$ be the valuation matrix of the inference from Γ to Δ, and let M_X be the consequence relation X matrix. An inference $\Gamma \models_X \Delta$ is valid if and only if $M_V(\Gamma, \Delta) \leq M_X$.

The proof follows directly from Definition 3.4 and Definition 3.5.

Summing up, we begin by assuming a set of semantic values that gives us a set of standards, each standard being a subset. Then, we take a matrix to be the representation of the relation that tells which combinations of mappings are allowed by a consequence relation: a mapping for premises relates to a mapping for conclusions if and only if it is permitted by the consequence relation. This gives us a binary representation in the form of a matrix. We can then build binary matrices that turn out to be Boolean matrices [8], so we can not only compare matrices, but also operate over them in a very familiar way:[16] we can obtain the conjunction and the disjunction of two matrices, and we can also obtain the negation of a matrix.[17]

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inference $\langle \Gamma, \Delta \rangle$ (that is $M_V(\Gamma, \Delta)$). In this definition, we assume that X could be represented with a mapping matrix, and then we can define its conditions for an inference being valid in X. But we are not saying that this is another way of defining validity for X. Nevertheless, for each consequence relation represented by a mapping matrix, we should be able to prove the equivalence of the corresponding instance of this definition and the validity conditions for an inference to be valid for this consequence relation.

[16] As an anonymous referee points out, the following operations and properties are derived from the fact that we are working with a Boolean algebra. However, we want to make them explicit, for clarity of the arguments.

[17] Sometimes the conjunction is called the "product" or "intersection", the disjunction, "sum" or "union", and the negation, "complement".

Definition 3.6. (Matrices conjunction)
$M_x \wedge M_y = M_x \cap M_y$

Definition 3.7. (Matrices disjunction)
$M_x \vee M_y = M_x \cup M_y$

Definition 3.8. (Matrices negation)
$\neg M = (\mathscr{P}^*\mathcal{SV} \times \mathscr{P}^*\mathcal{SV}) - M$

Given the order $\leq$ between matrices and the previous operations, we can state what follows:

Fact 3.3. For any matrix M, $M_0 \leq M \leq M_1$.

Fact 3.4. Let $M_V(\Gamma, \Delta)$ be the valuation matrix of the inference from Γ to Δ, and let M_X be the consequence relation X matrix. An inference $\Gamma \models_X \Delta$ is valid if and only if $M_V(\Gamma, \Delta) \vee M_X = M_X$.

Fact 3.5. Let $M_V(\Gamma, \Delta)$ be the valuation matrix of the inference from Γ to Δ, and let M_X be the consequence relation X matrix. An inference $\Gamma \models_X \Delta$ is valid if and only if $M_V(\Gamma, \Delta) \wedge M_X = M_V(\Gamma, \Delta)$.

We present the proof for Fact 3.4. The proof for Fact 3.5 is very similar to it.

Proof. By Fact 3.2, saying an inference $\Gamma \models_X \Delta$ is valid is equivalent to say that $M_V(\Gamma, \Delta) \leq M_X$, and then to $M_V(\Gamma, \Delta) \subseteq M_X$ (by Definition 3.5), which holds if and only if $M_V(\Gamma, \Delta) \cup M_X = M_X$, which in turn is equivalent to $M_V(\Gamma, \Delta) \vee M_X = M_X$ (by Definition 3.7). □

The re-interpretation of consequence relations, in terms of explicit constraints of permitted/prohibited mappings over premises and conclusions, gives us the possibility of analysing them under a new perspective. In the next section we will present some of the already analysed consequence relations as well as new ones. We will then exploit the mappings, in order to offer interpretations, not just for individual propositions, but for sets of propositions.

4. Common background: case analysis with mapping matrices

Some generalizations for consequence relations are easily translatable into the matrices that represent the permitted and prohibited mappings.

It will be useful to analyse first some elemental relations for the characterizations of sets of propositions.

If we continue with the three-valued setting, a case where a valuation v_a that makes true every proposition in a set Σ_a, can be stated as $\mathsf{map}(v_a, \Sigma_a, \{1\})$, and a case where a v_b that makes false every proposition in a set Σ_b, can be stated as $\mathsf{map}(v_b, \Sigma_b, \{0\})$. These are simple cases where a standard has only one value and the valuation maps every proposition to that value. Let's consider now more complex cases.

First, take a case where a valuation v_c makes non-true every proposition in Σ_c. This means that v_c makes false every proposition, or that it makes them indeterminate, or that it makes some propositions false and every other indeterminate. In "mapping" terms, $\mathsf{map}(v_c, \Sigma_c, \{0\})$ or $\mathsf{map}(v_c, \Sigma_c, \{\frac{1}{2}\})$ or $\mathsf{map}(v_c, \Sigma_c, \{0, \frac{1}{2}\})$. Any of the standards $\{0\}$, $\{\frac{1}{2}\}$ or $\{0, \frac{1}{2}\}$ has a non-true value, and a valuation that maps a set to one of them will be making non-true every proposition of the set. So, the "characterization" of the set of propositions in this case (i.e. the valuation v_c) corresponds to a set of mappings in disjunction. Notice that only one of the standards maps to a given valuation (as stated in Fact 3.1).

Next, take a case where a valuation v_d makes true some proposition of Σ_d. This means that v_d makes true every proposition, or that it makes true some of them and makes false every other proposition, or that it makes true some of them and makes the others indeterminate, or that it makes true some of them, makes false some others and makes indeterminate the rest. Every possibility corresponds to a single mapping, but the characterization corresponds to their disjunction.

Finally, something similar would happen to a valuation v_e that makes, for instance, non-true to some proposition of Σ_e. It will only ensure a value of non-true standard for a proposition, but not for all of them.

In a way, the characterizations with one single value are just special cases of the ones with many values. They can be separated into those that impose a characterization on every proposition, and those that on just some of them. The first kind will map the propositions to standards that have no values outside the indicated standard. The second kind will map the propositions to every standard that has at least one value of the indicated standard. This is captured by the following facts:

Fact 4.1. Given a valuation v, a set of propositions Σ and a standard S,

$$\forall\sigma \in \Sigma,\ v(\sigma) \in S, \text{ iff there is a } S_i \subseteq S, \text{ such that } \mathsf{map}(v, \Sigma, S_i)$$

Fact 4.2. Given a valuation v, a set of propositions Σ and a standard S,

$$\exists\sigma \in \Sigma,\ v(\sigma) \in S, \text{ iff there is a } S_i \cap S \neq \varnothing, \text{ such that } \mathsf{map}(v, \Sigma, S_i)$$

For proving both Fact 4.1 and Fact 4.2, we can define the following set A:

$$A = \{v(\sigma)|\ \sigma \in \Sigma\}$$

For it holds that for every proposition in the set, its valuation is in A, and that for every value in A, there is a proposition valued in that way:

$$\forall\sigma \in \Sigma(v(\sigma) \in A) \text{ and } \forall s \in A\ (\exists\sigma \in \Sigma(v(\sigma) = s))$$

This in turn shows that A is mapped to Σ by v (because, given the left side of the conjunction, for every proposition in Σ, there is an element from A that is equal to the valuation of the proposition).

In order to prove Fact 4.1, notice that $A \subseteq S$ if and only if $\forall\sigma \in \Sigma(v(\sigma) \in S)$, and to prove Fact 4.2, notice that $\varnothing \neq (A \cap S)$ if and only if $\exists\sigma \in \Sigma(v(\sigma) \in S)$.

The pure and mixed consequence relations, as said before, are defined by selecting a characteristic for the set of premises –every premise has a value of the premises-standard– and a characteristic for the set of conclusions –some conclusion has a value of the conclusions-standard. This is captured as a conditional clause for every valuation, which we can show as a direct consequence of Fact 4.1 and Fact 4.2:

Fact 4.3. Given a valuation v, two sets of propositions Γ and Δ, and two standards S_1 and S_2, if $\forall\gamma \in \Gamma(v(\gamma) \in S_1)$, then $\exists\delta \in \Delta(v(\delta) \in S_2)$, if and only if, if $\exists S_a \subseteq S_1(\mathsf{map}(v, \Gamma, S_a))$, then $\exists S_b \cap S_2 \neq \varnothing(\mathsf{map}(v, \Delta, S_b))$.

This result allows us to "translate" the standard clauses for pure and mixed consequence relations into mapping clauses. Fact 4.4 expresses these conditions more concisely:

Fact 4.4. Given a valuation v, two sets of propositions Γ and Δ, and two standards S_1 and S_2, if $\forall\gamma \in \Gamma(v(\gamma) \in S_1)$, then $\exists\delta \in \Delta(v(\delta) \in S_2)$, if and only if, $\exists\langle S_a, S_b\rangle$, such that if $S_a \subseteq S_1$ and $\mathsf{map}(v, \Gamma, S_a)$, then $(S_b \cap S_2) \neq \varnothing$ and $\mathsf{map}(v, \Delta, S_b)$.

The "respectable" mixed consequence relations presented in Section 2 can be presented as matrices, directly from the "translation" results:

$$M_{ss} = \begin{array}{c} \\ \textsc{f} \\ \textsc{i} \\ \textsc{t} \\ \textsc{nt} \\ \textsc{ni} \\ \textsc{nf} \\ \textsc{sv} \end{array} \begin{array}{c} \begin{array}{ccccccc} \textsc{f} & \textsc{i} & \textsc{t} & \textsc{nt} & \textsc{ni} & \textsc{nf} & \textsc{sv} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \quad M_{tt} = \begin{array}{c} \\ \textsc{f} \\ \textsc{i} \\ \textsc{t} \\ \textsc{nt} \\ \textsc{ni} \\ \textsc{nf} \\ \textsc{sv} \end{array} \begin{array}{c} \begin{array}{ccccccc} \textsc{f} & \textsc{i} & \textsc{t} & \textsc{nt} & \textsc{ni} & \textsc{nf} & \textsc{sv} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$M_{st} = \begin{array}{c} \\ \textsc{f} \\ \textsc{i} \\ \textsc{t} \\ \textsc{nt} \\ \textsc{ni} \\ \textsc{nf} \\ \textsc{sv} \end{array} \begin{array}{c} \begin{array}{ccccccc} \textsc{f} & \textsc{i} & \textsc{t} & \textsc{nt} & \textsc{ni} & \textsc{nf} & \textsc{sv} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \quad M_{ts} = \begin{array}{c} \\ \textsc{f} \\ \textsc{i} \\ \textsc{t} \\ \textsc{nt} \\ \textsc{ni} \\ \textsc{nf} \\ \textsc{sv} \end{array} \begin{array}{c} \begin{array}{ccccccc} \textsc{f} & \textsc{i} & \textsc{t} & \textsc{nt} & \textsc{ni} & \textsc{nf} & \textsc{sv} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

The order-theoretic relation is equivalent to the intersection of the pure relations ss and tt. The permitted mapping combinations for this relation, $ss{\cap}tt$, are those that are permitted by both ss and tt. The relation $ss{\cap}tt$ could be defined, then, as the conjunction of the ss-conditions and the tt-conditions for a mapping to be allowed, which can be done with previously-defined conjunction for matrices (Definition 3.6).

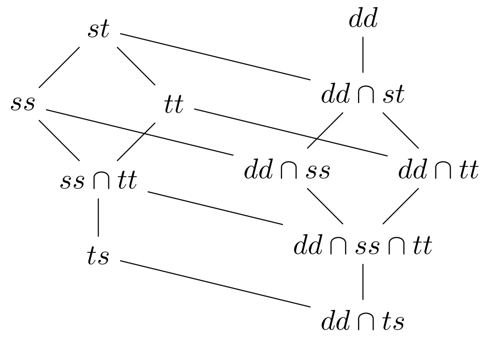
$$M_{ss\cap tt} = \begin{array}{c} \\ \textsc{f} \\ \textsc{i} \\ \textsc{t} \\ \textsc{nt} \\ \textsc{ni} \\ \textsc{nf} \\ \textsc{sv} \end{array} \begin{array}{c} \begin{array}{ccccccc} \textsc{f} & \textsc{i} & \textsc{t} & \textsc{nt} & \textsc{ni} & \textsc{nf} & \textsc{sv} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array} = M_{ss} \wedge M_{tt}$$

If we compare the matrices, $M_{ss\cap tt}$ has a distinctive feature: it restricts differently the mappings for premises, because it does not attend to a particular standard for premises, but it looks at the relationship between the characterization for both premises and conclusions. For instance, as in ss and tt, $ss \cap tt$ will allow any valuation that makes false one or more premises. For non-false premises, if any of them receives the value $\frac{1}{2}$, it will allow any valuation unless it makes it false every conclusion. And if every premise is true, it will allow just valuations that make true one or more conclusions.

Following this translation procedure, we can also give the matrices for any mixed consequence relation and for any intersection of them. Take for example, relations built from the standard d, linked to the *being-determinate* characterization.

$$M_{dd} = \begin{array}{c} \\ \text{F} \\ \text{I} \\ \text{T} \\ \text{NT} \\ \text{NI} \\ \text{NF} \\ \text{SV} \end{array} \begin{array}{c} \begin{array}{ccccccc} \text{F} & \text{I} & \text{T} & \text{NT} & \text{NI} & \text{NF} & \text{SV} \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \quad M_{dd\cap st} = \begin{array}{c} \\ \text{F} \\ \text{I} \\ \text{T} \\ \text{NT} \\ \text{NI} \\ \text{NF} \\ \text{SV} \end{array} \begin{array}{c} \begin{array}{ccccccc} \text{F} & \text{I} & \text{T} & \text{NT} & \text{NI} & \text{NF} & \text{SV} \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

At this point, an interesting question arises, regarding the order of these relations and others: which relationships are more restrictive and which more permissive? The more permissive relations are those that allow more standards combinations as permitted cases. So, the order between two arbitrary relations is the inclusion-order between the corresponding mapping constraints. We know the order for the "respectable" relations, and we can easily show, as an example, the order involving the relation dd (just looking at their matrices, representing the standards combinations that belong to each mapping constraint):



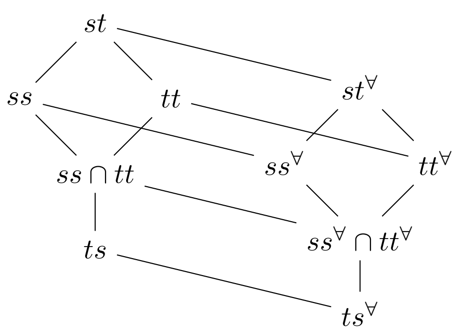
We can easily see that *dd* produces a *contra*logic with respect to those that each of the "respectable" relations also produces: there are some inferences from Γ_1 to Δ_1 such that $\Gamma_1 \models_{st} \Delta_1$ but $\Gamma_1 \not\models_{dd} \Delta_1$ and there are some inferences from Γ_2 to Δ_2 such that $\Gamma_2 \models_{dd} \Delta_2$ but $\Gamma_2 \not\models_{st} \Delta_2$ (there are inferences that a logic based on one relation makes valid while the other logic, based on the other relation, does not, and vice versa). Moreover, the intersection of *dd* and any of those relations is a sublogic of *dd*; its intersections even reflect the order of the "respectable" relations.

In the same way that we can vary the standards for premises or for conclusions, we have mentioned an alternative for the disjunctive/existential reading of the conclusions. Instead of interpreting an inference as stating that whenever the premises satisfy a given standard for premises, some conclusions will satisfy the standard for conclusions, we could read it as stating that *every* conclusion will satisfy the standard for conclusions. The clause for this reading will have a universal quantifier on each side of the conditional definition, and could be more restrictive than the "existential" reading. The matrix representation for the "respectable" mixed consequence relations, under the "universal" reading, would be the following:

$$M_{ss^\forall} = \begin{array}{c} \\ \text{F} \\ \text{I} \\ \text{T} \\ \text{NT} \\ \text{NI} \\ \text{NF} \\ \text{SV} \end{array} \begin{array}{c} \begin{array}{ccccccc} \text{F} & \text{I} & \text{T} & \text{NT} & \text{NI} & \text{NF} & \text{SV} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \qquad M_{tt^\forall} = \begin{array}{c} \\ \text{F} \\ \text{I} \\ \text{T} \\ \text{NT} \\ \text{NI} \\ \text{NF} \\ \text{SV} \end{array} \begin{array}{c} \begin{array}{ccccccc} \text{F} & \text{I} & \text{T} & \text{NT} & \text{NI} & \text{NF} & \text{SV} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$M_{st^\forall} = \begin{array}{c} \\ \text{F} \\ \text{I} \\ \text{T} \\ \text{NT} \\ \text{NI} \\ \text{NF} \\ \text{SV} \end{array} \begin{array}{c} \begin{array}{ccccccc} \text{F} & \text{I} & \text{T} & \text{NT} & \text{NI} & \text{NF} & \text{SV} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \quad M_{ts^\forall} = \begin{array}{c} \\ \text{F} \\ \text{I} \\ \text{T} \\ \text{NT} \\ \text{NI} \\ \text{NF} \\ \text{SV} \end{array} \begin{array}{c} \begin{array}{ccccccc} \text{F} & \text{I} & \text{T} & \text{NT} & \text{NI} & \text{NF} & \text{SV} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

These consequence relations are ordered in the same way as their "existential" counterparts and every "universal" interpretation is included in the corresponding "existential" interpretations. The most restrictive consequence relation is $ts^\forall$ and the most permissive is st:



Notice that these consequence relations are not comparable to those presented before, related to the consequence relation dd. We could obtain the "universal" reading for those, and it would complete the picture: every "universal" interpretation would be below its "existential" interpretation, and every consequence relation intersecting with dd (or $dd^\forall$) would be above the intersection.

Finally, there is another interesting example, concerning the interpretations of multiple propositions, specially regarding an "existential" interpretation, not for conclusions, but for premises. Imagine we are interested in a set of propositions and the conclusions we can draw *whenever at least one premise is true.* We can think of this consequence relation as resisting any falsehood, as long as some premise is true. Or we can imagine a consequence relation that tells us what we can draw from a set of propositions that contains at least one false premise. Let's call the former ${}^{\exists}xy$ and the latter ${}^{\exists}fx$ (where x could be s or t, depending on whether it is a standard for *being-strictly-true* or for *being-tolerantly-true*); below we show the matrices for the first class of relationships:

$$M_{\exists_{ss}} = \begin{array}{c} \\ \text{F} \\ \text{I} \\ \text{T} \\ \text{NT} \\ \text{NI} \\ \text{NF} \\ \text{SV} \end{array} \begin{array}{c} \begin{array}{ccccccc} \text{F} & \text{I} & \text{T} & \text{NT} & \text{NI} & \text{NF} & \text{SV} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array} \quad M_{\exists_{tt}} = \begin{array}{c} \\ \text{F} \\ \text{I} \\ \text{T} \\ \text{NT} \\ \text{NI} \\ \text{NF} \\ \text{SV} \end{array} \begin{array}{c} \begin{array}{ccccccc} \text{F} & \text{I} & \text{T} & \text{NT} & \text{NI} & \text{NF} & \text{SV} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$M_{\exists_{st}} = \begin{array}{c} \\ \text{F} \\ \text{I} \\ \text{T} \\ \text{NT} \\ \text{NI} \\ \text{NF} \\ \text{SV} \end{array} \begin{array}{c} \begin{array}{ccccccc} \text{F} & \text{I} & \text{T} & \text{NT} & \text{NI} & \text{NF} & \text{SV} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \quad M_{\exists_{ts}} = \begin{array}{c} \\ \text{F} \\ \text{I} \\ \text{T} \\ \text{NT} \\ \text{NI} \\ \text{NF} \\ \text{SV} \end{array} \begin{array}{c} \begin{array}{ccccccc} \text{F} & \text{I} & \text{T} & \text{NT} & \text{NI} & \text{NF} & \text{SV} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

As before, we can tell the order between these consequence relations and those with a "universal" interpretation for premises, by looking at the matrices: the "existential" interpretations are below the "universal" interpretation.

We will stop here because the number of consequence relations is immense though not all of them are of interest. There are, in a three-valued setting, 7 different non-empty standards, giving 49 different combinations of mappings for premises and conclusions. Each combination could be permitted or not. Therefore, there are 2^{49} different sets of permitted mappings (around 0.56×10^{15}), hence 2^{49} matrices defining them (including M_0 and M_1), linked to a consequence relation.[18]

[18] If we were not allowed to have multiple conclusions, our matrices would have just three columns, and we would be able to define 2^{21} relations (around 2.1×10^{6}). In this case, many of the above consequence relations would collapse into the same relation: we just have to consider the corresponding columns.

If we are permissive with the standards and their combinations for mixed relations, we can obtain 49 relations,[19] but we can give them different interpretations for their multiplicity of premises and conclusions: we can have an "existential" interpretation and a "universal" interpretation for any set of propositions. We can then obtain nearby 200 consequence relations. And we have not even considered their intersections yet!

This procedure works for translating the satisfaction conditions of a consequence relation to a matrix. But we can define a consequence relation by giving its corresponding matrix, which is equivalent to "list" in a disjunction the permitted mappings, i.e. the satisfaction conditions for an inference. There are 49 different matrices in which there exists only one element of the matrix equal to '1' (only one combination of mappings is permitted), but successive unions of these matrices give us every possible matrix (apart from M_0).

Before ending, we will like to address one of the criticisms that an anonymous referee has made to us. She says that our way of representing consequence relations scales badly as the number of truth-values increases, and that drawing matrices and comparing them can become a work that is difficult and error-prone. We recognize that she is right about the increasing difficulty as more truth-values became part of the picture. Nevertheless, on the one hand, our main interest is not *computational*. Our focus is not on how this way to represent logics—which we consider even more general than most of the semantic ways to define a consequence relation—can be run in a program, but to even define a schema as general as possible in order not only to represent mixed consequence relations, but also non-mixed consequence relations, as some intersective-mixed, order-theoretic and even multi-conclusion consequence relations with a universal reading for conclusions, just to mention that ones that we have explicitly talked about in this article. It is in this *philosophical* sense, that we find our approach superior to the others that we have discussed. And on the other hand, Ariel Roffé and Joaquín Toranzo Calderón have developed a software package [20] that determines whether a given inference is valid or not, whether a formula or inference is satisfied by a given valuation, and, for example, which valuations satisfy a given formula or inference, in any logic that is specified through a language and with a consequence relation defined through a matrix.

[19] See [15] for an analysis of these relations.

5. Conclusions

To sum up, we have presented a new and broader generalization for consequence relations that serves as a common ground for the analysis of a plethora of logics, including those with every mixed consequence relation and every intersective mixed consequence relation—thus being even more general than the analysis provided by [3]. This includes the "respectable" consequence relations, but also some other "non-respectable" relations such as *dd* and its intersections (as explored by [22]).

Moreover, we have shown how this approach deals with the usual existential reading for multiple conclusions, but also with the universal reading, as in [5]. It even allows the existential reading for multiple premises, and its combination with both existential and universal readings for multiple conclusions. This can be extended to the previous relations and their interactions.

Finally, every consequence relation presented in this work can be obtained as the result of some basic Boolean operations between the consequence relation matrices. With the "simplest" matrices relations—i.e. those with just one standards combination—, we can build up many others with successive unions.

Acknowledgments. We thank the members of the Buenos Aires Logic Group, as well as two anonymous referees, for their comments and constructive criticisms on earlier versions of this paper. While writing this paper, Federico Pailos enjoyed a Humboldt Research Fellowship for experienced researchers (March 2020 to July 2021). Joaquín Toranzo Calderón participates in the research project PUE-2020/1672 22920200100005CO (CONICET, Argentina). This research was also supported by CONICET and the University of Buenos Aires.

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