Koszul Calculus

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(joint work with Roland Berger, Thierry Lambre)

The idea of this joint work with Ronad Berger and Thierry Lambre is to present a calculus which is well-adapted to quadratic algebras. This calculus is defined in Koszul cohomology (homology) by cup products (cap products). Koszul homology and cohomology are interpreted in terms of derived categories. If the algebra is not Koszul, Koszul (co)homology provides different information than Hochschild (co)homology. Koszul homology is related to de Rham cohomology. If the algebra is Koszul, Koszul cohomology is related to the Calabi-Yau property. The calculus is made explicit on a non-Koszul example.

Quadratic algebras are associative algebras defined by homogeneous quadratic relations. Since their definition by Priddy [7], Koszul algebras form a widely studied class of quadratic algebras [6]. In his monograph [5], Manin brings out a general approach of quadratic algebras (not necessarily Koszul), including the fundamental observation that quadratic algebras form a category which should be a relevant framework for a noncommutative analogue of projective algebraic geometry. According to this general approach, non-Koszul quadratic algebras deserve certainly more attention.

The goal here is to introduce new general tools for studying quadratic algebras. These tools consist in a (co)homology, called Koszul (co)homology, together with products, called Koszul cup and cap products. They are organized in a calculus, called Koszul calculus. If two quadratic algebras are isomorphic in the sense of the Manin category, their Koszul calculus are isomorphic. If the quadratic algebra is Koszul, the Koszul calculus is isomorphic to Hochschild (co)homology endowed with usual cup and cap products – called Hochschild calculus. .

The Koszul homology $HK_\bullet(A,M)$ of a quadratic algebra A with coefficients in a bimodule M is defined by applying the functor $M \otimes_{A^e} -$ to the Koszul complex of A, analogously for the Koszul cohomology $HK^\bullet(A,M)$. If A is Koszul, the Koszul complex is a projective resolution of A, so that $HK_\bullet(A,M)$ (resp. $HK^\bullet(A,M)$) is isomorphic to the Hochschild homology $HH_\bullet(A,M)$ (resp. Hochschild cohomology $HH^\bullet(A,M)$). Restricting the Koszul calculus to $M = A$, we present a non-Koszul quadratic algebra A which is such that $HK_\bullet(A) \not\cong HH_\bullet(A)$ and $HK^\bullet(A) \not\cong HH^\bullet(A)$. So $HK_\bullet(A)$ and $HK^\bullet(A)$ provide further invariants associated to the Manin category, besides those provided by Hochschild (co)homology. We have proven that Koszul homology (cohomology) is isomorphic to a Hochschild hyperhomology (hypercohomology), showing that this new homology (cohomology) becomes natural in terms of derived categories.

For any unital associative algebra A, the Hochschild cohomology of A with coefficients in A itself, endowed with the cup product, has a richer structure provided by Gerstenhaber product $\circ$, called Gerstenhaber calculus [1]. When $\circ$ is replaced in the structure by the graded bracket associated to $\circ$, that is, the Gerstenhaber bracket $[-,-]$, the calculus becomes a Gerstenhaber algebra [1]. Next, the Gerstenhaber algebra and the Hochschild homology of A, endowed with cap products, are organized in a Tamarkin-Tsygan calculus [8], see also [4]. In the Tamarkin-Tsygan calculus, the Hochschild differential b is defined from the multiplication μ of A and the Gerstenhaber bracket by

$$b(f) = [\mu, f] \tag{1}$$

for any Hochschild cochain f.

The obstruction to see the Koszul calculus as a Tamarkin-Tsygan calculus is the following: the Gerstenhaber product $\circ$ *does not make sense on Koszul cochains.* However, this negative answer can be bypassed by the fundamental formula of the Koszul calculus

$$b_K(f) = -[e_A, f]_{\underset{K}{\smile}} \tag{2}$$

where b_K is the Koszul differential, e_A is the fundamental 1-cocycle and f is any Koszul cochain.

In formula (2), $[-,-]_{\underset{K}{\smile}}$ is the graded bracket associated to the Koszul cup product $\underset{K}{\smile}$. In other words, *the Koszul differential may be defined from the Koszul cup product.* Therefore, the Koszul calculus is simpler than the Tamarkin-Tsygan calculus, since no additional product such as $\circ$ is required to express the differential by means of a graded bracket. The Koszul calculus is more flexible since the formula (2) is valid for any bimodule M, while the definitions of Gerstenhaber product and bracket are meaningless when considering other bimodules of coefficients [2]; it is also more symmetric since there is an analogue of (2) in homology, where the Koszul cup product is replaced by the Koszul cap product.

In [3], Ginzburg mentions that the Hochschild cohomology algebras of A and its Koszul dual $A^!$ are isomorphic if the quadratic algebra A is Koszul. As an application of Koszul calculus, we obtain such a Koszul duality theorem linking the Koszul cohomology algebras of A and $A^!$ *for any quadratic algebra A, Koszul or not.* So the true nature of the Koszul duality theorem is independent of any assumption on quadratic algebras. The proof of our result lies on a Koszul duality at the level of Koszul cochains and uses standard facts on duality of finite dimensional vector spaces.

In the Tamarkin-Tsygan calculus, the Connes differential B defined on Hochschild homology is an essential ingredient. Although B does not send Koszul chains to Koszul chains, we have solved the question to find such a differential at the level of Koszul homology classes for some very particular cases, the general case being open. An important role is played by the Rinehart-Goodwillie operator whose Koszul analogue is the left Koszul cap product by the fundamental 1-cocycle e_A, leading to the higher Koszul homology of A. We conjecture that a quadratic algebra A is Koszul if and only if its higher Koszul homology annihilates in positive degrees.

References

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