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Abstract. We study a population involved in a cyclic game of three strategies – the rock-paper-scissors game – whose agents interact through groups of three individuals (triplets), considering the possibility that two weak agents cooperate and beat a strong one. In a wide range of parameters the system presents a stable heteroclinic cycle, which implies that in a finite population some of the strategies become extinct and others survive. We find that the cooperation within triplets only benefits the survival of the strategy if the cooperation probability is above a certain threshold. We study the survival probabilities of the different strategies as a function of the cooperation parameters through a analytic approximation and compare with simulations, obtaining a good agreement. Results are generalizable to other systems with heteroclinic cycles.

1 Introduction

The use of a graph or network for the representation of a pattern of interactions is a widespread paradigm in several branches of science. This representation, which assumes pairwise interactions, is useful to describe a variety of systems whose dynamical properties are affected by the structural characteristics of their interaction patterns [1–3]. However, there are systems where the basic interactions occur in groups of other sizes (triplets, quartets, etc.) and not just in pairs [4,5]. An example of this situation is the process of opinion formation or decision making in groups of people, where the evolution of individual opinions is driven by the collective exchange of views rather than by discussions in pairs, and is affected not only by the size of the group but also by its internal dynamics [6]. While an interaction network is specified by the list of all the pairs of interacting agents, a possible representation of those more complex structures is through a list of all the groups of different sizes present in the population.

These complex interactions that occur in larger groups also appear naturally, for example, as agreements or disputes between companies in an economical context, where they can be modeled using tools from game theory. In this sense it is important to understand the effects of these non-binary interactions in the collective behavior of social and economical systems where these coalitions, that allowed new strategies to the agents involved, are likely to happen.

In this work we study a population where agents adopting one of three possible strategies interact in groups of three agents (triplets), that are distributed

homogeneously over the population. The three strategies exhibit cyclic dominance, which makes this kind of dynamics known as rock-paper-scissors games [7,8]. Such cyclic interactions have been observed in physical [9] and in biological systems, the mating strategies of side-blotched lizards being one of the most well-known example [10]. Other instances are the overgrowth of certain marine sessile organisms [11], the competition between mutant strains of yeast [12], and possible cyclic interactions between outlaw genes [13].

A special aspect of the rock-paper-scissors game dynamics is the occurrence of stable heteroclinic cycles [14,15], which induce oscillatory orbits of increasing size, and the permanence of the system near pure states (with all agents in the same strategy) for increasingly long periods. In finite systems this behavior eventually leads to the extinction of a strategy and the temporal survival of the others; as one of the two remaining strategies dominates the other, finally just one survives. In the present paper, we focus the attention on how the cooperation within triplets affect the survival probability of a specific strategy. In the next section, we introduce our evolutionary model for interactions in pairs and triplets. Numerical simulations for the time evolution, the survival probabilities and the initial conditions dependence are presented in Section 3. The mean field approach is studied in Section 4, whereas an analytic approximation of the survival probabilities is provided in Section 5. Results are discussed in the last section.

2 The model

In the rock-paper-scissors game each agent i selects one of three strategies ($s_i = 0, 1, 2$), which dominate each other
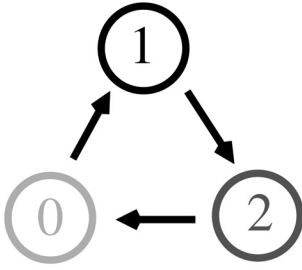
a e-mail: damian.g.h.l@gmail.com

Fig. 1. Diagram of the cyclic dominance between the three strategies. Strategy 0 beats 1, 1 beats 2, and 2 beats 0.

cyclically (Fig. 1). In the traditional model with binary interactions, two agents are selected at random from the whole population. As a result, the pairs $\{0,1\}$ and $\{1,0\}$ become $\{0,0\}$, $\{1,2\}$ and $\{2,1\}$ become $\{1,1\}$, and $\{2,0\}$ and $\{0,2\}$ become $\{2,2\}$.

In our version, on the other hand, agents interact in triplets, allowing us to explore richer dynamics in addition to the cyclic dominance of the strategies. In particular, we implement a process of cooperation within each triplet, where two agents with the same strategy can overcome the third one even if the latter adopts a stronger strategy. The third agent thus changes strategy to that of the two cooperators. For each strategy, we introduce a probability that agents form alliances, so that agents with a specific strategy may be more prone to cooperate than others. In this new context, we study if local cooperation results into a global benefit for the spread of a strategy, and how the asymmetry in the cooperation affects the asymptotic state of the population.

Considering a population of size N, we choose three agents at random at each evolution step. If all three agents have the same strategy or all have different strategies, their strategies are not changed. If two of the agents have a strategy that dominates over that of the third agent, the latter adopts the strategy of the other two agents. For example, the triplet $\{1,1,2\}$ becomes $\{1,1,1\}$. Finally, if two of the agents possessing a weak strategy face a strong agent, the weak agents can cooperate with a certain probability and beat the strong agent. For example, the triplet $\{1,2,2\}$ becomes $\{2,2,2\}$ with a given probability r_2. The whole set of possible transitions is

$$
\begin{aligned}
\{0,0,1\} &\rightarrow \{0,0,0\} \text{ with } p_t = 1 \\
\{0,1,1\} &\rightarrow \{0,0,1\} \text{ with } p_t = 1 - r_1 \\
&\qquad \{1,1,1\} \text{ with } p_t = r_1 \\
\{1,1,2\} &\rightarrow \{1,1,1\} \text{ with } p_t = 1 \\
\{1,2,2\} &\rightarrow \{1,1,2\} \text{ with } p_t = 1 - r_2 \\
&\qquad \{2,2,2\} \text{ with } p_t = r_2 \\
\{2,2,0\} &\rightarrow \{2,2,2\} \text{ with } p_t = 1 \\
\{2,0,0\} &\rightarrow \{2,2,0\} \text{ with } p_t = 1 - r_0 \\
&\qquad \{0,0,0\} \text{ with } p_t = r_0
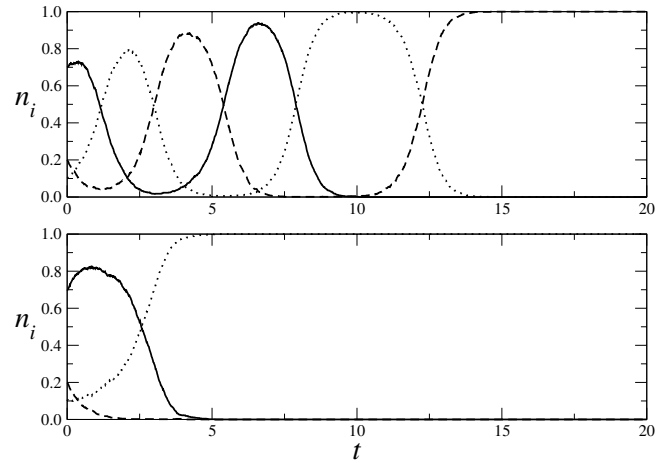\end{aligned}
\tag{1}
$$

Fig. 2. Fractions of the population adopting each of the three strategies as function of time (strategy 0 in dotted line, 1 in continuous line and 2 in dashed line), displaying the two typical behaviors of the system. In the upper panel the parameters are $r_0 = 0.12$, $r_1 = 0.09$, and $r_2 = 0.15$ while in the lower panel $r_0 = 0.60$, $r_1 = 0.45$, and $r_2 = 0.75$. In both cases we consider a total population of $N = 900$ and the initial condition $n_0 = 0.1$, $n_1 = 0.7$, and $n_2 = 0.2$.

where p_t stands for the transition probability. The system possesses cyclic symmetry only when the cooperation probabilities are all equal.

3 Simulations

Numerical simulations of the model with triplets are performed according to the game rules presented in the previous section. Figure 2 shows the fractions n_i of the total population (in this case, $N = 900$) with each of the strategies as a function of time, for two representative choices of the cooperation probabilities and the same initial conditions. The variables n_i are restricted to the simplex $n_0 + n_1 + n_2 = 1$, $n_i \geq 0$. In the simulations, the time unit corresponds to one interaction per agent on the average.

The cases shown in Figure 2 represent the two typical behaviors observed in our system. In the upper panel, the fractions n_i perform oscillations of increasing amplitude, and eventually reach one of the three possible pure states, with all the agents having the same strategy. In the lower panel, on the other hand, the evolution falls rapidly into one of the pure states. As we shall see in the following section, in the continuous version of the system these behaviors correspond, respectively, to the presence of a stable heteroclinic cycle and to the existence of one or more stable pure states. In the numerical simulations, the system ends up in an absorbing state in both cases. The final pure state is generally observed for all sets of cooperation probabilities. The only exception occurs when $r_i = 0$ for all i, in which case the system performs oscillations whose amplitude depends on the initial condition.

As it becomes clear from our analysis of the continuous version of the model (see next section) the falling of the system to different absorbing states at finite times
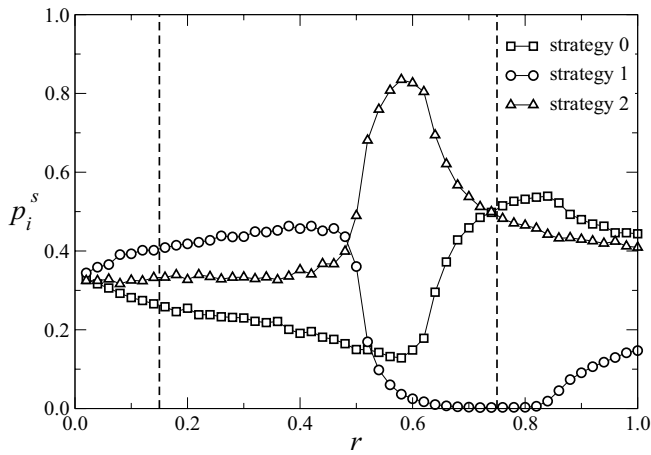
Fig. 3. Survival probabilities computed from simulations as a function of the cooperation parameter r, with $r_0 = \frac{4}{5}r$, $r_1 = \frac{3}{5}r$, $r_2 = r$, and a total population of $N = 9 \times 10^3$. For each value of r, we took 10^4 initial conditions uniformly distributed on the simplex. The vertical dashed lines represent the values of the cooperation parameters used in simulations of Figure 2.

is to be ascribed, in numerical simulations, to finite-size fluctuations. It is therefore important, for finite N, to determine the probability that the system ends in each pure state, as a function of the cooperation probabilities and the initial condition. In this sense, we proceed to measure the survival probability of each strategy over an ensemble of initial conditions taken from the whole simplex.

To explore the space of cooperation probabilities we vary them jointly, maintaining the ratios r_0/r_2 and r_1/r_2 fixed, and studying the survival probabilities as a function of $r_2 = r \in [0, 1]$. Then, for different values of the ratios r_0/r_2 and r_1/r_2, we analyze the possible behaviors of the system in the entire parameter space.

Figure 3 shows simulation results for the survival probabilities p_i^s ($i = 0, 1, 2$) as a function of r, with $r_0/r_2 = 4/5$ and $r_1/r_2 = 3/5$. In this case, maximal cooperation occurs between agents with strategy 2, followed in order by 0 and 1. For $0 \leq r \leq 1/2$, the survival probabilities vary smoothly, starting from the symmetric situation where all of them equal $1/3$ for $r = 0$. In this parameter range, rather surprisingly, the strategy with maximal cooperation (2) does not have the largest survival probability [16]. It is rather the strategy that dominates over the most cooperative agents (1) which has the largest chance to become the absorbing state. The qualitative behavior of the system is the same as in the upper panel of Figure 2, exhibiting oscillations of increasing amplitude. For $r > 1/2$, on the other hand, the behavior of the system changes, falling rapidly into one of the pure absorbing states as shown in the lower panel of the same figure.

Another aspect to consider from Figure 3 is that the survival probabilities for strategies 1 and 2 display a sharp variation for $r \approx 0.5$ ($r_2 \approx 0.5$), whereas for strategy 0 it maintains a smooth behavior in the same region. This critical value of the cooperation parameter, above which the survival probability of strategies 1 and 2, respectively
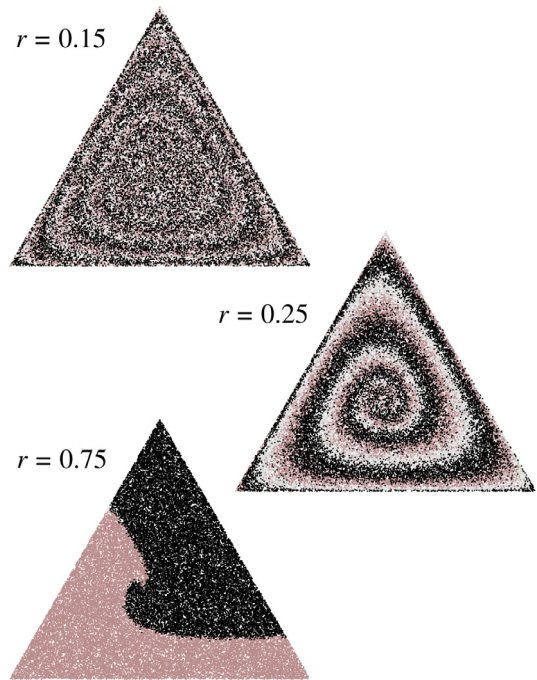

Fig. 4. Initial conditions on the simplex that lead to each final pure state (state 0 in brown, 1 in grey and 2 in black). Cooperation parameters are $r_0 = 0.8r$, $r_1 = 0.6r$ and $r_2 = r$. The total population size is $N = 9 \times 10^3$. The black crosses stands for the initial conditions taken for Figure 2.

decreases and increases, represents a threshold from which alliances benefit the strategy that cooperates – in this case, strategy 2. Within each triplet, it becomes more likely than not that two agents possessing strategy 2 ally to overcome an agent with strategy 1. The same effect is observed for the survival probabilities of strategies 0 and 2 when $r \approx 0.63$ ($r_0 \approx 0.5$), and for 0 and 1 when $r \approx 0.83$ ($r_1 \approx 0.5$). In general, thus, we find that the survival probability for i increases, while for $(i - 1)_3$ it decreases, above $r_i = 0.5$[1]. As we shall see in the next section, these sharp changes in the survival probabilities are related to the stabilization of a pure state.

Additionally, we have studied which initial conditions over the simplex lead to each of the final absorbing states. Figure 4 shows results for three values of r. The color of each dot indicates the pure state reached from that initial condition, with brown for strategy 0, grey for strategy 1, and black for strategy 2. Parameters are $r_0 = 0.8r_2$, $r_1 = 0.6r_2$, $N = 9 \times 10^3$ and the total number of initial conditions is 4×10^5. Regions of different colors are separated by spiral-like curves. Due to finite-size fluctuations, however, the boundaries are noisy. This effect of fluctuations is enhanced as r decreases and the cooperation probabilities become mutually similar. In this case, different strategies are less distinctive from each other, and the regions leading to each pure state tighten.

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[1] Due to the cyclic nature of our system, we use the notation $(j \pm 1)_3$ for addition and subtraction modulo 3. The index $(j + 1)_3$ refers to strategy 0 when $j = 2$, while $(j - 1)_3$ refers to strategy 2 when $j = 0$.

For $r = 0.75$, which implies $r_0 = 0.6$ and $r_2 = 0.75$ (both above the threshold at 0.5), we find that most of the initial conditions end up in one of two pure state, 0 or 2, depending on the location of the initial condition on the simplex. The temporal evolution for the simulations with $r = 0.75$ corresponds to a rapid fall into an absorbing state.

4 Mean field approach

Before starting a mean field analysis of the system with triplets, we first study the case with pair interactions. If the pair $\{i, (i+1)_3\}$ becomes $\{i, i\}$ with probability $\gamma_i > 0$ per time unit, the evolution equations for the fractions n_i of the population in each strategy are

$$\dot{n}_0 = \gamma_0 n_0 n_1 - \gamma_2 n_0 n_2$$
$$\dot{n}_1 = \gamma_1 n_1 n_2 - \gamma_0 n_1 n_0$$
$$\dot{n}_2 = \gamma_2 n_2 n_0 - \gamma_1 n_2 n_1 \quad (2)$$

that can be reduced to a system of two equations, given that $n_0 + n_1 + n_2 = 1$.

This dynamical system possesses four fixed points: the three vertices of the simplex, corresponding to the pure states, and an interior fixed point located at (n_0^*, n_1^*, n_2^*) where $n_i^* = \gamma_{(i+1)_3}/\sum_j \gamma_j$. The vertices are saddle points while the interior fixed point has marginal stability (its eigenvalues are pure imaginary). The orbits are cycles around the interior fixed point and the size of the cycles is determined by the initial condition.

In the case of the system with triplets, fixing $\gamma_i = 1$ for all i and including the cooperation parameters r_i, the equations for the fractions n_i are

$$\dot{n}_0 = n_0^2 n_1 + (1 - 2r_1)n_0 n_1^2 - (1 - 2r_0)n_0^2 n_2 - n_0 n_2^2$$
$$\dot{n}_1 = n_1^2 n_2 + (1 - 2r_2)n_1 n_2^2 - (1 - 2r_1)n_1^2 n_0 - n_1 n_0^2$$
$$\dot{n}_2 = n_2^2 n_0 + (1 - 2r_0)n_2 n_0^2 - (1 - 2r_2)n_2^2 n_1 - n_2 n_1^2. \quad (3)$$

These equations have seven fixed points. Three of them are the vertices of the simplex whose coordinates, written as (n_0, n_1, n_2), are $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. The vertex that correspond to the pure state i is stable if $r_i > 1/2$; otherwise, it is a saddle point. There are three other points that depending on the parameters lay on the edges of the simplex, in which case they are saddles, or on the prolongations of such edges, in which case they are stable. However, in this last case, these stable points are never reached when starting with an initial condition on the simplex, as we show below. The coordinates of these points are

$$\left(\frac{1}{2r_0}, 0, 1 - \frac{1}{2r_0}\right)$$
$$\left(1 - \frac{1}{2r_1}, \frac{1}{2r_1}, 0\right)$$
$$\left(0, 1 - \frac{1}{2r_2}, \frac{1}{2r_2}\right). \quad (4)$$

Finally there is an interior fixed point which is in general an unstable focus (we omit the complicated analytic expression of its coordinates).

If $r_i = 0$ for all i, the system of differential equations (3) is equivalent to equations (2) (with $\gamma_i = 1$ for all i), i.e. the orbits are cycles.

Generally, the dynamical system defined by equations (3) possesses two typical behaviors depending on the cooperation parameters. If $0 < r_i < 1/2$ for all i, the system has no stable fixed point on the simplex and exhibits a stable heteroclinic cycle, which implies that all the orbits are attracted to this cycle. We recall that a heteroclinic cycle is a collection of trajectories that links sequences of equilibria via saddle-sink connections, i.e. through the stable/unstable manifolds of the fixed points [7]. In our case, the heteroclinic cycle coincides with the boundary of the simplex, so that it is formed by the saddle points located at the vertices and by the edges of the simplex which are along the stable/unstable manifolds of these saddle points. The trajectories approach the boundary and spend longer and longer periods of time in the neighbourhood of the vertices, being not able to leave the simplex if starting with an initial condition on its interior.

The other typical behavior of the system occurs when $r_i > 1/2$ for one or more strategies. When this happens, the pure state i located at a vertex of the simplex becomes stable, and trajectories lead to a rapid fall to this absorbing state. At the same time a saddle point appears on the edge that connects the vertex i with $(i - 1)_3$. The unstable manifold of this saddle point is along the edge, and the basin of attraction of the pure state i increases at the expense of that of $(i - 1)_3$. Within this parameter range, the system behaves differently, depending on how many pure states are stable. If only one of the strategies satisfy $r_i > 1/2$, then the corresponding pure state i is stable and most orbits go to this vertex (there are three such regions, one for each i). If two strategies satisfy $r_i > 1/2$, two vertices of the simplex are stable and the remaining strategy is the less likely to be observed in the asymptotic state (again, there are three such regions). Finally, if $r_i > 1/2$ for all i, the three pure states are stable and the probabilities of ending in each pure state when initial conditions are taken at random from the simplex are proportional to the size of the corresponding basins of attraction.

Connections with finite populations

In this subsection we compare results from the numerical simulations of a population of agents with the analysis of the differential equations of the corresponding mean field approach. As triplets in the simulations are chosen homogeneously over the population, the numerical results converge to those obtained in the mean field analysis when the size of the population tends to infinity. Considering a finite population, on the other hand, some discrepancies emerge between the two approaches.

In particular, oscillations of increasing amplitude in the numerical system are related to the presence of a stable

heteroclinic cycle of the corresponding differential equations. However, in the simulations the orbit eventually reaches the boundary, i.e. one of the three strategies disappears due to fluctuations and, with two strategies left, the dominant strategy eventually becomes the only survivor. This effect is observed in Figure 2 (upper panel), where oscillations of increasing amplitude reach the boundary of the simplex, leading strategy 1 to extinction. Strategy 2 being the dominant of the two remaining strategies, it is the survivor.

Another link with the stochastic simulations is the appearance of sharp changes in the survival probabilities when $r_i \approx 0.5$, as shown in Figure 3. From the mean field analysis, we see that these changes in $p_i^s(r)$ and $p_{(i-1)_3}^s(r)$ correspond to the stability change of the pure state $n_i = 1$ from saddle to stable fixed point, i.e. to the appearance of a basin of attraction for strategy i.

In Section 3 we calculated the survival probabilities for each strategies through simulations of the stochastic model. An alternative way to obtain these results is to perform a numerical integration of the differential equations given by equation (3). In the first place, we establish a cutoff c as a density below which we consider a strategy extinct (in a finite system this parameter should equals $1/N$). From an initial condition we integrate the equations up to the time when, for some i, $n_i < c$, i.e. when the orbit is very close to the edge $\{(i-1)_3, (i+1)_3\}$ of the simplex. At this point, we declare the strategy i extinct. To decide which of the two remaining strategies will survive we have to consider two possible scenarios. If there is a saddle point on that edge ($r_{(i-1)_3} > 1/2$) and if the orbit ended in the segment that joints the saddle with the vertex $(i-1)_3$ ($n_{(i-1)_3} > 1/2r_{(i-1)_3}$), the strategy $(i-1)_3$ will survive. Otherwise the strategy $(i+1)_3$, which is the dominant of the two remaining strategies, will survive.

Another possible approach to obtain the survival probabilities is to use an approximation of the orbit when it is close enough to the boundary of the simplex. This can be done only when trajectories are attracted to the boundary, i.e. when the heteroclinic cycle of the corresponding mean field equations is stable. We study this approximation in the next section, and compare it with the simulations and with the numerical integration of equation (3).

5 Survival probabilities

As advanced, we wish here to evaluate analytically which strategy will survive in a finite population when the corresponding dynamical system possesses a stable heteroclinic cycle. As we previously observed, the initial conditions that reach the different pure states are typically strongly mixed and, due to stochastic fluctuations, the final state for a fixed initial condition is not unique. Therefore when using the corresponding dynamical system to calculate the survival probabilities, we will assume that these probabilities are obtained from the evolution of several initial conditions taken at random from the simplex (more precisely, uniformly distributed on the simplex).
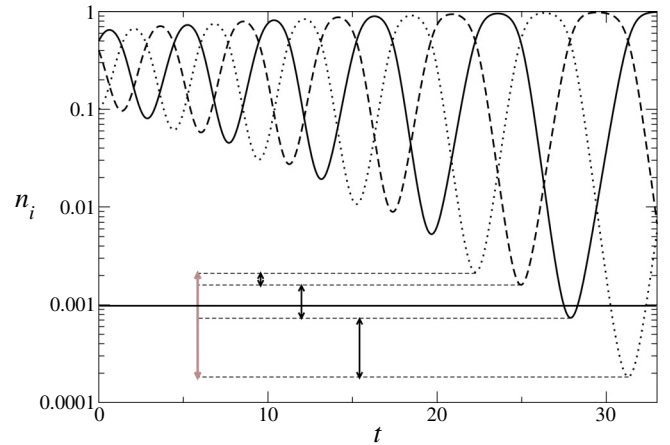

Fig. 5. Fractions of the population adopting each of the different strategies, on logarithmic scale, as a function of time for a specific selection of cooperation parameters obtained through the evolution of the dynamical system. We link the probability of extinction of certain strategy to the difference between the logarithm of a minimum of that strategy and the logarithm of the minimum of the previous strategy that reached a minimum (with the proper normalization). The horizontal line at $n_i = 10^{-3}$ corresponds to the cutoff for a population of $N = 10^3$. In this case, the strategy plotted in continuous line will go extinct as $n_i < 1/N$ for that strategy.

Firstly, we plot the evolution of the dynamical system when the heteroclinic cycle is stable in a specific case to see what relationship exist between the finite system and the corresponding mean field model.

Figure 5 shows the fractions n_i of the population adopting each of the different strategies, on logarithmic scale, as a function of time obtained through the evolution of the dynamical system for a specific selection of cooperation parameters. Let us assume that in the corresponding finite system we are working with a population of size N, so when $n_i \approx 1/N$ the strategy i will go extinct due to fluctuations (see horizontal line plotted in Fig. 5 at $n_i = 10^{-3}$). Thereby taking these previous ideas as a guide, we will associate the probability of extinction p_i^e, starting from random initial conditions, with the difference between the logarithm of a minimum of that strategy and the logarithm of the minimum of the previous strategy that reached a minimum (with the proper normalization). Considering which of the two remaining strategies is the dominant, we can calculate the survival probabilities: $p_i^s = p_{(i-1)_3}^e$.

We assume we are in the region of the parameter space where the orbit circles many times before reaching the boundary of the simplex due to fluctuations, which in our case is equivalent to ask the cooperation parameters r_i to be small. Under this condition the trajectory can be approximated taking local coordinates in the manifolds directions and linearizing the right-hand side of equations (3) [7]. Focusing on what we are interested, the orbit approximation establishes that the distance to the boundary of the simplex x changes to x^{σ_i} when the trajectory goes near the vertex i (far from the vertex, x changes

lineally). The exponent is $\sigma_i = \mu_i/\lambda_i$, where $(-\mu_i)$ is the eigenvalue of the stable manifold of vertex i, that as we previously stated is a saddle, while λ_i is the eigenvalue of the unstable manifold (in our case, $\sigma_i = 1/(1 - 2r_i)$).

Using the approximation previously described and considering that going from the vertex i to $(i-1)_3$ (the time evolution has the opposite sense of circulation to the dominance) the orbit reaches a minimum of $n_{(i+1)_3}$, we proceed to calculate the extinction probabilities. At first we start at a minimum of n_1, then we go through a minimum of n_0, then through a minimum of n_2 and finally we reach another minimum of n_1, so that the distance to the boundary changes as:

$$x \rightarrow x^{\sigma_2} \rightarrow x^{\sigma_2 \sigma_1} \rightarrow x^{\sigma_2 \sigma_1 \sigma_0}. \tag{5}$$

From this last equation we observe that the orbit approaches to the boundary if $\sigma_2 \sigma_1 \sigma_0 > 1$, which is the stability condition of the heteroclinic cycle. The total difference between the logarithm of the minima of n_1 is $\Delta = \log(x^{\sigma_2 \sigma_1 \sigma_0}) - \log(x) = (\sigma_2 \sigma_1 \sigma_0 - 1)\log(x)$, so when we start from a minimum of n_1 the extinction probabilities are:

$$p_0^{(1)} = \frac{\sigma_2 - 1}{\sigma_2 \sigma_1 \sigma_0 - 1}$$

$$p_2^{(1)} = \frac{\sigma_2(\sigma_1 - 1)}{\sigma_2 \sigma_1 \sigma_0 - 1}$$

$$p_1^{(1)} = \frac{\sigma_2 \sigma_1(\sigma_0 - 1)}{\sigma_2 \sigma_1 \sigma_0 - 1}. \tag{6}$$

However, we can also start from a minimum of n_2 or n_0. Taking average of these three cases we obtain the probabilities of extinction p_i^e due to fluctuations, which have a direct connection with the survival probabilities p_i^s as we previously stated

$$p_0^e = p_1^s = \frac{(1 + \sigma_0 + \sigma_0\sigma_1)(\sigma_2 - 1)}{3(\sigma_0\sigma_1\sigma_2 - 1)}$$

$$p_1^e = p_2^s = \frac{(1 + \sigma_1 + \sigma_1\sigma_2)(\sigma_0 - 1)}{3(\sigma_1\sigma_2\sigma_0 - 1)}$$

$$p_2^e = p_0^s = \frac{(1 + \sigma_2 + \sigma_2\sigma_0)(\sigma_1 - 1)}{3(\sigma_2\sigma_0\sigma_1 - 1)}. \tag{7}$$

If $\sigma_i < 1$, keeping $\sigma_2\sigma_1\sigma_0 > 1$, it implies that the strategy $(i-1)_3$ will not survive, and only the other two strategies have a chance to outlive. In this case, the survival probabilities should be calculated again and will have minor modifications (in our model this is not possible because it requires that $r_i < 0$).

In general, if we consider that the cooperation parameters are varied jointly as previously stated (r_0/r_2 and r_1/r_2 fixed with $r_2 = r$), in the limit $r \rightarrow 0$ we obtain:

$$p_0^s \approx \frac{r_1}{\sum_j r_j} + O(r)$$

$$p_1^s \approx \frac{r_2}{\sum_j r_j} + O(r)$$

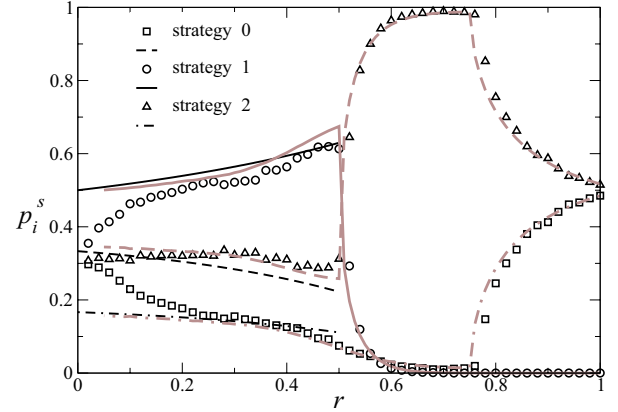$$p_2^s \approx \frac{r_0}{\sum_j r_j} + O(r) \tag{8}$$

Fig. 6. Survival probabilities, as a function of r, obtained from simulations (symbols), from the approximation given by (9) (black lines) and from the integration of the mean field differential equations (brown lines). We used $r_0 = \frac{2}{3}r$, $r_1 = \frac{1}{3}r$, $r_2 = r$ and a total population of $N = 9 \times 10^4$.

so the survival probability of a strategy is directly associated to the cooperation parameter of the corresponding dominated strategy, turning in our case strategy 1 the more likely to survive as we chose $r_2 = \max\{r_0, r_1, r_2\}$. As we can see, when $r \rightarrow 0$ we do not recover the symmetric case where all probabilities are the same ($p = 1/3$).

To analyze how the equations given by (7) work in a particular case, we vary the cooperation parameters keeping a fixed ratio between them so that $r_1 = \frac{1}{3}r$, $r_2 = r$ and $r_0 = \frac{2}{3}r$. With this choice, the survival probabilities are

$$p_1^s = \frac{27 - 24r + 8r^2}{54 - 66r + 24r^2} \approx \frac{1}{2} + \frac{r}{6} + \frac{7r^2}{54}$$

$$p_2^s = \frac{9 - 14r + 4r^2}{27 - 33r + 12r^2} \approx \frac{1}{3} - \frac{r}{9} - \frac{11r^2}{81}$$

$$p_0^s = \frac{9 - 14r + 8r^2}{54 - 66r + 24r^2} \approx \frac{1}{6} - \frac{r}{18} + \frac{r^2}{162}. \tag{9}$$

These equations hold only for $0 < r < 1/2$ where the heteroclinic cycle is stable. We compare these results with simulations and with those obtained through numerical integration of the equations given by (3).

Figure 6 shows the survival probabilities as a function of r, obtained from simulations, from the heteroclinic cycle approximation given by (9) and from the integration of the mean field differential equations (3) for each of the strategies. There is a good agreement between the simulations and mean field integration, and also with the analytic approximation (in the range of r where it is valid). In the limit $r \rightarrow 0$, the simulations deviate from the other two curves and go to the symmetric case where all probabilities are equal ($p_i^s = 1/3$). This can be understood considering that for small values of r the difference between the minima of $n_j(t)$ become smaller and any fluctuation due to the finite size of the population may make it disappear (this effect is clearly not included in the mean field integration nor in the analytic approximation). Sharp changes in the probabilities are observed for $r = 0.5$ and $r = 0.75$, which as we already studied are caused by the
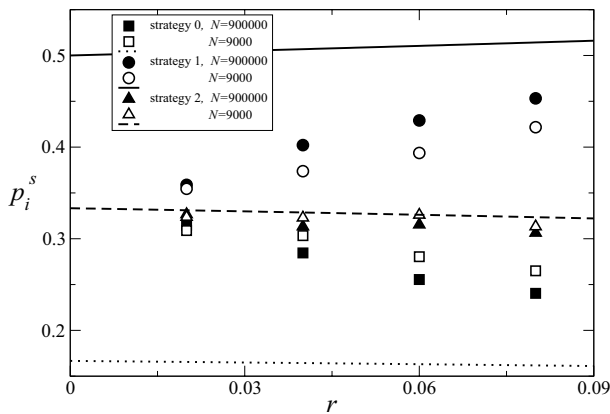
Fig. 7. Survival probabilities as a function of the cooperation parameter r obtained from simulations (symbols) and from the approximation given by (9) (lines). We used $r_0 = \frac{2}{3}r$, $r_1 = \frac{1}{3}r$ and $r_2 = r$. To study the effect of population size, we considered $N = 9 \times 10^3$ and $N = 9 \times 10^5$. For each value of r, 10^4 initial conditions were taken (uniformly distributed on the simplex).

appearance of a stable point on the vertex of the simplex, and by the increase of its basin of attraction.

Figure 7 shows the effect of increasing the population size (from $N = 9 \times 10^3$ to $N = 9 \times 10^5$) for small values of the cooperation parameter r. The points obtained from simulations approach the analytic approximation as N grows, but convergence is slow.

6 Discussion

We have studied a system where agents adopting one of three possible strategies interact in a cyclic game. The basic interaction events occur in triplets, which allowed us to consider cooperation amongst weak agents with the same strategy when facing a strong agent. The system was analyzed by numerical simulations and by a mean field approach, which gives a correct description as the triplets are homogeneously distributed through the population. The focus was placed on studying the survival probabilities of the different strategies as a function of the cooperation parameters.

An interesting fact is that cooperation within triplets of agents possessing the same strategy results in a final and global benefit of another strategy, in particular the one that dominates the first strategy. This conclusion, however, is valid only for low levels of cooperation. When the cooperation exceeds a specific threshold, enough to create a stable pure state, it becomes beneficial to the strategy that adopts it.

Secondly, we remark the importance of obtaining the survival probabilities of each strategy as a function of the properties of the heteroclinic cycle derived from the corresponding dynamic system, in particular the eigenvalues of the fixed points that form such cycle. We are

able to make predictions of the asymptotic behavior of the system only from the basic parameters of the dynamic, in this case the cooperation parameters. Moreover, this analytic approximation is easy to generalize to other system with planar heteroclinic cycles which is a robust behavior in biological and game theory models that possess some kind of symmetry or constraint.

Even when triplets interactions are not as frequent as binary ones, they may have an important role. For example if we consider a rock-paper-scissors game played in pairs in a homogeneous population, a model often used for biological systems, the collective response corresponds to oscillations, as we showed in Section 4. But if triplets interactions are introduced with a small probability, the system eventually goes to one of the pure states, changing the qualitative behavior of the population. So neglecting these complex interactions, even when they are rare, it may lead to wrong conclusions.

An interesting aspect to study in future works is how topology affects the results obtained. This generalization can be approached along two directions. In the first place, by considering complex triplets networks with different properties and, on the other side, by implementing this model on a population where the interactions occur in groups of different sizes (pairs, triplets, quartets, etc.).

References

1. R. Albert, A. Barabási, Rev. Mod. Phys. **74**, 47 (2002)
2. D.J. Watts, S.H. Strogatz, Nature **393**, 440 (1998)
3. M.E.J. Newman, SIAM Rev. **45**, 167 (2003)
4. J. Johnson, Multidimensional events in multilevel systems, in *The Dynamics of Complex Urban Systems* (Physica-Verlag, Germany, 2008)
5. K. Starkey, Ch. Barnatt, S. Tempest, Org. Sci. **11**, 299 (2000)
6. D.H. Zanette, Phil. Trans. R. Soc. A **367**, 3311 (2009)
7. J. Hofbauer, K. Sigmund, *Evolutionary Games and Population Dynamics* (Cambridge University Press, Cambridge, 1982)
8. J. Maynard Smith, *Evolution and the Theory of Games* (Cambridge University Press, Cambridge, 1982)
9. B. Kerr, M.A. Riley, M.W. Feldman, B.J. Bohannan, Nature **418**, 171 (2002)
10. B. Sinervo, C.M. Lively, Nature **380**, 240 (1996)
11. L.W. Buss, Proc. Natl. Acad. Sci. **77**, 5355 (1980)
12. C.E. Paquin, J. Adams, Nature **306**, 368 (1983)
13. K. Sigmund, *Games of Life: explorations in Ecology, Evolution and Behaviour* (Oxford University Press, Oxford, 1993)
14. R.M. May, W.J. Leonard, SIAM J. Appl. Math. **29**, 243 (1975)
15. J. Hofbauer, in *Equadiff 8, Proceedings of the Czech–Slovak Conference on Differential Equations and Their Applications, Bratislava, 1993*, Tatra Mountains Math. Publ. **4**, 105 (1994)
16. M. Frean, E.R. Abraham, Proc. R. Soc. London Ser. B **268**, 1323 (2001)