



Indefinite least-squares problems and pseudo-regularity



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ABSTRACT

Given two Krein spaces \mathcal{H} and \mathcal{K} , a (bounded) closed-range operator $C : \mathcal{H} \rightarrow \mathcal{K}$ and a vector $y \in \mathcal{K}$, the indefinite least-squares problem consists in finding those vectors $u \in \mathcal{H}$ such that

$$[Cu - y, Cu - y] = \min_{x \in \mathcal{H}} [Cx - y, Cx - y].$$

The indefinite least-squares problem has been thoroughly studied before under the assumption that the range of C is a uniformly J -positive subspace of \mathcal{K} . Along this article the range of C is only supposed to be a J -nonnegative pseudo-regular subspace of \mathcal{K} . This work is devoted to present a description for the set of solutions of this abstract problem in terms of the family of J -normal projections onto the range of C .

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1. Introduction

In signal processing applications it is frequently assumed that the mathematical model, describing the physical phenomena under study, satisfies the following equation:

$$z = Hx + \eta,$$

where $H \in \mathbb{R}^{m \times n}$ is known and $x \in \mathbb{R}^n$ is a parameter that needs to be determined. Sometimes, due to physical restrictions, it is not possible to measure x , and it is necessary to estimate this vector based on

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the measurement z , which is corrupted by noise η . According to the characteristics of the noise, different techniques may be used to estimate x . For instance, when no statistical information about the noise measurement is available, the \mathcal{H}^∞ -estimation technique has been proved to be an appropriate approach for several engineering problems. Given $\gamma > 0$, the \mathcal{H}^∞ -estimation technique in \mathbb{R}^n consists in finding an estimation \hat{x} of the vector x , such that:

$$\max_{x \in \mathbb{R}^n} \frac{\|x - \hat{x}\|^2}{\|z - Hx\|^2} \leq \gamma^2, \quad (1.1)$$

or equivalently,

$$\min_{x \in \mathbb{R}^n} \left(\|z - Hx\|^2 - \frac{1}{\gamma^2} \|x - \hat{x}\|^2 \right) \geq 0. \quad (1.2)$$

Note that the left hand side of (1.2) can be modeled as the minimization of an indefinite inner product on an affine manifold. In fact, \mathbb{R}^{m+n} can be endowed with the indefinite inner product $[x, y] := x^T J y$, $x, y \in \mathbb{R}^{m+n}$, where $J \in L(\mathbb{R}^{m+n})$ is the fundamental symmetry given by $J = \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}$. Then, considering $C := \begin{pmatrix} H \\ \gamma^{-1} I_n \end{pmatrix} \in L(\mathbb{R}^n, \mathbb{R}^{m+n})$ and $y := \begin{pmatrix} z \\ \gamma^{-1} \hat{x} \end{pmatrix} \in \mathbb{R}^{m+n}$, the \mathcal{H}^∞ -estimation problem is equivalent to finding a vector x (which depends on z) such that the following indefinite least-squares problem (ILSP) admits a solution:

$$\min_{x \in \mathbb{R}^n} [y - Cx, y - Cx], \quad (1.3)$$

and to show that this minimum is nonnegative, see [8].

This work is devoted to studying an abstract ILSP: Given arbitrary Krein spaces \mathcal{H} and \mathcal{K} , a closed-range operator $C \in L(\mathcal{H}, \mathcal{K})$ and a vector $y \in \mathcal{K}$, find the vectors $u \in \mathcal{H}$ such that

$$[y - Cu, y - Cu] = \min_{x \in \mathcal{H}} [y - Cx, y - Cx].$$

In finite-dimensional spaces, the ILSP has been exhaustively studied see e.g. [13,14,21,8,15,20,7]. In these papers, if J is the fundamental symmetry of \mathcal{K} , it is assumed that $C^T J C$ is a positive-definite matrix, which is a sufficient condition for the existence of a unique solution for the ILSP. This is equivalent to assuming that C is injective and the range of C (hereafter denoted by $R(C)$) is a uniformly J -positive subspace of \mathcal{K} . Then, the regularity of $R(C)$ plays an essential role, since it guarantees the existence of a J -selfadjoint projection onto $R(C)$, which determines the unique solution of the ILS problem (1.3).

Even for the general setting it is known that the ILSP admits a solution if and only if $R(C)$ is J -nonnegative and $y \in R(C) + R(C)^{\perp}$, see e.g. [6, Thm. 8.4]. Then, the ILSP is well-posed only for the vectors y in the (not necessarily closed) subspace $R(C) + R(C)^{\perp}$. Moreover, given $y \in R(C) + R(C)^{\perp}$, $u \in \mathcal{H}$ is a solution of the ILSP if and only if $y - Cu \in R(C)^{\perp}$ (see Lemma 3.1), i.e. if u is a solution of the normal equation associated to $Cx = y$:

$$C^\#(Cx - y) = 0,$$

where $C^\#$ stands for the J -adjoint operator of C .

The assumption that $R(C)$ is a uniformly J -positive subspace of \mathcal{K} implies that the ILSP is properly defined for every $y \in \mathcal{K}$, but this is a quite restrictive condition. Along this article (most of the time) it is assumed that $R(C)$ is a J -nonnegative pseudo-regular subspace of \mathcal{K} . Thus, the ILSP admits solutions for

every vector in the (proper) closed subspace $R(C) + R(C)^{\perp}$. The pseudo-regularity of $R(C)$ is equivalent to the closedness of $R(C\#C)$, see Lemma 3.4. Hence, under this assumption, the Moore–Penrose inverse $(C\#C)^\dagger$ of $C\#C$ is bounded and the solutions of the normal equation, and therefore of the ILSP, are exactly those

$$u \in u_y + N(C\#C),$$

where $u_y = (C\#C)^\dagger C\#y$ is the unique solution in $N(C\#C)^\perp$.

It is also worthy to mention that if \mathcal{K} is a Pontryagin space (i.e. $\kappa := \min\{\dim \mathcal{K}_+, \dim \mathcal{K}_-\} < \infty$ for any fundamental decomposition $\mathcal{K} = \mathcal{K}_+ [+] \mathcal{K}_-$) then every closed subspace turns out to be pseudo-regular. Therefore, in this case the assumption reduces to assume that $R(C)$ is just J -nonnegative.

Another advantage of considering an operator C with pseudo-regular range is that there is a family of J -normal projections onto $R(C)$. These projections, which have been previously studied in [19], are the main technical tool used along this work in order to characterize the set of solutions of the ILSP.

The article is organized as follows: Section 2 introduces the notation and terminology used along. It also contains some preliminaries on Krein spaces, mainly on pseudo-regularity and J -normal projections.

The indefinite least-squares problem is described in Section 3. After a brief reminder of the state of the art of the problem, it is studied under the assumption that the range of C is a J -nonnegative pseudo-regular subspace of \mathcal{K} . Also, some considerations are made in order to compare the ILSP associated to $Cx = y$ and the ILSP associated to another equation $C'x = y$, where C' is a closed-range operator such that $R(C')$ is a uniformly J -positive subspace of $R(C)$.

Until this point the Krein space structure of \mathcal{H} , the domain of C , was unnecessary. However, Section 4 is devoted to consider a minimization problem among the indefinite least-squares solutions of $Cx = y$. A minimal least-squares solution (MILSS) of $Cx = y$ is a vector $w \in u_y + N(C\#C)$ such that

$$[w, w] = \min_{u \in u_y + N(C\#C)} [u, u].$$

If the ILSP associated to $Cx = y$ admits solutions, in order to guarantee the existence of a MILSS of $Cx = y$ it is necessary and sufficient that $N(C\#C)$ is J -nonnegative and that the affine manifold $u_y + N(C\#C)$ intersects $N(C\#C)^{\perp}$, see Proposition 4.1. If it is also assumed that $N(C\#C)$ and $R(C)$ are pseudo-regular subspaces of \mathcal{H} and \mathcal{K} , respectively, then the set of MILSS can be computed in terms of the J -normal projections onto these subspaces and the Moore–Penrose inverse of C , see Theorem 4.3.

Finally, in Section 5 the operators used in Theorem 4.3 to describe the MILSS of $Cx = y$ are shown to be a family of generalized inverses of a fixed operator C' with regular range.

2. Preliminaries

Along this work \mathcal{H} denotes a complex (separable) Hilbert space. If \mathcal{K} is another Hilbert space then $L(\mathcal{H}, \mathcal{K})$ is the algebra of bounded linear operators from \mathcal{H} into \mathcal{K} and $L(\mathcal{H}) = L(\mathcal{H}, \mathcal{H})$.

If $T \in L(\mathcal{H}, \mathcal{K})$ then $R(T)$ stands for its range and $N(T)$ for its nullspace.

Given two closed subspaces \mathcal{S} and \mathcal{T} of a Hilbert space \mathcal{H} , $\mathcal{S} \dot{+} \mathcal{T}$ denotes the direct sum of them. Moreover, $\mathcal{S} \oplus \mathcal{T}$ stands for their (direct) orthogonal sum and $\mathcal{S} \ominus \mathcal{T} := \mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^\perp$.

If $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}$, $P_{\mathcal{S} // \mathcal{T}}$ denotes the (unique, bounded) projection onto \mathcal{S} along \mathcal{T} . In the particular case of $\mathcal{T} = \mathcal{S}^\perp$, the orthogonal projection onto \mathcal{S} is denoted by $P_{\mathcal{S}}$.

In what follows we present the standard notation and some basic results on Krein spaces. For a complete exposition on the subject see [6,2,1].

Given a Krein space $(\mathcal{H}, [\ , \])$ with a fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \dot{+} \mathcal{H}_-$, the direct (orthogonal) sum of the Hilbert spaces $(\mathcal{H}_+, [\ , \])$ and $(\mathcal{H}_-, -[\ , \])$ is denoted by $(\mathcal{H}, \langle \ , \ \rangle)$.

Observe that the inner products of \mathcal{H} are related by means of a *fundamental symmetry*, i.e. a unitary selfadjoint operator $J \in L(\mathcal{H})$ which satisfies:

$$[x, y] = \langle Jx, y \rangle, \quad x, y \in \mathcal{H}.$$

If \mathcal{H} and \mathcal{K} are Krein spaces, $L(\mathcal{H}, \mathcal{K})$ stands for the vector space of linear transformations which are bounded with respect to the associated Hilbert spaces $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ and $(\mathcal{K}, \langle \cdot, \cdot \rangle_{\mathcal{K}})$. Given $T \in L(\mathcal{H}, \mathcal{K})$, the J -adjoint operator of T is defined by $T^{\#} = J_{\mathcal{H}}T^*J_{\mathcal{K}}$, where $J_{\mathcal{H}}$ and $J_{\mathcal{K}}$ are the fundamental symmetries associated to \mathcal{H} and \mathcal{K} , respectively. An operator $T \in L(\mathcal{H})$ is J -selfadjoint if $T = T^{\#}$.

A vector $x \in \mathcal{H}$ is J -positive if $[x, x] > 0$. A subspace \mathcal{S} of \mathcal{H} is J -positive if every $x \in \mathcal{S}$, $x \neq 0$, is a J -positive vector. J -nonnegative, J -neutral, J -negative and J -nonpositive vectors and subspaces are defined analogously.

Given a subspace \mathcal{S} of a Krein space \mathcal{H} , the J -orthogonal subspace to \mathcal{S} is defined by

$$\mathcal{S}^{[\perp]} = \{x \in \mathcal{H} : [x, s] = 0, \text{ for every } s \in \mathcal{S}\}.$$

The isotropic part of \mathcal{S} , $\mathcal{S}^{\circ} := \mathcal{S} \cap \mathcal{S}^{[\perp]}$ can be a non-trivial subspace. It holds that

$$\mathcal{H} = \overline{\mathcal{S} + \mathcal{S}^{[\perp]}} \oplus J(\mathcal{S}^{\circ}),$$

see [2, Prop. 1.7.6]. A subspace \mathcal{S} of \mathcal{H} is J -non-degenerated if $\mathcal{S} \cap \mathcal{S}^{[\perp]} = \{0\}$. Otherwise, it is a J -degenerated subspace of \mathcal{H} .

A (closed) subspace \mathcal{S} of \mathcal{H} is *regular* if $\mathcal{S} \dot{+} \mathcal{S}^{[\perp]} = \mathcal{H}$. Equivalently, \mathcal{S} is regular if and only if there exists a (unique) J -selfadjoint projection E onto \mathcal{S} , see e.g. [2, Thm. 1.7.16].

On the other hand, a closed subspace \mathcal{S} of \mathcal{H} is called *pseudo-regular* if the algebraic sum $\mathcal{S} + \mathcal{S}^{[\perp]}$ is closed. Equivalently, \mathcal{S} is pseudo-regular if there exists a regular subspace \mathcal{M} such that $\mathcal{S} = \mathcal{S}^{\circ} [\dot{+}] \mathcal{M}$, where $[\dot{+}]$ stands for the J -orthogonal direct sum of the subspaces, see [9].

The importance of pseudo-regular subspaces lies in the fact that they enable to generalize some Pontryagin spaces arguments to general Krein spaces. They have also been used as a technical tool for the study of spectral functions (and distributions) for particular classes of operators in Krein spaces [10,11,17,18,22] and to extend the Beurling–Lax theorem for shifts in indefinite metric spaces [3,4].

Also, \mathcal{S} is pseudo-regular if and only if \mathcal{S} is the range of a J -normal projection, i.e. if there exists a projection $Q \in L(\mathcal{H})$ with $R(Q) = \mathcal{S}$ such that $QQ^{\#} = Q^{\#}Q$, see [19, Thm. 4.3]. In particular, given a pseudo-regular subspace \mathcal{S} , $Q_0 = P_{\mathcal{S} // \mathcal{S}^{[\perp]} \ominus \mathcal{S}^{\circ} + J(\mathcal{S}^{\circ})}$ is a J -normal projection onto \mathcal{S} . However, if $\mathcal{S}^{\circ} \neq \{0\}$ then there are infinitely many J -normal projections Q satisfying $R(Q) = \mathcal{S}$. In what follows, $\mathcal{Q}_{\mathcal{S}}$ stands for the set of J -normal projections onto the pseudo-regular subspace \mathcal{S} , i.e.

$$\mathcal{Q}_{\mathcal{S}} = \{Q \in L(\mathcal{H}) : Q^2 = Q, QQ^{\#} = Q^{\#}Q \text{ and } R(Q) = \mathcal{S}\}.$$

The next is a technical remark that will be frequently used along this work. It shows that, given a vector $y \in \mathcal{S} + \mathcal{S}^{[\perp]}$, the J -normal projections onto \mathcal{S} provide the different decompositions of y as a sum of a vector in \mathcal{S} and a vector in $\mathcal{S}^{[\perp]}$, i.e. if $Q \in \mathcal{Q}_{\mathcal{S}}$ then

$$y = Qy + (I - Q)y, \quad \text{where } Qy \in \mathcal{S} \quad \text{and} \quad (I - Q)y \in \mathcal{S}^{[\perp]}.$$

Remark 2.1. If \mathcal{S} is a pseudo-regular subspace of \mathcal{H} and $y \in \mathcal{S} + \mathcal{S}^{[\perp]}$, given any $Q \in \mathcal{Q}_{\mathcal{S}}$, then

$$Q^{\#}(I - Q)y = 0.$$

Indeed, if $P = Q(I - Q)^\#$ then $R(P) = \mathcal{S} \cap N(Q^\#) = \mathcal{S} \cap \mathcal{S}^{[\perp]} = \mathcal{S}^\circ$ and $N(P^\#) = R(P)^{[\perp]} = (\mathcal{S}^\circ)^{[\perp]} = \mathcal{S} + \mathcal{S}^{[\perp]}$. Therefore, if $y \in \mathcal{S} + \mathcal{S}^{[\perp]}$ then $Q^\#(I - Q)y = P^\#y = 0$. In particular, $(I - Q)y \in N(Q^\#) = R(Q)^{[\perp]} = \mathcal{S}^{[\perp]}$.

The following results belong to [19]. Their statements are included in order to make the paper self-contained.

Proposition 2.2. *A bounded projection Q acting on \mathcal{H} is J -normal if and only if there exist a J -selfadjoint projection $E \in L(\mathcal{H})$ and a projection $P \in L(\mathcal{H})$ satisfying $PP^\# = P^\#P = 0$ such that*

$$Q = E + P.$$

The projections E and P are uniquely determined by Q . More precisely, $E = QQ^\#$ and $P = Q(I - Q^\#)$.

Projections $P \in L(\mathcal{H})$ satisfying $PP^\# = P^\#P = 0$ were previously considered in [17,11], in connection with neutral dual companions. If \mathcal{S} is a fixed (closed) J -neutral subspace of \mathcal{H} , a *neutral dual companion* of \mathcal{S} is another (closed) J -neutral subspace \mathcal{T} of \mathcal{H} such that $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}^{[\perp]}$ holds. If \mathcal{T} is a neutral dual companion of \mathcal{S} then also $\mathcal{H} = \mathcal{T} \dot{+} \mathcal{S}^{[\perp]}$ holds. So, the pair of subspaces $(\mathcal{S}, \mathcal{T})$ is called a *neutral dual pair*. Note that in this case $\mathcal{S} \dot{+} \mathcal{T}$ is a regular subspace of \mathcal{H} .

A J -neutral subspace \mathcal{N} of \mathcal{H} is said to be a *hypermaximal J -neutral* subspace if it is simultaneously both maximal J -nonnegative and maximal J -nonpositive. Equivalently, \mathcal{N} is a hypermaximal J -neutral subspace if and only if $\mathcal{N} = \mathcal{N}^{[\perp]}$, see [2, Prop. 1.4.19].

Given $C \in L(\mathcal{H}, \mathcal{K})$, its restriction $C|_{N(C)^\perp} : N(C)^\perp \rightarrow R(C)$ admits a linear inverse $(C|_{N(C)^\perp})^{-1} : R(C) \rightarrow N(C)^\perp$. Then, the Moore–Penrose inverse of C is the linear operator $C^\dagger : R(C) + R(C)^\perp \rightarrow \mathcal{H}$ defined by

$$C^\dagger y = \begin{cases} (C|_{N(C)^\perp})^{-1}y & \text{if } y \in R(C); \\ 0 & \text{if } y \in R(C)^\perp. \end{cases}$$

Note that C^\dagger is densely-defined on \mathcal{K} , and it is well-known that $C^\dagger \in L(\mathcal{K}, \mathcal{H})$ if and only if $R(C)$ is closed.

Hereafter, given two Hilbert spaces \mathcal{H} and \mathcal{K} , let $CR(\mathcal{H}, \mathcal{K})$ denotes the set of bounded closed-range operators from \mathcal{H} into \mathcal{K} . The following are some properties of the Moore–Penrose inverse of a closed-range operator:

Proposition 2.3. *Given $C \in CR(\mathcal{H}, \mathcal{K})$,*

1. $CC^\dagger = P_{R(C)}$ and $C^\dagger C = P_{N(C)^\perp}$, the orthogonal projections onto $R(C)$ and $N(C)^\perp$, respectively. In particular, $CC^\dagger C = C$ and $C^\dagger CC^\dagger = C^\dagger$.
2. $C^* \in CR(\mathcal{K}, \mathcal{H})$ and $(C^*)^\dagger = (C^\dagger)^*$.
3. If $U \in L(\mathcal{K}), V \in L(\mathcal{H})$ are unitary operators, then $(UCV)^\dagger = V^*C^\dagger U^*$.

The Moore–Penrose inverse has been thoroughly studied along the years, see e.g. [5] for a complete exposition on this subject.

As a consequence of Proposition 2.3, if \mathcal{H} and \mathcal{K} are two Krein spaces and $C \in CR(\mathcal{H}, \mathcal{K})$ then $C^\# \in CR(\mathcal{K}, \mathcal{H})$ and $(C^\#)^\dagger = (C^\dagger)^\#$.

3. Indefinite least-squares problems

Along this work, the following indefinite least-squares problem is considered: Let \mathcal{H} and \mathcal{K} be two Krein spaces with fundamental symmetries $J_{\mathcal{H}}$ and $J_{\mathcal{K}}$, respectively. Given an operator $C \in CR(\mathcal{H}, \mathcal{K})$ and a vector $y \in \mathcal{K}$, find $u \in \mathcal{H}$ such that

$$[y - Cu, y - Cu]_{\mathcal{K}} = \min_{x \in \mathcal{H}} [y - Cx, y - Cx]_{\mathcal{K}}. \tag{3.1}$$

The next lemma shows necessary and sufficient conditions for the existence of indefinite least-squares solutions (ILSS) of the equation $Cx = y$. A proof can be found in [6, Theorem 8.4] or in [12, Lemma 3.1].

Lemma 3.1. *Let $C \in CR(\mathcal{H}, \mathcal{K})$ and $y \in \mathcal{K}$. Then, $u \in \mathcal{H}$ is an ILSS of the equation $Cx = y$ if and only if $R(C)$ is $J_{\mathcal{K}}$ -nonnegative and $y - Cu \in R(C)^{\perp\perp}$.*

Hence, in order to have a well-posed indefinite least-squares problem it is necessary that $y \in R(C) + R(C)^{\perp\perp}$. Note that the set of admissible points $R(C) + R(C)^{\perp\perp}$ is always dense in $(R(C)^{\circ})^{\perp\perp}$.

Proposition 3.2. *Let $C \in CR(\mathcal{H}, \mathcal{K})$. Then, $Cx = y$ admits an ILSS for every $y \in (R(C)^{\circ})^{\perp\perp}$ if and only if $R(C)$ is a $J_{\mathcal{K}}$ -nonnegative pseudo-regular subspace of \mathcal{K} .*

Proof. Note that $Cx = y$ admits an ILSS for every $y \in (R(C)^{\circ})^{\perp\perp}$ if and only if $(R(C)^{\circ})^{\perp\perp} \subseteq R(C) + R(C)^{\perp\perp}$ and $R(C)$ is $J_{\mathcal{K}}$ -nonnegative. But

$$(R(C)^{\circ})^{\perp\perp} = \overline{R(C) + R(C)^{\perp\perp}},$$

and the equivalence follows. \square

In particular, $Cx = y$ admits an ILSS for every $y \in \mathcal{K}$ if and only if $R(C)$ is a uniformly J -positive subspace of \mathcal{K} , see also [12, Proposition 3.2].

Before describing the indefinite least-squares solutions of $Cx = y$, observe that the minimum value of $L(x) = [y - Cx, y - Cx]$, $x \in \mathcal{H}$, is attained at the projections (by means of normal projectors) of y onto $R(C)$.

Lemma 3.3. *Given $C \in CR(\mathcal{H}, \mathcal{K})$ such that $R(C)$ is a J -nonnegative pseudo-regular subspace of \mathcal{K} and $y \in R(C) + R(C)^{\perp\perp}$,*

$$\min_{x \in \mathcal{H}} [y - Cx, y - Cx] = [(I - Q)y, (I - Q)y],$$

where $Q \in L(\mathcal{K})$ is any J -normal projection onto $R(C)$.

Proof. Since $R(C)$ is pseudo-regular, by [19, Thm. 4.3] there exists a J -normal projection $Q \in L(\mathcal{K})$ onto $R(C)$. Then, for any $x \in \mathcal{H}$,

$$\begin{aligned} [y - Cx, y - Cx] &= [(y - Qy) + (Qy - Cx), (y - Qy) + (Qy - Cx)] \\ &= [(I - Q)y, (I - Q)y] + 2 \operatorname{Re}[(I - Q)y, Qy - Cx] + [Qy - Cx, Qy - Cx] \\ &\geq [(I - Q)y, (I - Q)y] + 2 \operatorname{Re}[(I - Q)y, Qy - Cx], \end{aligned} \tag{3.2}$$

because $Qy - Cx \in R(C)$ which is a $J_{\mathcal{K}}$ -nonnegative subspace. Furthermore, by Remark 2.1, $y \in R(C) + R(C)^{\perp}$ implies that $Q^{\#}(I - Q)y = 0$ and

$$[(I - Q)y, Qy - Cx] = [(I - Q)y, Q(y - Cx)] = [Q^{\#}(I - Q)y, y - Cx] = 0.$$

Therefore,

$$[y - Cx, y - Cx] \geq [(I - Q)y, (I - Q)y]. \quad \square$$

Also, note that the pseudo-regularity of $R(C)$ is equivalent to the boundedness of the Moore–Penrose inverse of $C^{\#}C$:

Lemma 3.4. *Given $C \in CR(\mathcal{H}, \mathcal{K})$, $R(C)$ is pseudo-regular if and only if $R(C^{\#}C)$ is closed.*

Proof. Since $R(C)$ is closed, note that $R(C^{\#}C)$ is closed if and only if $R(C) + N(C^{\#}) = R(C) + R(C)^{\perp}$ is closed, see [16, Corollary 2.5]. Thus, $R(C^{\#}C)$ is closed if and only if $R(C)$ is a pseudo-regular subspace of \mathcal{K} . \square

Given $C \in CR(\mathcal{H}, \mathcal{K})$ and $y \in R(C) + R(C)^{\perp}$, observe that $C^{\#}y \in R(C^{\#}C)$. Then,

$$u_y := (C^{\#}C)^{\dagger}C^{\#}y, \tag{3.3}$$

is a solution of the normal equation:

$$C^{\#}(Cx - y) = 0. \tag{3.4}$$

In particular, u_y is the unique solution of the normal equation in $N(C^{\#}C)^{\perp}$ and the set of solutions of (3.4) is the affine manifold

$$u_y + N(C^{\#}C).$$

The following is the main result of this section. It shows that the solutions of the ILSP associated to the equation $Cx = y$ are the solutions of the normal equation $C^{\#}(Cx - y) = 0$, but it also characterizes them in terms of the J -normal projections onto $R(C)$.

Theorem 3.5. *Given $C \in CR(\mathcal{H}, \mathcal{K})$, if $R(C)$ is a J -nonnegative pseudo-regular subspace of \mathcal{K} and $y \in R(C) + R(C)^{\perp}$, the following conditions are equivalent:*

1. $u \in \mathcal{H}$ is an ILSS of $Cx = y$;
2. $u \in \mathcal{H}$ is a solution of the normal equation $C^{\#}(Cx - y) = 0$;
3. $Cu - Qy \in R(C)^{\circ}$ for any J -normal projection Q onto $R(C)$.

If $y \notin R(C)$ the above conditions are also equivalent to:

4. there exists a J -normal projection Q onto $R(C)$ such that $Cu = Qy$.

Moreover, the set of ILSS of $Cx = y$ coincides with the affine manifold

$$u_y + N(C^{\#}C),$$

where $u_y = (C^{\#}C)^{\dagger}C^{\#}y$.

Proof. By Lemma 3.1, assuming the J -nonnegativity of $R(C)$, u is an ILSS of $Cx = y$ if and only if $y - Cu \in R(C)^{\perp} = N(C^\#)$. Then, the equivalence 1. \leftrightarrow 2. follows.

2. \leftrightarrow 3.: By Remark 2.1, $(I - Q)y \in R(C)^{\perp} = N(C^\#)$ for any J -normal projection $Q \in L(\mathcal{K})$ onto $R(C)$. Hence, $u \in \mathcal{H}$ is a solution of $C^\#(Cx - y) = 0$ if and only if $C^\#(Cu - Qy) = 0$, or equivalently, $Cu - Qy \in R(C)^\circ$.

2. \leftrightarrow 4.: Assume that $y \notin R(C)$ and u is a solution of $C^\#(Cx - y) = 0$. Then, $y = Cu + z$ with $z \in R(C)^{\perp} \setminus R(C)$. So, there exists a regular subspace \mathcal{T} of $R(C)^{\perp}$ such that $z \in \mathcal{T}$ and $R(C)^{\perp} = \mathcal{T} \dot{+} R(C)^\circ$. Also, consider a regular subspace \mathcal{M} of $R(C)$ such that $R(C) = \mathcal{M} \dot{+} R(C)^\circ$. Then, note that $R(C)^\circ$ is a J -neutral subspace of the Krein space $\mathcal{K}' = (\mathcal{M} + \mathcal{T})^{\perp}$. So, it is well-known that there exists a neutral dual companion \mathcal{N} of $R(C)^\circ$ in \mathcal{K}' , see [11]. Furthermore, $R(C)^\circ$ is a hypermaximal neutral subspace of \mathcal{K}' [2, Prop. 1.4.19] because

$$\begin{aligned} (R(C)^\circ)^{\perp_{\mathcal{K}'}} &= (R(C)^\circ)^{\perp} \cap \mathcal{K}' \\ &= (R(C) + R(C)^{\perp}) \cap (\mathcal{M} + \mathcal{T})^{\perp} = \\ &= (\mathcal{M} \dot{+} \mathcal{T} \dot{+} R(C)^\circ) \cap (\mathcal{M} + \mathcal{T})^{\perp} = R(C)^\circ. \end{aligned}$$

Thus, $(\mathcal{M} + \mathcal{T})^{\perp} = \mathcal{K}' = \mathcal{N} \dot{+} (R(C)^\circ)^{\perp_{\mathcal{K}'}} = \mathcal{N} \dot{+} R(C)^\circ$ and the following decomposition of \mathcal{K} holds:

$$\mathcal{K} = \mathcal{M} \dot{+} (R(C)^\circ \dot{+} \mathcal{N}) \dot{+} \mathcal{T}.$$

Given the projection $Q = P_{R(C)/\mathcal{T} + \mathcal{N}} \in L(\mathcal{K})$, it is easy to see that $Q^\# = P_{\mathcal{M} + \mathcal{N}/R(C)^{\perp}}$. Therefore, Q is J -normal and it satisfies $Qy = Q(Cu + z) = Cu$.

Conversely, if $Cu = Qy$ for some J -normal projection $Q \in L(\mathcal{K})$ onto $R(C)$ then, by Remark 2.1, $y - Cu = (I - Q)y \in R(C)^{\perp} = N(C^\#)$. Therefore, $C^\#(Cu - y) = 0$.

Finally, recall that the set of solutions of the normal equation (which in this case coincides with the ILSS of $Cx = y$) is the affine manifold $u_y + N(C^\#C)$, where $u_y = (C^\#C)^\dagger C^\#y$. \square

Remark 3.6. Given $C \in CR(\mathcal{H}, \mathcal{K})$ with pseudo-regular range $R(C)$, the equivalences 2. \leftrightarrow 3. \leftrightarrow 4. in Theorem 3.5 holds independently of the (semi)definiteness of the range. Hence, Theorem 3.5 also characterizes the solutions of the normal equation $C^\#(Cx - y) = 0$ for $C \in CR(\mathcal{H}, \mathcal{K})$ with an arbitrary pseudo-regular range $R(C)$.

If $C \in CR(\mathcal{H}, \mathcal{K})$ and $R(C)$ is pseudo-regular, the set $\mathcal{Q}_{R(C)}$ of J -normal projections onto $R(C)$ is related to a family of inner inverses of C , where $X \in L(\mathcal{K}, \mathcal{H})$ is an inner inverse of C if $CXC = C$. Let \mathcal{I} denote the set of solutions $D \in L(\mathcal{K}, \mathcal{H})$ of the equations

$$CXC = C, \quad (CX)^\#CX = CX(CX)^\#. \tag{3.5}$$

Then, $D \in \mathcal{I}$ if and only if there exist $Q \in \mathcal{Q}_{R(C)}$ and $T \in L(\mathcal{K}, \mathcal{H})$ with $R(T) \subseteq N(C)$ such that

$$D = C^\dagger Q + T.$$

Indeed, if $D \in L(\mathcal{K}, \mathcal{H})$ is a solution of (3.5) then $Q := CD \in \mathcal{Q}_{R(C)}$ and $C^\dagger Q = C^\dagger CD = P_{N(C)^\perp} D$. So, $T := P_{N(C)} D \in L(\mathcal{K}, \mathcal{H})$ satisfies $R(T) \subseteq N(C)$ and $D = C^\dagger Q + T$.

Conversely, given $Q \in \mathcal{Q}_{R(C)}$ and $T \in L(\mathcal{K}, \mathcal{H})$ with $R(T) \subseteq N(C)$, consider $D := C^\dagger Q + T$. Then, $CD = CC^\dagger Q = P_{R(C)} Q = Q$ implies that D is a solution of (3.5).

The following result describes the solutions of the ILSP associated to $Cx = y$ in terms of these generalized inverses.

Proposition 3.7. *Given $C \in CR(\mathcal{H}, \mathcal{K})$, if $R(C)$ is a J -nonnegative pseudo-regular subspace of \mathcal{K} and $y \in R(C) + R(C)^{[\perp]}$, the following conditions are equivalent:*

1. $u \in \mathcal{H}$ is an ILSS of $Cx = y$;
2. $Dy - u \in N(C^\#C)$ for any solution $D \in L(\mathcal{K}, \mathcal{H})$ of (3.5).

If $y \notin R(C)$ the above conditions are also equivalent to:

3. there exists a solution of (3.5) such that $Dy = u$.

Proof. 1. \leftrightarrow 2.: Given a solution $D \in L(\mathcal{K}, \mathcal{H})$ of (3.5), consider $Q \in \mathcal{Q}_{R(C)}$ and $T \in L(\mathcal{K}, \mathcal{H})$ with $R(T) \subseteq N(C)$ such that $D = C^\dagger Q + T$. For $u \in \mathcal{H}$, follows that $Dy - u \in N(C^\#C)$ if and only if $C^\#(Qy - Cu) = 0$, or equivalently, $Qy - Cu \in R(C)^\circ$. Thus the equivalence follows from Theorem 3.5.

1. \leftrightarrow 3.: Given $y \in (R(C) + R(C)^{[\perp]}) \setminus R(C)$, suppose that $u = Dy$ where $D \in L(\mathcal{K}, \mathcal{H})$ is a solution of (3.5). It is easy to see that $Q = CD$ is a J -normal projection with $R(Q) = R(C)$. Furthermore, $Cu = CDy = Qy$. By Theorem 3.5, this implies that u is an ILSS of $Cx = y$.

Conversely, if $u \in \mathcal{H}$ is an ILSS of $Cx = y$, Theorem 3.5 states that $Cu = Qy$ for some J -normal projection $Q \in L(\mathcal{K})$. Then, $u = C^\dagger Qy + w$, where $w \in N(C)$. Consider $T \in L(\mathcal{K}, \mathcal{H})$ with $R(T) \subseteq N(C)$ such that $Ty = w$ and define $D = C^\dagger Q + T$. Thus, D is a solution of (3.5) and $Dy = C^\dagger Qy + Ty = C^\dagger Qy + w = u$. \square

In the following it is shown that the ILSP associated to the equation $Cx = y$ can be rewritten as an ILSP associated to another equation $C'x = y$, where $C' \in CR(\mathcal{H}, \mathcal{K})$ and $R(C')$ is a uniformly J -positive subspace of \mathcal{K} . But this is only true if the vector $y \in \mathcal{K}$ is admissible for the ILSP associated to the equation $Cx = y$ (recall that the ILSP associated to the equation $C'x = y$ is always well-posed).

If $R(C)$ is a $J_{\mathcal{K}}$ -nonnegative pseudo-regular subspace of \mathcal{K} and $y \in R(C) + R(C)^{[\perp]}$, then

$$u \in \mathcal{H} \text{ is an ILSS of } Cx = y \quad \Leftrightarrow \quad u \in \mathcal{H} \text{ is an ILSS of } (EC)x = y,$$

where $E = QQ^\#$ and Q is any J -normal projection onto $R(C)$.

First, observe that $R(EC) = E(R(C) + N(E)) = R(E)$ since $R(E) \subset R(C)$. Hence, $R(EC)$ is uniformly $J_{\mathcal{K}}$ -positive and the indefinite least-squares problem associated to the equation $ECx = y$ is well-posed. Then, by Theorem 3.5, $u \in \mathcal{H}$ is an ILSS of $Cx = y$ if and only if $Cu - Qy \in R(C)^\circ$. But, $R(C)^\circ \subset N(E)$ implies that

$$ECu = E(Cu - Qy) + EQy = Ey,$$

and E is the J -selfadjoint projection onto $R(EC)$. Then, u is an ILSS of $ECx = y$, see e.g. [12, Prop. 3.2].

Proposition 3.8. *Let $C \in CR(\mathcal{H}, \mathcal{K})$ be such that $R(C)$ is a pseudo-regular subspace of \mathcal{K} . Then, $C' = CP_{N(C^\#C)^\perp} \in CR(\mathcal{H}, \mathcal{K})$ has regular range and, if $y \in R(C) + R(C)^{[\perp]}$,*

$$u \in \mathcal{H} \text{ is an ILSS of } Cx = y \quad \Leftrightarrow \quad u \in \mathcal{H} \text{ is an ILSS of } C'x = y.$$

Proof. Given $C \in CR(\mathcal{H}, \mathcal{K})$, consider the operator $E_0 := C(C^\#C)^\dagger C^\#$. By Lemma 3.4, $E_0 \in L(\mathcal{K})$ and it is easy to check that $E_0^2 = E_0$. As a consequence of Proposition 2.3 the projection E_0 is J -selfadjoint, and $R(E_0)$ is obviously contained in $R(C)$. Then, $R(E_0C) = E_0(R(C) + N(E_0)) = R(E_0)$ and note that

$$E_0C = C(C^\#C)^\dagger C^\#C = CP_{N(C^\#C)^\perp} = C'.$$

Therefore, $R(C') = R(E_0C) = R(E_0)$ is regular.

Also, $R(E_0) \cap R(C)^\circ = \{0\}$ because $R(C)^\circ \subseteq R(C)^{[\perp]} = N(C^\#) \subseteq N(E_0)$. Since $C = C' + CP_{N(C^\#C)}$ and the range of $CP_{N(C^\#C)}$ coincides with $R(C)^\circ$, it follows that

$$R(C) = R(CP_{N(C^\#C)^\perp}) + R(C)^\circ = R(E_0C) + R(C)^\circ = R(E_0) \dot{+} R(C)^\circ.$$

Therefore, E_0 is a J -selfadjoint projection onto a regular complement of $R(C)^\circ$ in $R(C)$ and, by [19, Thm. 6.9] there exist (at least) a J -normal projection $Q \in L(\mathcal{K})$ such that $E_0 = QQ^\#$. Finally, if $y \in R(C) + R(C)^{[\perp]}$ the discussion above shows that the ILSS of $Cx = y$ and $C'x = y$ coincide. \square

4. Minimizers among indefinite least-squares solutions

The following paragraphs are devoted to consider a minimization problem among the indefinite least-squares solutions of $Cx = y$, where $C \in CR(\mathcal{H}, \mathcal{K})$ and $y \in R(C) + R(C)^{[\perp]}$.

Definition 1. A vector $w \in \mathcal{H}$ is a minimal least-squares solution (hereafter MILSS) of $Cx = y$ if w is an ILSS of $Cx = y$ and

$$[w, w]_{\mathcal{H}} \leq [u, u]_{\mathcal{H}}, \quad \text{for every ILSS } u \text{ of } Cx = y.$$

It follows from Theorem 3.5 that, if $R(C)$ is a pseudo-regular $J_{\mathcal{K}}$ -nonnegative subspace of \mathcal{K} and $y \in R(C) + R(C)^{[\perp]}$, the set of ILSS of $Cx = y$ coincides with

$$u_y + N(C^\#C),$$

where $u_y = (C^\#C)^\dagger C^\#y$. So, $w \in \mathcal{H}$ is a MILSS of $Cx = y$ if and only if

$$[w, w] = \min_{z \in N(C^\#C)} [u_y + z, u_y + z]. \tag{4.1}$$

Thus, if $P_{N(C^\#C)}$ is the orthogonal projection onto $N(C^\#C)$ and $w = u_y + z_w$ is the orthogonal decomposition of w according to $\mathcal{H} = N(C^\#C)^\perp \oplus N(C^\#C)$, note that (4.1) can be rewritten as

$$\begin{aligned} [u_y + z_w, u_y + z_w] &= \min_{z \in N(C^\#C)} [u_y + z, u_y + z] \\ &= \min_{x \in \mathcal{H}} [u_y + P_{N(C^\#C)}x, u_y + P_{N(C^\#C)}x]. \end{aligned}$$

Hence, if $w = u_y + z_w \in u_y + N(C^\#C)$,

$$w \text{ is a MILSS of } Cx = y \iff z_w \text{ is an ILSS of } P_{N(C^\#C)}x = -u_y. \tag{4.2}$$

By Lemma 3.1, the existence of an ILSS of $P_{N(C^\#C)}x = -u_y$ is equivalent to

$$u_y \in N(C^\#C) + N(C^\#C)^{[\perp]},$$

and the $J_{\mathcal{H}}$ -nonnegativity of $N(C^\#C)$. Therefore,

Proposition 4.1. *Let $C \in CR(\mathcal{H}, \mathcal{K})$ be such that $R(C)$ is a $J_{\mathcal{K}}$ -nonnegative pseudo-regular subspace of \mathcal{K} and consider $y \in R(C) + R(C)^{[\perp]}$. Then, there exists a MILSS $w \in \mathcal{H}$ of $Cx = y$ if and only if $N(C^\#C)$ is $J_{\mathcal{H}}$ -nonnegative and $u_y \in N(C^\#C) + N(C^\#C)^{[\perp]}$. In this case, the set of MILSS of $Cx = y$ coincides with*

$$(u_y + N(C^\#C)) \cap N(C^\#C)^{[\perp]}.$$

Proof. The equivalence between the existence of a MILSS for $Cx = y$ and the conditions on $N(C\#C)$ and u_y follows from the discussion above. Also, note that $u_y \in N(C\#C) + N(C\#C)^{\perp}$ if and only if

$$(u_y + N(C\#C)) \cap N(C\#C)^{\perp} \neq \emptyset.$$

Now, assume that $w \in \mathcal{H}$ is a MILSS of $Cx = y$. Then, there exists $z_w \in N(C\#C)$ such that $w = u_y + z_w$ and z_w is an ILSS of $P_{N(C\#C)}x = -u_y$. By Lemma 3.1, $-u_y - P_{N(C\#C)}z_w \in N(C\#C)^{\perp}$. So,

$$w = u_y + z_w = u_y + P_{N(C\#C)}z_w \in (u_y + N(C\#C)) \cap N(C\#C)^{\perp}.$$

Conversely, suppose that $w \in (u_y + N(C\#C)) \cap N(C\#C)^{\perp}$. Then, w is an ILSS of $Cx = y$ because $w \in u_y + N(C\#C)$. Also, there exists $z_w \in N(C\#C)$ such that $w = u_y + z_w$. Furthermore, since

$$-u_y - P_{N(C\#C)}z_w = -u_y - z_w = -w \in N(C\#C)^{\perp},$$

$z_w \in N(C\#C)$ is an ILSS of $P_{N(C\#C)}x = -u_y$. So, (4.2) implies that $w = u_y + z_w$ is a MILSS of $Cx = y$. \square

In the rest of this section it is assumed that $N(C\#C)$ is a $J_{\mathcal{H}}$ -nonnegative pseudo-regular subspace of \mathcal{H} , aiming to describe the set of MILSS of $Cx = y$ in terms of J -normal projections.

Let $C \in CR(\mathcal{H}, \mathcal{K})$ be such that $R(C)$ is pseudo-regular and consider $y \in R(C) + R(C)^{\perp}$. Then, note that

$$u_y = (C\#C)^{\dagger}C\#y = 0 \quad \text{if and only if} \quad y \in R(C)^{\perp}.$$

In this case, $u \in \mathcal{H}$ is an ILSS of $Cx = y$ if and only if $u \in N(C\#C)$. Moreover, by Proposition 4.1, $u \in \mathcal{H}$ is a MILSS of $Cx = y$ if and only if $u \in N(C\#C)^{\circ}$.

Lemma 4.2. *Let $C \in CR(\mathcal{H}, \mathcal{K})$ be such that $R(C)$ is a $J_{\mathcal{K}}$ -nonnegative pseudo-regular subspace of \mathcal{K} and consider $y \in (R(C) + R(C)^{\perp}) \setminus R(C)^{\perp}$. Assume also that $N(C\#C)$ is a $J_{\mathcal{H}}$ -nonnegative pseudo-regular subspace of \mathcal{H} . Then, $w \in \mathcal{H}$ is a MILSS of $Cx = y$ if and only if there exists $P \in \mathcal{Q}_{N(C\#C)}$ such that*

$$w = (I - P)u_y. \tag{4.3}$$

Proof. Given $C \in CR(\mathcal{H}, \mathcal{K})$ with $J_{\mathcal{K}}$ -nonnegative pseudo-regular range $R(C)$, let $y \in (R(C) + R(C)^{\perp}) \setminus R(C)^{\perp}$. By the above remark, $u_y \neq 0$.

If $w \in \mathcal{H}$ is a MILSS of $Cx = y$, consider its orthogonal decomposition $w = u_y + z$, where $z \in N(C\#C)$. Then, by (4.2), z is an ILSS of the equation $P_{N(C\#C)}x = -u_y$. Also $u_y \in N(C\#C)^{\perp}$ and, by Theorem 3.5, there exists $P \in \mathcal{Q}_{N(C\#C)}$ such that

$$z = P_{N(C\#C)}z = P(-u_y) = -Pu_y.$$

Thus, $w = u_y + z = u_y - Pu_y = (I - P)u_y$ for some $P \in \mathcal{Q}_{N(C\#C)}$.

Conversely, if $w = (I - P)u_y$ for some $P \in \mathcal{Q}_{N(C\#C)}$ then, since $u_y \in N(C\#C) + N(C\#C)^{\perp}$,

$$w = (I - P)P\#u_y + (I - P)(I - P)\#u_y = (I - P)(I - P)\#u_y,$$

because, by Proposition 4.1,

$$u_y \in R(P) + R(P)^{\perp} = R(P) + N(P\#) = N((I - P)P\#).$$

Then, $w \in N(C\#C)^{\perp}$ and, by Proposition 4.1, w is a MILSS of $Cx = y$. \square

If $R(C)$ is a pseudo-regular subspace of \mathcal{K} , consider $E_0 = C(C^\#C)^\dagger C^\#$. If $y \in R(C) + R(C)^{\perp\perp}$ then $Cu_y = E_0y$ and

$$u_y = C^\dagger Cu_y = C^\dagger E_0y,$$

because $u_y \in N(C^\#C)^\perp \subseteq N(C)^\perp$. Moreover, if $y \in (R(C) + R(C)^{\perp\perp}) \setminus R(C)^{\perp\perp}$, applying this identity in (4.3) it follows that if $w \in \mathcal{H}$ is a MILSS of $Cx = y$ then there exists $P \in \mathcal{Q}_{N(C^\#C)}$ such that

$$w = (I - P)u_y = (I - P)C^\dagger E_0y.$$

Furthermore, following the construction made in the proof of Theorem 3.5, it is easy to see that there exists $Q_0 \in \mathcal{Q}_{R(C)}$ such that $E_0 = Q_0^\#Q_0$. Hence, by Remark 2.1, $Q_0^\#(I - Q_0)y = 0$ and

$$w = (I - P)C^\dagger E_0y = (I - P)C^\dagger Q_0^\#y.$$

Theorem 4.3. *Let $C \in CR(\mathcal{H}, \mathcal{K})$ such that $R(C)$ is a $J_{\mathcal{K}}$ -nonnegative pseudo-regular subspace of \mathcal{K} and consider $y \in (R(C) + R(C)^{\perp\perp}) \setminus R(C)^{\perp\perp}$. Assume also that $N(C^\#C)$ is a $J_{\mathcal{H}}$ -nonnegative pseudo-regular subspace of \mathcal{H} . Then, $w \in \mathcal{H}$ is a MILSS of $Cx = y$ if and only if there exists $P \in \mathcal{Q}_{N(C^\#C)}$ and $Q \in \mathcal{Q}_{R(C)}$ such that*

$$w = (I - P)C^\dagger Q^\#y = (I - P)C^\dagger Ey, \tag{4.4}$$

where $E = QQ^\#$.

Proof. Under these assumptions, there exists a MILSS of $Cx = y$. Furthermore, in the discussion above it was shown that, if $w \in \mathcal{H}$ is a MILSS of $Cx = y$ then there exists $P \in \mathcal{Q}_{N(C^\#C)}$ and $Q_0 \in \mathcal{Q}_{R(C)}$ such that

$$w = (I - P)u_y = (I - P)C^\dagger Q_0^\#y = (I - P)C^\dagger E_0y.$$

Conversely, given $P \in \mathcal{Q}_{N(C^\#C)}$ and $Q \in \mathcal{Q}_{R(C)}$, consider the vector $w = (I - P)C^\dagger Q^\#y$. By Remark 2.1 it follows that $Q^\#(I - Q)y = 0$ and $x := C^\dagger Q^\#y = C^\dagger QQ^\#y$. Then, $Cx = P_{R(C)}QQ^\#y = Q^\#Qy$ and

$$Qy - Cx = Qy - QQ^\#y = Q(I - Q^\#)y \in R(C)^\circ.$$

So, by Theorem 3.5, $x \in u_y + N(C^\#C)$. Also, $w = (I - P)x = (I - P)u_y$ and, following the same arguments as in Lemma 4.2, $w \in N(C^\#C)^{\perp\perp}$. Therefore, by Proposition 4.1, w is a MILSS of $Cx = y$. \square

In the description obtained for the MILSS of $Cx = y$ in the above theorem, the family of operators

$$\{(I - P)C^\dagger E : P \in \mathcal{Q}_{N(C^\#C)}\}$$

appears, where E is the J -selfadjoint projection onto an arbitrary complement of $R(C)^\circ$ in $R(C)$. Along the next section, this family is related to some of the generalized inverses of $C' := EC$. Note that, under the assumptions of Theorem 4.3, $R(C') = R(E)$ is regular and $N(C') = N(C^\#C)$ is pseudo-regular.

5. Generalized inverses related to indefinite least-squares problems

The next result describes a family of generalized inverses of a closed-range operator with pseudo-regular range and nullspace.

Proposition 5.1. *Suppose that $C \in CR(\mathcal{H}, \mathcal{K})$ is such that $R(C)$ and $N(C)$ are pseudo-regular subspaces of \mathcal{K} and \mathcal{H} , respectively. Then, $D \in L(\mathcal{K}, \mathcal{H})$ is a solution of*

$$\begin{cases} CXC = C, \\ XCX = X, \\ (CX)(CX)^\# = (CX)^\#(CX), \\ (XC)(XC)^\# = (XC)^\#(XC), \end{cases} \tag{5.1}$$

if and only if there exist $Q \in \mathcal{Q}_{R(C)}$ and $P \in \mathcal{Q}_{N(C)}$ such that $D = (I - P)C^\dagger Q$.

Proof. Given $Q \in \mathcal{Q}_{R(C)}$ and $P \in \mathcal{Q}_{N(C)}$, consider $D = (I - P)C^\dagger Q$. Since $CP = 0$,

$$CD = C(I - P)C^\dagger Q = CC^\dagger Q = P_{R(C)}Q = Q.$$

Also,

$$DC = (I - P)C^\dagger QC = (I - P)C^\dagger C = (I - P)P_{N(C)^\perp} = I - P,$$

because $R(P) = N(C)$. Therefore, CD is a $J_{\mathcal{K}}$ -normal projection and DC is a $J_{\mathcal{H}}$ -normal projection. Furthermore,

$$CDC = (CD)C = QC = C \quad \text{and} \quad DCD = (DC)D = (I - P)D = D.$$

Conversely, assume that $D \in L(\mathcal{K}, \mathcal{H})$ satisfies the equations in (5.1). Then, note that $Q := CD \in \mathcal{Q}_{R(C)}$, $P := I - DC \in \mathcal{Q}_{N(C)}$ and

$$(I - P)C^\dagger Q = (DC)C^\dagger(CD) = D(CC^\dagger C)D = DCD = D. \quad \square$$

Let $E \in L(\mathcal{K})$ be a J -selfadjoint projection such that $R(E) \dot{+} R(C)^\circ = R(C)$. Applying the above proposition to $C' = EC$ it is possible to reinterpret the operators of the form $(I - P)C^\dagger E$ (with $P \in \mathcal{Q}_{N(C^\#C)}$) as a particular family of generalized inverses of C' .

Corollary 5.2. *Suppose that $C \in CR(\mathcal{H}, \mathcal{K})$ is such that $R(C)$ and $N(C^\#C)$ are pseudo-regular subspaces of \mathcal{K} and \mathcal{H} , respectively. Consider a J -selfadjoint projection $E \in L(\mathcal{K})$ such that $R(E) \dot{+} R(C)^\circ = R(C)$. If $C' = EC$ then the operators in the set*

$$\{(I - P)C^\dagger E : P \in \mathcal{Q}_{N(C^\#C)}\},$$

are the solutions in $L(\mathcal{K}, \mathcal{H})$ of

$$\begin{cases} C'XC' = C', \\ XC'X = X, \\ C'X = E, \\ (XC')(XC')^\# = (XC')^\#(XC'). \end{cases} \tag{5.2}$$

Proof. Consider a J -selfadjoint projection $E \in L(\mathcal{K})$ such that $R(E) \dot{+} R(C)^\circ = R(C)$. If $C' = EC$ note that

$$R(C') = R(E) \quad \text{and} \quad N(C') = N(C^\#C).$$

Then, apply Proposition 5.1 to C' . \square

Thus, the statement of [Theorem 4.3](#) can be rephrased as: $w \in \mathcal{H}$ is a MILSS of $Cx = y$ if and only if $u = Dy$ where $D \in L(\mathcal{K}, \mathcal{H})$ is a solution of Eq. (5.2).

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