Complete Calabi-Yau metrics from Kahler metrics in $D = 4$

Mauricio Leston[1,*] and Osvaldo P. Santillan[2,*]

[1]*Instituto de Astronomia y Fisica del Espacio (IAFE), Buenos Aires, Argentina Casilla de Correo 67 - Suc. 28 (C1428ZAA), Ciudad Autónoma de Buenos Aires, Argentina*

[2]*Departamento de Matematica, FCEyN, Universidad de Buenos Aires, Buenos Aires - 2010 Ciudad Universitaria - Pabellón I - (C1428EGA), Buenos Aires, Argentina*

(Received 19 July 2010; published 1 October 2010)

In the present work, a family of Calabi-Yau manifolds with a local Hamiltonian Killing vector is described in terms of a nonlinear equation whose solutions determine the local form of the geometries. The main assumptions are that the complex (3, 0)-form is of the form $e^{ik}\bar{\Psi}$, where $\bar{\Psi}$ is preserved by the Killing vector, and that the space of the orbits of the Killing vector is, for fixed value of the momentum map coordinate, a complex 4-manifold, in such a way that the complex structure of the 4-manifold is part of the complex structure of the complex 3-fold. The family considered here include the ones considered in A. Fayyazuddin, Classical Quantum Gravity **24**, 3151 (2007); O. P. Santillan, Classical Quantum Gravity **27**, 155013 (2010); H. Lu, Y. Pang, and Z. Wang, Classical Quantum Gravity **27**, 155018 (2010) as a particular case. We also present an explicit example with holonomy exactly SU(3) by use of the linearization introduced in A. Fayyazuddin, Classical Quantum Gravity **24**, 3151 (2007), which was considered in the context of D6 branes wrapping a complex 1-cycle in a hyperkahler 2-fold.

DOI: 10.1103/PhysRevD.82.085004 PACS numbers: 11.27.+d, 02.40.−k

I. INTRODUCTION

The development of the subject of Calabi-Yau (CY) manifolds is an illustrative example of the interplay between algebraic geometry and string theory. On the one hand, CY spaces are interpreted as internal spaces of string and M-theory giving supersymmetric field theories after compactification. In fact, CY 3 folds may provide compactifications which are more realistic than the ones corresponding to other Ricci-flat manifolds such as G_2 holonomy spaces, for which the generation of chiral matter and non-Abelian gauge symmetries seems harder (but not impossible) to achieve. On the other hand, string theory compactifications stimulated several new trends in the algebro-geometrical aspects of CY spaces; an example is the subject of mirror symmetry.

By definition, a CY manifold is a compact Kahler n-dimensional manifold with vanishing first Chern class. The Yau proof of the Calabi conjecture implies that these manifolds admit a Ricci-flat metric and their holonomy is reduced from SO(2n) to SU(n) [1]. Although compact Ricci-flat metrics exist, no explicit expressions have been found. The main technical problem for that is that a compact Ricci-flat metric does not admit globally defined Killing vectors (leaving aside the possibility to have trivial flat U(1) factors), and the absence of continuous symmetries makes the task of solving the Einstein equations explicitly really hard. For the noncompact case, the definition usually adopted is that a CY manifold is a Ricci-flat Kahler manifold, which also implies that the holonomy is reduced to SU(n) or to a smaller subgroup. In this case, several Calabi-Yau metrics with isometries have been found in [2–19]. Some of these metrics posses conical singularities, but in some cases these singularities have been resolved to give complete metrics.

Although noncompact Calabi-Yau metrics are not suitable for studying compactification in string theory, they have several applications in mathematical and theoretical physics. For instance, the localization techniques pioneered by Kontsevich [20,21] to calculate Gromov-Witten invariants is more easy to implement in the noncompact case and sometimes these invariants may been calculated for arbitrary genus. Also, it was conjectured in [22] that Chern-Simmons on S^3 is equivalent to topological strings on the resolved conifold T^*S^3, which is Calabi-Yau. These has been generalized in [23] where it is shown that for some three dimensional manifold M, the space T^*M is Calabi-Yau, and it was conjectured that Chern-Simmons on M is dual to topological strings propagating in T^*M (See [24] for a nice review).

In view of the above discussion, to find general methods for constructing noncompact CY metrics with isometries is a task of interest. An step in that direction was initiated by Fayyazuddin in [25] where the supergravity backgrounds corresponding to D6 branes wrapping a complex submanifold inside a 4-dimensional hyperkahler space were characterized in terms of a single linear equation. It was also shown in that reference that the uplift to 11 dimensions results in a purely geometrical background of the form $M_{1,4} \times Y_6$ where Y_6 is a Calabi-Yau space. The Ricci-flat Kahler metric on Y_6 is therefore determined by this linear equation, which is expressed in term of the laplacian over the curved hyperkahler space the branes wrap. For all these geometries, there is a U(1) isometry preserving the whole

*mauricio@iafe.uba.a
†firenzecita@hotmail.com and osantil@dm.uba.ar

SU(3) structure (which is, in particular, Hamiltonian, and therefore it defines a momentum map local coordinate) such that space formed by the orbits of the Killing vector is, for fixed values of the momentum map coordinate, a Kahler manifold. The Fayyazuddin construction was reconsidered in [26] where it was shown that the assumption that the quantities defining the geometry vary over a complex submanifold may be relaxed without violating the Calabi-Yau condition. The resulting geometries were described in terms of a nonlinear equation, which reduce to the Fayyazuddin one if the quantities describing the geometry vary over a complex submanifold. The nonlinear operator is defined in terms of the metric of the hyperkahler space; in fact, this method can be interpreted as a solution generating technique which starts with a hyperkahler metric and gives a noncompact Calabi-Yau metric as outcome.

The two approaches mentioned above have been used to find nontrivial Calabi-Yau metrics with holonomy exactly SU(3). Nevertheless, none of these examples were complete metrics. This situation was substantially improved in [27] where isometries which do not preserve the SU(3) structure, but just the metric g_6 and the Kahler two form ω_6, were considered. These authors showed that one may start with a hyperkahler structure as well and construct complete Calabi-Yau metrics. In particular, the resolution of the $Y^{p,q}$ cone found in [28–30] was rediscovered in these terms. The calculations made in [27] are impressive, but there is a striking fact there that motivates the present note, which is the following. The best results obtained in [27] are obtained in terms of the flat hyperkahler structure on R^4, in particular, the resolution of the Ricci-flat cone over $Y^{p,q}$. Instead, for a curved hyperkahler structure, the resulting equations seem harder to solve, and more restricted solutions are found, or even no solutions at all. One may wonder if a method for constructing Calabi-Yau metrics without the use of initial hyperkahler structures may be developed, which may allow us to avoid this kind of problems. In the present work, such a method will be presented and family of Calabi-Yau geometries characterized by a single nonlinear equation which is not necessarily related to a hyperkahler metric. It should be emphasized that there is *nothing* wrong with the use of hyperkahler structures as initial input. What the present letter shows is that this is just optional.

The organization of the present work is as follows. In Sec. II A generalities about SU(3) structures are reviewed. In Sec. II B the SU(3) structures with a Hamiltonian Killing vector, that is, a Killing vector preserving also the Kahler form are characterized. In Sec. II C a family of Calabi-Yau metrics of this type is presented, for which the complex (3, 0) form is of the form $\Psi = e^{ik}\bar{\Psi}$ in such a way that $\bar{\Psi}$ is preserved by the Killing vector but Ψ may not be preserved due to the phase factor. In Sec. 1 and III B it is explained that the metrics considered in [25,26] belong to the family of section II C. In Sec. III C, an example where the Fayyazuddin linearization [25] works properly is worked out explicitly and a nontrivial Calabi-Yau metric is obtained as outcome. In Sec. III D, we also show that the results of [27] belong to the family constructed here. Section IV contains the discussion of the results obtained.

II. CALABI-YAU METRICS WITH HAMILTONIAN ISOMETRIES

A. The general form of the $SU(3)$ structure

In this subsection, a large family of Calabi-Yau (CY) manifolds in dimension 6 with an isometry group with orbits of codimension 1 will be characterized. It will be assumed that the Killing vector V corresponding to this isometry preserves not only the metric, but the full Kahler two form ω_6. It will be convenient to give an operative definition of CY manifolds in six dimensions first; for further details, one may consult [31]. Roughly speaking, a Calabi-Yau manifold M_6 is a Kahler manifold, thus complex sympletic, which in addition admits a Ricci-flat metric g_6. This definition means that there exists an endomorphism of the tangent space $J: TM_6 \to TM_6$ such that $J^2 = -I_d$ and for which $g_6(X, JY) = -g_6(JX, Y)$ being X and Y arbitrary vector fields. It is commonly said that the metric g_6 is Hermitian with respect to J and the tensor $(g_6)_{\mu\alpha}J^{\alpha}_{\nu}$ is skew symmetric; therefore, locally it defines a 2-form

$$\omega_6 = \frac{1}{2}(g_6)_{\mu\alpha}J^{\alpha}_{\nu}dx^{\mu} \wedge dx^{\nu}. \tag{2.1}$$

Here, x^{μ} is a local choice of coordinates for M_6. The endomorphism J is called an almost complex structure. If the Nijenhuis tensor

$$N(X, Y) = [X, Y] + J[X, JY] + J[JX, Y] - [JX, JY],$$

vanishes identically, then the tensor J will be called a complex structure and M_6 a complex manifold. This is the case for any CY manifold. The Newlander-Niremberg theorem states that there is an atlas of charts for M_6 which are open subsets in C^n, in such a way that the transition maps are holomorphic functions. These local charts are parameterized by complex coordinates $(z_i, \bar{z}_i)$ with $i = 1, 2, 3$ for which the complex structure looks like

$$J^{j}_{i} = -J^{\bar{j}}_{\bar{i}} = i\delta^{j}_{i}, \qquad J^{\bar{j}}_{i} = J^{i}_{\bar{j}} = 0, \tag{2.2}$$

and for which the metric and the 2-form (2.1) are expressed as follows

$$g_6 = (g_6)_{i\bar{j}}dz_i \otimes d\bar{z}_j, \tag{2.3}$$

$$\omega_6 = \frac{i}{2}(g_6)_{i\bar{j}}dz_i \wedge d\bar{z}_j. \tag{2.4}$$

The form (2.4) is called of type (1, 1) with respect to J, while a generic 2-form containing only terms of the form $(dz_i \wedge dz_j)$ or $(d\bar{z}_i \wedge d\bar{z}_j)$ will be called of type (2, 0) or (0, 2), respectively. In addition, a Calabi-Yau manifold is sympletic with respect to ω_6; in other words, $d\omega_6 = 0$. A complex manifold which is sympletic with respect to (2.1) is known as a Kahler manifold; thus, CY spaces are all Kahler. The Kahler condition itself implies that the holomy is reduced from SO(6) to U(3). Furthermore, the fact that g_6 is Ricci-flat is equivalent to the existence of a 3-form

$$\Psi = \psi_+ + i\psi_-, \tag{2.5}$$

of type (3, 0) with respect to J, satisfying the compatibility conditions [32]

$$\begin{aligned} \omega_6 \wedge \psi_\pm &= 0, \\ \psi_+ \wedge \psi_- &= \frac{2}{3}\omega_6 \wedge \omega_6 \wedge \omega_6 \simeq dV(g_6), \end{aligned} \tag{2.6}$$

and which is closed, i.e,

$$d\psi_+ = d\psi_- = 0. \tag{2.7}$$

The relations (2.6) can be expressed in a more compact way as

$$\omega_6 \wedge \Psi = 0, \qquad \Psi \wedge \bar{\Psi} = \frac{1}{3}\omega_6 \wedge \omega_6 \wedge \omega_6 \simeq dV(g_6). \tag{2.8}$$

In the formula (2.8), $dV(g_6)$ denotes the volume form of g_6. In the situations described in (2.7), the holonomy is further reduced from U(3) to SU(3); thus, CY manifolds are of SU(3) holonomy. The converse of these statements are also true, that is, for any Ricci-flat Kahler metric in $D = 6$, there will exist an SU(3) structure (ω_6, Ψ) satisfying (2.8) and also

$$d\omega_6 = d\Psi = 0. \tag{2.9}$$

The knowledge SU(3) structure determines univocally metric g_6. In fact, the task to find complex coordinates for a given CY manifold may be not simple, but there always exists a tetrad basis e^a with $a = 1, \ldots, 6$ for which the SU (3) structure is expressed as

$$\omega_6 = \frac{i}{2}(E_1 \wedge \bar{E}_1 + E_2 \wedge \bar{E}_2 + E_3 \wedge \bar{E}_3,) \tag{2.10}$$

$$\Psi = E_1 \wedge E_2 \wedge E_3, \tag{2.11}$$

where $E_i \equiv e_j + ie_{j+1}$ $(j = 1, 3, 5)$, and for which the metric is

$$g_6 = E_1 \otimes \bar{E}_1 + E_2 \otimes \bar{E}_2 + E_3 \otimes \bar{E}_3. \tag{2.12}$$

Note that the multiplication by a phase factor $E_i \to e^{ik}E_i$ does not change the metric and induce the transformation $\Psi \to e^{3ik}\Psi$ on the (3, 0) form. This phase transformation does not alter the conditions (2.8). This fact will be important in the following.

B. Kahler structures with Hamiltonian isometries

The description given above just collects general facts about CY manifolds. In the following, we will assume that our CY manifold M_6 is equipped with a metric g_6 in such a way that there is a Killing vector V preserving g_6 and the Kahler form ω_6. In this situation, there exists a local coordinate system (α, x^i) with $i = 1, \ldots, 5$ for which $V = \partial_\alpha$ and for which the metric tensor g_6 takes the following form

$$g_6 = \frac{(d\alpha + A)^2}{H^2} + Hg_5, \tag{2.13}$$

where the function H, the one form A, and the metric tensor g_5 are independent on the coordinate α. Thus, these objects live in a 5-dimensional space which we denote M_5. The metric g_5 appearing in (2.13) can be expressed as $g_5 = e^a \otimes e^a$ with $a = 1, \ldots, 5$ for some basis of α-independent 1-forms e^a. Then, if V also preserves the Kahler form ω_6 (as we are assuming), one has the decomposition

$$\omega_6 = \omega_4 + \frac{1}{\sqrt{H}}e^5 \wedge (d\alpha + A). \tag{2.14}$$

Here the 1-form e^5 is by definition

$$\frac{e^5}{\sqrt{H}} = i_{\partial_\alpha}\omega_6, \tag{2.15}$$

i_V denoting the contraction with the vector field V. The elementary formula in differential geometry

$$d_5(i_{\partial_\alpha}\omega_6) = \mathcal{L}_{\partial_\alpha}\omega_6 - i_{\partial_\alpha}d\omega_6, \tag{2.16}$$

together with (2.15) implies that

$$d_5\left(\frac{e^5}{\sqrt{H}}\right) = \mathcal{L}_{\partial_\alpha}\omega_6 - i_{\partial_\alpha}d\omega_6. \tag{2.17}$$

Here, $d_5 = \partial_i dx^i$ and $\mathcal{L}_{\partial_\alpha}$ is the Lie derivate along the vector ∂_α. But the vector ∂_α, by assumption, preserves ω_6 and ω_6 is closed; thus, the right-hand side of (2.17) vanishes and

$$d_5\left(\frac{e^5}{\sqrt{H}}\right) = 0. \tag{2.18}$$

The last relation can be integrated, at least locally, to obtain that

$$e^5 = \sqrt{H}dy, \tag{2.19}$$

y being some function of the coordinates x^i parameterizing M_5, which is known as the momentum map of the isometry. At least locally, one can take the function y defined in (2.19) as one of the coordinates, which leads to the decomposition $M_5 = M_4 \times R_y$ and $d_5 = d_4 + \partial_y dy$. The metric (2.13) in this coordinates becomes

$$g_6 = \frac{(d\alpha + A)^2}{H^2} + H^2dy^2 + Hg_4(y), \tag{2.20}$$

where the tensor $Hg_4(y)$ will be determined below under certain additional assumptions. The Kahler form is

$$\omega_6 = \omega_4(y) + dy \wedge (d\alpha + A). \tag{2.21}$$

The next task will be to find specific examples of this type of structure.

C. Calabi-Yau metrics with Hamiltonian isometries

In this subsection, the generic Kahler structure described above will be extended to a specific family of Calabi-Yau structures. The main assumption will be that, for fixed values of the coordinates (α, y), the resulting 4-manifold is complex, and that the 2-form ω_4 appearing in (2.21) is of type (1, 1) with respect to a complex coordinate system for this manifold. This may be paraphrased by saying that the complex structure of the complex 4-manifold is part of the complex structure of the Ricci-flat Kahler 6-manifold. By denoting the complex coordinates as $(z_1, z_2, \bar{z}_1, \bar{z}_2)$, the main assumption implies that (2.20) may be expressed as

$$g_6 = \frac{(d\alpha + A)^2}{H^2} + H^2 dy^2 + Hg_4(y)_{z_i\bar{z}_j} dz_i \otimes d\bar{z}_j, \quad (2.22)$$

and the dependence on the coordinate y is only as a parameter.

In order to extend the Kahler structure given above to an SU(3) structure, an anzatz for the form Ψ of (2.5) is needed, in such a way that the compatibility conditions (2.8) are identically satisfied. By analogy with the choice [27], we propose the following form for Ψ

$$\Psi = e^{iK}\Omega_4 \wedge \left[Hdy + i\frac{(d\alpha + A)}{H}\right], \quad (2.23)$$

K being a function that may depend α and varying over M_5. The remaining quantities appearing in (2.23) are assumed to be α-independent. The compatibility conditions (2.8) are then satisfied if and only if

$$2\omega_4 \wedge \omega_4 = \Omega_4 \wedge \bar{\Omega}_4 = 4\det(Hg_4) dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2. \quad (2.24)$$

This relation is, for fixed value of the coordinate y, the same as the compatibility condition for SU(2) structures. It is a standard fact that if there is a complex coordinate system for which ω_4 is of type (1, 1), then Ω_4 is of type (2, 0) with respect to it. This means that

$$\Omega_4 = Hf dz_1 \wedge dz_2,$$

f being a function independent on α and varying over M_5 and the factor H in front is just by convenience. The compatibility condition (2.24) implies that

$$2\omega_4 \wedge \omega_4 = \Omega_4 \wedge \bar{\Omega}_4 = H^2 f^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2, \quad (2.25)$$

and by comparing (2.24) with (2.25), one obtains

$$H^2 f^2 = 4\det(Hg_4). \quad (2.26)$$

Taking into account all these relations and (2.23), it follows easily that

$$\Psi = e^{iK} H^2 f dz_1 \wedge dz_2 \wedge \left[dy + \frac{i(d\alpha + A)}{H^2}\right]. \quad (2.27)$$

The next task is to fix the unknown quantities A, H, f, and K by the Calabi-Yau condition (2.9). The first one applied to (2.21) gives

$$d_4\omega_4(y) = 0, \quad (2.28)$$

and

$$d_4 A = \partial_y \omega_4. \quad (2.29)$$

Note that the Eq. (2.28) implies, for fixed value of y, that Hg_4 is not only complex but also *Kahler*. The second (2.9) gives several equations, corresponding to the vanishing of each component of $d\Psi$. The vanishing of the terms with $(dz_1 \wedge dz_2 \wedge dy \wedge d\alpha)$ implies that

$$K_y = 0, \quad (2.30)$$

$$H^2 f \partial_\alpha K - f_y = 0. \quad (2.31)$$

The second equation implies that $K = K_0 + \alpha K_1$, with K_0 and K_1 independent of y. By combining this with the first one, it is obtained that

$$H^2 f K_1 = f_y. \quad (2.32)$$

The terms of the form $(dz_1 \wedge dz_2 \wedge d\alpha \wedge d\bar{z}_i)$ vanish if and only if

$$\bar{\partial} K_1 = 0, \quad (2.33)$$

$$-f\bar{\partial}K_0 + i\bar{\partial}f + fK_1\bar{A} = 0. \quad (2.34)$$

Since K_1 is real and y-independent, the first of these equations implies that it is a constant, which can be set to 0, 1 without loosing generality. The case $K_1 = 0$ correspond to a Killing vector preserving the whole SU(3) structure, which is the case considered in [26]. But for the moment, we focus in the case $K = 1$. In this case, the last equation implies that

$$d_4^c f = f d_4 K_0 - K_1 f A. \quad (2.35)$$

For these cases, the terms with $(dz_1 \wedge dz_2 \wedge dy \wedge d\bar{z}_i)$ vanish when

$$d_4^c f_y = -K_1 \partial_y (fA). \quad (2.36)$$

An immediate consequence the last two equation is

$$d_4 K_0 = 0. \quad (2.37)$$

Inserting this relation into (2.35) gives

$$d_4^c(\log f) = -A. \quad (2.38)$$

By taking d_4 in both sides of the last equation and using (2.29), it is seen that

$$d_4 d_4^c(\log f) = -\partial_y \omega_4. \quad (2.39)$$

But the condition (2.28) implies that the complex 4-dimensional manifold M_4 is also a *Kahler* manifold, with ω_4 being the Kahler form. Therefore, ω_4 has a Kahler potential G, that is, $\omega_4 = d_4 d_4^c G$. The Eq. (2.39) implies that

$$f = U(z_1, z_2) e^{-G_y}, \quad (2.40)$$

with $U(z_1, z_2)$ an arbitrary holomorphic function. In addition, Eq. (2.32) gives that $H^2 = G_{yy}$, and by combining this with (2.25) and (2.40), it is obtained that

$$U(z_1, z_2)(e^{-2G_y})_y = 32(G_{1\bar{1}}G_{2\bar{2}} - G_{(1/\mathrm{line}2)}G_{2\bar{1}}), \quad (2.41)$$

and that $H^2 = G_{yy}$, with $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} G$. But the holomorphic function can be absorbed by a holomorphic coordinate change $z_i' = f_i(z_1, z_2)$; thus, there exists always a local coordinate system such that (2.41) takes the form

$$(e^{-2G_y})_y = 32(G_{1\bar{1}}G_{2\bar{2}} - G_{1\bar{2}}G_{2\bar{1}}). \quad (2.42)$$

In this way, all the quantities appearing in the six dimensional metric are expressed in terms of G. Explicitly, the Calabi-Yau metric is

$$g_6 = \frac{(d\alpha + d_4^c G_y)^2}{G_{yy}} + G_{yy}dy^2 + 2G_{i\bar{j}}dz_i \otimes d\bar{z}_j. \quad (2.43)$$

For $K_1 = 0$, a calculation completely analogous to the one given above shows that the metric is still (2.43) but in this case G is given by

$$G_{yy} = 8(G_{1\bar{1}}G_{2\bar{2}} - G_{1\bar{2}}G_{2\bar{1}}). \quad (2.44)$$

Note that in both cases $K_1 = 0, 1$, the metric is determined in terms of a single function G.

It should be mentioned that the method described by (2.42) or (2.44) may be generalized to arbitrary complex dimensions in straightforward manner. The resulting metrics will be described by (2.43), but the function G will depend on n-complex coordinates z_i with $i = 1, \ldots, n$ and will be the solution of

$$(e^{-2G_y})_y = 2^{2n+1}\det(G_{i\bar{j}}), \quad (2.45)$$

for $K_1 = 1$ and of

$$G_{yy} = 2^{n+1}\det(G_{i\bar{j}}), \quad (2.46)$$

for $K_1 = 0$, $\det(G_{i\bar{j}})$ being the determinant of the matrix whose entries are the second derivatives of G of type (1, 1). The resulting metric (2.43) will have $(n+1)$ complex dimensions, but in the following, we will keep considering the case $n = 2$.

III. SOLUTIONS RELATED TO HYPERKAHLER STRUCTURES

In the following sections, the connection between the solution generating technique given by (2.42), (2.43), and (2.44) and the known ones given in [25–27] will be detailed. The assumptions for obtaining the CY metrics (2.42), (2.43), and (2.44) were the following: there is an isometry preserving the CY metric and the Kahler 2-form; the complex 3-form has the generic expression (2.23); the manifold obtained for fixed values of y and α is complex, in such a way that the metric is of the form (2.22) and such that the 2-form ω_4 appearing in (2.21) is of type (1, 1) with respect to the complex coordinates. The last assumption automatically implies that the complex (3, 0) form is given by (2.27). These, together with the Calabi-Yau condition, determined completely the local form of the Calabi-Yau metric (2.42), (2.43), and (2.44). The task is now to show that the metrics [25–27] are under these hypothesis, and therefore they are a particular case of (2.42), (2.43), and (2.44).

A. Examples with isometries preserving the whole SU(3) strucuture

In this subsection, the results of [25,26] are briefly reported; for more details about the proofs, we refer the reader to the original references. The solution generating techniques of [25,26] start with a hyperkahler structure $\tilde{\omega}_i$ with $i = 1, 2, 3$, and one of these closed 2-forms, say $\tilde{\omega}_i$, is deformed to a new y-dependent 2-form

$$\omega_4(y) = \tilde{\omega}_1 - d_4 d_4^c G, \quad (3.1)$$

while $\tilde{\omega}_2$ and $\tilde{\omega}_3$ are kept intact. This 2-form plays the role of $\omega_4(y)$ in (2.21). Here, the operator $d^c = J_1\, d$ is constructed in terms of the complex structure J_1 which is defined by $\tilde{\omega}_1$ and the hyperkahler metric by the relation (2.3). In the expression (3.1), G denotes an unknown function which varies on M_4 and which, in a generic situation, may depend also on the coordinate y. If there is a Killing vector preserving the whole SU(3) structure, which corresponds to the case $K_1 = 0$ in (2.23), then the SU(3) structure (2.21) and (2.27) takes the following form

$$\begin{aligned} \omega_6 &= \tilde{\omega}_1 - d_4 d_4^c G + dy \wedge (d\alpha + A), \\ \psi_+ &= H^2 \tilde{\omega}_3 \wedge dy + \tilde{\omega}_2 \wedge (d\alpha + A), \\ \psi_- &= -H^2 \tilde{\omega}_2 \wedge dy + \tilde{\omega}_3 \wedge (d\alpha + A), \end{aligned} \quad (3.2)$$

with $\Psi = \psi_- + i\psi_+$. Given the deformed structure (3.1), the compatibility condition (2.6) implies that

$$(\tilde{\omega}_1 - d_4 d_4^c G) \wedge (\tilde{\omega}_1 - d_4 d_4^c G) = H^2 \tilde{\omega}_2 \wedge \tilde{\omega}_2, \quad (3.3)$$

and, as the wedge products appearing in the last equality are all proportional to the volume form $dV(g_4)$ of the initial hyperkahler metric g_4, the relation

$$(\tilde{\omega}_1 - d_4 d_4^c G) \wedge (\tilde{\omega}_1 - d_4 d_4^c G) = M(G)\tilde{\omega}_1 \wedge \tilde{\omega}_1, \quad (3.4)$$

defines a nonlinear expression $M(G)$ involving G. The CY condition (2.9) applied to (3.2) imposes further constraints, which are explained in detail in [26] and which we will not reproduce here. The result is that the geometry is described in terms of a nonlinear differential equation determining the function G and which involves the operator $M(G)$. This equation is[1]

$$G_{yy} = M(G). \quad (3.5)$$

[1]This equation strongly resembles the one found in [33] for the G2 holonomy case.

In addition, the explicit expression for the SU(3) structure is completely determined in terms of G as

$$\begin{aligned}\omega_6 &= \tilde{\omega}_1 - d_4 d_4^c G + dy \wedge (d\alpha - d_4^c G_y),\\ \psi_+ &= G_{yy}\tilde{\omega}_3 \wedge dy + \tilde{\omega}_2 \wedge (d\alpha - d_4^c G_y),\\ \psi_- &= -G_{yy}\tilde{\omega}_2 \wedge dy + \tilde{\omega}_3 \wedge (d\alpha - d_4^c G_y).\end{aligned} \tag{3.6}$$

The generic form of the 6-dimensional Calabi-Yau metric corresponding to this structure is given by

$$g_6 = g_4(y) + G_{yy}dy^2 + \frac{(d\alpha - d_4^c G_y)^2}{G_{yy}}, \tag{3.7}$$

where $g_4(y)$ is the Kahler 4-dimensional metric corresponding to the deformed Kahler structure $\omega_1(y) = \tilde{\omega}_1 - d_4 d_4^c G$.

It is important to remark that the metrics of this subsection are under the hypothesis leading to (2.42), (2.43), and (2.44). First of all, the 2-form $\omega_1(y)$ introduced in (3.1) is of type (1, 1) with respect to the complex coordinates which diagonalize J_1. This follows from the fact that ω_1 is of type (1, 1) with respect to these coordinates, and the term $d_4 d_4^c G$ is also of this type. The form $\tilde{\omega}_2 + i\tilde{\omega}_3$ is kept intact and, for a closed hyperkahler structure, is of type (2, 0). Moreover, (3.1) is closed with respect to d_4, which leads immediately to the condition (2.28). In addition, (3.2) is of the type (2.44). All this implies that the metrics (3.6) and (3.7) are a subcase of the family of Calabi-Yau metrics described in Sec. II C.

B. The Fayyazuddin linearization

The family of SU(3) structures (3.6) and (3.7) found above are completely determined in terms of a single function G which is a solution of (3.5). This is a nonlinear equation, and the general solution is not known, but it can be solved in some specific examples. The source of the nonlinearity of the operator $M(G)$ defined in (3.4) and (3.5) is the quadratic term

$$Q(G) = d_4 d_4^c G \wedge d_4 d_4^c G. \tag{3.8}$$

Therefore, the operator $M(G)$ will reduce to a linear one if $Q(G)$ vanishes. This will be the case when the function G is of the form $G = G(w, \bar{w})$ where $w = f(z_1, z_2)$ is a holomorphic function of the coordinates (z_1, z_2) which diagonalize the complex structure J_1 [34]. This affirmation may be justified as follows. By use of the simple expression

$$dd^c G = 2iG_{i\bar{j}}dz_i \wedge d\bar{z}_j, \tag{3.9}$$

the quadratic term (3.8) may be rewritten as

$$Q(G) = -4(G_{1\bar{1}}G_{2\bar{2}} - G_{1\bar{2}}G_{2\bar{1}})dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2. \tag{3.10}$$

But the functional dependence $G = G(w, \bar{w})$ implies that

$$G_{i\bar{j}} = w_i \bar{w}_{\bar{j}} G_{w\bar{w}},$$

and by inserting this into (3.10) gives $Q(G) = 0$. This result may be paraphrased as follows. If the function G depends only on two complex coordinates $(w, \bar{w})$, then the quantity $d_4 d_4^c G$ is essentially a 2-form in two dimensions; therefore, the wedge product (3.8) vanishes identically.

The situation described above is essentially the one considered by Fayyazuddin in Ref. [25] and, if suitable boundary conditions are imposed, the resulting metrics give a dual description of D6 branes wrapping a complex submanifold in a hyperkahler manifold. A simple example is obtained when the initial hyperkahler structure is the flat metric on R^4 and G_{yy} varies over an arbitrary set of 2-dimensional hyperplanes inside R^4. There, it was shown in [25] that the resulting metrics are the direct sum of the flat metric in $R^2 \simeq C$ and a general Gibbons-Hawking metric in dimension four [35]. These metrics are of holonomy SU(2), which is a subgroup of SU(3). Our aim in the following is to improve this situation and find Calabi-Yau metrics with holonomy *exactly* SU(3) by use of this linearization.

C. Calabi-Yau extensions of the 4-dimensional Bando, Kobayashi, Tian and Yau (BKTY) metrics

In the present subsection, Fayyazuddin linearization explained above will be illustrated with an explicit example. This linearization is performed in terms of an initial hyperkahler structure, and the one considered in Refs. [5,6] will be chosen by simplicity, namely, the distance element

$$g_4 = zdz^2 + z(dx^2 + du^2) + \frac{1}{z}(dt - xdu)^2. \tag{3.11}$$

By denoting $V = z$ and $A = -xdy$, it is seen that (3.11) takes the usual Gibbons-Hawking form [35], which means that it is hyperkahler and with a triholomorphic Killing vector $K = \partial_t$. The solution (3.11) corresponds to a superposition of 6-branes, which results in a linearly growing potential independent on the coordinates (x, u). In fact, $V = z$ is the electric potential for a infinite plane with constant density charge at $z = 0$, for which the electric field is constant.

The first difficulty for (3.11) is that crossing the plane $z = 0$ implies a change in its signature. Something similar happens, for instance, for the Taub-Nut metric with negative mass parameter. The last one corresponds to a potential $V = 1 - 1/r$ and has a change of signature when crossing the region $r = 1$. For the Taub-Nut metric, the explanation is that it is asymptotic to the Atiyah-Hitchin metric, which is complete and regular. The change of the signature is an indication that the Taub-Nut approximation breaks down for $r > 1$. It is plausible to think that something similar happens for (3.11). In fact, there have been several approaches to interpret its meaning. The authors of [6] proposed to replace z by $|z|$ in (3.11). They justify this procedure by interpreting the region $z = 0$ as a source plane, and the regions $z > 0$ and $z < 0$ are the sides of a domain wall. The problem is that the metric in the surface $z = 0$ is singular. Another idea was introduced in [5]. In that reference, the authors were able to identify an exact hyperkahler metric for which (3.11) is the asymptotic form. These authors observed that the coordinates (x, u) may parameterize a torus T^2 by making the coordinate t periodic such that the periods satisfy

$$n = \frac{T_x T_u}{T_t},$$

being n an integer. The resulting manifold is a nilmanifold for which the curvature of the connection pulled back to the T^2 satisfies the Dirac quantization condition

$$\frac{1}{T_t}\int_{T^2} F = n.$$

By defining the "proper time" $w = 2z^{3/2}/3$, one can write (3.11) as

$$g_4 = dw^2 + \left(\frac{3w}{2}\right)^{-(2/3)}(\sigma^3)^2 + \left(\frac{3w}{2}\right)^{2/3}((\sigma^1)^2 + (\sigma^2)^2) \tag{3.12}$$

where σ_k are left invariant forms on the Heisenberg group. The metric (3.12) is in fact of the Gibbons-Hawking form for $n = 2$, and it was conjectured in [5] that they describe the asymptotic form of some specific CY metrics found in [7] by Bando, Kobayashi, Tian and Yau (BTKY metrics). These metrics arise as a degenerate limit of a K3 surfaces. The point is that K3 surfaces have 58 parameter moduli spaces, and as one moves to the boundary of the moduli space, the metric may decompactify while remaining complete and nonsingular. The metric (3.12) is believed to describe the asymptotic metric of a K3 surface in one of those limits of the parameters.

Our task is now to extend (3.11) to a CY six metric. We do not expect the resulting metric to be complete as the initial hyperkahler is just valid as an asymptotic expression. But this example illustrates clearly how the Fayyazuddin linearization applies in a generic case. In order to use the linearization, a complex coordinate system for (3.11) should be found. A Kahler 2-form for this metric is

$$\omega = dt \wedge du - z dz \wedge dx, \tag{3.13}$$

and the corresponding complex structure has the following nonzero components

$$\begin{aligned} J_t^t &= \frac{x}{z}, \qquad J_t^u = \frac{1}{z}, \qquad J_x^z = -1, \\ J_u^t &= \frac{z^2 + x^2}{z}, \qquad J_u^u = \frac{x}{z}, \qquad J_z^x = 1. \end{aligned} \tag{3.14}$$

A complex coordinate system z_i with $i = 1, 2$ is then any choice for which the components of the complex structure take the form $\tilde{J}_{\bar{j}}^{\bar{i}} = -\tilde{J}_j^i = \delta_i^j$. This amounts to finding a coordinate change for which

$$\frac{\partial x^a}{\partial z^i} J_a^b \frac{\partial z^j}{\partial x^b} = \delta_i^j,$$

and the last equation is equivalent following the following system:

$$(J_b^a - i\delta_b^a)\partial_a z^i = 0, \qquad i = 1, 2. \tag{3.15}$$

It can be checked from (3.14) that Eqs. (3.15) are equivalent to the two following independent equations

$$\partial_z z^i = -i\partial_x z^i, \qquad i\partial_u z^i = (z - ix)\partial_t z^i.$$

Two independent solutions of the last system are given by $z^1 = -x + iz$ and $z^2 = iu(z - ix) + t$.

Now let us suppose that the function G in (3.5) is of the form $G = u^2 + U(w, \bar{w}, u)$ and we choose $w = z_1$. Let us denote $U_{uu} = H^2$. If we further assume that U does not depend on x, then by taking the derivative of (3.5) with respect to u twice gives an equation for H^2, namely,

$$\left(\frac{1}{z}\partial_z^2 + \partial_u^2\right)H^2 = 0, \tag{3.16}$$

with solution

$$H^2 = 1 + \frac{m}{(4z^3 + 9u^2)^{1/6}}. \tag{3.17}$$

By integrating twice with respect to the variable u and remembering that $U_{uu} = H^2$, it follows that

$$\begin{aligned} G = u^2 - \frac{(\sqrt{2})^5}{15} m z^{5/2}\Bigg[-1 + \left(1 + \frac{9u^2}{4z^3}\right)^{5/6} \\ - \frac{15u^2}{4z^3}\, {}_2F_1\left[\left(\frac{1}{6}, \frac{1}{2}\right), \left(\frac{3}{2}\right), -\frac{9u^2}{4z^3}\right]\Bigg] \end{aligned} \tag{3.18}$$

where ${}_2F_1$ denotes a generalized hypergeometric function. Now, a simple calculation shows that $A = d_4^c G_u = -G_{uz} dx$, and this together with (3.18) gives

$$\begin{aligned} A = -\frac{mu}{2z(9u^2 + 4z^3)^{1/6}}\Bigg[-3 + 2^{2/3}\left(1 + \frac{9u^2}{4z^3}\right)^{1/6} \\ \times {}_2F_1\left[\left(\frac{1}{6}, \frac{1}{2}\right), \left(\frac{3}{2}\right), -\frac{9u^2}{4z^3}\right]\Bigg] dx. \end{aligned} \tag{3.19}$$

Also, a simple calculation shows that (3.1) is in this case

$$\begin{aligned} \omega_1(u) &= \omega_1 - dd^c G = \omega_1 - G_{1\bar{1}} dz_1 \wedge d\bar{z}_1 \\ &= \omega_1 + G_{zz} dz_1 \wedge d\bar{z}_1, \end{aligned} \tag{3.20}$$

with ω_1 given in (3.13), and in the last step, we took into account that $z = i\bar{z}_1 - iz_1$. The explicit expression of (3.20) is obtained from (3.18). The result is

$$\omega_1(u) = \omega_1 + \frac{m}{2(4+\frac{9u^2}{z^3})^{5/6}z^2(9u^2+4z^3)^{1/6}}\Big[92^{2/3}u^2 + 2\Big(22^{2/3} - \Big(4+\frac{9u^2}{z^3}\Big)^{5/6}\Big)z^3\Big]dz_1\wedge d\bar{z}_1. \tag{3.21}$$

The metric $g_4(u)$ in (3.7) is the one that corresponds to the modified Kahler potential (3.21) namely

$$g_4(u) = \frac{1}{z}(dt - xdu)^2 + z(du^2+dz^2+dx^2) + \frac{m}{2(4+\frac{9u^2}{z^3})^{5/6}z^2(9u^2+4z^3)^{1/6}}\Big[92^{2/3}u^2 + 2\Big(22^{2/3} - \Big(4+\frac{9u^2}{z^3}\Big)^{5/6}\Big)z^3\Big](dz^2+dx^2). \tag{3.22}$$

By collecting the results (3.17), (3.18), (3.19), (3.20), (3.21), and (3.22), it follows that the Calabi-Yau extension (3.7) of the BTKY metric is

$$g_6 = \Big(1+\frac{m}{(4z^3+9u^2)^{1/6}}\Big)^{-1}(d\alpha + A)^2 + \Big(1+\frac{m}{(4z^3+9u^2)^{1/6}}\Big)du^2 + \frac{1}{z}(dt-xdu)^2 + zdu^2 + \Big\{\frac{m}{2(4+\frac{9u^2}{z^3})^{5/6}z^2(9u^2+4z^3)^{1/6}}\Big[9\,2^{2/3}u^2 + 2\Big(2\,2^{2/3} - \Big(4+\frac{9u^2}{z^3}\Big)^{5/6}\Big)z^3\Big] + z\Big\}(dz^2+dx^2). \tag{3.23}$$

with A given in (3.19). This example is a nontrivial Ricci-flat and Kahler metric in six dimensions, with holonomy exactly SU(3). Nevertheless, in the region near $z = 0$, we do not expect our solution to be valid, as the approximation (3.11) breaks down.

D. Complete examples with Hamiltonian isometries

The explicit examples presented in the previous sections do possess isometries preserving the full SU(3) structure; in other words, they correspond to the case $K_1 = 0$ of Sec. 1. The remaining case $K_1 = 1$ was considered in [27]. These authors propose an anzatz which is given in terms of an initial hyperkahler structure which is deformed as in (3.1). In addition, they propose a sympletic form ω_6 of the form (3.2). The unique difference with the case considered in Sec. 1 is that the complex 3-form is now α dependent and is given by $\Psi = e^{i\alpha}(\psi_- + i\psi_+)$, with $\psi_\pm$ given (3.2). This implies that the isometry preserves the Kahler 2-form but not Ψ. The compatibility and the Calabi-Yau conditions were worked out explicitly in [27], and the outcome is again that the metric and the $SU(3)$ structure is completely determined by G, which is now a solution of the equation

$$(e^{-(1/2)G_y})_y = M(G), \tag{3.24}$$

$M(G)$ being the nonlinear operator defined by (3.4). The CY metric is again given by (3.7), but now G is a solution of (3.24). It has been shown in [27] that complete metrics may be obtained when the initial hyperkahler structure is the flat one. In this case, (3.24) becomes

$$(e^{-(1/2)G_y})_y = 2(1 + G_{1\bar{1}} + G_{2\bar{2}} + G_{1\bar{1}}G_{2\bar{2}} - G_{1\bar{2}}G_{2\bar{1}}). \tag{3.25}$$

By parameterizing

$$z_1 = r\cos\frac{\theta}{2}\exp\Big(\frac{i(\psi+\phi)}{2}\Big),\qquad z_2 = r\sin\frac{\theta}{2}\exp\Big(\frac{i(\psi-\phi)}{2}\Big), \tag{3.26}$$

and assuming that G is a function of r and y, Eq. (3.26) reduces to

$$(e^{-(1/2)G_y})_y = \frac{1}{2r^3}\partial_r\Big[r^4\Big(1+\frac{1}{2r}\partial_r G\Big)^2\Big], \tag{3.27}$$

which is Eq. (61) of Ref. [27]. Particular solutions of this equation have been found in that reference and which, after appropriate coordinate transformations and different rescalings, give the resulting family of Calabi-Yau metrics [27]

$$g_6 = \frac{dy^2}{W} + \frac{1}{4}Wy^2(d\alpha - s^2\sigma_3)^2 + y^2\Big(\frac{ds^2}{V} + \frac{1}{4}Vs^2\sigma_3^2 + \frac{1}{4}s^2(\sigma_1^2+\sigma_2^2)\Big) \tag{3.28}$$

with

$$W = 1 - \frac{a}{y^6}\qquad V = 1 - s^2 - \frac{b}{s^4}.$$

The metric with $b = 0$ describes a higher dimensional generalization of the Eguchi-Hanson instanton [36,37], with $R^2\times CP^2$ topology and an asymptotic R^6/Z_3 [38]. For $a = 0$, the metric is a cone of $Y^{p,q}$. The general solution describes a resolution of the $Y^{p,q}$ cone, and the detailed global analysis can be found in [28–30]. More details of this calculation can be found in the original Ref. [27].

It is important to remark that Eq. (3.25), which corresponds to the flat metric, is completely equivalent to (2.42). This may be seen by making the redefinition $G_{i\bar{j}} \to \delta_{i\bar{j}} + G_{i\bar{j}}$ in (2.42), which gives (3.25) as a result, and *vice versa*. In addition, the complex coordinates z_i appearing in (2.43) are locally given by (3.26). But although the starting point is the flat hyperkahler structure, it is not necessarily true that (3.26) parameterizes R^4 globally; in fact, there may appear singularities in the resulting Calabi-Yau metric which can be avoided by changing the periodicity of the angular variables or the range of the radial coordinate. In any case, the above reasoning shows that metrics (3.28) are special solutions of (2.42) and (2.43).

A priori, it may be expected that the use of curved hyperkahler backgrounds will enhance the number of solutions of (3.24). In particular, it may sound plausible that if one starts with a gravitational instanton admitting a flat limit (such as the Taub-Nut one), then the resulting Calabi-Yau metrics obtained by solving (3.24) will contain the ones arising from (3.25) as a particular case, and, moreover, the families described by (3.25) such as (3.28) will be reobtained by taking the corresponding flat limit. As (3.25) is equivalent to (2.42), this reasoning will imply that (3.24) describes a more general family than (2.42). But what the results of the present work are showing is that the situation is the opposite, that is, any Calabi-Yau metric found in terms of a curved hyperkahler space by solving (3.24) can be obtained from solutions of the "flat" Eq. (2.42) as well. Thus, the number of solutions of (3.24) is less or equal to the solutions of (2.42). The arguments showing this are the same as in Sec. 1, namely, that all the metrics described by (3.24) are under the hypothesis givin in Eq. (2.42).[2] Although this conclusion may sound a bit odd, there is further evidence for that, which is the following. If one starts with a curved hyperkahler metric with triaxial symmetry instead of the flat one, then Calabi-Yau metrics resulting from (3.24) are the ones with $R^2 \times CP^2$ topology and an asymptotic R^6/Z_3 together with the resolution of the cone over $T^{1,1}/Z_2$ [27]. But $T^{1,1}/Z_2$ is a particular case of the $Y^{p,q}$ Einstein-Sasaki manifolds; thus, the solutions obtained with the triaxial metrics are a special subcase of (3.28). For other curved manifolds, the system becomes harder to solve, and no new solutions have been found. Although formally there is nothing wrong with the use of hyperkahler structures to guess new solutions, it may be the case that the use of curved geometries complicates the task instead of helping to solve it. For this reason, it is perhaps convenient to find a formalism which avoids this problem, and the one developed here in (2.42), (2.43), and (2.44) possesses these advantages, as these equations do not make any reference to any vielbein of a curved hyperkahler metric.

IV. DISCUSSION

In the present work, a family of Calabi-Yau manifolds with a local Hamiltonian Killing vector, i.e, a Killing vector which preserves the metric together with the Kahler form, was characterized. It was assumed that the complex (3, 0)-form is of the form $e^{ik}\bar{\Psi}$, where $\bar{\Psi}$ is preserved by the Killing vector as well, and that the space of the orbits of the Killing vector is, for fixed value of the momentum map coordinate, a complex manifold, in such a way that the complex structure of the 2-fold is part of the complex structure of the 3-fold. Under these assumptions, it was shown that the local form of the geometry is completely determined in terms of a function G satisfying the nonlinear Eq. (2.42) if the phase k is nontrivial or (2.44) if the phase k is zero. It has been also pointed out that the constructions given in [25–27] are included in this family.

The advantages of this method are that, unlike the ones presented in [25–27], it does not require a hyperkahler structure as initial input. As it was discussed in Sec. II, it is only required that the 4-dimensional manifold defined by the orbits of the Killing vector for fixed momentum map coordinate is a complex 2-fold, and the Calabi-Yau conditions imply automatically that it is Kahler. In fact, Eqs. (2.42) and (2.44) for the function G defining the six dimensional metric do not contain any reference to the vielbein of the complex 2-fold. In this form, one may avoid the complications in the calculation of the local form of the geometry due to the nontrivial curvature of an initial hyperkahler geometry.

It is perhaps better to compare this situation with known results in four dimensions. Consider a 4-dimensional Calabi-Yau (hyperkahler) space, such that the Killing vector preserves the Kahler form ω_4 but not Ω_4. As is well known, the general local form of the Ricci-flat Kahler 4-metric is [39,40]

$$\begin{aligned} g_4 &= u_z[e^u(dx^2+dy^2)+dz^2] \\ &\quad + u_z^{-1}[dt+(u_x dy - u_y dx)]^2, \end{aligned} \tag{4.1}$$

where u is the solution of the equation

$$(e^u)_{zz} + u_{yy} + u_{xx} = 0. \tag{4.2}$$

Equation (4.2) is known as the continuum limit of the sl(n) Toda equation and is called SU(∞) Toda equation. The three dimensional base metric, namely,

$$g_3 = e^u(dx^2+dy^2)+dz^2,$$

is Einstein-Weyl [41–43]. But the general Einstein-Weyl equation is not related to a Toda system, so these base metrics are Einstein-Weyl spaces of restricted type. One may try to find solutions of (4.2) by perturbing around a solution related to a known Einstein-Weyl structure. This is not wrong, but optional. In the same way the 4-dimensional metric (2.43) is Kahler with Kahler potential G, but G is of restricted type, given by solutions of (2.42) or (2.44). One may try to find a solution to these equations by perturbing around a known hyperkahler one, as it was done in [25–27], but this is optional as well.

We also presented in Sec. III C an example which is obtained by means of the Fayyazuddin linearization. This example has holonomy exactly SU(3), but it is not complete. It may be interesting to see if it is possible to find complete metrics by means of this linearization, which will correspond to D6 branes wrapping a complex 1-cycle inside a hyperkahler. We hope to answer this question in the near future.

[2] See the last paragraph of Sec. 1; in fact, it is not difficult to see that the α-dependent phase does not change these arguments at all.

ACKNOWLEDGMENTS

M. L. and O. P. S. are supported by CONICET (Argentina) and by the ANPCyT Grant No. PICT-2007-00849. When we were finishing the present note, the work [44] appeared, which probably has overlap with the results presented here.

[1] S. T. Yau, Commun. Pure Appl. Math. **31**, 339 (1978).
[2] P. Candelas and X. de la Ossa, Nucl. Phys. **B342**, 246 (1990).
[3] G. Gibbons, H. Lu, C. Pope, and K. Stelle, Nucl. Phys. **B623**, 3 (2002).
[4] M. F. Atiyah and N. J. Hitchin, Phys. Lett. A **107**, 21 (1985).
[5] G. Gibbons and P. Rychenkova, J. Geom. Phys. **32**, 311 (2000).
[6] I. Lavrinenko, H. Lu, and C. Pope, Classical Quantum Gravity **15**, 2239 (1998); E. Bergshoeff, M. de Roo, M. Green, G. Papadopoulos, and P. Townsend, Nucl. Phys. **B470**, 113 (1996).
[7] S. Bando and R. Kobayashi, Math. Ann. **287**, 175 (1990); *Proc. 21st Int. Taniguchi Symp, Lecture Notes in Pure Math* (Springer, Toyobo Co., Katata, Japan, 1988), Vol. 1339, p. 20.
[8] G. Tian and S. T. Yau, Prog. Theor. Phys. **81**, 17 (1989); J. Am. Math. Soc. **3**, 579 (1990).
[9] J. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, Adv. Theor. Math. Phys. **8**, 711 (2004); **8**, 987 (2004).
[10] C. Boyer, K. Galicki, and M. Nakamaye, Trans. Am. Math. Soc. **354**, 2983 (2002).
[11] C. Boyer and K. Galicki, J. Diff. Geom. **57**, 443 (2001); International Journal of Mathematics **11**, 873 (2000).
[12] A. Balasubramanian, S. Govindarajan, and C. Gowdigere, Classical Quantum Gravity **24**, 6393 (2007).
[13] G. Gibbons, S. Hartnoll, and Y. Yasui, Classical Quantum Gravity **21**, 4697 (2004).
[14] D. Martelli and J. Sparks, J. High Energy Phys. 04 (2008) 067; J. Geom. Phys. **59**, 1175 (2009); Phys. Lett. B **621**, 208 (2005).
[15] M. Cvetic, H. Lu, D. Page, and C. Pope, Phys. Rev. Lett. **95**, 071101 (2005); J. High Energy Phys. 07 (2009) 082.
[16] H. Lu, C. N. Pope, and J. F. Vazquez-Poritz, Phys. Rev. D **75**, 026005 (2007).
[17] W. Chen, H. Lu, C. N. Pope, and J. F. Vazquez-Poritz, Classical Quantum Gravity **22**, 3421 (2005).
[18] J. Gomis, Nucl. Phys. **B606**, 3 (2001).
[19] J. Edelstein and C. Nunez, J. High Energy Phys. 04 (2001) 028; J. Edelstein, A. Paredes, and A. Ramallo, Phys. Lett. B **554**, 197 (2003); J. High Energy Phys. 12 (2002) 075.
[20] M. Kontsevich, in *The Moduli Space of Curves* (Birkhauser, Texel Island, 1995), p. 335.
[21] T. M. Chiang, A. Klemm, S. T. Yau, and E. Zaslow, Adv. Theor. Math. Phys. **3**, 495 (1999).
[22] R. Gopakumar and C. Vafa, Adv. Theor. Math. Phys. **3**, 1415 (1999).
[23] M. Aganagic and C. Vafa, arXiv:hep-th/0012041.
[24] M. Marino, arXiv:hep-th/0210145.
[25] A. Fayyazuddin, Classical Quantum Gravity **24**, 3151 (2007).
[26] O. P. Santillan, Classical Quantum Gravity **27**, 155013 (2010).
[27] H. Lu, Y. Pang, and Z. Wang, Classical Quantum Gravity **27**, 155018 (2010).
[28] T. Oota and Y. Yasui, Phys. Lett. B **639**, 54 (2006).
[29] H. Lu and C. N. Pope, Nucl. Phys. **B782**, 171 (2007).
[30] W. Chen, M. Cvetic, H. Lu, C. N. Pope, and J. F. Vazquez-Poritz, Nucl. Phys. **B785**, 74 (2007).
[31] A. Besse, *Einstein Manifolds* (Springer Verlag, New York, Berlin, Heidelberg, 2002), 2nd ed.
[32] S. Chiossi and S. Salamon, *Proc. Conf. Differential Geometry Valencia 2001* (World Sci. Publishing, River Edge, NJ, 2002), p. 115-133.
[33] V. Apostolov and S. Salamon, Commun. Math. Phys. **246**, 43 (2004).
[34] E. Bedford and M. Kalka, Commun. Pure Appl. Math. **30**, 543 (1977).
[35] G. Gibbons and S. Hawking, Phys. Lett. B **78**, 430 (1978).
[36] T. Eguchi and A. Hanson, Phys. Lett. B **74**, 249 (1978); Ann. Phys. (N.Y.) **120**, 82 (1979); Gen. Relativ. Gravit. **11**, 315 (1979).
[37] G. Gibbons and C. Pope, Commun. Math. Phys. **66**, 267 (1979).
[38] D. N. Page and C. N. Pope, Classical Quantum Gravity **4**, 213 (1987).
[39] J. Gegenberg and A. Das, Gen. Relativ. Gravit. **16**, 817 (1984).
[40] C. Boyer and J. Finley, J. Math. Phys. (N.Y.) **23**, 1126 (1982).
[41] P. E. Jones and K. P. Tod, Classical Quantum Gravity **2**, 565 (1985).
[42] K. P. Tod, *The SU*(∞) *Toda Equation and Special Four Dimensional Metrics in Geometry and Physics Aarhaus*, edited by J. E. Andersen, J. Dupont, H. Pedersen, and A. Swann (1995); *Lecture Notes in Pure and Appl. Maths* (Marcel Dekker, New York, 1997), Vol. 184.
[43] R. S. Ward Einstein-Weyl, Classical Quantum Gravity **7**, L95 (1990); N. Woodhouse, Classical Quantum Gravity **6**, 933 (1989).
[44] H. Lu and Zhao-Long Wang, J. Geom. Phys. **60**, 1741 (2010).