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Proper splittings of Hilbert space operators

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ABSTRACT

Proper splittings of operators are commonly used to study the convergence of iterative processes. In order to approximate solutions of operator equations, in this article we deal with proper splittings of closed range bounded linear operators defined on Hilbert spaces. We study the convergence of general proper splittings of operators in the infinite dimensional context. We also propose some particular splittings for special classes of operators and we study different criteria of convergence and comparison for them. In some cases, these criteria are given under hypothesis of operator order relations. In addition, we relate these results with the concept of the symmetric approximation of a frame in a Hilbert space.

1. Introduction

Given an invertible matrix $T \in M^n(\mathbb{C})$, a splitting of T is a partition $T = U - V$, where $U, V \in M^n(\mathbb{C})$ with U invertible. The theory of regular splittings for an invertible matrix $T \in M^n(\mathbb{R})$, i.e., a splitting $T = U - V$ such that $U, V \in M^n(\mathbb{R})$ and U^{-1} has all its entries nonnegative, started in a work due to Varga [36], where an iterative process to get the unique solution of the system $Tx = w$ was given. To be more precise, given $T \in M^n(\mathbb{R})$ invertible and $T = U - V$ a regular splitting of T, Varga defined the iterative process

$$x^{i+1} = U^{-1}Vx^i + U^{-1}w,$$

and he proved that this process converges for every initial x^0 if and only if the spectral radius of $U^{-1}V$, $\rho(U^{-1}V) < 1$, is less than 1. Moreover, in such a case, the process converges to $T^{-1}w$.

The idea of Varga was extended by Berman and Plemmons [7] in order to define an iterative process that allows to get the minimum norm least square solution of a system $Tx = w$, for $T \in M^{m\times n}(\mathbb{R})$. For

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this purpose, the concept of proper splitting of $T \in M^{m\times n}(\mathbb{R})$ was introduced: a proper splitting of T is a decomposition $T = U - V$ where $\mathcal{R}(U) = \mathcal{R}(T)$ and $\mathcal{N}(U) = \mathcal{N}(T)$. Then, Berman and Plemmos proved that given a proper splitting of $T \in M^{m\times n}(\mathbb{R})$ the iterative process

$$x^{i+1} = U^\dagger V x^i + U^\dagger w, \tag{1}$$

converges for every initial x^0 if and only if $\rho(U^\dagger V) < 1$ and, in such a case, it converges to $T^\dagger w$, the minimum norm least square solution of $Tx = w$.

In the literature different kinds of splittings and proper splittings of a matrix can be found. Some of them are regular splittings [36,14], nonnegative splittings with first and second type [34,35], weak regular splittings with first and second type [37,10,19], proper splittings [30,7,5], P-regular splittings [24,26], P-proper splittings [28], weak nonnegative splittings of the first and the second type [11], among others. For each class of splitting there exist several results that guarantee the convergence of the associated iterative process. Also there exist results that compare the speed of convergence of the iterative method for different splittings of a matrix.

Arias and Gonzalez [5] used the concept of proper splitting of a matrix $T \in M^{m\times n}(\mathbb{C})$ to extend the iterative method (1) and so to define an iterative process that converges to a reduced solution of the solvable matrix equation $TX = W$. There, criteria of convergence and comparison for proper splittings were given under hypothesis of positivity (according to Löwner order) of certain matrices. Also in [5] it was proposed a particular proper splitting which is defined for every matrix in $M^{m\times n}(\mathbb{C})$; namely, the polar proper splitting. There, the convergence of the polar proper splitting was studied and it was shown that this particular splitting is "better" than others on certain classes of matrices.

Along this article we will be interested in extending the treatment of proper splittings of matrices to the context of Hilbert space operators in order to approximate solutions of an operator equation $TX = W$. On one hand we generalize to the infinite dimensional context results of [5] like a criterion of convergence for general proper splittings. Also, we study the polar proper splitting of a closed range operator and we characterize its convergence. We introduce particular proper splittings for the classes of split and Hermitian operators, we analyze their convergence and we establish different criteria of comparison. In addition, throughout the article we focus in comparing proper splittings of operators which are related under different factorizations. In particular some of the factorizations considered are given by means of the star order, the minus order, the sharp order. Also different proper splittings of operators in classes as the product of orthogonal projections, the product of an orthogonal projection by an Hermitian operator and the product of an orthogonal projection by a positive operator, are investigated.

The reader is referred to [16,31,29] among others, for results on splittings, iterative methods and criteria of convergence for splittings on infinite dimensional spaces. For example, Nacevsca [31] studied iterative methods for computing generalized inverses of linear operators by means of the method of splitting of operators. Liu and Huang [29] studied the convergence of an iterative method to find solutions of a linear system $Tx = w$ by means of a proper splitting of the operator GT, where G is an operator with range and nullspace prescribed.

The paper is organized as follows. Section 2 contains notations and preliminaries results. In particular, we recall definitions of some operator orders and generalized inverses that will be used along the paper. Also, an extension to the infinite dimensional case of a characterization of the Löwner order for positive operators due to Baksalary, Liski and Trenkler [6,5] is given. In Section 3 the study of general proper splittings given in [5] for the finite dimensional case is now generalized for Hilbert space operators. Here the main result is Theorem 3.7, where we provide sufficient conditions for the convergence of proper splittings under certain compacity hypothesis. Section 4 is devoted to study the polar proper splitting. We characterize its convergence in Theorem 4.3. In Theorem 4.5 the polar proper splittings of two operators related by the star order, are compared. Also we show that the star order condition can not be replaced for a sharp order nor a minus order condition. In Section 5 we present two particular proper splittings for a split operator, i.e., a bounded linear operator T defined on a Hilbert space $\mathcal{H}$ which satisfies the condition $\mathcal{R}(T)\dot{+}\mathcal{N}(T) = \mathcal{H}$. The splittings for such a T are defined by means of operators that emerge from that decomposition of $\mathcal{H}$; namely, the Q-proper splitting and the group proper splitting. For each one of them we give conditions that guarantee its convergence. We also provide some criteria of convergence for the splittings of two split operators related with the star and the sharp order. In Section 6 we study particular proper splittings associated to a closed range Hermitian operator T. Here we consider the MP-proper splitting and the projection proper splitting, defined by means of the Moore-Penrose operator of T and the orthogonal projection onto the range of T, respectively. We characterize the convergence of these proper splittings and we establish a comparison criterion between them and the polar proper splitting. On the other hand, we apply the projection proper splitting to induce a proper splitting of an operator that can be factorized in terms of some Hermitian operator (Theorem 6.7 and its corollaries). In subsection 6.1 we illustrate how we can apply the theory of splittings operators to get a symmetric approximation of a given frame.

Finally, given two bounded linear operators S, T on $\mathcal{H}$, in Section 7 we study the induced splittings by T on S in the particular case that S and T are related by an invertible operator.

2. Preliminaries

Along this article $\mathcal{H}$ is a complex Hilbert space with inner product $\langle .,. \rangle$ and $\mathcal{L}(\mathcal{H})$ is the algebra of bounded linear operators defined from $\mathcal{H}$ to $\mathcal{H}$. The norm of $T \in \mathcal{L}(\mathcal{H})$ is $\|T\| = \sup\{\|Tx\| : \ \|x\| = 1\}$. By $\mathcal{K}$, $\mathcal{L}^h$, $\mathcal{L}^+, \mathcal{U}$, $\mathcal{Q}$ and $\mathcal{P}$ we denote the classes of compact operators, Hermitian operators, positive operators, unitary operators, oblique projections and orthogonal projections of $\mathcal{L}(\mathcal{H})$, respectively. Given $T \in \mathcal{L}(\mathcal{H})$, T^* denotes the adjoint operator of T, $\mathcal{R}(T)$ denotes the range of T and $\mathcal{N}(T)$ denotes the nullspace of T. The direct and orthogonal sum between subspaces are denoted by $\dot{+}$ and $\oplus$, respectively. Given closed subspaces $\mathcal{S}, \mathcal{T} \subseteq \mathcal{H}$ such that $\mathcal{S}\dot{+}\mathcal{T} = \mathcal{H}$, $Q_{\mathcal{S}//\mathcal{T}}$ indicates the oblique projection onto $\mathcal{S}$ along $\mathcal{T}$. In particular, the orthogonal projection onto $\mathcal{S}$ is denoted by $P_{\mathcal{S}}$. We will also use the notation P_T to indicate the orthogonal projection onto $\mathcal{R}(T)$, for a closed range operator $T \in \mathcal{L}(\mathcal{H})$. The spectrum and the spectral radius of $T \in \mathcal{L}(\mathcal{H})$ are denoted by $\sigma(T)$ and $\rho(T)$, respectively. It is well known that $\rho(T) \leq \|T\|$ and, if T is a normal operator (i.e., $T^*T = TT^*$) then $\rho(T) = \|T\|$.

The classical Löwner order for Hermitian operators is denoted by $\leq$. Given $S, T \in \mathcal{L}^h$, it holds that $S \leq T$ if and only if $0 \leq T - S$, or equivalently $0 \leq \langle (T - S)x, x\rangle$ for all $x \in \mathcal{H}$.

The following result on range inclusion and operator factorization is due to Douglas [17] and [2]:

Douglas' theorem. Consider $S, T \in \mathcal{L}(\mathcal{H})$. The following conditions are equivalent:

1. $\mathcal{R}(S) \subseteq \mathcal{R}(T)$;
2. there exists a number $\lambda > 0$ such that $SS^* \leq \lambda TT^*$;
3. there exists $R \in \mathcal{L}(\mathcal{H})$ such that $TR = S$.

Moreover, if any of the above conditions holds and $\mathcal{M} \subseteq \mathcal{H}$ is a closed subspace such that $\mathcal{M}\dot{+}\mathcal{N}(T) = \mathcal{H}$ then there exists a unique operator $X_{\mathcal{M}} \in \mathcal{L}(\mathcal{H})$ such that $TX_{\mathcal{M}} = S$ and $\mathcal{R}(X_{\mathcal{M}}) \subseteq \mathcal{M}$. The operator $X_{\mathcal{M}}$ is called the reduced solution for $\mathcal{M}$ of the equation $TX = S$ and also satisfies that $\mathcal{N}(X_{\mathcal{M}}) = \mathcal{N}(S)$.

In the case that $\mathcal{M} = \mathcal{N}(T)^{\perp}$, the unique solution $X_r \in \mathcal{L}(\mathcal{H})$ such that $TX_r = S$ and $\mathcal{R}(X_r) \subseteq \mathcal{N}(T)^{\perp}$ is called *the Douglas reduced solution* of the equation $TX = S$. In addition X_r satisfies that $\mathcal{N}(X_r) = \mathcal{N}(S)$ and $\|X_r\| = inf\{\lambda : \ SS^* \leq \lambda TT^*\}$.

Remember that a general inverse (or pseudoinverse) of $T \in \mathcal{L}(\mathcal{H})$ is an operator which satisfies the equations $TXT = T$ and $XTX = X$. It is well known that a general inverse of $T \in \mathcal{L}(\mathcal{H})$ is bounded if and only if T has closed range. There exist different kinds of pseudoinverses of an operator according to the restriction imposed. In this article we deal with the Moore-Penrose inverse and with the group inverse of an operator. Next we collect some results about them.

Given $T \in \mathcal{L}(\mathcal{H})$ with closed range, $T^\dagger \in \mathcal{L}(\mathcal{H})$ denotes the *Moore-Penrose inverse* of T. The following two characterizations for the Moore-Penrose inverse of a closed range operator are known and they can be found in [23], [2]:

1. $T^\dagger$ is the unique operator that satisfies simultaneously the following four equations:

$$TXT = T, \quad XTX = X, \quad TX = P_T, \quad XT = P_{T^*};$$

2. $T^\dagger$ is the Douglas reduced solution of the equation $TX = P_T$.

If $S, T \in \mathcal{L}(\mathcal{H})$ are invertible operators it is well known that $(ST)^{-1} = T^{-1}S^{-1}$. The extension of this identity to non invertible operators is known as "the reverse order law" for the Moore-Penrose inverse; i.e., $(ST)^\dagger = T^\dagger S^\dagger$. However this equality does not hold in general. A classical result due to Greville [22] provides a characterization of the reverse order law for matrices. Next, we state its extension to bounded linear operators in Hilbert spaces proved in [8,25].

Proposition 2.1. *Consider $S, T \in \mathcal{L}(\mathcal{H})$ with closed ranges such that ST has closed range. Then $(ST)^\dagger = T^\dagger S^\dagger$ if and only if $\mathcal{R}(S^*ST) \subseteq \mathcal{R}(T)$ and $\mathcal{R}(TT^*S^*) \subseteq \mathcal{R}(S^*)$.*

Corollary 2.2. *Let $S, T \in \mathcal{L}^h$ be closed range operators such that $ST = TS$ has closed range. Then $(ST)^\dagger = T^\dagger S^\dagger$.*

Proof. It is straightforward from Proposition 2.1. □

The following result characterizes the antitonicity property for the Moore-Penrose operator on the set $\mathcal{L}^+$. Its proof can be found in [20].

Theorem 2.3. *Consider $S, T \in \mathcal{L}^+$ closed range operators. Then any two of the following conditions imply the third condition:*

1. $S \leq T$;
2. $T^\dagger \leq S^\dagger$;
3. $\mathcal{R}(S) \cap \mathcal{N}(T) = \mathcal{R}(T) \cap \mathcal{N}(S) = \{0\}$.

As a consequence of Douglas' theorem we can characterize the Löwner order between positive operators by means of the spectral radius of certain product of operators and a range inclusion condition. The next result is the extension to the infinite dimensional case of a result due to Baksalary, Liski and Trenkler [6] and [5].

Proposition 2.4. *Consider $S, T \in \mathcal{L}^+$ such that $\mathcal{R}(T)$ is closed. Then $S \leq T$ if and only if $\rho(S^{1/2}T^\dagger S^{1/2}) \leq 1$ and $\mathcal{R}(S^{1/2}) \subseteq \mathcal{R}(T)$.*

Proof. Let $S, T \in \mathcal{L}^+$. If $S \leq T$ then, by Douglas' theorem, $\mathcal{R}(S^{1/2}) \subseteq \mathcal{R}(T^{1/2}) = \mathcal{R}(T)$ and $\|(T^{1/2})^\dagger S^{1/2}\| = \inf\{\lambda : S \leq \lambda T\}$. Therefore, $\|(T^{1/2})^\dagger S^{1/2}\| \leq 1$ and $\rho(S^{1/2}T^\dagger S^{1/2}) = \|S^{1/2}T^\dagger S^{1/2}\| = \|(T^{1/2})^\dagger S^{1/2}\|^2 \leq 1$. Conversely, suppose that $\rho(S^{1/2}T^\dagger S^{1/2}) \leq 1$ and $\mathcal{R}(S^{1/2}) \subseteq \mathcal{R}(T) = \mathcal{R}(T^{1/2})$. Then

$\|(T^{1/2})^\dagger S^{1/2}\|^2 = \|S^{1/2}T^\dagger S^{1/2}\| = \rho(S^{1/2}T^\dagger S^{1/2}) \leq 1$. Hence, applying again Douglas' theorem we get that, $\inf\{\lambda : S \leq \lambda T\} = \|(T^{1/2})^\dagger S^{1/2}\| \leq 1$ and $S \leq T$. □

On the other hand, the group inverse of an operator is defined for the class of split operators; $T \in \mathcal{L}(\mathcal{H})$ is a *split operator* if $\mathcal{R}(T) \dot{+} \mathcal{N}(T) = \mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$ is a split operator then there exists a unique $T^\sharp \in \mathcal{L}(\mathcal{H})$ such that the following equations are simultaneously satisfied:

$$TXT = T, \qquad XTX = X, \qquad TX = XT.$$

The operator $T^\sharp$ is called the *group inverse* of T. In [32] necessary and sufficient conditions for the existence of this kind of pseudoinverse are shown:

Theorem 2.5. *Consider $T \in \mathcal{L}(\mathcal{H})$, then the following statements are equivalent:*

1. $\mathcal{H} = \mathcal{R}(T) \dot{+} \mathcal{N}(T)$;
2. $\mathcal{R}(T^2) = \mathcal{R}(T)$ *and* $\mathcal{N}(T^2) = \mathcal{N}(T)$;
3. $T^\sharp$ *exists.*

It is well known that if T admits a group inverse then it has closed range, $\mathcal{R}(T^\sharp) = \mathcal{R}(T)$, $\mathcal{N}(T^\sharp) = \mathcal{N}(T)$ and $TT^\sharp = T^\sharp T = Q_{\mathcal{R}(T)//\mathcal{N}(T)}$. In addition, T admits a group inverse if and only if T^* admits a group inverse. On the other hand, if $T \in \mathcal{L}(\mathcal{H})$ is an EP operator (i.e., T has closed range and $\mathcal{R}(T) = \mathcal{R}(T^*)$) then it admits a group inverse. Moreover, $T^\sharp = T^\dagger$ if and only if T is an EP operator.

Remember that every $T \in \mathcal{L}(\mathcal{H})$ admits a factorization $T = U|T|$, where U is a partial isometry and $|T| = (T^*T)^{1/2}$. Moreover, this factorization is unique if the condition $\mathcal{N}(U) = \mathcal{N}(T)$ is imposed. The factorization $T = U_T|T|$, where U_T is the partial isometry such that $\mathcal{N}(U_T) = \mathcal{N}(T)$ is called the polar decomposition of T. It is not difficult to see that $T^\dagger = |T|^\dagger U_T^*$. In addition, if $T \in \mathcal{L}^h$ then $T \leq |T|$.

We finish this section by introducing some order relations on $\mathcal{L}(\mathcal{H})$ that will be used throughout the article.

Definition 2.6. Let $S, T \in \mathcal{L}(\mathcal{H})$. Then:

1. $S \overset{*}{\leq} T$ if there exist $P, Q \in \mathcal{P}$ such that $S = PT$ and $S^* = QT^*$. The orthogonal projections can be chosen such that $P = P_S$ and $Q = P_{S^*}$.
2. $S \overset{-}{\leq} T$ if there exist $P, Q \in \mathcal{Q}$ such that $S = PT$ and $S^* = QT^*$. The ranges of P and Q can be fixed as $\mathcal{R}(P) = \overline{\mathcal{R}(S)}$ and $\mathcal{R}(Q) = \overline{\mathcal{R}(S^*)}$.
3. $S \overset{\sharp}{\leq} T$ if $S = T$ or if there exists $Q \in \mathcal{Q}$ with $\mathcal{R}(Q) = \mathcal{R}(S)$, $\mathcal{N}(Q) = \mathcal{N}(S)$ such that $S = QT = TQ$.

The above order relations are known as *star order, minus order* and *sharp order*, respectively. The reader can be referred to [1,15,18,27] and references therein for different treatments on these order relations. The following result gives characterizations of these orders by means of operator range decompositions. The proof can be found in [15] and [20].

Proposition 2.7. *Consider $S, T \in \mathcal{L}(\mathcal{H})$. Then,*

1. $S \overset{-}{\leq} T$ *if* $\mathcal{R}(T) = \mathcal{R}(S) \dot{+} \mathcal{R}(T - S)$ *and* $\mathcal{R}(T^*) = \mathcal{R}(S^*) \dot{+} \mathcal{R}(T^* - S^*)$.
2. $S \overset{*}{\leq} T$ *if and only if* $\mathcal{R}(T) = \mathcal{R}(S) \oplus \mathcal{R}(T - S)$ *and* $\mathcal{R}(T^*) = \mathcal{R}(S^*) \oplus \mathcal{R}(T^* - S^*)$.

3. If S,T are group invertible then $S \overset{\sharp}{\leq} T$ if and only if $\mathcal{R}(T) = \mathcal{R}(S) \dot{+} \mathcal{R}(T-S)$ where $\mathcal{R}(T-S) \subseteq \mathcal{N}(S)$ and $\mathcal{R}(S) \subseteq \mathcal{N}(T-S)$.

3. Proper splittings of operators and an iterative process

In [7] the concept of proper splitting of a matrix T is applied to approximate the minimum least square solution of a linear system $Tx = w$. In [5] this kind of splitting is used to guarantee the convergence of an iterative process to the reduced solution for $\mathcal{M}$ of a solvable matrix equation $TX = W$, where $T, W \in M^{m\times n}(\mathbb{C})$ and $\mathcal{M}$ is a subspace of $\mathbb{C}^n$ such that $\mathcal{M} \dot{+} \mathcal{N}(T) = \mathbb{C}^n$. There, sufficient conditions for the convergence of the iterative process were given. Our goal is to extend the study made in [5] to the infinite dimensional context.

Definition 3.1. Consider $T \in \mathcal{L}(\mathcal{H})$ with closed range. A proper splitting of T is a decomposition $T = U - V$, where $U, V \in \mathcal{L}(\mathcal{H})$ and $\mathcal{R}(U) = \mathcal{R}(T)$ and $\mathcal{N}(U) = \mathcal{N}(T)$.

Consider $T \in \mathcal{L}(\mathcal{H})$ with closed range, $W \in \mathcal{L}(\mathcal{H})$ such that $\mathcal{R}(W) \subseteq \mathcal{R}(T)$ and a closed subspace $\mathcal{M} \subseteq \mathcal{H}$ such that $\mathcal{M} \dot{+} \mathcal{N}(T) = \mathcal{H}$. Let $T = U - V$ be a proper splitting of T. Consider $Y_{\mathcal{M}}, Z_{\mathcal{M}} \in \mathcal{L}(\mathcal{H})$ the reduced solutions for $\mathcal{M}$ of $UY = V$ and $UZ = W$, respectively. We define, as in [5], *the iterative process for $\mathcal{M}$ of the proper splitting $T = U - V$ with respect to W*:

$$X^{i+1} = Y_{\mathcal{M}} X^i + Z_{\mathcal{M}}. \tag{2}$$

As in the finite dimensional context [5], in the infinite dimensional case it can be proved that if the iteration (2) converges, then it converges to the reduced solution for $\mathcal{M}$ of $TX = W$. We omit the proof of the following two results because they are similar to those given in [5].

Theorem 3.2. *Consider $T \in \mathcal{L}(\mathcal{H})$ with closed range, $W \in \mathcal{L}(\mathcal{H})$ such that $\mathcal{R}(W) \subseteq \mathcal{R}(T)$ and $\mathcal{M}$ a closed subspace of $\mathcal{H}$ such that $\mathcal{M} \dot{+} \mathcal{N}(T) = \mathcal{H}$. Consider the proper splitting $T = U - V$ of T. Then the iterative process (2) converges for all X^0 to $X_{\mathcal{M}}$, the reduced solution for $\mathcal{M}$ of the equation $TX = W$, if and only if $\rho(Y_{\mathcal{M}}) < 1$.*

Corollary 3.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be a closed range operator and $T = U - V$ a proper splitting of T. $W \in \mathcal{L}(\mathcal{H})$ such that $\mathcal{R}(W) \subseteq \mathcal{R}(T)$. Then, the iterative process of the proper splitting $T = U - V$ converges for some closed subspace $\mathcal{M}$ such that $\mathcal{M} \dot{+} \mathcal{N}(T) = \mathcal{H}$ if and only if the iterative process of the proper splitting $T = U - V$ converges for all closed subspace $\mathcal{M}$ such that $\mathcal{M} \dot{+} \mathcal{N}(T) = \mathcal{H}$. In particular, the iterative process (2) is convergent if and only if $\rho(U^\dagger V) < 1$.*

Given $T \in \mathcal{L}(\mathcal{H})$ with closed range, the aim of this section is to obtain sufficient conditions that guarantee the convergence of a general proper splitting $T = U - V$. We first collect some properties of proper splittings that will be useful.

Proposition 3.4. *Consider $T \in \mathcal{L}(\mathcal{H})$ a closed range operator. If $T = U - V$ is a proper splitting of T then the following assertions follow:*

1. $(U^\dagger T)^\dagger = T^\dagger U$;
2. $\mathcal{N}(U^\dagger V) = \mathcal{N}(V)$;
3. $T^\dagger = (I - U^\dagger V)^{-1} U^\dagger$.

Proof. 1. Since $\mathcal{R}(U) = \mathcal{R}(V)$ and $\mathcal{N}(U) = \mathcal{N}(V)$, the assertion follows by Proposition 2.1.

2. Since $U^\dagger V$ is the Douglas' reduced solution of the equation $UX = V$, the assertion follows by Douglas' theorem.

3. Let $T = U - V$ be a proper splitting of T. First, let us see that $I - U^\dagger V$ is an invertible operator. In fact, observe that $U^\dagger V = U^\dagger(U - T) = P_{T^*} - U^\dagger T$, then $I - U^\dagger V = P_{\mathcal{N}(T)} + U^\dagger T$. In order to see that this operator is invertible it is sufficient to note that $(P_{\mathcal{N}(T)} + (U^\dagger T)^\dagger)(P_{\mathcal{N}(T)} + U^\dagger T) = (P_{\mathcal{N}(T)} + U^\dagger T)(P_{\mathcal{N}(T)} + (U^\dagger T)^\dagger) = I$. Therefore, $(I - U^\dagger V)^{-1} = P_{\mathcal{N}(T)} + (U^\dagger T)^\dagger$. Now, let us see that $T^\dagger = (I - U^\dagger V)^{-1}U^\dagger$. In fact, since $T(I - U^\dagger V)^{-1}U^\dagger = T(P_{\mathcal{N}(T)} + (U^\dagger T)^\dagger)U^\dagger = T(U^\dagger T)^\dagger U^\dagger = TT^\dagger UU^\dagger = P_T$ and $\mathcal{R}((I - U^\dagger V)^{-1}U^\dagger) = \mathcal{R}((P_{\mathcal{N}(T)} + (U^\dagger T)^\dagger)U^\dagger) = \mathcal{R}((U^\dagger T)^\dagger U^\dagger) = \mathcal{R}(T^\dagger) = \mathcal{N}(T)^\perp$ then the assertion follows. □

Proposition 3.5. *Let $T \in \mathcal{L}(\mathcal{H})$ be a closed range operator. If $T = U - V$ is a proper splitting of T then the following assertions are equivalent:*

1. $T^\dagger V \in \mathcal{L}^+$;
2. $U^\dagger V \in \mathcal{L}^+$;
3. $0 \le U^\dagger V \le P_{V^*}$;
4. $0 \le U^\dagger T \le P_{T^*}$;
5. *The equation $UX = V$ has a positive solution;*
6. $U^\dagger T \in \mathcal{L}^h$ *and* $(P_{T^*} - U^\dagger T)^2 \le \lambda(P_{T^*} - U^\dagger T)$, *for some* $\lambda \ge 0$.

Proof. $1 \leftrightarrow 2$. Suppose that $T^\dagger V \in \mathcal{L}^+$. Since $T^\dagger V = T^\dagger(U - T) = T^\dagger U - P_{T^*}$ then we get that $P_{T^*} \le T^\dagger U$. Now, as $\mathcal{R}(P_{T^*}) \cap \mathcal{N}(T^\dagger U) = \{0\}$ and $\mathcal{R}(T^\dagger U) \cap \mathcal{N}(P_{T^*}) = \{0\}$ then, by Theorem 2.3, $U^\dagger T = (T^\dagger U)^\dagger \le P_{T^*}$. Therefore, $U^\dagger V = U^\dagger(U - T) = P_{T^*} - U^\dagger T \in \mathcal{L}^+$. The converse is similar.

$1 \leftrightarrow 3$. Let $T = U - V$ be a proper splitting of T. Since $T^\dagger = (I - U^\dagger V)^{-1}U^\dagger$ then

$$\begin{aligned} 0 \le T^\dagger V &\leftrightarrow 0 \le (I - U^\dagger V)^{-1}U^\dagger V \leftrightarrow 0 \le \left((I - U^\dagger V)^{-1}U^\dagger V\right)^\dagger \\ &\leftrightarrow 0 \le \left(U^\dagger V\right)^\dagger (I - U^\dagger V) \leftrightarrow 0 \le \left(U^\dagger V\right)^\dagger - P_{\mathcal{N}(U^\dagger V)^\perp} \\ &\leftrightarrow 0 \le P_{V^*} \le \left(U^\dagger V\right)^\dagger, \end{aligned} \tag{3}$$

where the third equivalence follows by Corollary 2.2 and by equivalence 1. $\leftrightarrow$ 2. of this proposition. Now, if $0 \le P_{V^*} \le \left(U^\dagger V\right)^\dagger$ then by Theorem 2.3, we get $0 \le U^\dagger V \le P_{V^*}$. Conversely, if $0 \le U^\dagger V \le P_{V^*}$ then by Theorem 2.3, $0 \le P_{V^*} \le \left(U^\dagger V\right)^\dagger$. Hence, by the equivalences (3), the assertion follows.

$2 \leftrightarrow 4$. Observe that $0 \le U^\dagger T \le P_{T^*}$ if and only if $U^\dagger V = P_{T^*} - U^\dagger T \in \mathcal{L}^+$.

$2 \leftrightarrow 5$. If $U^\dagger V \in \mathcal{L}^+$ then, by Douglas' theorem, the equation $UX = V$ has a positive solution. Conversely, if $UX = V$ has a positive solution then $VV^* \le \lambda UV^*$ for some $\lambda \ge 0$, see [33] or [3]. Now, multiplying on the left and right by $U^\dagger$ and $(U^\dagger)^*$, respectively, the assertion follows.

$5 \leftrightarrow 6$. If $UX = V$ has a positive solution then $VV^* \le \lambda UV^*$ for some $\lambda \ge 0$, see [33] or [3]. Now, multiplying on the left and right by $U^\dagger$ and $(U^\dagger)^*$, respectively, it follows that $U^\dagger VV^*(U^\dagger)^* \le \lambda U^\dagger UV^*(U^\dagger)^*$. Then $0 \le U^\dagger V(U^\dagger V)^* \le \lambda U^\dagger UV^*(U^\dagger)^* = \lambda P_{U^*}V^*(U^\dagger)^* = \lambda P_{T^*}V^*(U^\dagger)^* = \lambda(U^\dagger V)^= \lambda U^\dagger V$. Since $U^\dagger V = P_{T^*} - U^\dagger T$, the assertion follows. The converse is similar. □

In order to state a sufficient condition for the convergence of a proper splitting we prove the following result. Recall that $T \in \mathcal{L}(\mathcal{H})$ is a compact operator, i.e. $T \in \mathcal{K}$, if the image by T of the closed unit ball in $\mathcal{H}$ has compact closure in $\mathcal{H}$.

Proposition 3.6. *Consider $T \in \mathcal{L}(\mathcal{H})$ with closed range. If $T = U - V$ is a proper splitting of T then the following statements are equivalent:*

1. $T^\dagger V \in \mathcal{K}$*;*
2. $U^\dagger V \in \mathcal{K}$*;*
3. $V \in \mathcal{K}$*.*

Proof. $1 \leftrightarrow 2$. If $T^\dagger V = T^\dagger(U - T) = T^\dagger U - P_{T^*} \in \mathcal{K}$ then $(T^\dagger U)^\dagger(T^\dagger U - P_{T^*}) \in \mathcal{K}$. Now observe that $(T^\dagger U)^\dagger(T^\dagger U - P_{T^*}) = P_{T^*} - U^\dagger T = U^\dagger V$. The converse is similar.

$1 \leftrightarrow 3$ Suppose $T^\dagger V \in \mathcal{K}$. Since $\mathcal{R}(V) \subseteq \mathcal{R}(T)$ then $TT^\dagger V = V \in \mathcal{K}$. Conversely, if $V \in \mathcal{K}$ it is clear that $T^\dagger V \in \mathcal{K}$. □

We can now prove a sufficient condition to guarantee the convergence of a general proper splitting of a closed range operator. The proof is similar to the one given for the finite dimensional case [5], however we include it for the sake of completeness.

Theorem 3.7. *Consider* $T \in \mathcal{L}(\mathcal{H})$ *with closed range and* $T = U - V$ *a proper splitting of* T*. If* $U^\dagger V \in \mathcal{L}^+ \cap \mathcal{K}$ *then the proper splitting of* T *converges. Moreover, it holds that* $\rho(U^\dagger V) = \dfrac{\rho(T^\dagger V)}{1 + \rho(T^\dagger V)} < 1$.

Proof. Note that $U^\dagger V \in \mathcal{L}^+ \cap \mathcal{K}$ if and only if $T^\dagger V \in \mathcal{L}^+ \cap \mathcal{K}$. Now, let $0 \neq \lambda$. Let us see that $\lambda \in \sigma(T^\dagger V)$ if and only if $\frac{\lambda}{1+\lambda} \in \sigma(U^\dagger V)$. In fact,

$$T^\dagger V x = \lambda x \leftrightarrow (I - U^\dagger V)^{-1} U^\dagger V x = \lambda x \leftrightarrow U^\dagger V x = (I - U^\dagger V)\lambda x$$
$$\leftrightarrow U^\dagger V(1 + \lambda)x = \lambda x \leftrightarrow U^\dagger V x = \frac{\lambda}{1 + \lambda} x.$$

Since $T^\dagger V \in \mathcal{L}^+ \cap \mathcal{K}$, then $\rho(T^\dagger V) \in \sigma(T^\dagger V)$ and $\dfrac{\lambda}{1 + \lambda}$ achieves its maximum when $\lambda = \rho(T^\dagger V)$. Therefore $\rho(U^\dagger V) = \dfrac{\rho(T^\dagger V)}{1 + \rho(T^\dagger V)} < 1$. □

4. The polar proper splitting

The polar proper splitting for rectangular matrices was defined in [5]. Next we extend this particular proper splitting for Hilbert space operators.

Definition 4.1. Consider $T \in \mathcal{L}(\mathcal{H})$ a closed range operator. The polar proper splitting of T is $T = U_T - V$, where U_T is the partial isometry of the polar decomposition of T.

By Theorem 3.7, given $T \in \mathcal{L}(\mathcal{H})$ such that $T = U - V$ is a proper splitting of T and $U^\dagger V \in \mathcal{K}$, the positivity of $U^\dagger V$ is a sufficient condition to guarantee the convergence of the proper splitting of T. In the particular case that $T = U_T - V$ is the polar proper splitting of T, note that $U_T^\dagger = U_T^*$. In the next result we see that condition $U_T^* V \in \mathcal{L}^+$ can be characterized by means of the norm of T.

Proposition 4.2. *Consider* $T \in \mathcal{L}(\mathcal{H})$ *with closed range. If* $T = U_T - V$ *is the polar proper splitting of* T *then the following assertions are equivalent:*

1. $U_T^* V \in \mathcal{L}^+$*;*
2. $\|T\| \leq 1$*;*
3. $(P_T - |T^*|)^2 \leq \lambda(|T^*| - P_T)$ *for some* $\lambda \geq 0$*.*

Proof. 1. $\leftrightarrow$ 2. Recall that $U_T^*V = P_{T^*} - |T|$. If $U_T^*V \geq 0$ then $|T| \leq P_{T^*}$, so that $\|T\| = \||T|\| \leq 1$. Conversely, if $\|T\| \leq 1$ then $\rho(|T|^{1/2}P_{T^*}|T|^{1/2}) = \||T|^{1/2}P_{T^*}|T|^{1/2}\| = \||T|\| = \|T\| \leq 1$. Moreover, $\mathcal{R}(|T|^{1/2}) = \mathcal{R}(P_{T^*})$. Then, by Proposition 2.4, it follows that $|T| \leq P_{T^*}$. Therefore, $U_T^*V \in \mathcal{L}^+$.

1. $\leftrightarrow$ 3. If $U_T^*V \in \mathcal{L}^+$ then the equation $U_TX = V$ has a positive solution. Then by [33] or [3, Theorem 2.3], we get that $VV^* \leq \lambda U_TV^*$ for some $\lambda \geq 0$. So that, $(P_T - |T^*|)^2 \leq \lambda(|T^*| - P_T)$ for some $\lambda \geq 0$. Conversely, since $VV^* = (P_T - |T^*|)^2 \leq \lambda(|T^*| - P_T) = \lambda U_TV^*$ for some $\lambda \geq 0$, then $U_TV^* \geq 0$. Hence $U_T^*U_TV^*U_T = V^*U_T \geq 0$, so that $U_T^*V \in \mathcal{L}^+$. □

Now, we get a characterization of the convergence of the polar proper splitting. Before that, remember that given $0 \neq T \in \mathcal{L}(\mathcal{H})$, the reduced minimum modulus of T is $\gamma(T) = \inf\{\|Tx\|,\ dist(x, \mathcal{N}(T)) = 1\}$. It is well-known that $\gamma(T) > 0$ if and only if T has closed range and in this case, $\gamma(T) = \|T^\dagger\|^{-1}$.

Theorem 4.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be a closed range operator and $T = U_T - V$ be the polar proper splitting of T. Then the following assertions are equivalent:*

1. *The polar proper splitting of T converges;*
2. $\|U_T - T\| < 1$*;*
3. $\|P_{T^*} - |T|\| < 1$*;*
4. $\|T\| < 2$.

Proof. Since $U_T^*V = P_{T^*} - |T| \in \mathcal{L}^h$ then $\rho(U_T^*V) = \|U_T^*V\|$. Now, the equivalences 1. $\leftrightarrow$ 2. $\leftrightarrow$ 3. follow from the fact that $\|U_T^*V\| = \|P_{T^*} - |T|\| = \|U_T^*(U_T - T)\| = \|U_T - T\|$. To prove equivalence 2. $\leftrightarrow$ 4. observe that by [9, Lemma 2.1] it holds that $\|U_T - T\| = \max\{1 - \gamma(T), \|T\| - 1\}$. Then the assertion follows noticing that $\gamma(T) > 0$ because T has closed range. □

Example 4.4. Consider $T \in \mathcal{L}(\mathcal{H})$ a closed range operator. Observe that $T^*T = P_{T^*} - V$ is the polar splitting of T^*T. By Theorem 4.3, $T^*T = P_{T^*} - V$ converges if and only if $\|T^*T\| < 2$ or equivalently, $\|T\| < \sqrt{2}$.

In the next result we show that given $S, T \in \mathcal{L}(\mathcal{H})$ with closed ranges such that $S \overset{*}{\leq} T$ then the convergence of the polar proper splitting of T guarantees the convergence of the polar proper splitting of S. In addition, in this case, the polar proper splitting of T induces the polar proper splitting of S.

Theorem 4.5. *Consider $S, T \in \mathcal{L}(\mathcal{H})$ with closed range such that $S \overset{*}{\leq} T$. If the polar proper splitting of T converges then the polar proper splitting of S converges. Moreover, $\rho(U_S^*W) \leq \rho(U_T^*V)$, where $S = U_S - W$ and $T = U_T - V$ are the polar proper splitting of S and T, respectively. In addition, the polar proper splitting of S can be obtained from the polar proper splitting of T as follows: $S = P_SU_T - P_SV$.*

Proof. If $S \overset{*}{\leq} T$ then $S = P_ST$. Now, if the polar proper splitting of T converges then, by Theorem 4.3, we get that $\|T\| < 2$. Then, $\|S\| = \|P_ST\| \leq \|P_S\|\|T\| = \|T\| < 2$ and so that, the polar proper splitting of S converges. Moreover, since $S \overset{*}{\leq} T$ then by [1, Theorem 2.15], it holds that $|S| \overset{*}{\leq} |T|$. Therefore, $|S| = P_{S^*}|T|$. Then $\rho(U_S^*W) = \|P_{S^*} - |S|\| = \|P_{S^*} - P_{S^*}|T|\| = \|P_{S^*}(I - |T|)\| = \|P_{S^*}P_{T^*}(I - |T|)\| \leq \|P_{T^*}(I - |T|)\| = \rho(U_T^*V) < 1$. Finally, since $S \overset{*}{\leq} T$ then, again by [1, Theorem 2.15], it holds that $U_S \overset{*}{\leq} U_T$ i.e., $U_S = P_SU_T = U_TP_{S^*}$. Now, $S = U_S - W = P_SU_T - W$ then $W = P_SU_T - P_ST = P_S(U_T - T) = P_SV$. Hence the assertion follows. □

In Theorem 4.5 the convergence of the polar proper splitting of S can be strictly faster than the convergence of the polar proper splitting of T. We illustrate this assertion with both infinite and finite dimensional examples.

Example 4.6. Consider $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, with $\mathcal{H}_k$ Hilbert spaces for $k = 1, 2, 3$. Let $T \in \mathcal{H}$ with matrix representation $T = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, according to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, where $T_k \in \mathcal{L}^+(\mathcal{H}_k)$ are closed range operators such that $\|T_k\| < 2$ for $k = 1, 2$ and $\|P_{T_1} - T_1\| < \|P_{T_2} - T_2\|$. Take $S = P_{T_1} T = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. From Definition 2.6, it holds that $S \overset{*}{\leq} T$. Let $S = U_S - W$ and $T = U_T - V$ be the polar proper splittings of S and T, respectively. Recall that $\rho(U_S^* W) = \|P_S - |S|\|$ and $\rho(U_T^* V) = \|P_T - |T|\|$. By [9, Lemma 2.1], it follows that $\|P_{T_k} - T_k\| = \max\{1 - \gamma(T_k), \|T_k\| - 1\} < 1$ for $k = 1, 2$. Since $\|P_S - |S|\| = \|P_{T_1} - T_1\|$ and $\|P_T - |T|\| = \|P_T - T\| = \max\{\|P_{T_1} - T_1\|, \|P_{T_2} - T_2\|\}$, it holds that the polar proper splittings of S and T converge. Moreover, it follows that $\rho(U_S^* W) < \rho(U_T^* V)$.

Remark 4.7. Note that the above example is achievable in the context of frames on Hilbert spaces. Remember that a sequence $\mathcal{F} = \{f_i\}_{i\in\mathbb{N}}$ is a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{i=1}^{\infty} |\langle x, f_i\rangle|^2 \leq B\|x\|^2, \tag{4}$$

for all $x \in \mathcal{H}$. The numbers A and B are called the frame bounds. The optimal constants which satisfy (4) are denoted by $A_\mathcal{F}$ and $B_\mathcal{F}$, respectively. The frame is called tight if $A_\mathcal{F} = B_\mathcal{F}$. If $\{e_i\}_{i\in\mathbb{N}}$ is the standard orthonormal basis of ℓ^2 then the bounded linear operator $T_\mathcal{F} : \ell^2 \to \mathcal{H}$ defined by $T_\mathcal{F}(e_i) = f_i$ is called the synthesis operator associated to the frame $\mathcal{F}$. The operator $S_\mathcal{F} = T_\mathcal{F} T_\mathcal{F}^* \in \mathcal{L}^+$ is invertible and it is called the frame operator. Moreover, it holds that $A_\mathcal{F} = \|S_\mathcal{F}^{-1}\|^{-1}$ and $B_\mathcal{F} = \|S_\mathcal{F}\|$. Now, returning to the example, we can consider $T_1 = S_\mathcal{F} \in \mathcal{L}^+$ and $T_2 = S_\mathcal{G} \in \mathcal{L}^+$ the frame operators of two tight frames $\mathcal{F} = \{f_i\}_{i\in\mathbb{N}}$ and $\mathcal{G} = \{g_i\}_{i\in\mathbb{N}}$ such that $A_\mathcal{F} = B_\mathcal{F} = \frac{7}{6}$ and $A_\mathcal{G} = B_\mathcal{G} = \frac{5}{4}$. Then, note that $\|P_{T_1} - T_1\| = \max\{1 - \gamma(T_1), \|T_1\| - 1\} = \max\{1 - \frac{1}{\|T_1^{-1}\|}, \|T_1\| - 1\} = \frac{1}{6}$. Analogously it can be checked that $\|P_{T_2} - T_2\| = \frac{1}{4}$.

Example 4.8. Consider $\mathcal{H} = \mathbb{C}^3$, $S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$. Since $\mathcal{R}(T) = \mathcal{R}(S) \oplus \mathcal{R}(T - S)$ and $\mathcal{R}(T^*) = \mathcal{R}(S^*) \oplus \mathcal{R}(T^* - S^*)$ then $S \overset{*}{\leq} T$. Now, the polar proper splitting of S is $S = U_S - W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2}/2 - 1/2 & 0 \\ 0 & \sqrt{2}/2 - 1/2 & 0 \end{pmatrix}$ and the polar splitting of T is $T = U_T - V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 & 0 \end{pmatrix} - \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & \sqrt{2}/2 - 1/2 & 0 \\ 0 & \sqrt{2}/2 - 1/2 & 0 \end{pmatrix}$. Then it holds that $\rho(U_S^* W) = \frac{2-\sqrt{2}}{2} \lneq \rho(U_T^* V) = \frac{1}{2}$.

Remark 4.9. Condition $S \overset{*}{\leq} T$ in Theorem 4.5 can not be replaced by a weaker order condition like $S \overset{-}{\leq} T$. In fact, consider $\mathcal{H} = \mathbb{C}^3$. Take $S = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. It is clear that $\mathcal{R}(T) = \mathcal{R}(S) \dot{+} \mathcal{R}(T - S)$ and $\mathcal{R}(T) = \mathcal{R}(S^*) \dot{+} \mathcal{R}(T - S^*)$ (remember that $T = T^*$). So that, $S \overset{-}{\leq} T$ but they are not related with the star order. Now, since $P_T - |T| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ then $\|P_T - |T|\| < 1$ and thus the polar proper splitting of T converges. On the other hand, $P_{S^*} - |S| =$

$\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} - \begin{pmatrix}\sqrt{5} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}1-\sqrt{5} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$. Then $\|P_{S^*} - |S|\| = |1-\sqrt{5}| > 1$. Then, the polar proper splitting of S does not converge.

Remark 4.10. Condition $S \overset{*}{\leq} T$ in Theorem 4.5 can not be replaced by condition $S \overset{\sharp}{\leq} T$. In fact, consider $T = \begin{pmatrix}3/2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 3/2\end{pmatrix}$ and $S = \begin{pmatrix}3/2 & 0 & 9/2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$. Then $S \overset{\sharp}{\leq} T$ because $S = QT = TQ$, where $Q = \begin{pmatrix}1 & 0 & 3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$. Also, note that the polar proper splitting of T converges because $\|T\| = \frac{3}{2} < 2$. However, the polar proper splitting of S does not converge because $\|S\| > 2$.

Proposition 4.11. *Let $S \in \mathcal{P} \cdot \mathcal{L}^h$. If $S = P_S T$, where $T \in \mathcal{L}^h$ and the polar proper splitting of T converges then the polar proper splitting of S converges. Conversely, if the polar proper splitting of S converges then there exists $T_0 \in \mathcal{L}^h$ such that $S = P_S T_0$ and the polar proper splitting of T_0 converges.*

Proof. Let $S = P_S T$, where $T \in \mathcal{L}^h$. Consider the set $\mathcal{A}_S = \{B \in \mathcal{L}^h : S = P_S B\}$. By [4, Theorem 3.2], $\|S\| = \min_{B \in \mathcal{A}_S} \|B\|$. Therefore, if the polar proper splitting of T converges then, by Theorem 4.3, $\|S\| \leq \|T\| < 2$ and so, the polar proper splitting of S converges. Conversely, if the polar proper splitting of S converges then $\|S\| < 2$. Then applying again [4, Theorem 3.2], there exists $T_0 \in \mathcal{A}_S$ such that $\|T_0\| = \|S\|$. Therefore, the polar proper splitting of T_0 converges. □

5. Splittings for split operators

In this section we put the focus on splittings of split operators. Recall that $T \in \mathcal{L}(\mathcal{H})$ is a split operator if T has closed range and $\mathcal{R}(T) \dot{+} \mathcal{N}(T) = \mathcal{H}$. For this class of operators we consider the following proper splittings:

Definition 5.1. Given a split operator $T \in \mathcal{L}(\mathcal{H})$, we say that:

1. $T = Q_T - V$ is the Q_T-proper splitting of T, where $Q_T = Q_{\mathcal{R}(T)//\mathcal{N}(T)}$.
2. $T = T^\sharp - V$ is the group proper splitting of T, where $T^\sharp$ is the group inverse of T.

Proposition 5.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be a split operator such that $\|P_{T^*}(I - T)\| < 1$. Then the Q_T-proper splitting of T converges. In addition, if $P_{T^*} T \in \mathcal{L}^h$ and the Q_T-proper splitting of T converges then $\|P_{T^*}(I-T)\| < 1$.*

Proof. Let $T = Q_T - V$ be the Q_T proper splitting of T. By [12, Theorem 4.1], it holds that $Q_T^\dagger V = P_{T^*} P_T V = P_{T^*} V = P_{T^*}(Q_T - T) = P_{T^*} - P_{T^*} T$. Now, since $\rho(Q_T^\dagger V) \leq \|P_{T^*}(I-T)\| < 1$ then $T = Q_T - V$ converges. The converse follows from the above argument plus the fact that since $Q_T^\dagger V \in \mathcal{L}^h$ then it holds $\rho(Q_T^\dagger V) = \|Q_T^\dagger V\|$. □

Next, we study conditions for the convergence of the Q_T-proper splitting for different classes of operators. Namely, we will consider the sets $\mathcal{P} \cdot \mathcal{P} = \{T \in \mathcal{L}(\mathcal{H}) : T = PQ, \text{ for } P, Q \in \mathcal{P}\}$, $\mathcal{P} \cdot \mathcal{L}^+ = \{T \in \mathcal{L}(\mathcal{H}) : T = PA, \text{ for } P \in \mathcal{P} \text{ and } A \in \mathcal{L}^+\}$ and $\mathcal{P} \cdot \mathcal{L}^h = \{T \in \mathcal{L}(\mathcal{H}) : T = PB, \text{ for } P \in \mathcal{P} \text{ and } B \in \mathcal{L}^h\}$. The sets $\mathcal{P} \cdot \mathcal{P}$, $\mathcal{P} \cdot \mathcal{L}^+$ and $\mathcal{P} \cdot \mathcal{L}^h$ have been studied in [13], [3] and [4], respectively. From the definitions is evident that the following inclusions hold: $\mathcal{P} \cdot \mathcal{P} \subseteq \mathcal{P} \cdot \mathcal{L}^+ \subseteq \mathcal{P} \cdot \mathcal{L}^h$. Moreover, these inclusions are strict in general [3,4]. They are not subsets of the class of split operators, in general. However, under certain extra conditions, their elements are split operators. In fact, it holds that: $T \in \mathcal{P} \cdot \mathcal{P}$ is a split operator if and only if $\mathcal{R}(T)$ is closed [13, Theorem 3.2]; if $\mathcal{R}(T)$ is closed then $T \in \mathcal{P} \cdot \mathcal{L}^+$ if and only if T is a split operator and $TP_T \in \mathcal{L}^+$

[3, Theorem 3.3]; if $\mathcal{R}(T)$ is closed and $T \in \mathcal{P} \cdot \mathcal{L}^h$ then, there exists $B \in \mathcal{L}^h$ with $\mathcal{N}(B) = \mathcal{N}(T)$ such that $T = P_T B$ if and only if T is a split operator [4, Corollary 2.13].

Corollary 5.3. *The following assertions hold:*

1. *Let $T \in \mathcal{L}(\mathcal{H})$ be a split operator such that $T^* \in \mathcal{P} \cdot \mathcal{L}^h$. Then, the Q_T-proper splitting of T converges if and only if $\|P_{T^*}(I - T)\| < 1$.*
2. *Let $T \in \mathcal{L}(\mathcal{H})$ be a closed range operator such that $T^* \in \mathcal{P} \cdot \mathcal{L}^+$. Then, the Q_T-proper splitting of T converges if and only if $\|P_{T^*}(I - T)\| < 1$.*
3. *Let $T \in \mathcal{L}(\mathcal{H})$ be a closed range operator such that $T \in \mathcal{P} \cdot \mathcal{P}$. Then, the Q_T-proper splitting of T converges.*

Proof. 1. It follows from Proposition 5.2 and equivalence $a \leftrightarrow b$ of [4, Theorem 2.2].

2. It follows from Proposition 5.2 and by equivalence $1 \leftrightarrow 3$ of [3, Theorem 3.3].

3. If $T \in \mathcal{P} \cdot \mathcal{P}$ then $T = P_T P_{T^*}$, see [13, Theorem 3.1]. Therefore, $Q_T^\dagger V = P_{T^*} - P_{T^*} P_T P_{T^*}$. Observe that $P_{T^*} P_T P_{T^*} \geq 0$ and $\mathcal{R}(P_{T^*} P_T P_{T^*}) = P_{T^*}\mathcal{R}(T) = \mathcal{R}(P_{T^*} P_T) = \mathcal{R}(T^*)$. Then P_{T^*} is the partial isometry of $P_{T^*} P_T P_{T^*}$, so by [9, Lemma 2.1], it holds that $\|P_{T^*} - P_{T^*} P_T P_{T^*}\| = \max\{1 - \gamma(P_{T^*} P_T P_{T^*}), \|P_{T^*} P_T P_{T^*}\| - 1\}$. Since $\mathcal{R}(P_{T^*} P_T P_{T^*})$ is closed, then $\gamma(P_{T^*} P_T P_{T^*}) > 0$, so that $1 - \gamma(P_{T^*} P_T P_{T^*}) < 1$. Also, $\|P_{T^*} P_T P_{T^*}\| \leq 1$ so that $\|P_{T^*} P_T P_{T^*}\| - 1 \leq 0$. Hence $\rho(Q_T^\dagger V) = \|P_{T^*} - P_{T^*} P_T P_{T^*}\| < 1$ and the assertion follows. □

Example 5.4. We now describe the Q_T-proper splitting for a closed range operator $T \in \mathcal{P} \cdot \mathcal{P}$. For such T it holds that $T = P_T P_{T^*}$ and so $T^* = P_{T^*} P_T$, see [13, Theorem 3.1]. Then by [12, Theorem 4.1], it follows that $(T^*)^\dagger = Q_{\mathcal{R}(T)//\mathcal{N}(T)}$. Therefore the Q_T-proper splitting of T is given by $T = (T^*)^\dagger - V$, which is convergent by item 3 of the above proposition.

Proposition 5.5. *Let $T \in \mathcal{L}(\mathcal{H})$ be a split operator. If $\|P_{T^*}(I - T^2)\| < 1$ then, the group proper splitting $T = T^\sharp - V$ converges. For the converse, if $P_{T^*}T^2 \in \mathcal{L}^h$ and the group proper splitting of T converges then $\|P_{T^*}(I - T^2)\| < 1$.*

Proof. Note that $(T^\sharp)^\dagger V = P_{T^*} - (T^\sharp)^\dagger T = P_{T^*} - (T^\sharp)^\dagger T^\sharp T T = P_{T^*}(I - T^2)$. Therefore, if $\|P_{T^*}(I - T^2)\| < 1$ then $T = T^\sharp - V$ converges because $\rho((T^\sharp)^\dagger V) \leq \|P_{T^*}(I - T^2)\| < 1$. Conversely, if $P_{T^*}T^2 \in \mathcal{L}^h$ and $T = T^\sharp - V$ converges then $\|P_{T^*}(I - T^2)\| = \rho(P_{T^*}(I - T^2)) = \rho((T^\sharp)^\dagger V) < 1$. □

We finish this section by studying different criteria for the convergence of the Q-proper splitting and the group proper splitting of two operators related by means of the star and sharp orders.

Proposition 5.6. *Consider $S, T \in \mathcal{L}(\mathcal{H})$ split operators such that $\|P_{T^*}(I - T)\| < 1$. The following assertions hold:*

1. *If $S \overset{*}{\leq} T$ then the Q_T-proper splitting of T and the Q_S-proper splitting of S converge.*
2. *If $S \overset{\sharp}{\leq} T$ then the Q_T-proper splitting of T and the Q_S-proper splitting of S converge.*

Proof. 1. Note that by Proposition 5.2, it holds that the Q_T-proper splitting of T converges. Let $S = Q_S - W$ be the Q_S-proper splitting of S. Recall that $Q_S^\dagger W = P_{S^*} P_S W = P_{S^*} W = P_{S^*}(I - S)$. Since $S \overset{*}{\leq} T$ then $S = P_S T = T P_{S^*}$. Therefore, $\rho(Q_S^\dagger W) \leq \|P_{S^*}(I - S)\| = \|P_{S^*}(I - T)P_{S^*}\| = \|P_{S^*} P_{T^*}(I - T)P_{S^*}\| \leq \|P_{T^*}(I - T)\| < 1$, where the last inequality follows by hypothesis. Hence, the Q_S-proper splitting of S converges.

2. Since $S \overset{\sharp}{\leq} T$, then $\mathcal{R}(S) \subseteq \mathcal{R}(T)$, $\mathcal{R}(S^*) \subseteq \mathcal{R}(T)$ and $S = Q_S T = T Q_S$. Therefore, $\rho(Q_S^\dagger W) \leq \|Q_S^\dagger W\| = \|Q_S^\dagger Q_S(I-T)\| = \|P_{S^*}(I-T)\| = \|P_{S^*}P_{T^*}(I-T)\| \leq \|P_{T^*}(I-T)\| < 1$, where the last inequality follows by hypothesis. Hence the Q_S-proper splitting of S converges. □

Proposition 5.7. *Let $S, T \in \mathcal{L}(\mathcal{H})$ be split operators such that $\|P_{S^*}(I - ST)\| < 1$. Then the following assertions hold:*

1. *If $S \overset{*}{\leq} T$ then the group proper splitting of S converges.*
2. *If $S \overset{\sharp}{\leq} T$ then the group proper splitting of S converges.*

Proof. Let $T = T^\sharp - V$ and $S = S^\sharp - W$ the group proper splittings of T and S, respectively.

1. Since $S \overset{*}{\leq} T$ then $S = P_S T = T P_{S^*}$. Observe that $(S^\sharp)^\dagger W = P_{S^*}(1 - S^2) = P_{S^*}(1 - STP_{S^*}) = P_{S^*}(1 - ST)P_{S^*}$. Hence $\rho((S^\sharp)^\dagger W) \leq \|P_{S^*}(I - ST)\| < 1$, so that the group proper splitting of S also converges.

2. Since $S \overset{\sharp}{\leq} T$ then $S = Q_S T = T Q_S$. Observe that $(S^\sharp)^\dagger W = (S^\sharp)^\dagger(S^\sharp - S) = P_{S^*} - (S^\sharp)^\dagger Q_S T = P_{S^*} - (S^\sharp)^\dagger S^\sharp S T = P_{S^*}(I - ST)$. Hence $\rho((S^\sharp)^\dagger W) \leq \|P_{S^*}(I - ST)\| < 1$, so that the group proper splitting of S converges. □

6. Splittings for Hermitian operators

In this section we introduce two new types of proper splittings for closed range Hermitian operators.

Definition 6.1. Given $T \in \mathcal{L}^h$ with closed range we define the following splittings of T:

1. the MP-proper splitting of T is $T = T^\dagger - W$;
2. the projection proper splitting of T is $T = P_T - Z$.

Remark 6.2. Note that for $T \in \mathcal{L}^h$, the MP-proper splitting and the projection proper splitting of T are particular cases of the group proper splitting and the Q_T-proper splitting of T, respectively. In addition, if $T \in \mathcal{L}^+$ the projection proper splitting of T coincides with the polar proper splitting of T.

Remark 6.3. Given $T \in \mathcal{L}^h$ with closed range it could be natural to consider the proper splitting of T, $T = |T| - Y$. However, as in the finite dimensional case [5] it holds that if this proper splitting converges then $T \in \mathcal{L}^+$, so that $T = |T|$. In fact, since $T \in \mathcal{L}^h$ then $T \leq |T|$ and so that $Y \in \mathcal{L}^+$. Now, if the proper splitting $T = |T| - Y$ converges then $\rho(|T|^\dagger Y) = \rho(Y^{1/2}|T|^\dagger Y^{1/2}) < 1$. In addition it holds that $\mathcal{R}((|T| - T)^{1/2}) \subseteq \overline{\mathcal{R}(|T| - T)} \subseteq \overline{\mathcal{R}(|T|) + \mathcal{R}(T)} = \overline{\mathcal{R}(T) + \mathcal{R}(T)} = \overline{\mathcal{R}(T)} = \mathcal{R}(T)$. Then, by Proposition 2.4, we get that $|T| - T \leq |T|$. Therefore, $T \in \mathcal{L}^+$.

The following result gives a characterization for the convergence of the MP-proper splitting and the projection proper splitting.

Proposition 6.4. *Consider $T \in \mathcal{L}^h$ a closed range operator. Then the following assertions hold:*

1. *The MP-proper splitting of T converges if and only if $\|P_T - T^2\| < 1$.*
2. *The projection proper splitting of T converges if and only if $\|P_T - T\| < 1$.*

Proof. Let $T = T^\dagger - W$ the MP- proper splitting of T and let $T = P_T - Z$ the projection proper splitting of T.

1. Since $TW = P_T - T^2 \in \mathcal{L}^h$ then $\rho(TW) = \|P_T - T^2\|$. Therefore, the assertion follows.
2. Since $P_T Z = P_T(P_T - T) = P_T - T \in \mathcal{L}^h$ then $\rho(P_T Z) = \|P_T - T\|$. Hence, the assertion follows. □

The next result is a comparison criterion between the MP-proper splitting, the projection proper splitting and the polar proper splitting for Hermitian operators.

Proposition 6.5. *Consider $T \in \mathcal{L}^h$ a closed range operator and $T = U_T - V = T^\dagger - W = P_T - Z$, the polar proper splitting, the MP-proper splitting and the projection proper splitting of T, respectively. Then the following assertions hold:*

1. *If the projection proper splitting of T converges then the polar proper splitting of T converges. Moreover, $\rho(U_T^* V) \leq \rho(P_T Z) < 1$.*
2. *If $\|T\| \leq 1$ and the MP-proper splitting of T converges then $\rho(U_T^* V) \leq \rho(TW) < 1$.*
3. *If $\|P_T + T\| \leq 1$ and the projection proper splitting of T converges then the MP-proper splitting of T converges. Moreover, $\rho(TW) \leq \rho(P_T Z) < 1$.*

Proof. Let $T = U_T - V = T^\dagger - W = P_T - Z$, the polar proper splitting, the MP-proper splitting and the projection proper splitting of T, respectively.

1. Suppose that the projection splitting of T converges. Recall that $\rho(P_T Z) = \|P_T - T\|$ and $\rho(U_T^* V) = \|P_T - |T|\|$. Now, by [9, Corollary 2.5], it follows that $\|P_T - |T|\| \leq \|U_T - |T|\| = \|U_T(P_T - U_T^*|T|)\| = \|U_T(P_T - T)\| \leq \|P_T - T\|$, and so the assertions follow.

2. Since $\|T\| \leq 1$, by Theorem 4.3, the convergence of the polar proper splitting of T is guaranteed. Also, note that $\rho(|T||T|^\dagger|T|) = \rho(|T|) = \||T|\| = \|T\| \leq 1$, where the second equality holds because $|T| \in \mathcal{L}^+$. Then, by Proposition 2.4, it holds that $|T|^2 \leq |T|$. Now, observe that $TW = P_T - |T|^2 \geq P_T - |T| = U_T^* V \geq 0$, where the last inequality holds by Proposition 4.2. Hence, $\rho(U_T^* V) \leq \rho(TW) < 1$.

3. Since $T \in \mathcal{L}^h$ then $P_T - T$, $P_T - T^2 \in \mathcal{L}^h$. Then observe that, $\rho(TW) = \|P_T - T^2\| = \|(P_T - T)(P_T + T)\| \leq \|P_T - T\| = \rho(P_T Z) < 1$. □

Given $T \in \mathcal{L}^h$ with closed range and $W \in \mathcal{L}(\mathcal{H})$, the equation $TX = W$ is solvable if and only if the equation $|T|X = W$ is solvable. Therefore, it is interesting to establish some relationships between proper splittings of T and proper splittings of $|T|$.

Proposition 6.6. *Consider $T \in \mathcal{L}^h$ a closed range operator. Then the polar proper splitting of T converges if and only if the projection proper splitting of $|T|$ converges. Moreover, if $T = U_T - V$ is the polar proper splitting of T and $|T| = P_{|T|} - Z$ is the projection proper splitting of $|T|$ then $\rho(U_T^* V) = \rho(P_{|T|} Z)$.*

Proof. It is immediate. □

Next, we build a proper splitting of an operator in $\mathcal{P} \cdot \mathcal{L}^h$ in terms of the projection proper splitting of an Hermitian associated factor.

Theorem 6.7. *Let $S \in \mathcal{P} \cdot \mathcal{L}^h$ be a split operator with closed range. Then $S = P_S P_{S^*} - W$ is a proper splitting of S. Moreover, the proper splitting $S = P_S P_{S^*} - W$ converges if and only if $\|P_{S^*} - Q^* S\| < 1$, where $Q = Q_{\mathcal{R}(S)//\mathcal{N}(S)}$.*

Proof. Since $S \in \mathcal{P} \cdot \mathcal{L}^h$ is a split operator with closed range then, by [4, Theorem 2.12 and item *a*) of Proposition 2.17] there exists a unique $T \in \mathcal{L}^h$ with $\mathcal{N}(T) = \mathcal{N}(S)$ such that $S = P_S T$. Namely, $T = (S + S^* - P_S S^*)Q$, where $Q = Q_{\mathcal{R}(S)//\mathcal{N}(S)}$. Let $T = P_T - V$ be the projection proper splitting of T. Let us see that $S = P_S P_T - P_S V$ is a proper splitting of S. In fact, $\mathcal{R}(P_S P_T) = P_S \mathcal{R}(T) = \mathcal{R}(P_S T) = \mathcal{R}(S)$. In addition, $\mathcal{N}(S) = \mathcal{N}(T) = \mathcal{R}(T)^\perp \subseteq \mathcal{N}(P_S P_T)$. Now, if $P_S P_T x = 0$ then $P_T x \in \mathcal{R}(T) \cap \mathcal{R}(S)^\perp = \mathcal{N}(T)^\perp \cap \mathcal{R}(S)^\perp = \mathcal{N}(S)^\perp \cap \mathcal{R}(S)^\perp = (\mathcal{N}(S) + \mathcal{R}(S))^\perp = \{0\}$, so that $x \in \mathcal{R}(T)^\perp = \mathcal{N}(S)$. Then, $\mathcal{N}(P_S P_T) = \mathcal{N}(S)$ and therefore $S = P_S P_T - P_S V$ is a proper splitting of S. Finally, since $(P_S P_{S^*})^\dagger W = Q^*(P_S P_{S^*} - S) = Q^*(P_S P_{S^*} - P_S T) = P_{S^*} - T \in \mathcal{L}^h$ then the proper splitting $S = P_S P_{S^*} - W$ converges if and only if $\|P_{S^*} - T\| < 1$. The assertion follows replacing T by $(S + S^* - P_S S^*)Q$, where $Q = Q_{\mathcal{R}(S)//\mathcal{N}(S)}$. □

Remark 6.8. With the same notation that in the proof of Theorem 6.7 it holds that the projection proper splitting of T converges if and only if the proper splitting $S = P_S P_T - P_S V$ converges. In fact, by [12, Theorem 4.1] we get that $(P_S P_T)^\dagger P_S V = Q^* P_S V = Q^* V = Q^*(P_T - T) = P_T - T$, because $\mathcal{R}(T) = \mathcal{R}(S^*)$. Then $\rho((P_S P_T)^\dagger P_S V) = \|P_T - T\|$ and the assertion follows from Proposition 6.4.

Corollary 6.9. *Consider $S \in \mathcal{P} \cdot \mathcal{L}^+$ with closed range. Then there exists $T \in \mathcal{L}^+$ with $\mathcal{N}(T) = \mathcal{N}(S)$ such that $S = P_S T$. Moreover, the projection proper splitting of T, $T = P_T - V$, induces a proper splitting of S, namely, $S = P_S P_T - P_S V$. In addition, the projection proper splitting of T converges if and only if $S = P_S P_T - P_S V$ converges.*

Proof. The existence of $T \in \mathcal{L}^+$ with $\mathcal{N}(T) = \mathcal{N}(S)$ such that $S = P_S T$ is guarantee by [3, Proposition 4.1]. Then the proof follows as in the above theorem. □

Consider $S \in \mathcal{L}(\mathcal{H})$ with closed range such that $S = P_S T$, where $T \in \mathcal{L}^+$ and $\mathcal{N}(T) = \mathcal{N}(S)$. The next result shows that the proper splitting of S induced by the projection proper splitting of T in Corollary 6.9, converges faster than the polar proper splitting of S.

Corollary 6.10. *Consider $S \in \mathcal{L}(\mathcal{H})$ with closed range. Suppose that there exists $T \in \mathcal{L}^+$ with $\|T\| \le 1$, $\mathcal{N}(T) = \mathcal{N}(S)$ and $\mathcal{R}(T)$ closed such that $S = P_S T$. Consider $T = P_T - V$ the projection proper splitting of T and $S = P_S P_T - P_S V = U_S - W$ the proper splitting of S induced by T and the polar proper splitting of S, respectively. Then it holds that $\rho((P_S P_T)^\dagger P_S V) \le \rho(U_S^* W)$.*

Proof. Since $S = P_S T$ with $T \in \mathcal{L}^+$ and $\mathcal{N}(T) = \mathcal{N}(S)$ then $\mathcal{R}(T) = \mathcal{R}(S^*)$. Therefore $(P_S P_T)^\dagger P_S V = P_{S^*} - T \ge 0$ by Proposition 2.4. Now, note that $|S| = (T P_S T)^{1/2} \le T$. Then $P_{S^*} - |S| \ge P_{S^*} - T \ge 0$ and so $\rho(U_S^* W) = \|P_{S^*} - |S|\| \ge \|P_{S^*} - T\| = \rho((P_S P_T)^\dagger P_S V)$. Hence the assertion follows. □

Remark 6.11. Let us see that in Corollary 6.10 the inequality can be strict. Take $S = P_S T \in \mathcal{P} \cdot \mathcal{L}^+$, where $P_S = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ and $T = \begin{pmatrix} 1/4 & 0 \\ 0 & 0 \end{pmatrix}$. It holds that $T \in \mathcal{L}^+$ and $\mathcal{N}(S) = \mathcal{N}(T)$. The projection proper splitting of T is $T = P_T - V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 3/4 & 0 \\ 0 & 0 \end{pmatrix}$. Then the proper splitting of S induced by the projection proper splitting of T is $S = P_S P_T - P_S V = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 3/4 & 0 \\ 0 & 0 \end{pmatrix}$.

Now, $(P_S P_T)^\dagger P_S V = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix}^\dagger \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 3/4 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 3/4 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3/4 & 0 \\ 0 & 0 \end{pmatrix}$. Then $\rho((P_S P_T)^\dagger P_S V) = 3/4$. On the other hand, the polar proper splitting of S is $S = U_S - W = \begin{pmatrix} \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & 0 \end{pmatrix} - \begin{pmatrix} \sqrt{2}/2 - 1/8 & 0 \\ \sqrt{2}/2 - 1/8 & 0 \end{pmatrix}$, so that, $U_S^* W = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 - 1/8 & 0 \\ \sqrt{2}/2 - 1/8 & 0 \end{pmatrix} = \begin{pmatrix} (8 - \sqrt{2})/8 & 0 \\ 0 & 0 \end{pmatrix}$. Hence, $\rho(U_S^* W) = 1 - \sqrt{2}/8$ and therefore $\rho((P_S P_T)^\dagger P_S V) < \rho(U_S^* W)$.

Proposition 6.12. *Let $S \in \mathcal{L}(\mathcal{H})$ be a split operator. If $T \in \mathcal{L}^h$ then the following assertions hold:*

1. If $S \overset{*}{\leq} T$ and the projection proper splitting of T converges then the Q_S-proper splitting of S converges.
2. If $S \overset{\sharp}{\leq} T$ and the projection proper splitting of T converges then the Q_S-proper splitting of S converges.

Proof. Consider $T = P_T - Z$ the projection proper splitting of T and $S = Q_S - W$ the Q_S-proper splitting of S:

1. Since $S \overset{*}{\leq} T$ and $T \in \mathcal{L}^h$ then $\mathcal{R}(S) \subseteq \mathcal{R}(T)$, $\mathcal{R}(S^*) \subseteq \mathcal{R}(T)$ and $S = P_S T = T P_{S^*}$. Observe that $\rho(Q_S^\dagger W) \leq \|Q_S^\dagger W\| = \|P_{S^*}(1 - S)\| = \|P_{S^*} P_T (1 - T) P_{S^*}\| \leq \|P_T(I - T)\|$. Since the projection proper splitting of T converges then, by Proposition 6.4, it holds that $\|P_T(I - T)\| < 1$. Therefore, the Q_S-proper splitting of S converges.

2. It is similar to the proof of item 1. □

The next result allows to give an extension of the above proposition for the case of the minus order as we will see in Remark 6.14.

Proposition 6.13. *Let $S \in \mathcal{L}(\mathcal{H})$ be a split operator and $T \in \mathcal{L}^h$ such that $\mathcal{R}(S^*) \subseteq \mathcal{R}(T)$ and $S^* = QT$, for some $Q \in \mathcal{Q}$ with $\mathcal{R}(Q) = \mathcal{R}(S^*)$ and $\|Q\| < \frac{1}{\|P_T - T\|}$. If the projection proper splitting of T converges then the Q_S-proper splitting of S converges.*

Proof. Let $T \in \mathcal{L}^h$ such that $\mathcal{R}(S^*) \subseteq \mathcal{R}(T)$ and $S^* = QT$, where $Q \in \mathcal{Q}$ and $\mathcal{R}(Q) = \mathcal{R}(S^*)$. The Q_S proper splitting of S is $S = Q_S - W$, where $Q_S = Q_{\mathcal{R}(S)//\mathcal{N}(S)}$. Then $Q_S^\dagger W = P_{S^*} P_S (Q_S - S) = P_S^*(I - S)$. Now, since $\mathcal{R}(S^*) \subseteq \mathcal{R}(T)$ we get that $\|P_{S^*}(I - S)\| = \|(I - S^*)P_{S^*}\| = \|(I - QT)P_{S^*}\| = \|(Q - QT)P_{S^*}\| = \|Q(I - T)P_{S^*}\| = \|QP_T(I - P_T)P_{S^*}\| \leq \|Q\|\|P_T - T\| < 1$. □

Remark 6.14. Suppose that $S \in \mathcal{L}(\mathcal{H})$ is a split operator and $T \in \mathcal{L}^h$ are such that $S \overset{-}{\leq} T$. This means that $S = Q_1 T$ and $S^* = Q_2 T$, for some $Q_1, Q_2 \in \mathcal{Q}$ with $\mathcal{R}(Q_1) = \mathcal{R}(S)$, $\mathcal{R}(Q_2) = \mathcal{R}(S^*)$. Observe that if $\|Q_2\| < \frac{1}{\|P_T - T\|}$ then we are in the case of Proposition 6.13.

Proposition 6.15. *Consider $S, T \in \mathcal{L}^h$ closed range operators such that $S \overset{*}{\leq} T$. If the MP-proper splitting of T converges then the MP-proper splitting of S converges.*

Proof. Since $S \overset{*}{\leq} T$ then $S = P_S T = T P_S$ and $\mathcal{R}(S) \subseteq \mathcal{R}(T)$. If the MP-splitting of T converges then, by Proposition 6.4, it holds that $\|P_T - T^2\| < 1$. Now, $\|P_S - S^2\| = \|P_S - P_S T^2 P_S\| = \|P_S(I - T^2)P_S\| = \|P_S P_T (I - T^2) P_S\| \leq \|P_T(I - T^2)\| < 1$. Therefore the assertion follows by Proposition 6.4. □

6.1. An application to symmetric approximations of frames

In this section we will apply the results obtained to find the synthesis operator of the symmetric approximation of a frame in a Hilbert space. First, we state a consequence of Proposition 3.6 for the polar proper splitting of T that will be useful in this subsection:

Proposition 6.16. *Consider $T \in \mathcal{L}(\mathcal{H})$ with closed range. If $T = U_T - V$ is the polar proper splitting of T then the following statements are equivalent:*

1. $U_T^* V \in \mathcal{K}$;
2. $|T|^\dagger - P_{T^*} \in \mathcal{K}$;
3. $|T| - P_{T^*} \in \mathcal{K}$;
4. $U_T - T \in \mathcal{K}$.

Proof. $1 \leftrightarrow 4$. It follows from Proposition 3.6.

$2 \leftrightarrow 3$. It follows from the fact that $|T| - P_{T^*} = |T|(P_{T^*} - |T|^\dagger)$ and $|T|^\dagger - P_{T^*} = |T|^\dagger(P_{T^*} - |T|)$.

$3 \leftrightarrow 4$. It follows from the fact that $(U_T - T) = U_T(P_{T^*} - |T|)$ and $P_{T^*} - |T| = U_T^*(U_T - T)$. □

A sequence $\{f_i\}_{i\in\mathbb{N}} \subseteq \mathcal{H}$ is a frame for a subspace $\mathcal{S} \subseteq \mathcal{H}$ if there exist constants $A, B > 0$ such that the inequalities (4) hold for all $x \in \mathcal{S}$. If $A = B = 1$ the frame is said normalized tight. Two frames $\{f_i\}_{i\in\mathbb{N}}$ and $\{g_i\}_{i\in\mathbb{N}}$ for subspaces $\mathcal{S}$ and $\mathcal{T}$ respectively, are called weakly similar if there exists a bounded linear invertible operator $T : \mathcal{S} \to \mathcal{T}$ such that $T(f_i) = g_i$ for all $i \in \mathbb{N}$.

A normalized tight frame $\{v_i\}_{i\in\mathbb{N}}$ in $\mathcal{T} \subseteq \mathcal{H}$ is said to be a symmetric approximation of $\{f_i\}_{i\in\mathbb{N}}$ if the inequality

$$\sum_{i=1}^{\infty} \|u_i - f_i\|^2 \geq \sum_{i=1}^{\infty} \|v_i - f_i\|^2$$

holds for all normalized tight frames $\{u_i\}_{i\in\mathbb{N}}$ which are weakly similar to $\{f_i\}_{i\in\mathbb{N}}$ and the sum of the right side is finite.

If F denotes the synthesis operator of a given frame $\{f_i\}_{i\in\mathbb{N}} \subseteq \mathcal{H}$, it is known [21, Theorem 2.3] that there exists a symmetric approximation of this frame if $P_{F^*} - |F| \in \mathcal{L}^2$, where $\mathcal{L}^2$ is the Hilbert-Schmidt class of $\mathcal{L}(\mathcal{H})$. Moreover, the symmetric approximation is given by the frame $\{U_F e_i\}_{i\in\mathbb{N}}$, where $\{e_i\}$ is the canonical orthonormal basis of ℓ^2 and U_F is the partial isometry of the polar decomposition of F, $F = U_F|F|$. Now, if we consider the equation $|F|X = F^*$ it is well known that U_F^* is its Douglas' reduced solution. Then, we can consider the projection proper splitting $|F| = P_{F^*} - V$ of $|F|$ (observe that this proper splitting is the polar proper splitting of $|F|$) to obtain the operator U_F^*. We summarize in the following result this fact:

Proposition 6.17. *Consider $\{f_i\}_{i\in\mathbb{N}}$ a frame for a subspace $\mathcal{S} \subseteq \mathcal{H}$ and $|F| = P_{F^*} - V$ the projection proper splitting of $|F|$. If $P_{F^*} - |F| \in \mathcal{L}^2$ and $\|P_{F^*} - |F|\| < 1$ then the iterative process*

$$X^{i+1} = (P_{F^*} - |F|)X^i + F^*,$$

converges to the adjoint of the synthesis operator of the symmetric approximation of $\{f_i\}_{i\in\mathbb{N}}$.

Proof. It follows from [21, Theorem 2.3], Theorem 3.2 and Proposition 6.4. □

Remark 6.18. Observe that from Theorem 3.7 if $P_{F^*} - |F| \in \mathcal{L}^+ \cap \mathcal{L}^2$ then the projection proper splitting $|F| = P_{F^*} - V$ of $|F|$, converges. So that, $\|P_{F^*} - |F|\| < 1$. However, condition $\|P_{F^*} - |F|\| < 1$ does not imply $P_{F^*} - |F| \in \mathcal{L}^+$, in general. Therefore, with this particular proper splitting we get a weaker condition that the one given in Theorem 3.7 to guarantee the convergence.

7. A few more on induced splittings

In this section we study induced splittings for the case of two operators related by an invertible operator.

Proposition 7.1. *Consider $S \in \mathcal{L}(\mathcal{H})$ and $T \in \mathcal{L}^h$ such that $S = TG$ for some $G \in \mathcal{G}$. Then the projection proper splitting $T = P_T - V$ of T induces a proper splitting of S, namely, $S = P_T G - VG$. Moreover, the projection proper splitting of T converges if and only if the $S = P_T G - VG$ converges.*

Proof. Observe that $S = TG = P_T G - VG$. Let us see that $\mathcal{R}(P_T G) = \mathcal{R}(S)$ and $\mathcal{N}(P_T G) = \mathcal{N}(S)$. In fact, $\mathcal{R}(P_T G) = \mathcal{R}(P_T) = \mathcal{R}(T) = \mathcal{R}(S)$. In addition, since $P_T G = T^\dagger TG$ we get that $\mathcal{N}(S) = \mathcal{N}(TG) \subseteq \mathcal{N}(P_T G)$. Now, if $0 = P_T Gx = T^\dagger TGx$ then $TGx \in \mathcal{R}(T) \cap \mathcal{R}(T)^\perp = \{0\}$, so that $x \in \mathcal{N}(TG) = \mathcal{N}(S)$.

Thus, $\mathcal{N}(P_T G) = \mathcal{N}(S)$. Then $S = P_T G - VG$ is a proper splitting of S. Observe that $P_T G = T^\dagger TG = T^\dagger S$. Finally, $(T^\dagger S)^\dagger VG = S^\dagger TVG = S^\dagger T(P_T - T)G = S^\dagger(TG - T^2G) = S^\dagger(S - TS) = S^\dagger(P_T - T)S$, where the first equality holds by Proposition 2.1. Now, as $\sigma((T^\dagger S)^\dagger VG) \cup \{0\} = \sigma(S^\dagger(P_T - T)S) \cup \{0\} = \sigma((P_T - T)SS^\dagger) \cup \{0\} = \sigma((P_T - T)P_T) \cup \{0\} = \sigma((P_T - T)) \cup \{0\}$ then we get that $\rho((T^\dagger S)^\dagger VG) = \rho(P_T - T)$, and the assertion follows. □

Proposition 7.2. *Consider $S, T \in \mathcal{L}(\mathcal{H})$ such that $S = TW$, for some $W \in \mathcal{U}$. If $T = U - V$ is a proper splitting of T then $S = UW - VW$ is a proper splitting of S. Moreover, it holds that the proper splitting of T converges if and only if the proper splitting $S = UW - VW$ converges.*

Proof. Let us see that $S = UW - VW$ is a proper splitting of S. Note that $\mathcal{R}(UW) = \mathcal{R}(U) = \mathcal{R}(T) = \mathcal{R}(S)$. In addition, $\mathcal{N}(UW) = \mathcal{N}(TW) = \mathcal{N}(S)$. Indeed, $UWx = 0$ if and only if $Wx \in \mathcal{N}(U) = \mathcal{N}(T)$ if and only if $TWx = 0$. Therefore $S = UW - VW$ is a proper splitting of S. Now, since $\mathcal{R}(U^*UW) \subseteq \mathcal{R}(W) = \mathcal{H}$ and $\mathcal{R}(WW^*U^*) = \mathcal{R}(U^*)$ then, by Proposition 2.1 $(UW)^\dagger = W^*U^\dagger$. Finally, since $(UW)^\dagger VW = W^*U^\dagger VW$ then $\rho((UW)^\dagger VW) = \rho(U^\dagger V)$. Then the assertion follows. □

Corollary 7.3. *Let $S, T \in \mathcal{L}(\mathcal{H})$ such that $S = TW$, for some $W \in \mathcal{U}$. Then the following assertions hold:*

1. *If $T = U_T - V$ is the polar proper splitting of T then the induced proper splitting of S, $S = U_T W - VW$ is the polar proper splitting of S.*
2. *If $S, T \in \mathcal{L}^h$ and $T = T^\dagger - V$ is the MP-proper splitting of T then the induced proper splitting of S, $S = T^\dagger W - VW$ is the MP-proper splitting of S.*

Proof. 1. By Proposition 7.2 it is sufficient to note that $U_T W = U_S$. If fact, $\mathcal{R}(U_T W) = \mathcal{R}(S)$, $\mathcal{N}(U_T W) = \mathcal{N}(S)$ and $U_T WW^*U_T^* = U_T^* U_T = P_T = P_S$. So that, $U_T W = U_S$.

2. By Proposition 7.2 it is sufficient to note that $T^\dagger W = S^\dagger$. Since $S = S^*$, by Proposition 2.1 it holds that $S^\dagger = (W^*T)^\dagger = T^\dagger W$. So, the assertion follows. □

Corollary 7.4. *Let $S, T \in \mathcal{L}(\mathcal{H})$ such that $S = TW$, for some $W \in \mathcal{U}$. Then the following assertions hold:*

1. *The polar proper splitting of T converges if and only if the polar proper splitting of S converges.*
2. *If $S, T \in \mathcal{L}^h$ then the MP-proper splitting of T converges if and only if the MP-proper splitting of S converges.*

Proof. It follows from Proposition 7.2 and Corollary 7.3. □

The last result shows that the polar proper splitting of all operators in the unitary orbit of an $T \in \mathcal{L}(\mathcal{H})$ has the same behavior with respect to the convergence. Moreover, the speed of convergence of these polar proper splittings coincides.

Proposition 7.5. *Consider $S, T \in \mathcal{L}(\mathcal{H})$ such that $S = UTU^*$, where $U \in \mathcal{U}$. Then, the polar proper splitting of S converges if and only if the polar proper splitting of T converges. Moreover, if $S = U_S - W$ and $T = U_T - V$ are the polar proper splittings of S and T, respectively, then $\rho(U_S^* W) = \rho(U_T^* V)$.*

Proof. Since $S = UTU^*$ then $U^*SU = T = U^*U_S U - U^*WU$. Let us see that $U^*U_S U = U_T$. In fact, $|S| = U|T|U^*$ then $|S|^\dagger = U|T|^\dagger U^*$. Now, $U_S = S|S|^\dagger = UTU^*U|T|^\dagger U^* = UT|T|^\dagger U^* = UU_T U^*$. So that $U^*U_S U = U_T$ and then $U^*WU = V$. Therefore, $\rho(U_T^* V) = \|U_T^* V\| = \|U^*U_S^* UV\| = \|UU^*U_S^* UVU^*\| = \|U_S^* UVU^*\| = \|U_S^* W\| = \rho(U_S^* W)$. □

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