Obviously Strategy-proofness and Generalized Median Voter Schemes

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Abstract: The set of all strategy-proof and onto social choice functions on the domain of single-peaked preferences over a finite and linearly ordered set of alternatives coincides with the class of all generalized median voter schemes. Our objective in this paper is to characterize the subclass of generalized median voter schemes that, in addition of being strategy-proof, are also obviously strategy-proof. Our proof is constructive: for each obviously strategy-proof generalized median voter scheme we define an extensive game form that implements it in obviously dominant strategies.

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1 Introduction

A social choice function (mapping preference profiles into alternatives) is strategy-proof if it is always in the agents' interest to reveal their preferences truthfully. However, the Gibbard-Satterthwaite Theorem ([5] and [10] indicates the difficulties in designing non-trivial and strategy-proof social choice functions. Yet, and despite this negative result, there is abundant literature studying and characterizing classes of strategy-proof social choice functions for specific settings where the Gibbard-Satterthwaite Theorem does not apply. For instance, the class of generalized median voter schemes on the domain of ordinal and single-peaked preferences over a linearly ordered set of alternatives is large.

Nevertheless, the mechanism design literature has mainly neglected two features of direct revelation mechanisms when used to implement strategy-proof social choice functions on restricted domains of preferences. The first one is related to the ease with which agents can realize that their truth-telling strategies are indeed weakly dominant i.e., how much contingent reasoning is required to do so.

The second one is related to the degree of bilateral commitment of the designer who, after collecting the revealed profile of agents' preferences, will supposedly implement the alternative that the social choice function would have chosen at the revealed profile, regardless of whether he likes it or not (see [3] is an example of a paper that considers a setting where the designer does not have commitment power at all.) Li (2017) proposes the notion of obvious strategy-proofness to deal simultaneously with both concerns.

We now ask: what are the properties that a generalized median voter scheme has to satisfy to be obviously strategy-proof? We identify the two properties that together answer this question for the general case and, for each generalized median voter scheme satisfying them, we exhibit an extensive game form that implements it in obviously dominant strategies.

We want to emphasize that, 0ur proofs are constructive: for each obviously strategy-proof social choice function, we exhibit (and show how to construct) an extensive game form that implements the social choice function in obviously dominant strategies (at this point our paper differers from [9], [2] and [7]).

The paper is organized as follows. Section 2 contains the definition of generalized median voter. Sections 3 contains the corresponding analysis of obviously strategy proof. Section 4 concludes with the main results.

2 Generalized median voter schemes

Consider a social choice problem where the set of alternatives X is a finite and linearly ordered set. Without loss of generality we may assume that X is a finite subset of integers $\{1, \ldots, M\}$, where $M > 1$.

Agent i's preference P_i is *single-peaked* over X if (i) there exists $t(P_i) \in X$, called the top of P_i, such that $t(P_i)P_i x$ for all $x \in X \backslash \{t(P_i)\}$ and (ii) for all $x, y \in X$, $x < y \leq t(P_i)$ or $t(P_i) \leq y < x$ implies $yP_i x$. Let $\mathcal{P}$ be the set of all single-peaked preferences over X. Define $\mathcal{P}^n = \mathcal{P} \times \cdots \times \mathcal{P}$ as the set of single-peaked preference profiles. Given $P = (P_1, \ldots, P_n) \in \mathcal{P}^n$, we denote the vector of tops at P by $t(P) = (t(P_1), \ldots, t(P_n))$.

Let 2^N denote the family of all subsets of N, referred as coalitions, and let $x \in X$ be given. A non-empty family $\mathcal{C}_x \subset 2^N \backslash \{\emptyset\}$ of non-empty coalitions is a *committee for* x if it is (coalition) monotonic in the sense that $S \in \mathcal{C}_x$ and $S \subsetneq T$ imply $T \in \mathcal{C}_x$. Coalitions in $\mathcal{C}_x$ are called *winning* coalitions (for x). Denote by $\mathcal{C}_x^m$ the family of *minimal* winning coalitions $\mathcal{C}_x$. A family of committees $\{\mathcal{C}_x\}_{x \in X}$, one for each alternative in X, is a *coalition system* if it is (outcome) monotonic in the sense that for all $x < M$, $S \in \mathcal{C}_x$ implies $S \in \mathcal{C}_{x+1}$ and $\mathcal{C}_M = 2^N \backslash \{\emptyset\}$.

Definition 1 *A social choice function $f : \mathcal{P}^n \to X$ is a generalized median voter scheme if there exists a coalition system $\{\mathcal{C}_x\}_{x \in X}$ such that, for all $P \in \mathcal{P}^n$,*

$$f(P) = x \textit{ if and only if } \quad \textit{(i)} \ \{i \in N \mid t(P_i) \leq x\} \in \mathcal{C}_x \textit{ and}$$
$$\textit{(ii) for all } x' < x, \ \{i \in N \mid t(P_i) \leq x'\} \notin \mathcal{C}_{x'}.$$

A social choice function $f : \mathcal{P}^n \to X$ is onto and strategy-proof if and only if f is a generalized median voter scheme.[1]

3 Obviously strategy-proofness

We adapt [6] to our ordinal voting setting with no uncertainty. An *extensive game form with consequences in* X requires basic notation that is provided in Table 1 .

Object	Notation	Generic element
Players	N	i
Outcomes	X	x
Nodes	Z	z
Partial order on Z	$\prec$	
Initial node	z_0	
Terminal nodes	Z_T	
Non-terminal nodes	Z_{NT}	
Nodes where i plays	Z_i	z_i
Available actions at $z \in Z_{NT}$	$\mathcal{A}(z)$	
Outcome at $z \in Z_T$	$g(z)$	

Table 1: Notation for Extensive Game Forms

An extensive game form with consequences in X (or simply, a *game*) is denoted by Γ. A *strategy* of i in Γ is a function $\sigma_i : Z_i \to A$ such that for each $z \in Z_i$, $\sigma_i(z) \in \mathcal{A}(z)$.

Definition 2 *Let Γ be a game and $P_i \in \mathcal{D}_i$ be a preference for agent $i \in N$. We say that σ_i is obviously dominant in Γ for i with P_i if for all $\sigma_i' \neq \sigma_i$, $g(\sigma_i, \sigma_{-i}) R_i = g(\sigma_i', \sigma_{-i})$.* [2]

Definition 3 *The SCF $f : \mathcal{D} \to X$ is obviously strategy-proof (OSP) if there exists a game Γ such that implements f in obviously dominant strategy.*

[1] See [8] and [4].

[2] $g(\sigma_i, \sigma_{-i})$ and $g(\sigma_i', \sigma_{-i})$ are the outcomes of the game when the players play according to σ and (σ_i', σ_{-i}) By [7] we know that obviously dominant is equivalent to dominant on extensive game with no uncertainty.

4 INCREASING INTERSECTION PROPERTY AND MAIN RESULT

For any $k \geq 1$, denote by I_x^k the intersection of the coalitions in $\mathcal{C}_x^m$ with cardinality greater or equal than k.

For each $x \in \{1, \ldots, M\}$, let k^x denote the cardinality of the coalitions in $\mathcal{C}_x^m$ with maximal cardinality.

We are now ready to present the key definition of the paper.

Definition 4 *A coalition system $\{\mathcal{C}_x\}_{x \in X}$ satisfies the increasing intersection propertyif for each $x \in X$ such that $k^x > 1$,*
(a) $\left|I_x^k\right| \geq k-1$ for all $k \leq k^x$, and
(b) there exists $i \in I_x^2$ such that $I_{x+1}^1 \cup \{i\} \in \mathcal{C}_{x+1}^m$.
The agent identified in part (b) is not necessarily unique, and we will denote one of such agents by i^x.

Theorem 1 *A social choice function $f : \mathcal{SP}^n \to X$ is onto and obviously strategy-proof if and only if f is a generalized median voter scheme whose associated coalition system $\mathcal{C} = \{\mathcal{C}_x\}_{x \in X}$ satisfies the increasing intersection property.*

4.1 THE SUBGAME BETWEEN x AND $x+1$

Fix f and its coalition system $\{\mathcal{C}_x\}_{x \in X}$ satisfying the increasing intersection property. The full game that implements f will be based on a collection of subgames played between x and $x+1$, denoted by Γ^x,.

Let $1 < k_1 < \ldots < k_T \leq n$ be the cardinalities of the coalitions in $\mathcal{C}_x^m$ that are strictly larger than 1.[3]

Assume $k \in \{k_1, \ldots, k_{T-1}\}$. Then, by (a) of the increasing intersection property,

$$\left|I_x^k\right| = k-1.$$

Then, each $S \in \mathcal{C}_x^m$ with $|S| = k$ contains I_x^k and it is "completed" by an agent i such that $i \notin I_x^k$. The set of these agents that complete the minimal wining coalitions of cardinality k will be denoted by F_x^k.

If $k = k_T$ and there are two or more coalitions with maximal cardinality, the previous definitions also apply. However, if $\mathcal{C}_x^m$ has only one coalition with maximal cardinality, call it S, then $S = I_x^k$. In this case, define $F_x^k = \emptyset$.

The subgame

Input: A committee $\mathcal{C}_x$ for x, for which (a) holds.

Step I. Agents that are singleton winning coalitions in $\mathcal{C}_x$ (if any) are called to play sequentially, in any order; they have to choose either x or $x+1$. If at least one of these agents chooses x, the subgame Γ^x ends immediately with outcome x. Otherwise, go to Step II.

Step II. Agents in $I_x^{k_1}$ are called to play sequentially, starting with i^x and followed by the remainder agents, in any order; they have to choose either x or $x+1$. If at least one of these agents chooses $x+1$, the subgame Γ^x ends immediately with outcome $x+1$. Otherwise, go to Step III. [4]

Step III. Agents in $F_x^{k_1}$ are called to play sequentially, starting with i^{x-1} (if this agent has not played yet) and folowed by the remainder agents, in any order;[5] they have to choose either x or $x+1$. If at least one of these agents chooses x, the subgame ends immediately with outcome x. Otherwise, go to Step IV.1.[6]

[3]For $t \in \{1, \ldots, T\}$, k_t depends on x but as x is fixed we omit it.

[4]As $k_1 > 1$, the agent i^x identified in (b) of the increasing intersection property is in $I_x^{k_1}$.

[5]If $x = 1$, i^{x-1} denotes any fixed agent that completes a minimal wining coalition in $\mathcal{C}_x^m$ with cardinality k_1.

[6]If $\mathcal{C}_x$ contains a single coalition, $I_x^1 = \emptyset$ and then by part b) of the increasing intersection property, i^{x-1} is a single coalition in $\mathcal{C}_x$.

If $\mathcal{C}_x$ no contains a single coalition, by part b) of the increasing intersection property, the agent i^{x-1} "completes" a wining coalition of cardinality k_1 in $\mathcal{C}_x$. Therefore $i^{x-1} \in F_x^{k_1}$

Step IV.t **(**$t \geq 1$**).** Agents in $I_x^{k_{t+1}} \setminus I_x^{k_t}$ are called to play sequentially in any order; they have to choose either x or $x+1$. If at least one of these agents chooses $x+1$, the subgame ends immediately with outcome $x+1$. Otherwise, go to Step V.t.

Step V.t **(**$t \geq 1$**).** Agents in $F_x^{k_{t+1}}$ are called to play sequentially in any order; they have to choose either x or $x+1$. If at least one of these agents chooses x, the subgame ends immediately with outcome x. Otherwise, go to Step IV.(t+1).

The algorithm stops at any step where the subgame ends (if any) or else at the end of Step V.$T-1$.

Output: Γ^x.

4.2 The general extensive game form $\Gamma^{\mathcal{C}}$

Input: A coalition system $\{\mathcal{C}_x\}_{x\in X}$ satisfying the increasing intersection property.

Let $x^* \in X$ be the smallest alternative with the property that its committee $\mathcal{C}_{x^*}$ has a singleton set as a minimal winning coalition.

Step 1: The first agent to play is i, the agent with the smallest index among all agents such that $\{i'\} \in \mathcal{C}_{x^*}^m$ if $x^*=1$ and $i = i^{x^*-1}$. [7] Agent i has to choose an action in the set A_i, where

$$A_i = \begin{cases} \{x^*, x^*+1\} & \text{if } x^*=1 \\ \{x^*-1, x^*, x^*+1\} & \text{if } 1 < x^* < M \\ \{x^*-1, x^*\} & \text{if } x^* = M. \end{cases}$$

If i chooses x^*, the game ends immediately with outcome x^*.

If i chooses x^*+1, go to Step A.1.

If i chooses x^*-1, go to Step B.1.

Step Up.t **(**$t \geq 1$**).** Let $x = x^*+(t-1)$. Agents play the subgame Γ^x with only one difference. At Step II, if agent i^x chooses $x+1$ and $x+2 \leq M$ go (immediately) to Step Up.$t+1$.

Step Down.t **(**$t \geq 1$**).** Let $x = x^*-t$. Agents play the subgame Γ^x with the only one difference. At Step III, if agent i^x chooses $x-1$ and $x-1 \geq 1$ go (immediately) to Step Down.$t+1$.

The algorithm stops in Step 1 when i chooses x^* or else at the end of either Step Up.t when $x = x^*+(t-1) = M$ or Step Down.t when $x = x^*-t = 1$.

Output: Γ^x.

The extensive game form defined above for the coalition system $\mathcal{C} = \{\mathcal{C}_x\}_{x\in X}$ is denoted by $\Gamma^{\mathcal{C}}$.

Theorem 2 *If $f : \mathcal{P}^n \to X$ is generalized median voter scheme satisfying the increasing intersection property. The game $\Gamma^{\mathcal{C}}$ implements f in obviously dominant strategies.*

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[7] Aagent i^{x^*-1} is the agent that $i^{x^*-1} \in I_{x^*-1}^2$ and $I_{x^*}^1 \cup \{i^{x^*-1}\} \in \mathcal{C}_{x^*}$, identified in part (b) of the increasing intersection property.