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Toric Geometry

Organised by
Jürgen Hausen, Tübingen
Diane Maclagan, Coventry
Hal Schenck, Urbana

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Abstract. Toric geometry is a subfield of algebraic geometry with deep intersections with combinatorics. This workshop brought together researchers working in toric geometry, applying toric geometry elsewhere in algebraic geometry, and applying toric geometry elsewhere inside and outside mathematics.

Mathematics Subject Classification (2010): 14M25.

Introduction by the Organisers

Toric geometry is a subfield of algebraic geometry with deep intersections with combinatorics. A toric variety X is a partial compactification of the algebraic torus $T \cong (\mathbb{C}^*)^n$ with an action of T that extends the action of T on itself. Behind this simple definition, however, is a striking combinatorial dictionary that relates algebro-geometric invariants of the variety X to geometric-combinatorial invariants of an associated lattice polytope or polyhedral fan. This bridge between the two fields has made toric geometry to an important source of examples and counterexamples in algebraic geometry.

Toric techniques also have applications in other areas, both inside algebraic geometry, in other areas of mathematics, and outside mathematics. Examples inside algebraic geometry include the study of Mori Dream Spaces, varieties with torus actions, Newton-Okounkov bodies, tropical geometry, and degenerations to toric varieties. There are also strong connections to string theory and symplectic geometry, and increasing ties to arithmetic geometry and commutative algebra. Finally, toric varieties also have applications outside mathematics, in areas as diverse as statistics, coding theory, computer modelling, and chemistry.

This workshop brought together researchers working in all aspects of the subject. The talks presented current developments and recent results in "classical" toric geometry, toric-inspired topics, and the use of toric tools in other fields ranging from algebraic geometry via commutative algebra, topology and arithmetic geometry to applications.

Some of the broad themes covered were:

(1) Applications to combinatorics (Huh, Katz, Lasoń)
(2) Connections to number theory and topology (Gubler, De Cataldo)
(3) Applications outside mathematics: Dickenstein (biochemistry), He (physics), Michalek (statistics)
(4) Toric inspired algebraic geometry (Brion, Karu, Laface, Satriano)
(5) Algebraic aspects (Hering, Kaveh, Smith)
(6) Classical toric questions (Altmann, Arzhantsev, Brown, Di Rocco, Grassi, Ilten, Mustaţă, Teissier)

One aspect that we would like to highlight was an evening session on Tuesday of five minute talks by largely junior participants. The session was lively, and began with a five minute talk by Sturmfels, and finished with one by Batyrev. As with most Oberwolfach workshops, the informal conversations that followed the talks, over meals, during coffee breaks, and in the evening, also contributed to a rich scientific week, and we are grateful to Oberwolfach for facilitating that.

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Toric MESSI biochemical systems

Alicia Dickenstein
(joint work with Mercedes Pérez Millán)

Many processes within cells involve some kind of post-translational modification of proteins. A subclass of these mechanisms which present modifications of type Enzyme-Substrate or Swap with Intermediates, has attracted considerable theoretical attention due to its abundance in nature and the special characteristics in the topologies. In my talk, I introduced a general framework for these biological systems, developed in joint work with Mercedes Pérez Millán [9], which we call MESSI networks.

MESSI biochemical systems are MESSI chemical reaction networks endowed with mass-action kinetics. The set of species can be partitioned into a subset $S^{(0)}$ of *intermediate* species and different subsets $S^{(1)}, \dots, S^{(m)}$ of *core* species, in such a way that the associated autonomous polynomial dynamical system is linear in the variables of each $S^{(i)}$ union some subset $S_i^{(0)}$ of the intermediate variables. The union of these subsets $S_i^{(0)}$ equals $S^{(0)}$, but they are in general not disjoint, which accounts for several important properties of the systems. We characterize with algebro-geometric and combinatorial tools general properties of MESSI systems (as compactness of invariant subspaces and permanence) and we concentrate on the important question of multistationarity, that is, on the ocurrence of more than one positive steady state with the same conserved quantities.

Many post-translational modification networks are MESSI networks. For example: the motifs in [2], sequential distributive multisite phosphorylation networks [10], sequential processive multisite phosphorylation networks [1], phosphorylation cascades or the bacterial EnvZ/OmpR network in [12]. Our work is inspired by and extends some results in several previous articles [3, 4, 5, 6, 7, 8, 10, 11, 13].

We show that the steady states of most popular MESSI systems (including all those recalled above) present a toric structure, and we give in this case a characterization of the capacity for multistationarity, which leads to an algorithmic approach that we implemented with tools from oriented matroid theory. The statement of our precise results would need a long glossary together with clarifying examples, that we omit in this account.

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Hodge theory in combinatorics

Eric Katz
(joint work with Karim Adiprasito, June Huh)

This abstract describes recent work [1] resolving Rota's conjecture on the log-concavity of the characteristic polynomial of a matroid.

1. The Characteristic Polynomial

We begin by considering the case of realizable matroids. Let $\mathbf{k}$ be a field. Let $V \subset \mathbf{k}^{n+1}$ be an $(r+1)$-dim linear subspace not contained in any coordinate hyperplane. We would like to use inclusion/exclusion to express $[V \cap (\mathbf{k}^*)^{n+1}]$ as a linear combination of $[V \cap L_I]$ where L_I is the coordinate subspace given by

$$L_I = \{x_{i_1} = x_{i_2} = \cdots = x_{i_l} = 0\}$$

for $I = \{i_1, i_2, \ldots, i_l\} \subset \{0, \ldots, n\}$. You may interpret the brackets as sets of geometric points.

To identify the intersections, we have to discuss flats. A subset $I \subset \{0, \ldots, n\}$ is said to be a *flat* if for any $J \supset I$, we have $V \cap L_J \neq V \cap L_I$. The rank of a flat is $\rho(I) = \mathrm{codim}(V \cap L_I \subset V)$. The flats uniquely label the intersections $V \cap L_I$. We can now write for unique choices $\nu_I \in \mathbb{Z}$,

$$[V \cap (\mathbf{k}^*)^{n+1}] = \sum_{\text{flats } I} \nu_I [V \cap L_I].$$