Root Systems in Lie Theory: From the Classic Definition to Nowadays

Iván Angiono

1. Root Systems: The Origin

The purpose of this article is to discuss the role played by root systems in the theory of Lie algebras and related objects in representation theory, with focus on the combinatorial description and properties.

1.1. Semisimple Lie algebras. The study of Lie algebras began toward the end of the 19th century. They emerged as the algebraic counterpart of a purely geometric object: *Lie groups*, which we can briefly define as groups that admit a differentiable structure such that multiplication and the function that computes inverses are differentiable. Lie algebras appeared as some algebraic structure attached to the tangent space of the unit of this group.

Initially Lie algebras were only considered over complex or real numbers, but the abstraction of the definition led to Lie algebras over arbitrary fields.

Definition 1.1. Let $\Bbbk$ be a field. A *Lie algebra* is a pair $(\mathfrak{g}, [\,,\,])$, where $\mathfrak{g}$ is a $\Bbbk$-vector space and $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a bilinear map (called the *bracket*) such that the following equalities hold for all $x, y, z \in \mathfrak{g}$:

$$\begin{aligned} [x, y] &= -[y, x], && \textit{antisymmetry}, \\ [x, [y, z]] &= [[x, y], z] + [y, [x, z]], && \textit{Jacobi identity}. \end{aligned}$$

There is a subtle difference when the field is of characteristic two: the antisymmetry is replaced by $[x, x] = 0$ for all $x \in \mathfrak{g}$ (which implies the former one). From now on all Lie algebras considered here are assumed to be finite-dimensional.

An easy example is to pick a vector space $\mathfrak{g}$ together with trivial bracket $[x, y] = 0$ for all $x, y \in \mathfrak{g}$; these Lie algebras are called *abelian*.

Iván Angiono is a professor of mathematics at Universidad Nacional de Córdoba, Argentina. His email address is `ivan.angiono@unc.edu.ar`.

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There is a general way to move from an associative algebra A to a Lie algebra: take $\mathfrak{g} = A$ as vector space and set $[a, b] := ab - ba$ for each pair $a, b \in A$. A prominent example of this construction is the general linear algebra $\mathfrak{gl}(V)$, which is the set of linear endomorphisms of a finite-dimensional vector space V. Other classical examples appear as Lie subalgebras (that is, subspaces closed under the bracket) of $\mathfrak{gl}(V)$:

- $\mathfrak{sl}(V)$, those endomorphisms whose trace is zero; if $V = \Bbbk^n$, then we simply denote $\mathfrak{sl}(V)$ by $\mathfrak{sl}(n, \Bbbk)$, or $\mathfrak{sl}(n)$ when the field $\Bbbk$ is clear from the context.
- The orthogonal and symplectic Lie subalgebras $\mathfrak{so}(V, b)$, respectively $\mathfrak{sp}(V, b)$, of those endomorphisms T such that
$$b\,(T(v), w) + b\,(v, T(w)) = 0 \qquad \text{for all } v, w \in V,$$
where b is a symmetric, respectively antisymmetric, nondegenerate bilinear form on V.

Analogously, we may start with $A = \Bbbk^{n \times n}$, the algebra of $n \times n$ matrices, and take some subalgebras, as the subspaces of upper triangular matrices, those of trace 0, the orthogonal matrices, between others.

Once we have a notion of *algebra*, it is natural to ask for ideals: in the case of Lie algebras, these are subspaces $\mathfrak{I} \subseteq \mathfrak{g}$ such that $[\mathfrak{I}, \mathfrak{g}] \subseteq \mathfrak{I}$. This leads to consider *simple* Lie algebras, those Lie algebras $\mathfrak{g}$ such that $\dim \mathfrak{g} > 1$ and the unique ideals are the trivial ones: 0 and $\mathfrak{g}$. In addition, we say that a Lie algebra $\mathfrak{g}$ is *semisimple* if $\mathfrak{g}$ is isomorphic to the direct sum of simple Lie algebras.

For the rest of this section we fix $\Bbbk = \mathbb{C}$. We know that a Lie algebra $\mathfrak{g}$ is simple if and only if $\mathfrak{g}$ is (isomorphic to) $\mathfrak{sl}(V)$, $\mathfrak{so}(V, b)$, $\mathfrak{sp}(V, b)$, and a few exceptional examples E_k, $k = 6, 7, 8$, F_4, G_2. That is, up to 5 exceptions, all the complex simple Lie algebras are subalgebras of matrices. Thus one may wonder if some properties of the algebras of matrices still hold for simple Lie algebras. We will recall some of them by the end of this section, following [Hum78].

As for associative algebras, we can study modules over Lie algebras. A $\mathfrak{g}$-*module* is a pair $(V, \cdot)$, where V is a

$\Bbbk$-vector space and $\cdot : \mathfrak{g} \otimes V \to V$ is a linear map such that

$$[x,y]\cdot v = x\cdot(y\cdot v) - y\cdot(x\cdot v), \qquad x,y\in\mathfrak{g}, v\in V.$$

For example, the bracket gives an action of $\mathfrak{g}$ over itself, called the *adjoint action*.

For each $x\in\mathfrak{g}$ we look at the *inner derivation*

$$\operatorname{ad} x : \mathfrak{g}\to\mathfrak{g}, \qquad \operatorname{ad} x(y) = [x,y], \qquad y\in\mathfrak{g},$$

associated to the adjoint action. These endomorphisms induce a symmetric bilinear form on $\mathfrak{g}$, called the *Killing form*:

$$\kappa : \mathfrak{g}\times\mathfrak{g}\to\Bbbk, \quad \kappa(x,y) := \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y), \quad x,y\in\mathfrak{g}.$$

The Killing form and the $\mathfrak{g}$-modules give other characterizations of semisimplicity: $\mathfrak{g}$ is semisimple if and only if κ is nondegenerate if and only if every module is semisimple, i.e., every $\mathfrak{g}$-submodule admits a complement which is a $\mathfrak{g}$-submodule.

When $\mathfrak{g}$ is one of the Lie algebras of matrices above, the action of diagonal matrices is, in fact, diagonalizable. Mimicking this fact we look for subalgebras such that the action of their elements is diagonalizable, called *toral* subalgebras.

From now on assume that $\mathfrak{g}$ is also semisimple. It can be shown that toral algebras are abelian, and we pick a maximal one $\mathfrak{h}$. Thus $\mathfrak{g}$ decomposes as the direct sum of the $\mathfrak{h}$-eigenspaces:

$$\mathfrak{g} = \oplus_{\alpha\in\mathfrak{h}^*}\mathfrak{g}_\alpha, \qquad \text{where } \mathfrak{g}_\alpha := \{x\in\mathfrak{g} \mid [h,x] = \alpha(h)x\}.$$

As $\mathfrak{h}$ is abelian, we have that $\mathfrak{h}\subseteq\mathfrak{g}_0$: one can show that we have an equality, $\mathfrak{h}=\mathfrak{g}_0$. Thus, if we set $\Delta := \{\alpha\in\mathfrak{h}^* \mid \alpha\neq 0, \mathfrak{g}_\alpha\neq 0\}$, then Δ, a finite set called the *root system* of $\mathfrak{g}$, gives a decomposition of $\mathfrak{g}$ into $\mathfrak{h}$-eigenspaces as follows:

$$\mathfrak{g} = \mathfrak{h}\oplus(\oplus_{\alpha\in\Delta}\mathfrak{g}_\alpha).$$

This decomposition is compatible with the bracket,

$$[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]\subseteq\mathfrak{g}_{\alpha+\beta} \qquad \text{for all } \alpha,\beta\in\mathfrak{h}^*,$$

and the Killing form

$$\kappa_{\mathfrak{g}_\alpha\times\mathfrak{g}_\beta} = 0 \qquad \text{if } \alpha+\beta\neq 0.$$

We can derive that $\kappa_{\mathfrak{h}\times\mathfrak{h}}$ is nondegenerate, thus it induces a symmetric nondegenerate bilinear form $(\cdot,\cdot) : \mathfrak{h}^*\times\mathfrak{h}^*\to\mathbb{C}$.

Example 1.2. If $\mathfrak{g}=\mathfrak{sl}(4)$, the Lie algebra of 4×4 matrices with trace 0, then $\mathfrak{h}$ is the subspace of diagonal matrices, with basis $h_i := E_{ii} - E_{i+1,i+1}$, $i=1,2,3$. Here, E_{ij} is the matrix with 1 in the (i,j)-entry and 0 otherwise. Let $A := \left[\begin{smallmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2\end{smallmatrix}\right]$, and set $\alpha_j\in\mathfrak{h}^*$ as the element such that $\alpha_j(h_i) = a_{ij}$. Then

- $e_i := E_{i,i+1}\in\mathfrak{g}_{\alpha_i}$, $i=1,2,3$;
- $E_{13}\in\mathfrak{g}_{\alpha_1+\alpha_2}$, $E_{24}\in\mathfrak{g}_{\alpha_2+\alpha_3}$, $E_{14}\in\mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3}$;
- for all $i<j$, if $E_{ij}\in\mathfrak{g}_\alpha$, then $E_{ji}\in\mathfrak{g}_{-\alpha}$. In particular, $f_i := E_{i+1,i}\in\mathfrak{g}_{-\alpha_i}$, $i=1,2,3$.

Thus, if we set $\alpha_{ij} := \sum_{k=i}^{j}\alpha_k$, $i\le j$, then

$$\Delta = \{\pm\alpha_{ij} \mid 1\le i\le j\le 3\}.$$

This example has a straightforward generalization to $\mathfrak{sl}(n)$ for any $n\geq 2$.

1.2. Root systems for Lie algebras.

We may derive strong properties of the root system Δ using the representation theory of $\mathfrak{sl}(2)$, we refer to [Bou02, Hum78] for more details.

(i) $\mathfrak{h}^*$ is spanned by Δ.

(ii) If $\alpha\in\Delta$, then $-\alpha\in\Delta$. Moreover, for each $\alpha\in\Delta$, $\Delta\cap\mathbb{C}\alpha = \{\pm\alpha\}$.

(iii) For each $\alpha\in\Delta$, the eigenspace $\mathfrak{g}_\alpha$ is one-dimensional. Moreover, $S_\alpha := \mathfrak{g}_\alpha\oplus\mathfrak{g}_{-\alpha}\oplus[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]$ is a subalgebra isomorphic to $\mathfrak{sl}(2)$. Notice that $[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]\subset\mathfrak{h}$.

(iv) If $\alpha,\beta,\alpha+\beta\in\Delta$, then $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

(v) Let $\alpha,\beta\in\Delta$ be such that $\alpha\neq\pm\beta$. Then there exist $q,r\in\mathbb{N}_0$ such that

$$\{i\in\mathbb{Z} \mid \beta+i\alpha\in\Delta\} = \{-r\le i\le q\}.$$

Moreover, $r-q = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$. That is, the *root string* over β in the direction of α has no holes.

By (i) there exists a basis B of $\mathfrak{h}^*$ contained in Δ. We can check that all the coefficients of any $\beta\in\Delta$, written in terms of B, are rational numbers, so we may consider the $\mathbb{Q}$-linear subspace $\mathfrak{h}^*_{\mathbb{Q}}$ generated by Δ and take the extension to $\mathbb{R}$: we get a finite-dimensional $\mathbb{R}$-vector space V which *contains* all the information and the geometry of Δ.

Remark 1.3. V becomes an Euclidean vector space with the scalar product induced by the Killing form.

For each $\alpha\in\Delta$ set $s_\alpha : V\to V$,

$$s_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha, \qquad \beta\in V.$$

Then s_α is a linear automorphism of the Euclidean space V such that $s_\alpha^2 = \mathrm{id}$, and by (v), $s_\alpha(\Delta)=\Delta$.

2. Classical Root Systems

From the information above one may wonder if there exists an abstract notion of root system. The answer is *yes*, and we will recall it following [Bou02], see also [Hum78]. We can classify all finite root systems in terms of so-called finite Cartan matrices. We will also recall a way to come back from (abstract) root systems to complex Lie algebras.

2.1. Abstract definition.

Definition 2.1 ([Bou02]). Let V be a finite-dimensional $\mathbb{R}$-vector space. A finite subset $\Delta\subset V$ is a *root system* in V if

(RS1) $0\notin\Delta$ and V is spanned by Δ.

(RS2) For each $\alpha \in \Delta$, there exists $\alpha^\vee \in V^*$ such that $\alpha^\vee(\alpha) = 2$ and the reflection

$$s_\alpha : V \to V, \qquad s_\alpha(\beta) = \beta - \alpha^\vee(\beta)\alpha, \qquad \beta \in V,$$

satisfies that $s_\alpha(\Delta) = \Delta$.

(RS3) For all $\alpha, \beta \in \Delta$, $\alpha^\vee(\beta) \in \mathbb{Z}$.

The elements of Δ are called *roots*, and $\dim V$ is the *rank* of Δ. The (finite) subgroup $\mathcal{W}$ of $\mathrm{Aut}(V)$ generated by s_α, $\alpha \in \Delta$, is the *Weyl group* of Δ.

In [Hum78] one also requires that for each $\alpha \in \Delta$, $\Delta \cap \mathbb{R}\alpha = \{\pm\alpha\}$. In other references, root systems with this extra property are called *reduced*.

The reflections s_α, $\alpha \in \Delta$ are univocally determined and there exists a symmetric invariant nondegenerate bilinear form $(\cdot|\cdot) : V \times V \to \mathbb{R}$, which is moreover invariant by $\mathcal{W}$ and positive definite. Now, the elements $\alpha^\vee$ are recovered using this form:

$$\alpha^\vee(\beta) = \tfrac{2(\beta,\alpha)}{(\alpha,\alpha)} \text{ for all } \alpha, \beta \in \Delta.$$

Also, the set $\Delta^\vee := \{\alpha^\vee : \alpha \in \Delta\}$ is a root system of V^*, with $(\alpha^\vee)^\vee = \alpha$. There are four examples of reduced root systems in rank 2: $A_1 \times A_1$, A_2, B_2 and G_2, with 2, 6, 8, and 12 roots, respectively. The third one is depicted in Figure 1.

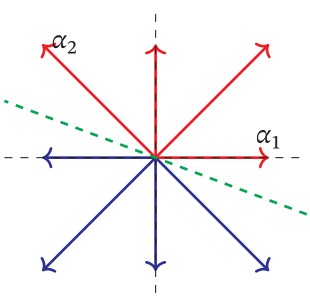

Figure 1. Root system of type B_2.

Let $\alpha, \beta \in \Delta$ be such that $\alpha \neq \pm\beta$. One may check that $\alpha + \beta \in \Delta$ (respectively, $\alpha - \beta \in \Delta$) if $(\alpha, \beta) < 0$ (respectively, $(\alpha, \beta) > 0$). This is the starting point, together with (RS2) and (RS3), to check that an analogue of (v) holds for (abstract) root systems.

Another key point is the existence of a *base* of a root system. It means a subset $B \subset \Delta$ such that B is a basis of V (as a vector space), and every $\beta \in \Delta$ is written, in terms of B, as a linear combination whose coefficients are all nonnegative integers, or all nonpositive integers.

The proof of existence of bases gives the geometric flavor behind root systems. We take a vector γ such that the orthogonal hyperplane P to γ does not contain any root. Indeed γ belongs to $V - \cup_{\alpha\in\Delta} H_\alpha$, where H_α is the kernel of $\alpha^\vee$, i.e., the hyperplane orthogonal to α: the connected components of $V - \cup_{\alpha\in\Delta} H_\alpha$ are called the *Weyl chambers*. Thus $\Delta = \Delta^+(\gamma) \cup \Delta^-(\gamma)$, where

$$\Delta^\pm(\gamma) = \{\beta \in \Delta | \pm(\beta, \gamma) > 0\}.$$

A base is made by those *indecomposable* roots in $\Delta^+(\gamma)$: those $\beta \in \Delta^+(\gamma)$ which cannot be written as a sum $\beta = \beta_1 + \beta_2$, with $\beta_i \in \Delta^+(\gamma)$. Moreover every base can be constructed in this way.

For example, in Figure 1 we take the green hyperplane: the positive roots are the red ones, the negative are the blue ones, and $B = \{\alpha_1, \alpha_2\}$ is a base.

The Weyl group $\mathcal{W}$ permutes bases (and Weyl chambers as well), and the action is simply transitive. We check then that any root $\alpha \in \Delta$ belongs to a base, and for each base B, $\mathcal{W}$ is generated by s_α, $\alpha \in B$ (we reduce the number of generators of $\mathcal{W}$ to the rank of the root system). This leads to the study of *groups generated by reflections* and *Coxeter groups* considered in [Bou02], which became an important subject of research on its own, and remains active until now.

2.2. The classification. As for algebraic objects, we may ask for *irreducible* root systems: those which cannot split into two orthogonal subsets (otherwise each subset is itself a root system). Every root system Δ of V decomposes uniquely as a union of irreducible root systems Δ_i corresponding to the subspaces V_i of V spanned by Δ_i. Thus, in order to classify root systems, we can restrict to the irreducible ones.

Assume now that Δ is an irreducible root system of rank θ. Set $A^\Delta \in \mathbb{Z}^{\theta\times\theta}$ as the matrix with entries

$$a_{ij} := \alpha_i^\vee(\alpha_j) = \tfrac{2(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)}, \qquad 1 \le i, j \le \theta,$$

where $B = \{\alpha_i\}_{1\le i\le\theta}$ is a base. One can check that A^Δ is well-defined; i.e., it does not depend on the chosen base. In addition, A is indecomposable: for all $i < j$ there exist $i_k \in \{1, \cdots, \theta\}$ such that $a_{ii_1} a_{i_1 i_2} \cdots a_{i_t j} \neq 0$. Moreover,

(GCM1) $a_{ii} = 2$ for all $1 \le i \le \theta$,
(GCM2) $a_{ij} = 0$ if and only if $a_{ji} = 0$,
(GCM3) for all $i \neq j$, $a_{ij} \le 0$.

Any $A \in \mathbb{Z}^{\theta\times\theta}$ satisfying (GCM1)–(GCM3) is called a *generalized Cartan matrix* (GCM) [Kac90]. The information of GCM is encoded in a graph called the *Dynkin diagram*: it has θ vertices, labelled with $1, 2, \dots, \theta$, and for each pair $1 \le i < j \le \theta$,

- if $a_{ij}a_{ji} \le 4$, then we add $\max\{|a_{ij}|, |a_{ji}|\}$ edges between vertices i and j, with an arrow from j to i (respectively i to j) if $|a_{ij}| > 1$ (respectively, $|a_{ji}| > 1$); in particular, if $a_{ij} = 0$ (so $a_{ji} = 0$ as well) then we draw no edges between i and j, and if $a_{ij} = a_{ji} = -1$, then we draw just a line;
- if $a_{ij}a_{ji} > 4$, then we draw a thick line between i and j labelled with $(|a_{ij}|, |a_{ji}|)$.

For example, the Dynkin diagrams of $\left[\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 2 & -3 \\ -2 & 2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{smallmatrix}\right]$ are, respectively

$$\circ \Longleftrightarrow \circ \qquad \circ \overset{(3,2)}{\text{———}} \circ \qquad \circ \text{———} \circ \text{———} \circ \, .$$

One reason to differentiate between $a_{ij}a_{ji} \le 4$ and $a_{ij}a_{ji} > 4$ is all finite and affine Dynkin diagrams satisfy the first condition, and these are probably the most studied cases. We refer to [Bou02, Hum78] for the definition of affine

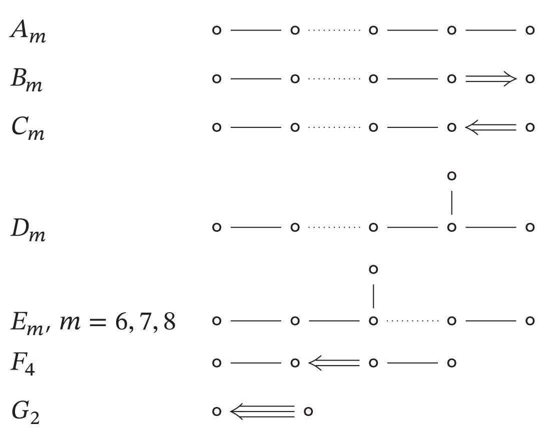

Figure 2. Finite connected Dynkin diagrams.

Dynkin diagrams while finite ones are depicted in Figure 2, in connection with finite-dimensional complex Lie algebras.

One may define the Weyl group $\mathcal{W}^A$ of a GCM A as the subgroup of $\mathrm{Aut}(V)$ generated by reflections $s_i : V \to V$, $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$, where $(\alpha_i)_{1\le i\le\theta}$ is the canonical basis of $V = \mathbb{R}^\theta$: if A is the Cartan matrix of a Lie algebra as above, then the Weyl group of Δ is generated by these s_i's. Analogously, we can define

$$\Delta^A = \{w(\alpha_i) | w \in \mathcal{W}^A, 1 \le i \le \theta\}.$$

Then one can prove that $\mathcal{W}^A$ is finite if and only if Δ^A is finite, which is equivalent to the notion of *finite GCM*. Finite GCM are parametrized by *finite* Dynkin diagrams, i.e., those in Figure 2.

Theorem 2.2. *Reduced irreducible root systems are parametrized by Dynkin diagrams in Figure 2.*

Up to now we deal with three notions:

(i) Simple Lie algebras over $\mathbb{C}$,
(ii) Irreducible root systems,
(iii) Finite Cartan matrices, or the corresponding Dynkin diagrams.

We moved first from (i) to (ii), and then state a correspondence (ii)$\longleftrightarrow$ (iii). Now we need to come back to (i). We can check that $\mathfrak{sl}(n+1)$ has Cartan matrix of type A_n (see Example 1.2), while matrices of types B_n, C_n and D_n appear for orthogonal and symplectic Lie algebras. For each one of the exceptional finite Cartan matrices A in Figure 2 we can construct *by hand* a simple Lie algebra with Cartan matrix A. The natural question is if there exists a *systematic* way to build these Lie algebras. We will recall it in the next subsection, i.e., a correspondence (iii)$\longrightarrow$(i).

2.3. Back to Lie algebras: Kac-Moody construction. Looking at Example 1.2, the Cartan matrix of $\mathfrak{sl}(4)$ can be recovered from the action of the Cartan subalgebra $\mathfrak{h}$ on eigenvectors of a base of the root system Δ. In addition the decomposition $\Delta = \Delta^+ \cup \Delta^-$ into positive and negative roots for the chosen base corresponds in this case to the upper and lower triangular matrices $\mathfrak{n}_\pm$ of $\mathfrak{sl}(4)$ (recall that $\mathfrak{h}$ is spanned by the set of all the diagonal matrices in $\mathfrak{sl}(4)$).

As for associative algebras, we have a notion of a Lie algebra *presented by generators and relations* as the appropiate quotient of a *free* Lie algebra. We will attach a Lie algebra $\mathfrak{g} := \mathfrak{g}(A)$ to each matrix $A \in \mathbb{C}^{\theta\times\theta}$; these algebras were introduced by Serre in 1966 for finite matrices A, and by Kac and Moody in two independent and simultaneous works in the late sixties, see [Kac90] and the references therein. For the sake of simplicity of the exposition we assume that $\det A \neq 0$.

Let $\tilde{\mathfrak{g}} := \tilde{\mathfrak{g}}(A)$ be the Lie algebra presented by generators e_i, h_i, f_i, $1 \le i \le \theta$, and relations

$$\begin{aligned} [h_i, h_j] &= 0, & [h_i, e_j] &= a_{ij} e_j, \\ [e_i, f_j] &= \delta_{ij} h_i, & [h_i, f_j] &= -a_{ij} f_j. \end{aligned} \tag{1}$$

Let $\mathfrak{h}$ be the subspace spanned by $(h_i)_{1\le i\le\theta}$, $\widetilde{\mathfrak{n}}_\pm$ the subalgebra generated by $(e_i)_{1\le i\le\theta}$, respectively $(f_i)_{1\le i\le\theta}$. We have the following facts:

(a) $\widetilde{\mathfrak{n}}_\pm$ is a free Lie algebra in θ generators.
(b) As a vector space, $\tilde{\mathfrak{g}} = \widetilde{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \widetilde{\mathfrak{n}}_-$.
(c) The adjoint action of $\mathfrak{h}$ on $\widetilde{\mathfrak{n}}_\pm$ is diagonalizable.
(d) Among all the ideals of $\tilde{\mathfrak{g}}$ intersecting trivially $\mathfrak{h}$, there exists a maximal one $\mathfrak{r}$, which satisfies
$$\mathfrak{r} = (\mathfrak{r} \cap \widetilde{\mathfrak{n}}_+) \oplus (\mathfrak{r} \cap \widetilde{\mathfrak{n}}_-).$$

Definition 2.3. The *contragredient* Lie algebra $\mathfrak{g}(A)$ associated to A (sometimes called the *Kac-Moody algebra*) is the quotient $\mathfrak{g}(A) := \tilde{\mathfrak{g}}(A)/\mathfrak{r}$.

Because of the definition of $\mathfrak{r}$, $\mathfrak{g}(A)$ is generated by e_i, f_i, h_i, $1 \le i \le \theta$, has a triangular decomposition

$$\mathfrak{g}(A) = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-,$$

where $\mathfrak{n}_\pm$ is the image of $\widetilde{\mathfrak{n}}_\pm$ under the projection $\pi : \tilde{\mathfrak{g}}(A) \twoheadrightarrow \mathfrak{g}(A)$, i.e., the subalgebra generated by (the image of) $(e_i)_{1\le i\le\theta}$, respectively $(f_i)_{1\le i\le\theta}$, and any other Lie algebra with a triangular decomposition as above, generated by the same set of generators satisfying (1), projects onto $\mathfrak{g}(A)$.

Theorem 2.4. (A) *Let A be a finite Cartan matrix. Then $\mathfrak{g}(A)$ is a finite-dimensional simple Lie algebra, with Cartan matrix A.*
(B) *The list of Dynkin diagrams in Figure 2 provides a classification of all finite-dimensional simple Lie algebras over $\mathbb{C}$.*

When the generalized Cartan matrix A is not of finite type, the associated Kac-Moody Lie algebra $\mathfrak{g}(A)$ is infinite-dimensional. Although for the purposes of this exposition we are interested in the finite-dimensional examples, the infinite-dimensional Lie algebras $\mathfrak{g}(A)$ (or at least some of them, mainly the affine ones) are quite important since they have appeared in connection either with other areas of mathematics, especially representation theory, or theoretical physics, for example in conformal field theory.

3. Root Systems for Other Kinds of Lie Algebras

Next we deal with contragredient Lie algebras over fields of positive characteristic and later with Lie superalgebras over any field. We will recall the main differences with the picture of Lie algebras over $\mathbb{C}$ which leads to a more general notion of root system. This root system still captures the combinatorics of these Lie theoretic objects.

3.1. Lie algebras over fields of positive characteristic.

Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$. The study of simple Lie algebras becomes more and more complicated as far as p is smaller, see, e.g., [Str04]. A main difference with the case of complex numbers is that not all simple Lie algebras have a triangular decomposition as above, and the Cartan subalgebra plays a weaker role in the structure of the whole Lie algebra.

On the other hand, Definition 2.3 still holds over $\Bbbk$, so we may ask about the classification of finite-dimensional contragredient Lie algebras. A subtle difference is that we restrict to $\mathbb{Z}$-homogeneous ideals intersecting trivially $\mathfrak{h}$, where each e_i has degree 1, each f_i has degree -1 and each h_i has degree 0. Thus, there exists a finer grading of the Lie algebra $\mathfrak{g}(A)$ by $\mathbb{Z}^\theta$, where $\deg e_i = \alpha_i$ (the i-th element of the canonical basis), $\deg f_i = -\alpha_i$ and $\deg h_i = 0$ as well. Let $\Delta^A \subset \mathbb{Z}^\theta$ be the subset of all nonzero degrees whose homogeneous components are nontrivial.

Remark 3.1. From the triangular decomposition,

$$\Delta^A \subset \mathbb{N}_0^\theta \cup (-\mathbb{N}_0^\theta),$$

that is, the coefficients of each $\alpha \in \Delta$ are all nonnegative, or else all nonpositive. Also, there exists an involution ω of $\mathfrak{g}(A)$ (called the Chevalley involution) such that $e_i \mapsto f_i$, $f_i \mapsto e_i$, $h_i \mapsto -h_i$. As $\omega\left(\mathfrak{g}(A)\right)_\beta = \mathfrak{g}(A)_{-\beta}$ for all $\beta \in \mathbb{Z}^\theta$, we have that

$$\Delta^A = -\Delta^A.$$

For example we can consider the finite Cartan matrices over $\Bbbk$, since the entries of these Cartan matrices are integer numbers, and show that the associated Lie algebras are finite-dimensional. But, even for contragredient Lie algebras, there are significant differences with the case of complex numbers. As shown in [VK71], there are examples of finite-dimensional Lie algebras with diagonal entries $a_{ii} = 0$, and two different matrices can give place to isomorphic contragredient Lie algebras. The classification shown in [VK71] was incomplete: there was a missing example for $p = 3$, the 29-dimensional Brown algebra $\mathfrak{br}(3)$, discovered by Brown in the eighties, whose realization as contragredient Lie algebra with two different matrices was shown in [Skr93]:

Theorem 3.2. *Fix $p = 3$. Let*

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then there exists an isomorphism $\Phi : \mathfrak{g}(A) \to \mathfrak{g}(B)$ such that

$$\Phi(e_1) = \underline{e}_1, \qquad \Phi(e_2) = (\operatorname{ad} \underline{e}_3)^2 \underline{e}_2, \qquad \Phi(e_3) = \underline{f}_3,$$
$$\Phi(f_1) = \underline{f}_1, \qquad \Phi(f_2) = (\operatorname{ad} \underline{f}_3)^2 \underline{f}_2, \qquad \Phi(f_3) = \underline{e}_3.$$

The expression of Φ is close to that for the action of reflections of the Weyl group on complex Lie algebras, but here Φ relates two "different" contragredient data.

Remark 3.3. We fix the following GCM

$$C^A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{bmatrix}, \quad C^B = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -2 & 2 \end{bmatrix},$$

and set $s_i^A, s_i^B : \mathbb{Z}^3 \to \mathbb{Z}^3$ as the corresponding reflections defined by C^A, C^B, respectively, $i = 1, 2, 3$. Notice that $s_3^A = s_3^B$ and $\Phi\left(\mathfrak{g}(A)_\beta\right) = \mathfrak{g}(B)_{s_3^A(\beta)}$ for all $\beta \in \mathbb{Z}^3$. Thus $\Delta^B = s_3^A(\Delta^A)$.

In addition, there exist automorphisms

$$\Phi_i^A : \mathfrak{g}(A) \to \mathfrak{g}(A), \quad \Phi_i^B : \mathfrak{g}(B) \to \mathfrak{g}(B), \quad i = 1, 2,$$

such that $\Phi_i^A\left(\mathfrak{g}(A)_\beta\right) = \mathfrak{g}(A)_{s_i^A(\beta)}$, $\Phi_i^B\left(\mathfrak{g}(B)_\beta\right) = \mathfrak{g}(B)_{s_i^B(\beta)}$ for all $\beta \in \mathbb{Z}^3$. This implies that

$$\Delta^A = s_i^A(\Delta^A), \qquad \Delta^B = s_i^B(\Delta^B), \qquad i = 1, 2.$$

3.2. Lie superalgebras.

Recall that a *Lie superalgebra* is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ ($\mathfrak{g}_0$ is the *even* part and $\mathfrak{g}_1$ is the *odd* part) together with a linear $\mathbb{Z}_2$-graded map $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying analogous versions of antisymmetry and Jacobi identity:

$$[x, y] = -(-1)^{|x||y|}[y, x],$$
$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]],$$

for all homogeneous elements $x, y, z \in \mathfrak{g}$, see, e.g., [Kac77]. Here, $|x| \in \{0, 1\}$ denotes the degree of x. We have examples from associative algebras, analogous to those of Lie algebras: given $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ a $\mathbb{Z}_2$-graded associative algebra, set

$$[x, y] = xy - (-1)^{|x||y|} yx.$$

In particular we have, for $V = V_0 \oplus V_1$, the Lie superalgebra $\mathfrak{gl}(V) = \operatorname{End}(V)$, with

$$\mathfrak{gl}(V)_i = \{T \in \mathfrak{gl}(V) : T(V_j) \subseteq V_{i+j}\}.$$

For each $T \in \mathfrak{gl}(V)$ set $\operatorname{str}(T) := \operatorname{tr}\left(T_{|V_0}\right) - \operatorname{tr}\left(T_{|V_1}\right)$, the *super trace* of T. We can consider the subalgebra

$$\mathfrak{sl}(V) = \{T \in \mathfrak{gl}(V) : \operatorname{str}(T) = 0\}.$$

Here we consider contragredient data $(A, \mathbf{p})$, where $A = (a_{ij})_{1 \le i, j \le \theta} \in \Bbbk^{\theta \times \theta}$ is still the matrix of scalars determining the action of the generators h_i on the remaining generators, and $\mathbf{p} = (p_i)_{1 \le i \le \theta} \in \mathbb{Z}_2^\theta$ gives the $\mathbb{Z}_2$-grading: $e_i, f_i \in \mathfrak{g}(A, \mathbf{p})_{p_i}$ for all i. Notice that all h_i are necessarily even.

As observed in [Kac77] when $\Bbbk = \mathbb{C}$, different contragredient data can give isomorphic Lie superalgebras. But, similar to Lie algebras in positive characteristic, we can give isomorphisms between some pairs $\mathfrak{g}(A,\mathbf{p})$, $\mathfrak{g}(A',\mathbf{p}')$ with formulas close to the action of the Weyl group for complex simple Lie algebras. In this direction, Serganova [Ser96] introduced the notion of *odd reflection* relating two different pairs by a kind of reflection but on a simple odd root e_i such that $a_{ii} = 0$ (called *isotropic*). This is consistent with one of the differences with Lie algebras: there exists a symmetric bilinear form on $\mathfrak{h}$, but either the bilinear form can have isotropic roots α (i.e., $(\alpha,\alpha) = 0$) or else the matrix $A \in \mathbb{C}^{\theta\times\theta}$ can take nonintegral values. We have to distinguish the matrix A from the GCM $C^A \in \mathbb{Z}^{\theta\times\theta}$ responsible for the odd reflections.

Example 3.4. Let $V = (\mathbb{C}^2)_0 \oplus \mathbb{C}_1$, that is a $\mathbb{Z}_2$-graded vector space with even component of dimension 2, and odd component of dimension 1. Here $\mathfrak{sl}(V)$ is denoted simply by $\mathfrak{sl}(2|1)$: as for Lie algebras we identify $\mathfrak{sl}(2|1)$ with matrices $\mathrm{A} \in \mathbb{C}^{3\times3}$, here with zero supertrace, i.e., $\mathrm{a}_{11} + \mathrm{a}_{22} - \mathrm{a}_{33} = 0$. The Lie superalgebra $\mathfrak{sl}(2|1)$ is $\mathbb{Z}^2$-graded, with $\mathfrak{h}$ (the diagonal matrices) in degree 0, and one-dimensional components of degrees $\pm\alpha_1$ (even roots, since the corresponding spaces are E_{12} and E_{21}), $\pm\alpha_2$, $\pm(\alpha_1+\alpha_2)$ (these four roots are odd). The contragredient datum is $A = \left[\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, $\mathbf{p}_2 = (0,1)$. The odd reflection in α_2 moves to the pair $B = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, $\mathbf{p}_{12} = (1,1)$. Thus we may apply the odd reflection in α_1 to $(B,\mathbf{p}_{12})$ and obtain $(C,\mathbf{p}_1)$, where $C = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 2 \end{smallmatrix}\right]$, $\mathbf{p}_1 = (1,0)$. These are all the possible movements between pairs whose associated Lie superalgebra is isomorphic to $\mathfrak{sl}(2|1)$.

We see that the set of $\mathbb{Z}^2$-degrees of the nontrivial components is the *same*, the Cartan matrix is $\left[\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right]$ for the three pairs, but the parity of the elements is not the same. For example, for $(B,\mathbf{p}_{12})$, $\pm(\alpha_1+\alpha_2)$ are even roots while $\pm\alpha_1$, $\pm\alpha_2$ are odd.

If we study Lie superalgebras over fields of positive characteristic, then we can have more and more exceptional examples. Finite-dimensional contragredient Lie algebras over fields of prime characteristic were classified in [BGL09]. The picture is the same: several pairs of contragredient data $(A,\mathbf{p})$ give isomorphic Lie superalgebras.

The question is then how to handle uniformly all possible pairs giving isomorphic Lie superalgebras, and their corresponding roots (i.e., the $\mathbb{Z}^\theta$-degrees of the nontrivial components). This will be done with a groupoid, i.e., a category where all the morphisms are invertible. As we look for a generalization of the Weyl group, we will consider a groupoid generated by *reflections*.

3.3. Generalized root systems. There exist different notions of *generalized* root systems in the literature. They try to capture different situations, as for example the one by Serganova in [Ser96] for complex finite-dimensional Lie superalgebras. A nice axiomatic version was given in [HY08], see also [HS20] for a refined version of these ideas.

Fix $\theta \in \mathbb{N}$, $\mathbb{I} = \{1,\cdots,\theta\}$. Let $\mathcal{X} \neq \emptyset$ a set (which will correspond to the different contragredient data). A *semi-Cartan graph* $\mathcal{G}(\mathbb{I},\mathcal{X},(C^X)_{X\in\mathcal{X}},(\rho_i)_{i\in\mathbb{I}})$ ($\mathcal{G}$ for short) of *rank* θ over $\mathcal{X}$ consist of

- functions $\rho_i : \mathcal{X} \to \mathcal{X}$, $i \in \mathbb{I}$, such that $\rho_i^2 = \mathrm{id}_{\mathcal{X}}$,
- GCM $C^X = (c_{ij}^X)_{i,j\in\mathbb{I}} \in \mathbb{Z}^{\theta\times\theta}$, $X \in \mathcal{X}$,

such that $c_{ij}^X = c_{ij}^{\rho_i(X)}$ for all $X \in \mathcal{X}$, $i,j \in \mathbb{I}$.

As for Lie algebras, we set $s_i^X \in \mathrm{GL}(\mathbb{Z}^\theta)$ as the reflection $s_i^X(\alpha_j) = \alpha_j - c_{ij}^X\alpha_i$.

Let $\mathcal{M}$ be a monoid. There exists a small category $\mathcal{D}(\mathcal{X},\mathcal{M})$ whose set of objects is $\mathcal{X}$ and the set of morphisms between any two objects is $\mathcal{M}$. Given $f \in \mathcal{M}$ and $X,Y \in \mathcal{X}$ we write (Y,f,X) for f viewed as an element of $\mathrm{Hom}(X,Y)$, so the composition becomes

$$(Z,f,Y)\circ(Y,g,X) = (Z,fg,X),$$

for any $X,Y,Z \in \mathcal{X}$, $f,g \in \mathcal{M}$.

We are interested in the case $\mathcal{M} = \mathrm{GL}(\mathbb{Z}^\theta)$, the group of automorphisms of $\mathbb{Z}^\theta$.

Definition 3.5. The *Weyl groupoid* of $\mathcal{G}$ is the full subcategory $\mathcal{W}(\mathbb{I},\mathcal{X},(A_x)_{x\in\mathcal{X}},(\rho_i)_{i\in\mathbb{I}})$ of $\mathcal{D}(\mathcal{X},\mathrm{GL}(\mathbb{Z}^\theta))$ generated by

$$\sigma_i^X := (\rho_i(X), s_i^X, X), \qquad i \in \mathbb{I}, X \in \mathcal{X}.$$

Notice that $\mathcal{W}$ is indeed a groupoid, since

$$\sigma_i^{\rho_i(X)}\sigma_i^X = (X,\mathrm{id}_X,X) \text{ for all } i \in \mathbb{I}, X \in \mathcal{X}.$$

Fix $X \in \mathcal{X}$. Then $\Delta^{X,\mathrm{re}}$ is the set of all elements of the form $w(\alpha_i) \in \mathbb{Z}^\theta$, where $i \in \mathbb{I}$, $Y \in \mathcal{X}$, $(X,w,Y) \in \mathrm{Hom}_{\mathcal{W}}(X,Y)$. This is the set of *real roots* of $\mathcal{G}$. As for roots of Lie (super)algebras, we consider the subsets

$$\Delta_+^{X,\mathrm{re}} := \Delta^{X,\mathrm{re}} \cap \mathbb{N}_0^\theta, \qquad \Delta_-^{X,\mathrm{re}} := \Delta^{X,\mathrm{re}} \cap (-\mathbb{N}_0^\theta),$$

of positive and negative real roots, and set

$$m_{ij}^X := |\Delta^{X,\mathrm{re}} \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)| \in \mathbb{N}\cup\{\infty\}.$$

If $\Delta^{X,\mathrm{re}}$ is finite for all $X \in \mathcal{X}$ (equivalently, for some $X \in \mathcal{X}$), then we say that $\mathcal{G}$ is *finite*.

A semi-Cartan graph $\mathcal{G}$ is a *Cartan graph* if the following conditions hold for all $X \in \mathcal{X}$:

- $\Delta^{X,\mathrm{re}} = \Delta_+^{X,\mathrm{re}} \cup \Delta_-^{X,\mathrm{re}}$;
- for all $i \neq j$ such that $m_{ij}^X < \infty$, $(\rho_i\rho_j)^{m_{ij}^X}(X) = X$.

Mimicking what happens for Lie algebras, see Remarks 3.1 and 3.3, we introduce the following notion:

Definition 3.6. A *root system* over $\mathcal{G}$ is a family $\mathcal{R} = (\Delta^X)_{X\in\mathcal{X}}$ of subsets $\Delta^X \subset \mathbb{Z}^\theta$ such that

$$0 \notin \Delta^X, \qquad \Delta^X \subset \mathbb{N}_0^\theta \cup (-\mathbb{N}_0^\theta),$$
$$\alpha_i \in \Delta^X, \qquad s_i^X(\Delta^X) = \Delta^{\rho_i(X)},$$

for all $i \in \mathbb{I}$ and all $X \in \mathcal{X}$.

We say that $\mathcal{R}$ is *reduced* if $\mathbb{Z}\alpha \cap \Delta^X = \{\pm\alpha\}$ for all $\alpha \in \Delta^X$, $X \in \mathcal{X}$. $\mathcal{R}$ is *finite* if every Δ^X is so.

Cuntz and Heckenberger obtained the classification of finite root systems [CH15]. The proof involves the bijection between root systems of rank θ and crystallographic arrangements (certain subsets of hyperplanes) in $\mathbb{R}^\theta$. About the list of finite root systems, in rank $\theta = 2$ there are infinitely many examples, in bijection with triangulations of n-gons for any $n \geq 3$. For $\theta \geq 9$ we only have families corresponding to Lie superalgebras and Lie algebras of types A, B, C, D, while for $3 \leq \theta \leq 8$ we have members of the families of Lie (super)algebras and several exceptions.

The definition of a root system seems to carry the possibility to have several examples attached to the same Cartan graph $\mathcal{G}$. But this is not the case when $\mathcal{G}$ is finite. Indeed, by [HS20, 10.4.7], if $\mathcal{G}$ is a finite Cartan graph, then $\mathcal{R} = (\Delta^{X,\mathrm{re}})_{X\in\mathcal{X}}$ is the only reduced root system over $\mathcal{G}$.

Remark 3.7. In [HY08] the authors state the existence of a Weyl groupoid for finite-dimensional complex Lie superalgebras, the ones coming from the $\mathbb{Z}^\theta$-grading as above. Moreover, Andruskiewitsch and Angiono proved that the same holds for Lie superalgebras over fields of arbitrary characteristic, in a work in progress. In the same work they derived the classification of finite-dimensional Lie superalgebras from the classification of finite root system in [CH15].

It should be noted that not all finite root systems come from a Lie superalgebra.

Once we show the existence of a finite root system for a Lie superalgebra, there are many strong properties derived from the combinatorics of the Weyl groupoid. For example:

- $\dim \mathfrak{g}(A, \mathbf{p})_\alpha = 1$ for all $\alpha \in \Delta^{(A,\mathbf{p})}$.
- There might exist roots $\alpha \in \Delta^{(A,\mathbf{p})}$ such that $2\alpha \in \Delta^{(A,\mathbf{p})}$, which are the odd nonisotropic roots. All of them are the image of simple odd nonisotropic roots of some pair $(A', \mathbf{p}')$ obtained up to odd reflections, and $\dim \mathfrak{g}(A, \mathbf{p})_{2\alpha} = 1$, as shown by Andruskiewitsch-Angiono.
- The whole set $\Delta^{(A,\mathbf{p})}$ is obtained up to reflections of the simple roots, attaching 2α for each odd nonisotropic root.

In the same line we may wonder if there exists a *geometric-combinatoric* side on these Lie superalgebras (Weyl chambers and so on) coming from the associated crystallographic arrangements.

Example 3.8. We continue with the study of $\mathfrak{br}(3)$. Here $\mathcal{X} = \{A, B\}$, with $\rho_3(A) = B$ and $\rho_1 = \rho_2 = \mathrm{id}$. The associated GCM are those in Remark 3.3. Thus we get all the roots applying repeatedly the reflections. For example, $s_2^B(\alpha_1) = \alpha_1 + 2\alpha_2 \in \Delta^B$, so

$$s_3^B(\alpha_1 + 2\alpha_2) = \alpha_1 + 2\alpha_2 + 4\alpha_3 \in \Delta^A.$$

Using the notation $1^a2^b3^c := a\alpha_1 + b\alpha_2 + c\alpha_3$, we can check that

$$\begin{aligned}\Delta_+^A =&\{1, 12, 123, 1^22^33^4, 12^23^2, 12^23^3, 12^23^4,\\ &12^33^4, 123^2, 2, 23^2, 23, 3\},\\ \Delta_+^B =&\{1, 12^2, 12, 123^2, 12^33^2, 1^22^33^2, 12^23^2,\\ &123, 12^23, 2, 23^2, 23, 3\}.\end{aligned}$$

Thus, $\dim \mathfrak{n}_\pm = 13$, so $\dim \mathfrak{br}(3) = 29$. In addition one can show that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ for every pair $\alpha, \beta \in \Delta_+^A$ such that $\alpha + \beta \in \Delta_+^A$. Thus we can obtain recursively a nonzero element $e_\alpha \in \mathfrak{g}_\alpha$:

$$e_{12} := [e_1, e_2], \quad e_{123} := [e_{12}, e_3], \quad e_{123^2} := [e_{123}, e_3],$$

and so on.

4. Other Contexts and Problems

We finish by recalling other algebraic structures where these generalized root systems appear, as Nichols algebras, and posing some related problems where they could play a key role: Lie algebras in a broad sense and their representations. We are not going to introduce all the involved concepts, we refer to the corresponding papers for more information.

4.1. Nichols algebras.

Quantized enveloping algebras are certain deformations of enveloping algebras of semisimple Lie algebras introduced in the eighties by Drinfeld and Jimbo, depending on a parameter q. Later on, Lusztig considered Hopf algebras obtained by evaluation of q at a root of unity, which lead to some finite-dimensional examples, usually called *Frobenius-Lusztig kernels*. These examples have a triangular decomposition whose zero part is a group algebra of copies of finite cyclic groups, and the positive (also, the negative) part is a kind of Hopf algebra.

In the denomination currently used, these positive parts are examples of Nichols algebras. Nichols algebras are Hopf algebras in the category of Yetter-Drinfeld modules over a group algebra (or more precisely, over a Hopf algebra), which play a fundamental role in the classification of finite-dimensional Hopf algebras. Following the line of work of Andruskiewitsch and Schneider, joined by Heckenberger, one can define Hopf algebras with triangular decomposition, whose positive part is a Nichols algebra and which have generalised root systems, see the book [HS20], and also [AHS10].

The list of all finite-dimensional Nichols algebras is known when the group is finite and abelian, thanks to the work of Heckenberger, and almost complete when the group is finite but nonabelian, by Heckenberger-Vendramin. Both works explode the existence of the generalized root system.

We can see that the list of all generalized root systems appearing for some Nichols algebras contains properly the list of all those appearing for Lie superalgebras, but there are some root systems not attached to any Nichols algebras.

4.2. Lie algebras in symmetric tensor categories and representations. One can extend the definition of Lie algebra to symmetric tensor categories. Indeed Lie superalgebras are essentially Lie algebras in the category of super vector spaces. When $\Bbbk$ is of characteristic zero, Deligne proved that any symmetric tensor category (under a mild condition) fibers over the category of supervector spaces, so any Lie algebra over these symmetric tensor categories can be considered as a Lie superalgebra. When $\Bbbk$ is of characteristic $p > 0$, Coulembier-Etingof-Ostrik proved recently [CEO23] that any symmetric tensor category (under a mild condition) fibers over the Verlinde category Ver_p. This category is the semisimplification of the category of representations of $\mathbb{Z}_p$ over $\Bbbk$ and contains properly the category of super vector spaces. Thus, in this case, the consideration of Lie algebras in symmetric tensor categories essentially reduces to Lie algebras in Ver_p. One may ask about the existence of contragredient Lie algebras in Ver_p, and root systems.

In the classical case (that is, over $\mathbb{C}$), the root system controls the representation theory of simple Lie algebras, or more precisely a quite interesting subcategory called the *category $\mathcal{O}$*. For example, finite-dimensional modules are parametrized by nonnegative *weights* associated to the root system, and the Weyl group describes a *character formula* for these simple modules. The situation is a bit more complicated for Lie superalgebras, and a character formula exists for certain weights. Recently Sergeev and Veselov used what they called a Weyl groupoid (which is not clearly related to the one considered here) to describe strong properties on the representations. Also, Yamane described very recently character formulas for the so-called atypical weights of quantized enveloping Lie superalgebras by means of the Weyl groupoid. So one may wonder if the Weyl groupoid plays a key role in the description of the representations of Lie algebras in a broad sense.

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Iván Angiono

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