ASYMPTOTIC INSIGHTS FOR PROJECTION, GORDON–LEWIS AND SIDON CONSTANTS IN BOOLEAN CUBE FUNCTION SPACES

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ABSTRACT. The main aim of this work is to study important local Banach space constants for Boolean cube function spaces. Specifically, we focus on $\mathscr{B}^N_{\mathscr{S}}$, the finite-dimensional Banach space of all real-valued functions defined on the N-dimensional Boolean cube $\{-1,+1\}^N$ that have Fourier–Walsh expansions supported on a fixed family $\mathscr{S}$ of subsets of $\{1,\dots,N\}$. Our investigation centers on the projection, Sidon and Gordon–Lewis constants of this function space. We combine tools from different areas to derive exact formulas and asymptotic estimates of these parameters for special types of families $\mathscr{S}$ depending on the dimension N of the Boolean cube and other complexity characteristics of the support set $\mathscr{S}$. Using local Banach space theory, we establish the intimate relationship among these three important constants.

1. INTRODUCTION

We study important Banach space invariants such as projection, Sidon and Gordon–Lewis constants of certain natural subspaces of the Banach space of all real-valued functions defined on the N-dimensional Boolean cube $\{-1,+1\}^N$. Recall that every function $f\colon \{-1,+1\}^N \to \mathbb{R}$ has a Fourier–Walsh expansion of the form

$$f(x) = \sum_{S\subset\{1,\dots,N\}} \widehat{f}(S)x^S,$$

where for each $(x_1,\dots,x_N)\in\{-1,+1\}^N$, $x^S := \prod_{k\in S} x_k$ is a Walsh function. The set of all S for which $\widehat{f}(S)\neq 0$ is called the spectrum of f.

More precisely, given a set $\mathscr{S}$ of subsets in $\{1,\dots,N\}$, we consider the space $\mathscr{B}^N_{\mathscr{S}}$ of all functions $f\colon\{-1,+1\}^N\to\mathbb{R}$ with Fourier–Walsh expansions supported on $\mathscr{S}$, that is, $\widehat{f}(S)\neq 0$ only if $S\in\mathscr{S}$. Endowed with the supremum norm on $\{-1,+1\}^N$, this is a finite dimensional Banach space. Our main goal then is to find asymptotically correct estimates for the projection constant $\boldsymbol{\lambda}\big(\mathscr{B}^N_{\mathscr{S}}\big)$, and to link this invariant with other important invariants from Fourier analysis and local Banach space theory like the Sidon, unconditional basis or Gordon–Lewis constants.

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We note that the Fourier analysis of functions on Boolean cubes is essential in theoretical computer science, and plays a key role in combinatorics, random graph theory, statistical physics, Gaussian geometry, the theories of metric spaces/Banach spaces, learning theory, or social choice theory (see, e.g., [33] and [34] and the references therein). Moreover, the last decades show growing interest for applications of Boolean functions in the context of quantum algorithms complexity and quantum information [2]. For recent important developments and applications in this direction see also [1, 5, 10, 14, 15, 38, 42, 47] .

We mainly focus on the Banach space $\mathscr{B}^N_{\leq d} := \mathscr{B}^N_{\{S:\,|S|\leq d\}}$, that is all real-valued functions f on the compact abelian group $\{-1,+1\}^N$ with Fourier transforms $\hat{f}$ supported on all subsets of $[N]$ with cardinality $\leq d$, and similarly on the Banach space $\mathscr{B}^N_{=d} := \mathscr{B}_{\{S:\,|S|=d\}}$.

The study of complemented subspaces X of a Banach space Y and their projection constants has a long history going back to the beginning of operator theory in Banach spaces. For a general overview of the state of art of the theory of projection constants in Banach spaces, we refer to the excellent monograph [45] by Tomaczak-Jaegermann and references therein.

We recall that if X is a subspace of a Banach space Y, then the relative projection constant of X in Y is defined by

$$\boldsymbol{\lambda}(X,Y) = \inf\big\{\|P\|:\ P\in\mathscr{L}(Y,X),\ P|_X = \mathrm{id}_X\big\},$$

where id_X is the identity operator on X, and the (absolute) projection constant of X is given by

$$\boldsymbol{\lambda}(X) := \sup\,\boldsymbol{\lambda}(I(X),Z)\,, \tag{1}$$

where the supremum is taken over all Banach spaces Z and isometric embeddings $I\colon X\to Z$. The following straightforward result shows the intimate link between projection constants and extensions of linear operators: For every Banach space Y and its subspace X one has

$$\boldsymbol{\lambda}(X,Y) = \inf\big\{c>0:\ \forall\, T\in\mathscr{L}(X,Z)\ \ \exists\ \text{ an extension }\ \widetilde{T}\in\mathscr{L}(Y,Z)\ \text{ with }\|\widetilde{T}\|\leq c\,\|T\|\big\}.$$

Drawing from Rudin's averaging technique from [39] for estimating the projection constant (dating back to the 1960s), an adapted technique for spaces of trigonometric polynomials on compact abelian groups was devised in [7]. We apply this new perspective into the framework of functions on Boolean cubes. As a consequence, we in Theorem 3.1 see that

$$\boldsymbol{\lambda}\big(\mathscr{B}^N_{\mathscr{S}}\big) = \frac{1}{2^N}\sum_{x\in\{-1,+1\}^N}\Big|\sum_{S\in\mathscr{S}} x^s\Big|.$$

In principle, this integral can be calculated with a computer - at least for concrete well-structured families $\mathscr{S}$ in $[N] := \{1,2,\ldots,N\}$. Nevertheless, if N is large or the set $\mathscr{S}$ is 'too big', this might get unfeasible. Therefore, it is important to study the asymptotic order of $\boldsymbol{\lambda}\big(\mathscr{B}^N_{\mathscr{S}}\big)$ in the dimension N and/or other parameters quantifying the complexity of $\mathscr{S}$.

Among others, we show in Theorem 4.1 that

$$\lim_{N\to\infty} N^{-d/2}\boldsymbol{\lambda}\big(\mathscr{B}^N_{\mathscr{S}}\big) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{|h_d(t)|}{d!}\exp(-t^2/2)\,dt\,,$$

where h_d is the d-th Hermite polynomial.

While this outcome can be derived from a very interesting identity by Beckner [3] from the mid-seventies—utilizing it as a black box—alongside the central limit theorem, we derived it (without prior knowledge of it) from a specific technique to study the combinatorial structure of the index set $\mathscr{S}$ involved. This technique is grounded on ideas introduced in [16] and [31], where it was employed to examine sets of monomial convergence of spaces of holomorphic functions in high dimensions. We highlight that this combinatorial tool holds significant potential, extending its applicability to diverse contexts. In this manuscript, we present both methods for obtaining the result.

The Sidon constant plays a pivotal role in Fourier analysis, providing crucial insights into the behavior and convergence properties of Fourier series and related transformations. Recall that, given $\mathscr{S}\subset[N]$, the Sidon constant of the characters $(\chi_S)_{S\in\mathscr{S}}$ in the group $\{-1,1\}^N$ is the best constant $C>0$ such that for all $f\in\mathscr{B}^N_{\mathscr{S}}$,

$$\sum_{S\in\mathscr{S}}|\hat{f}(S)|\le C\|f\|_\infty\,, \tag{2}$$

and we in the following are going to denote this constant by $\mathbf{Sid}(\mathscr{B}^N_{\mathscr{S}})$. In the context of spaces of Boolean functions, this invariant establishes a connection between the Fourier coefficients and the supremum norm of the function. Essentially, it quantifies how the distribution of the Fourier coefficients across different frequencies influences the overall behavior of the function, as reflected by its supremum norm. This relationship provides valuable insights into the harmonic structure and complexity of Boolean functions, aiding in the analysis and understanding of their properties and computational characteristics.

We demonstrate how to estimate the Sidon constant of spaces of Boolean functions for a specific set $\mathscr{S}$ and the projection constant of another space associated with the 'reduced form' of $\mathscr{S}$. To achieve this, we employ various techniques from local analysis of Banach spaces, including the connection with the so-called Gordon–Lewis constant, tensorial techniques, and certain symmetrization/desymmetrization ideas.

The main result here is Theorem 5.1 (and more generally Theorem 5.2), where we show that in the homogeneous case the Sidon constant of $\mathscr{B}^N_{=d}$ may be estimated by $\boldsymbol{\lambda}\big(\mathscr{B}^N_{=d-1}\big)$ up to a constant C^d, where $C>0$ is universal. The proof goes a detour relating first the Sidon constant with the Gordon–Lewis constant (Theorem 5.4) and then in a second step the Gordon–Lewis constant with the projection constant (Theorem 5.7).

Summarizing, we use local Banach space theory to show that the Sidon, Gordon–Lewis and projection constant of every Banach space $\mathscr{B}^N_{\mathscr{S}}$ are intimately linked – a connection of which all three of these fundamental constants benefit abundantly.

Using a recent variant of the Bohnenblust-Hille inequality for functions on Boolean cubes from [10], we conclude the manuscript by relating the Sidon constant of the space $\mathscr{B}^N_{\mathscr{S}}$ with the 'size' of the support set $\mathscr{S}$.

2. Preliminaries

Standard notation from Banach space theory as e.g., used in the monographs [12, 30, 37, 45, 48] is going to be needed. In this note, we basically only consider real Banach spaces.

2.1. **Functions on Boolean cubes.**

Throughout the paper we for $N \in \mathbb{N}$ call $\{-1,+1\}^N$ the N-dimensional Boolean cube. A Boolean function is any function $f\colon \{-1,+1\}^N \to \{-1,+1\}$, more generally, functions $f : \{-1,+1\}^N \to \mathbb{R}$ are said to be 'real valued functions on the N-dimensional Boolean cube'.

The study of real valued functions on Boolean cubes is deeply influenced by Fourier analysis. Considering the N-dimensional Boolean cube $\{-1,+1\}^N$ as a compact abelian group endowed with the coordinatewise product and the discrete topology (so the Haar measure is given by the normalized counting measure), we may apply the machinery given by abstract harmonic analysis.

In particular, the integral or expectation of each function $f : \{-1,+1\}^N \to \mathbb{R}$ is given by

$$\mathbb{E}[f] := \frac{1}{2^N} \sum_{x \in \{-1,+1\}^N} f(x)\,.$$

The characters on $\{-1,+1\}^N$ are the so-called Walsh functions defined as

$$\chi_S\colon \{-1,+1\}^N \to \{-1,1\}\,, \quad \chi_S(x) = x^S := \prod_{k \in S} x_k \quad \text{for} \quad x \in \{-1,+1\}^N,$$

where $S \subset [N] := \{1,\dots,N\}$ and $\chi_\emptyset(x) := 1$ for each $x \in \{-1,+1\}^N$. This allows to associate to each function $f : \{-1,+1\}^N \to \mathbb{R}$ its Fourier–Walsh expansion

$$f(x) = \sum_{S \subset [N]} \widehat{f}(S)\, x^S\,, \quad x \in \{-1,+1\}^N\,,$$

where $\widehat{f}(S) = \mathbb{E}[f \cdot \chi_S]$ are the Fourier coefficients.

Given $d \in \mathbb{N}$, we say that f has degree d whenever $\widehat{f}(S) = 0$ for all $|S| > d$, and f is d-homogeneous whenever additionally $\widehat{f}(S) = 0$ provided $|S| \neq d$; where, as usual, $|S|$ stands for the cardinality of the set S.

For every set $\mathscr{S}$ of subsets $S \subset [N]$, we define the linear space $\mathscr{B}^N_{\mathscr{S}}$ of all functions $f\colon \{-1,+1\}^N \to \mathbb{R}$, which have Fourier–Walsh expansions supported on $\mathscr{S}$. Endowed with the supremum norm $\|\cdot\|_\infty$ on the N-dimensional Boolean cube, this space turns into a Banach space. We mainly concentrate on the following special classes of functions on the N-dimensional Boolean cube:

- $\mathscr{B}^N :=$ all functions $f : \{-1,+1\}^N \to \mathbb{R}$,
- $\mathscr{B}^N_{=d} :=$ all d-homogeneous $f : \{-1,+1\}^N \to \mathbb{R}$,
- $\mathscr{B}^N_{\le d} :=$ all $f : \{-1,+1\}^N \to \mathbb{R}$ with degree less or equal d.

Obviously, we have the isometric identity

$$\mathscr{B}^N = \ell_\infty\big(\{-1,+1\}^N\big), \quad f \mapsto (f(x))_{x\in\{-1,+1\}^N}. \tag{3}$$

2.2. **Functions on Boolean cubes vs tetrahedral polynomials.** As usual we call the elements $\alpha = (\alpha_i)$ in $\mathbb{N}_0^{(\mathbb{N})}$ (all finite sequences in $\mathbb{N}_0$) multi indices, and $|\alpha| = \sum \alpha_i$ is their so-called order. For $N \in \mathbb{N}$ and $d \in \mathbb{N}_0$

$$\Lambda(d,N) := \{\alpha \in \mathbb{N}_0^N : |\alpha| = d\}, \qquad \Lambda(\le d,N) := \{\alpha \in \mathbb{N}_0^N : |\alpha| \le d\}$$

denote the sets of multi indices which are d-homogeneous and of order $\le d$, respectively. A multi index $\alpha = (\alpha_i) \in \mathbb{N}_0^{(\mathbb{N})}$ is said to be tetrahedral whenever each entry α_i is either 0 or 1. We denote by Λ_T the subset of all tetrahedral multi indices in $\mathbb{N}_0^{(\mathbb{N})}$. For $d \le N$ we let

$$\Lambda_T(d,N) := \Lambda(d,N) \cap \Lambda_T\,, \qquad \Lambda_T(\le d,N) := \Lambda(\le d,N) \cap \Lambda_T.$$

It turns out to be convenient to have an equivalent description of $\Lambda(d,N)$. We write

$$\begin{aligned} &\mathscr{M}(d,N) = [N]^d\,,\\ &\mathscr{J}(d,N) = \big\{\mathbf{j} = (j_1,\ldots,j_d) \in \mathscr{M}(d,N) :\; j_1 \le \ldots \le j_d\big\}. \end{aligned}$$

Then there obviously is a canonical bijection between $\mathscr{J}(d,N)$ and $\Lambda(d,N)$. Indeed, assign to $\mathbf{j} \in \mathscr{J}(d,N)$ the multi index $\alpha \in \Lambda(d,N)$ given by $\alpha_r = |\{k : \mathbf{j}_k = r\}|$, $1 \le r \le N$, and conversely assign to each $\alpha \in \Lambda(d,N)$ the index $\mathbf{j} \in \mathscr{J}(d,N)$, where $j_1 = \ldots = j_{\alpha_1} = 1$, $j_{\alpha_1+1} = \ldots = j_{\alpha_1+\alpha_2} = 2, \ldots$

On $\mathscr{M}(d,N)$ we consider the equivalence relation: $\mathbf{i} \sim \mathbf{j}$ if there is a permutation σ on $\{1,\ldots,d\}$ such that $(i_1,\ldots,i_d) = (j_{\sigma(1)},\ldots,j_{\sigma(d)})$. The equivalence class of $\mathbf{i} \in \mathscr{M}(d,N)$ is denoted by $[\mathbf{i}]$, and its cardinality by $|[\mathbf{i}]|$. We write $|[\alpha]| := |[\mathbf{j}]|$ provided that $\mathbf{j}$ is associated with α, and in this case have that

$$|[\alpha]| = |[\mathbf{j}]| = \frac{d!}{\alpha!}, \tag{4}$$

where $\alpha! := \alpha_1! \cdot \ldots \cdot \alpha_N!$.

We consider finite polynomials $P:\mathbb{R}^N \to \mathbb{R}$, i.e., polynomials of the form $P(x) = \sum_{\alpha\in F} c_\alpha x^\alpha$, $x \in \mathbb{R}$, where F is a finite set of multi indices in $\mathbb{N}_0^{(\mathbb{N})}$. We write $\mathscr{P}_d(\ell_\infty^N)$ for all d-homogenous polynomials, so polynomials of the form $P(x) = \sum_{\alpha\in\Lambda(d,N)} c_\alpha x^\alpha$, $x \in \mathbb{R}$. The space $\mathscr{P}_d(\ell_\infty^N)$, together with the supremum norm taken on the unit ball of ℓ_∞^N, forms a Banach space. A polynomial is said to be tetrahedral, whenever $c_\alpha = 0$ for all $\alpha \notin \Lambda_T$. The subspace of all tetrahedral d-homogeneous polynomials is denoted by $\mathscr{T}_d(\ell_\infty^N)$, that is, all polynomials of the form $P(x) = \sum_{\alpha\in\Lambda_T(d,N)} c_\alpha x^\alpha$, $x \in \mathbb{R}$. Analogously, we define the Banach spaces $\mathscr{P}_{\le d}(\ell_\infty^N)$ and $\mathscr{T}_{\le d}(\ell_\infty^N)$.

Obviously, for each $f\colon \{-1,+1\}^N \to \mathbb{R}$ there is a unique tetrahedral polynomial $P_f : \mathbb{R}^N \to \mathbb{R}$ for which the following diagram commutes:

$$\begin{array}{ccc} \{-1,+1\}^N & \hookrightarrow & \mathbb{R}^N\,, \\ & {\scriptstyle f}\searrow \quad \swarrow {\scriptstyle P_f} & \\ & \mathbb{R} & \end{array}$$

and in this case

$$\|f\|_\infty := \sup_{x\in\{-1,+1\}^N} |f(x)| = \sup_{x\in\{-1,+1\}^N} |P_f(x)| = \sup_{x\in[-1,1]^N} |P_f(x)| =: \|P_f\|_\infty.$$

Moreover, each subset $S \subset [N]$ may be identified with a tetrahedral multi index $\alpha^S \in \mathbb{N}_0^N$ given by $\alpha^S(k) = 1, k \in S$ and $\alpha^S(k) = 0, k \notin S$. Conversely, every tetrahedral multi index $\alpha \in \mathbb{N}_0^N$ defines the subset $S = \operatorname{supp}\alpha \subset [N]$. We write

$$\Lambda(\mathscr{S}) = \{\alpha^S \in \mathbb{N}_0^N \colon S \in \mathscr{S}\}\,,$$

and $\mathscr{P}_{\Lambda(\mathscr{S})}(\ell_\infty^N(\mathbb{R}))$ for the Banach space of all polynomials on ℓ_∞^N, which are generated by functions $f \in \mathscr{B}_{\mathscr{S}}^N$. This all together leads to the isometric identity

$$\mathscr{B}_{\mathscr{S}}^N = \mathscr{P}_{\Lambda(\mathscr{S})}(\ell_\infty^N)\,, \quad f \mapsto P_f\,. \tag{5}$$

In view of this identification, it from time to time is convenient to use the usual monomial notation, that is, for $S \subset [N]$ we identify the Boolean function χ_S with x^{α_S}.

2.3. **Sidon, unconditional basis and Gordon–Lewis constants.**

• Recall that the unconditional basis constant of a basis $(e_i)_{i\in I}$ of a Banach space X is given by the infimum over all $K > 0$ such that for any finitely supported family $(\alpha_i)_{i\in I}$ of scalars and for any finitely supported family $(\varepsilon_i)_{i\in I}$ with $\varepsilon_i \in \{-1,+1\}$, $i \in I$ we have

$$\Big\|\sum_{i\in I} \varepsilon_i \alpha_i e_i\Big\| \le K \Big\|\sum_{i\in I} \alpha_i e_i\Big\|\,. \tag{6}$$

We denote the unconditional basis constant of $(e_i)_{i\in I}$ by $\boldsymbol{\chi}((e_i)_{i\in I}) = \boldsymbol{\chi}((e_i)_{i\in I}; X)$. We also write $\boldsymbol{\chi}((e_i)_{i\in I}) = +\infty$, whenever $(e_i)_{i\in I}$ is not unconditional, and say that $(e_i)_{i\in I}$ is a 1-unconditional basis, whenever $\boldsymbol{\chi}((e_i)_{i\in I}) = 1$. The unconditional basis constant $\boldsymbol{\chi}(X)$ of X is defined to be the infimum of $\boldsymbol{\chi}((e_i)_{i\in I})$ taken over all possible unconditional bases $(e_i)_{i\in I}$ of X.

It should be noted that, from the Banach space point of view, the Sidon constant $\mathrm{Sid}(\mathscr{B}^N_{\mathscr{S}})$ is nothing else than the unconditional basis constant of the Walsh functions $(\chi_S)_{S\in\mathscr{S}}$, that is

$$\mathbf{Sid}(\mathscr{B}^N_{\mathscr{S}}) = \boldsymbol{\chi}((\chi_S)_{S\in\mathscr{S}}).$$

• Given Banach spaces X, Y and $1 \le p \le \infty$, an operator $u \in \mathscr{L}(X,Y)$ is said to be p-factorable whenever there exist a measure space (Ω,Σ,μ) and operators $v \in \mathscr{L}(X, L_p(\mu))$, $w \in \mathscr{L}(L_p(\mu), Y^{**})$, satisfying the following factorization $\kappa_Y u\colon X \xrightarrow{v} L_p(\mu) \xrightarrow{w} Y^{**}$; here, as usual, $\kappa_Y\colon Y \to Y^{**}$ is the canonical embedding. In this case the γ_p-norm of the p-factorable operator u is given by

$$\gamma_p(u) = \inf \|v\|\,\|w\|,$$

where the infimum is taken over all possible factorizations. We are mainly interested in the norms γ_p for operators acting between finite dimensional Banach spaces X and Y. In this case, the infimum in (2.3) is realized considering all possible factorizations of the more simple form

$$\begin{array}{ccc} X & \xrightarrow{\quad u \quad} & Y\,, \\ & {}_{v}\searrow \quad \nearrow_{w} & \\ & \ell_p^n & \end{array}$$

where n is arbitrary.

• An operator $u \in \mathscr{L}(X,Y)$ is said to be 1-summing if there is a constant $C>0$ such that for each choice of finitely many $x_1,\dots,x_N \in X$ one has

$$\sum_{j=1}^N \|u x_j\|_Y \le C \sup\Big\{ \sum_{j=1}^N |x^*(x_j)| \,:\, \|x^*\|_{X^*} \le 1 \Big\}.$$

By $\pi_1(u\colon X \to Y)$ we denote the least such C satisfying this inequality.

• A Banach space X has the Gordon–Lewis property if every 1-summing operator $u\colon X \to \ell_2$ is 1-factorable. In this case, there is a constant $C>0$ such that for all 1-summing operators $u\colon X \to \ell_2$

$$\gamma_1(u) \le C\,\pi_1(u),$$

and the best such C is called the Gordon–Lewis constant of X and denoted by $\boldsymbol{gl}(X)$.

A fundamental tool for the study of unconditionality in Banach spaces is the Gordon–Lewis inequality from [19] (see also [12, 17.7] or [8, Proposition 21.13]): For every Banach space X with an unconditional basis $(e_i)_{i\in I}$ one has

$$\boldsymbol{gl}(X) \le \boldsymbol{\chi}(X) \le \boldsymbol{\chi}((e_i)_{i\in I}). \tag{7}$$

In contrast to the unconditional basis constant, the Gordon–Lewis constant has the useful (ideal) property that

$$\boldsymbol{gl}(X) \leq \|u\|\,\|v\|\,\boldsymbol{gl}(Y)\,, \tag{8}$$

whenever $\mathrm{id}_X = uv$ for appropriate operators $u\colon X \to Y$ and $v\colon Y \to X$.

2.4. **Projection constants.** General bounds for projection constants of various finite dimensional Banach spaces were studied by many authors (see again (1) for the definition).

The projection constant of a Banach space X can be formulated in terms of the ∞-factorization norm of the identity operator id_X. More precisely, if X is a Banach space and X_0 is any subspace of some $L_\infty(\mu)$ isometric to X, then

$$\boldsymbol{\lambda}(X) = \gamma_\infty(\mathrm{id}_X) = \boldsymbol{\lambda}(X_0, L_\infty(\mu)). \tag{9}$$

Recall the following fundamental estimate due to Kadets-Snobar [24]: For every n-dimensional Banach space X_n one has

$$\boldsymbol{\lambda}(X_n) \leq \sqrt{n}\,. \tag{10}$$

In contrast, König and Lewis [26] showed that for any Banach space X_n of dimension $n \geq 2$ the strict inequality $\boldsymbol{\lambda}(X_n) < \sqrt{n}$ holds, and this estimate was later improved by Lewis [29].

The exact values of $\boldsymbol{\lambda}(\ell_2^n)$ and $\boldsymbol{\lambda}(\ell_1^n)$ were computed by Grünbaum [20] and Rutovitz [40]: In the complex case

$$\boldsymbol{\lambda}\big(\ell_2^n(\mathbb{C})\big) = n\int_{\mathbb{S}_{n-1}^{\mathbb{C}}} |x_1|\,d\sigma(x) = \frac{\sqrt{\pi}}{2}\frac{n!}{\Gamma(n+\frac{1}{2})}\,,$$

where $d\sigma$ stands for the normalized surface measure on the sphere $\mathbb{S}_{n-1}^{\mathbb{C}}$ in $\mathbb{C}^n$, and

$$\boldsymbol{\lambda}\big(\ell_1^n(\mathbb{C})\big) = \int_{\mathbb{T}^n}\Big|\sum_{k=1}^n z_k\Big|\,dz = \int_0^\infty \frac{1-J_0(t)^n}{t^2}\,dt\,,$$

where dz denotes the normalized Lebesgue measure on the distinguished boundary $\mathbb{T}^n$ in $\mathbb{C}^n$ and J_0 is the zero Bessel function defined by $J_0(t) = \frac{1}{2\pi}\int_0^\infty \cos(t\cos\varphi)\,d\varphi$. The corresponding real constants are different:

$$\boldsymbol{\lambda}\big(\ell_2^n(\mathbb{R})\big) = n\int_{\mathbb{S}_{n-1}^{\mathbb{R}}} |x_1|\,d\sigma = \frac{2}{\sqrt{\pi}}\frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{n+1}{2})}$$

and

$$\boldsymbol{\lambda}\big(\ell_1^n(\mathbb{R})\big) = \begin{cases} \boldsymbol{\lambda}\big(\ell_2^n(\mathbb{R})\big), & n \text{ odd} \\ \boldsymbol{\lambda}(\ell_2^{n-1}(\mathbb{R})), & n \text{ even}. \end{cases} \tag{11}$$

Additionally, Gordon [18] and Garling-Gordon [17] determined the asymptotic growth of $\boldsymbol{\lambda}(\ell_p^n)$ for $1 < p < \infty$ with $p \neq 1, 2, \infty$ showing that

$$\boldsymbol{\lambda}(\ell_p^n) \sim n^{\min\left\{\frac{1}{2}, \frac{1}{p}\right\}},$$

and König, Schütt and Tomczak-Jagermann [27] proved that for $1 \leq p \leq 2$

$$\lim_{n\to\infty} \frac{\boldsymbol{\lambda}(\ell_p^n)}{\sqrt{n}} = \gamma, \tag{12}$$

where $\gamma = \sqrt{\frac{2}{\pi}}$ in the real and $\gamma = \frac{\sqrt{\pi}}{2}$ in the complex case. For an extensive treatment on all of this and more see [45].

3. Integral formula

The following integral formula for the projection constant of $\mathscr{B}_{\mathscr{S}}^N$, where $\mathscr{S}$ is an arbitrary family of subsets in $[N]$, is fundamental for our purposes.

Theorem 3.1. *For each family $\mathscr{S}$ of subsets in $[N]$*

$$\boldsymbol{\lambda}(\mathscr{B}_{\mathscr{S}}^N) = \mathbb{E}\Big[\Big|\sum_{S\in\mathscr{S}} \chi_S\Big|\Big].$$

This immediately follows from a result regarding arbitrary compact abelian groups as presented in [7, Theorem 2.1], which finds its inspiration and roots in Rudin's work [39] (also see [48, Theorem III.B.13]). In fact, for a compact abelian group G (with Haar measure m) and a finite set $E := \{\gamma_1, \ldots, \gamma_N\} \subset \widehat{G}$ of characters, we denote by $\mathrm{Trig}_E(G)$ the finite dimensional Banach space of all trigonometric polynomials formed by the span of E in $C(G)$. Then the projection $P\colon C(G) \to C(G)$, given by $Pf = \sum_{j=1}^N \widehat{f}(\gamma_j)\gamma_j$ for all $f \in C(G)$, is a minimal projection onto $\mathrm{Trig}_E(G)$ and

$$\boldsymbol{\lambda}\big(\mathrm{Trig}_E(G)\big) = \|P\colon C(G) \to C(G)\| = \int_G \Big|\sum_{j=1}^N \gamma_j(x)\Big| \, d\mathrm{m}(x).$$

Taking for G the N-dimensional Boolean cube $\{-1,+1\}^N$ and recalling that all its characters are given by the functions χ_S, $S \subset [N]$, we see that Theorem 3.1 indeed is a very special case.

Clearly, by the Kadets-Snobar theorem (recall again (10)) one has

$$1 \leq \boldsymbol{\lambda}(\mathscr{B}_{\mathscr{S}}^N) \leq \sqrt{|\mathscr{S}|}. \tag{13}$$

Note that this estimate is also a straight forward consequence of Theorem 3.1 and the orthogonality of the Fourier–Walsh functions χ_S, since both imply

$$\boldsymbol{\lambda}(\mathscr{B}_{\mathscr{S}}^N) = \mathbb{E}\Big[\Big|\sum_{S\in\mathscr{S}} \chi_S\Big|\Big] \leq \Big(\mathbb{E}\Big[\Big|\sum_{S\in\mathscr{S}} \chi_S\Big|^2\Big]\Big)^{1/2} = \sqrt{|\mathscr{S}|}.$$

We show that the estimates of (13), the upper as well as the lower one, are attained. Indeed, the two possible extreme cases for (13) are as follows: For the lower bound we take the family $\mathscr{S}$ consisting of all possible subsets of $[N]$, and for the upper bound any family $\mathscr{S}$ of one-point sets only. In the first case the identity from (3) gives that for all N

$$\boldsymbol{\lambda}\big(\mathscr{B}^N\big) = 1\,, \tag{14}$$

and in the second case by Kintchine's inequality (as proved by Szarek in [43])

$$\frac{1}{\sqrt{2}}\sqrt{|\mathscr{S}|} \leq \mathbb{E}\Big[\Big|\sum_{\{x\}\in\mathscr{S}} \chi_{\{x\}}\Big|\Big] \leq \sqrt{|\mathscr{S}|}\,. \tag{15}$$

For the latter case of one-point sets we have a more precise formula. Indeed, for each family $\mathscr{S}$ of one-point sets of $[N]$ by Theorem 3.1 and an integral form of Rademacher averages proved by Haagerup in [21] one has

$$\boldsymbol{\lambda}\big(\mathscr{B}^N_{\mathscr{S}}\big) = \frac{2}{\pi}\int_0^\infty t^{-2}\Big(1-\prod_{k=1}^{|\mathscr{S}|}\cos t\Big)\,dt\,. \tag{16}$$

Moreover, if $(\mathscr{S}_N)_{N\in\mathbb{N}}$ is a sequence of support sets which are finite and consists only of one-point sets in $\mathbb{N}$ such that the cardinality $|\mathscr{S}_N| \to \infty$ as $N\to\infty$, then

$$\lim_{N\to\infty}\frac{\boldsymbol{\lambda}\big(\mathscr{B}^N_{\mathscr{S}_N}\big)}{\sqrt{|\mathscr{S}_N|}} = \sqrt{\frac{2}{\pi}}\,. \tag{17}$$

This follows by a standard duality argument, since the mapping

$$\mathscr{B}^N_{\mathscr{S}_N} \ni \sum_{\{i_k\}\in\mathscr{S}_N} \alpha_k\chi_{\{i_k\}} \longmapsto (\alpha_k)_{k=1}^N$$

is a linear isometric isomorphism of $\mathscr{B}^N_{\mathscr{S}_N}$ onto $\ell_1^{|\mathscr{S}_N|}$, and then the conclusion follows from the case $p=1$ in (12).

The following consequences of Theorem 3.1 collect a few extensions of (15). The substitute for Kintchine's inequality in the case $d=1$ we need, is the following hypercontractivity estimate for homogeneous functions on Boolean cubes due to Ivanisvili and Tkocz [23, Theorem 2] which shows that, for $d>1$ and every $f\in\mathscr{B}^N_{=d}$,

$$\|f\|_{L_2(\{-1,+1\}^N)} \leq e^{\frac{d}{2}}\|f\|_{L_1(\{-1,+1\}^N)}\,. \tag{18}$$

More generally, in [13, Theorem 13] Eskenazis and Ivanisvili showed that for every $f\in\mathscr{B}^N_{\leq d}$

$$\|f\|_{L_2(\{-1,+1\}^N)} \leq (2.69076)^d\|f\|_{L_1(\{-1,+1\}^N)}\,. \tag{19}$$

The previous two inequalities are recent improvements of the classical hypercontractive Bonami-Weissler inequality for the Boolean cube (see e.g., [34, Theorem 9.22]).

Combining Theorem 3.1 with (19) and (18) gives the following extension of (15).

Corollary 3.2. *Let $\mathscr{S} \subset 2^{[N]}$. If $|S| = d$ for all $S \in \mathscr{S}$, then*

$$\frac{1}{e^{\frac{d}{2}}}\sqrt{|\mathscr{S}|} \le \boldsymbol{\lambda}\big(\mathscr{B}^N_{\mathscr{S}}\big) \le \sqrt{|\mathscr{S}|},$$

and if $|S| \le d$ for all $S \in \mathscr{S}$, then

$$\frac{1}{(2.69076)^d}\sqrt{|\mathscr{S}|} \le \boldsymbol{\lambda}\big(\mathscr{B}^N_{\mathscr{S}}\big) \le \sqrt{|\mathscr{S}|}.$$

Calculating cardinalities, yields more concrete estimates for $\mathscr{B}^N_{=d}$ and $\mathscr{B}^N_{\le d}$: For $1 \le d \le N$

$$\frac{1}{e^{\frac{d}{2}}}\binom{N}{d}^{\frac{1}{2}} \le \boldsymbol{\lambda}\big(\mathscr{B}^N_{=d}\big) \le \binom{N}{d}^{\frac{1}{2}} \tag{20}$$

$$\frac{1}{(2.69076)^d}\left(\sum_{k=0}^{d}\binom{N}{k}\right)^{\frac{1}{2}} \le \boldsymbol{\lambda}\big(\mathscr{B}^N_{\le d}\big) \le \left(\sum_{k=0}^{d}\binom{N}{k}\right)^{\frac{1}{2}}, \tag{21}$$

and as a consequence

$$\frac{1}{e^{\frac{d}{2}}}\Big(\frac{N}{d}\Big)^{\frac{d}{2}} \le \boldsymbol{\lambda}\big(\mathscr{B}^N_{=d}\big) \le e^{\frac{d}{2}}\Big(\frac{N}{d}\Big)^{\frac{d}{2}} \tag{22}$$

$$\frac{1}{(2.69076)^d}\Big(\frac{N}{d}\Big)^{\frac{d}{2}} \le \boldsymbol{\lambda}\big(\mathscr{B}^N_{\le d}\big) \le e^{\frac{d}{2}}\Big(\frac{N}{d}\Big)^{\frac{d}{2}}. \tag{23}$$

Indeed, the preceding two estimates follow immediately from (20) and (21) taking into account the elementary estimates

$$\Big(\frac{N}{d}\Big)^{d} \le \binom{N}{d}, \tag{24}$$

$$\binom{N}{d} \le \sum_{k=0}^{d}\binom{N}{k} \le \sum_{k=0}^{d}\frac{N^k}{k!} = \sum_{k=0}^{d}\frac{d^k}{k!}\Big(\frac{N}{d}\Big)^{k} \le e^{d}\Big(\frac{N}{d}\Big)^{d}. \tag{25}$$

Note that applying a remarkable formula due to McKay [32], we have (see also [9, Lemma 5.7]): For each $N \in \mathbb{N}$ and each $0 \le \alpha < N$ with $N - \alpha$ being an odd integer, there exists $0 \le c_{\alpha,N} \le \sqrt{\pi/2}$ such that

$$\sum_{k \le \frac{N-\alpha-1}{2}}\binom{N}{k} = \sqrt{N}\binom{N-1}{\frac{N-\alpha-1}{2}}\,Y\left(\frac{\alpha+1}{\sqrt{N}}\right)\exp\left(\frac{c_{\alpha,N}}{\sqrt{N}}\right),$$

where Y is given by

$$Y(x) = e^{\frac{x^2}{2}}\int_x^{\infty} e^{-\frac{t^2}{2}}\,dt, \quad x \ge 0.$$

In particular, taking $\alpha = 0$, we obtain a nice asymptotic formula for $\sum_{k=0}^{d}\binom{N}{k}$, whenever N is odd and $d = \frac{N-1}{2}$.

It worth noting that the following estimates hold (see [44, Proposition 3])

$$\frac{2}{x+(x^2+4)^{1/2}} \le Y(x) \le \frac{4}{3x+(x^2+8)^{1/2}}, \quad x \ge 0.$$

We conclude with an observation showing that the upper bound in (25) can be improved as follows whenever $2d-1<N$:

$$\sum_{k=0}^{d}\binom{N}{k} \le \binom{N}{d}\frac{N-(d-1)}{N-(2d-1)}. \tag{26}$$

In fact, we have

$$\frac{1}{\binom{N}{d}}\left(\binom{N}{d}+\binom{N}{d-1}+\binom{N}{d-2}+\cdots+1\right) = 1+\frac{d}{N-d+1}+\frac{d(d-1)}{(N-d+1)(N-d+2)}+\cdots.$$

Thus bounding the right-hand side from above by the geometric series

$$1+\frac{d}{N-d+1}+\left(\frac{d}{N-d+1}\right)^2+\cdots = \frac{N-(d-1)}{N-(2d-1)},$$

we get the required estimate (26).

In view of the preceding estimates for the projection constants of $\mathscr{B}^N_{=d}$ and $\mathscr{B}^N_{\le d}$, we add another useful result comparing both constants.

Proposition 3.3. *For every $N,d\in\mathbb{N}$ with $d\le N$*

$$\boldsymbol{\lambda}\big(\mathscr{B}^N_{=d}\big) \le (1+\sqrt{2})^d\,\boldsymbol{\lambda}\big(\mathscr{B}^N_{\le d}\big).$$

Proof. Indeed, this follows from an important fact proved by Klimek in [25] (see also [10, Lemma1,(iv)]): If $f\in\mathscr{B}^N_{\le d}$ and $f_k=\sum_{|S|=k}\hat{f}(S)\chi_S$ is the k-homogeneous part of f for $0\le k\le d$, then

$$\|f_k\|_\infty \le (1+\sqrt{2})^d\,\|f\|_\infty. \tag{27}$$ □

We finish with two more remarks on Theorem 3.1, which have a number theoretical flavour. The first one is

$$\lim_{N\to\infty}\frac{\boldsymbol{\lambda}\big(\mathscr{B}^N_{\mathscr{P}_N}\big)}{\sqrt{\frac{N}{\log N}}} = \sqrt{\frac{2}{\pi}}, \tag{28}$$

where $\mathscr{P}_N$ stands the family of one-point sets $\{p\}$ generated by all primes $p\le N$. Of course this is an immediate consequence of the prime number theorem and (17) applied to the sequence (p_k) of all primes.

For the second remark we denote by $\mathscr{P}^{sf}_N$ the family of all finite subsets A of primes in $[N]$ such that $n=\prod_{p\in A}p\le N$ (the prime number decomposition of n is square-free). Observe, that since every $n\in[N]$ has a unique prime number decomposition, there is a one to one correspondence between the set of all square-free numbers $n\in[N]$ and $\mathscr{P}^{sf}_N$.

Based on a recent deep result of Harper from [22], for large integers N with universal constants

$$\boldsymbol{\lambda}\Big(\mathscr{B}^N_{\mathscr{P}^{sf}_N}\Big) \sim \frac{\sqrt{N}}{(\log\log N)^{\frac{1}{4}}}. \tag{29}$$

For the proof note that all projections $\chi_{\{k\}}\colon \{-1,+1\}^{\mathbb{N}} \to \mathbb{R}$, $k \in \mathbb{N}$ form a sequence of independent Rademacher random variables. Moreover, every square-free number $n \in [N]$ defines a random variable $f_n\colon \{-1,+1\}^{\mathbb{N}} \to \mathbb{R}$ given by

$$f_n = \prod_{\substack{p|n \\ p\,\text{prime}}} \chi_{\{p\}}\,.$$

Then Harper's result mentioned above states that

$$\mathbb{E}\Big|\sum_{\substack{1\le n\le N \\ n\text{ square-free}}} f_n\Big| \sim \frac{\sqrt{N}}{(\log\log N)^{\frac{1}{4}}}.$$

But since the one to one correspondence between the square-free numbers $n \in [N]$ and the family of sets $A \in \mathscr{P}^{sf}_N$ identifies the random variables f_n and χ_A, we see that

$$\mathbb{E}\Big|\sum_{\substack{1\le n\le N \\ n\text{ square-free}}} f_n\Big| = \mathbb{E}\Big|\sum_{A\in\mathscr{S}} \chi_A\Big|.$$

Consequently, Theorem 3.1 finishes the argument for (29).

4. THE LIMIT CASE

The spaces $\mathscr{B}^N_{=1}$ and $\ell_1^N(\mathbb{R})$ identify as Banach spaces whenever we interpret the N-tuple $\sum_{k=1}^N x_k e_k$ as the function $\sum_{k=1}^N x_k \chi_{\{k\}}$ on the N-dimensional Boolean cube. Then by the result of Grünbaum mentioned in (11) we know that

$$\lim_{N\to\infty} \frac{\boldsymbol{\lambda}(\mathscr{B}^N_{=1})}{\sqrt{N}} = \lim_{N\to\infty} \frac{\boldsymbol{\lambda}(\ell_1^N(\mathbb{R}))}{\sqrt{N}} = \sqrt{\frac{2}{\pi}}.$$

In the following we show a procedure that allows to extend this result to $\mathscr{B}^N_{=d}$ and $\mathscr{B}^N_{\le d}$, where the degree d is arbitrary. Our main result here is as follows.

Theorem 4.1. *For each $d \in \mathbb{N}$,*

$$\lim_{N\to\infty} \frac{\boldsymbol{\lambda}(\mathscr{B}^N_{=d})}{N^{d/2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |P_d(t)| e^{-\frac{t^2}{2}}\, dt, \tag{30}$$

where $P_d(t) = \frac{t^d}{d!} - \sum_{k=1}^{\lfloor d/2 \rfloor} \frac{1}{k!2^k} P_{d-2k}(t)$ with $P_0(t) = 1$, $P_1(t) = t$. Moreover,

$$\lim_{N\to\infty} \frac{\boldsymbol{\lambda}(\mathscr{B}^N_{=d})}{N^{d/2}} = \lim_{N\to\infty} \frac{\boldsymbol{\lambda}(\mathscr{B}^N_{\le d})}{N^{d/2}}.$$

The following observation is important, since it helps to understand the preceding result in a larger context: For each $d \in \mathbb{N}_0$, we have that

$$P_d = \frac{h_d}{d!}, \tag{31}$$

where h_n for each $n \in \mathbb{N}_0$ denotes the n-th (probabilist's) Hermite polynomial given by

$$h_n(t) := (-1)^n e^{\frac{t^2}{2}} \frac{d^n}{dt^n} e^{-\frac{t^2}{2}}, \quad t \in \mathbb{R}.$$

In order to prove this, note first that for $d \in \{0,1\}$ this equality holds trivially. For arbitrary d's we use induction. Suppose that $P_k = \frac{h_k}{k!}$ for every $0 \le k \le d-1$, and let us show that $P_d = \frac{h_d}{d!}$. By the so-called 'inverse explicit expression for Hermite polynomials' we know that for all $t \in \mathbb{R}$

$$t^d = d! \sum_{k=0}^{\lfloor d/2 \rfloor} \frac{1}{k! 2^k} \frac{h_{d-2k}(t)}{(d-2k)!} = d! \sum_{k=1}^{\lfloor d/2 \rfloor} \frac{1}{k! 2^k} \frac{h_{d-2k}(t)}{(d-2k)!} + h_d(t).$$

Thus, by the inductive hypothesis,

$$\frac{h_d(t)}{d!} = \frac{t^d}{d!} - \sum_{k=1}^{\lfloor d/2 \rfloor} \frac{1}{k! 2^k} \frac{h_{d-2k}(t)}{(d-2k)!} = \frac{t^d}{d!} - \sum_{k=1}^{\lfloor d/2 \rfloor} \frac{1}{k! 2^k} P_{d-2k}(t) = P_d(t).$$

Note that from (20) for $\mathscr{S} = \{S \subset [N] : \mathrm{card}(S) = d\}$ we have

$$\frac{1}{e^{\frac{d}{2}}} \le \liminf_{N\to\infty} \frac{\boldsymbol{\lambda}\big(\mathscr{B}^N_{\mathscr{S}}\big)}{\sqrt{\dim(\mathscr{B}^N_{\mathscr{S}})}} \le \limsup_{N\to\infty} \frac{\boldsymbol{\lambda}\big(\mathscr{B}^N_{\mathscr{S}}\big)}{\sqrt{\dim(\mathscr{B}^N_{\mathscr{S}})}} \le 1,$$

and in the case $\mathscr{S} = \{S \subset [N] : \mathrm{card}(S) \le d\}$ the constant $e^{-\frac{d}{2}}$ has to be changed by $(2.69076)^{-d}$. The following result is a considerable improvement.

Corollary 4.2. *For $\mathscr{S} = \{S \subset [N] : card(S) = d\}$ or $\mathscr{S} = \{S \subset [N] : card(S) \le d\}$*

$$\lim_{N\to\infty} \frac{\boldsymbol{\lambda}\big(\mathscr{B}^N_{\mathscr{S}}\big)}{\sqrt{\dim(\mathscr{B}^N_{\mathscr{S}})}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{|h_d(t)|}{\sqrt{d!}} e^{-\frac{t^2}{2}}\, dt = \frac{2^{7/4}}{\pi^{5/4}} \frac{1}{d^{1/4}} \Big(1 + O\Big(\frac{1}{d^2}\Big)\Big).$$

Indeed, this follows from Theorem 4.1, equation (31) and a result of Larsson-Cohn [28, Remark 2.6 and 3.2], which says that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |h_d(t)| e^{-\frac{t^2}{2}}\, dt = \frac{2^{7/4}}{\pi^{5/4}} \frac{\sqrt{d!}}{d^{1/4}} \Big(1 + O\Big(\frac{1}{d^2}\Big)\Big).$$

Now, if $\mathscr{S} = \{S : \mathrm{card}(S) = d\}$, then $\dim(\mathscr{B}^N_{\mathscr{S}}) = \sqrt{\binom{N}{d}}$, and if $\mathscr{S} = \{S : \mathrm{card}(S) \le d\}$, then in the case $2d-1 < N$ (so in particular for large N), we have (see again (26))

$$\binom{N}{d} \le \dim(\mathscr{B}^N_{\mathscr{S}}) = \sum_{k=0}^{d} \binom{N}{k} \le \binom{N}{d} \frac{N-(d-1)}{N-(2d-1)}.$$

Since $\lim_{N\to\infty}\sqrt{\binom{N}{d}}/N^{d/2} = 1/\sqrt{d!}$, we in both considered cases have

$$\lim_{N\to\infty}\frac{\sqrt{\dim(\mathscr{B}^N_{\mathscr{S}})}}{N^{d/2}} = \frac{1}{\sqrt{d!}},$$

which gives the required statement.

To show a few examples, with the use of a computational platform, we get the following limits:

$$\lim_{N\to\infty}\frac{\boldsymbol{\lambda}(\mathscr{B}^N_{=2})}{\sqrt{\dim(\mathscr{B}^N_{=2})}} = \lim_{N\to\infty}\frac{\boldsymbol{\lambda}(\mathscr{B}^N_{\leq 2})}{\sqrt{\binom{N}{2}}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|\frac{t^2-1}{\sqrt{2}}\right|e^{-\frac{t^2}{2}}dt = \frac{2^{7/4}}{\sqrt{e}\sqrt{2\pi}} \approx \frac{0.814}{2^{1/4}}$$

$$\lim_{N\to\infty}\frac{\boldsymbol{\lambda}(\mathscr{B}^N_{=3})}{\sqrt{\dim(\mathscr{B}^N_{=3})}} = \lim_{N\to\infty}\frac{\boldsymbol{\lambda}(\mathscr{B}^N_{\leq 3})}{\sqrt{\binom{N}{3}}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|\frac{t^3-3t}{\sqrt{6}}\right|e^{-\frac{t^2}{2}}dt = \frac{1}{3\sqrt{2\pi}}\left(1+\frac{4}{e^{3/2}}\right) \approx \frac{0.811}{3^{1/4}}$$

$$\lim_{N\to\infty}\frac{\boldsymbol{\lambda}(\mathscr{B}^N_{=4})}{\sqrt{\dim(\mathscr{B}^N_{=4})}} = \lim_{N\to\infty}\frac{\boldsymbol{\lambda}(\mathscr{B}^N_{\leq 4})}{\sqrt{\binom{N}{4}}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|\frac{t^4-6t^2+3}{\sqrt{24}}\right|e^{-\frac{t^2}{2}}dt \approx \frac{0.808}{4^{1/4}}$$

$$\lim_{N\to\infty}\frac{\boldsymbol{\lambda}(\mathscr{B}^N_{=5})}{\sqrt{\dim(\mathscr{B}^N_{=5})}} = \lim_{N\to\infty}\frac{\boldsymbol{\lambda}(\mathscr{B}^N_{\leq 5})}{\sqrt{\binom{N}{5}}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|\frac{t^5-10t^3+15t}{\sqrt{120}}\right|e^{-\frac{t^2}{2}}dt \approx \frac{0.807}{5^{1/4}}$$

$$\lim_{N\to\infty}\frac{\boldsymbol{\lambda}(\mathscr{B}^N_{=6})}{\sqrt{\dim(\mathscr{B}^N_{=6})}} = \lim_{N\to\infty}\frac{\boldsymbol{\lambda}(\mathscr{B}^N_{\leq 6})}{\sqrt{\binom{N}{6}}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|\frac{t^6-15t^4+45t^2-15}{\sqrt{720}}\right|e^{-\frac{t^2}{2}}dt \approx \frac{0.806}{6^{1/4}}.$$

For the proof of Theorem 4.1 we use a probabilistic point of view, treating the coordinate functions $(\chi_{\{i\}})_{1\leq i\leq N}$ on the Boolean cube as independent Bernoulli random variables (taking the values ± 1 with equal probability $1/2$); for the random variable $\chi_{\{i\}}$ we shortly write x_i. From this perspective, any Walsh function χ_S is itself a random variable, being a product of coordinate functions. Consequently, any function on the Boolean cube may also be seen as a random variable.

By Theorem 3.1 the projection constant of $\mathscr{B}^N_{=d}$ can be computed as the expectation

$$\mathbb{E}\Big|\sum_{|S|=d, S\subset[N]}\chi_S\Big|.$$

Moreover we know from the central limit theorem that

$$\frac{\sum_{i=1}^N x_i}{\sqrt{N}} \longrightarrow Z, \tag{32}$$

where Z is a normal random variable with mean 0 and variance 1, and the convergence is in distribution. Based on this, the main idea of our procedure is to rewrite the random variable $\sum_{|S|=d,S\subset[N]}\chi_S$ in a suitable way into another random variable involving $\sum_{i=1}^N x_i/\sqrt{N}$, for which we manage to control its mean.

We use the notation $Y_n \xrightarrow{D} Y$, whenever a sequence (Y_n) converges in distribution to a random variable Y. Additionally to the notion of convergence in distribution, it will be necessary to consider convergence in probability. We write $Y_n \xrightarrow{P} Y$ if the sequence (Y_n) converges in probability to a random variable Y. Of course, convergence in probability implies convergence in distribution but, in general, the converse is not true. We recall that these two notions of convergence coincide, provided the limit is a constant.

Moreover, we frequently need a classical theorem of Slutsky. It states that, given two sequences $(X_n)_n$ and $(Y_n)_n$ of random variables such that $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{P} c$ (where X is a random variable and $c \in \mathbb{R}$ a constant), then $X_n + Y_n \xrightarrow{D} X + c$ and $X_n Y_n \xrightarrow{D} cX$.

Another result used at several places is that convergence in distribution is inherited under continuous functions in the sense that $f(Y_n) \xrightarrow{D} f(Y)$, whenever $Y_n \xrightarrow{D} Y$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Finally, recall that a sequence of random variables $(Y_n)_n$ is said to be uniformly integrable, whenever

$$\lim_{a\to\infty} \sup_{n\geq 1} \int_{|Y_n|\geq a} |Y_n| \, dP = 0 .$$

A sufficient condition is that $\sup_n \mathbb{E}(|Y_n|^{1+\varepsilon}) \leq C$ for some $\varepsilon, C > 0$, since then

$$\lim_{a\to\infty} \sup_{n\geq 1} \int_{|Y_n|\geq a} |Y_n| \, dP \leq \lim_{a\to\infty} \frac{1}{a^{\varepsilon}} C . \tag{33}$$

Uniform integrability is of particular importance to us due to the fact that Y is integrable and

$$\mathbb{E}(Y_n) \to \mathbb{E}(Y) , \tag{34}$$

provided $(Y_n)_n$ is a uniformly integrable and $Y_n \xrightarrow{D} Y$ (see for example [4, Theorem 3.5]).

4.1. **The 2-homogeneous case.** To keep our later calculations for the proof of Theorem 4.1 more transparent, we deal in detail first with the 2-homogeneous case.

Theorem 4.3.

$$\lim_{N\to\infty} \frac{\boldsymbol{\lambda}(\mathscr{B}^N_{=2})}{N} = \sqrt{\frac{2}{\pi e}}$$

Proof. In a first step we expand the square of the Boolean function $x \mapsto \sum_{i=1}^N x_i$ and rearrange the terms using $x_i^2 = 1$, to get

$$\sum_{1\leq i<j\leq N} x_i x_j = \sum_{1\leq i<j\leq N} x_{\{i,j\}} = \frac{1}{2}\left[\left(\sum_{i=1}^N x_i\right)^2 - N\right] .$$

By the central limit theorem the sequence of random variables (Z_N) given by

$$Z_N := \frac{1}{\sqrt{N}} \sum_{i=1}^N x_i$$

converges in distribution to a normal random variable Z with mean 0 and variance 1. Since the function $f(x) = \frac{|x^2-1|}{2}$ is continuous, we have

$$\frac{1}{N}\Big|\sum_{1\le i<j\le N} x_i x_j\Big| = \frac{|Z_N^2-1|}{2} \xrightarrow{D} \frac{|Z^2-1|}{2}.$$

Now note that by the orthogonality of the Fourier–Walsh basis get

$$\mathbb{E}\Big[\Big|\frac{1}{N}\sum_{1\le i<j\le N} x_i x_j\Big|^2\Big] = \frac{\binom{N}{2}}{N^2} \le 1\,,$$

and hence the uniform integrability of the random variable sequence $\left(\left|\sum_{1\le i<j\le N} x_i x_j\right|\right)_{N\ge 1}$ (see the remark done in (33)). Then, thanks to Theorem 3.1 and to what we explained in (34), we see that

$$\lim_{N\to\infty}\frac{\boldsymbol{\lambda}(\mathscr{B}^N_{=2})}{N} = \lim_{N\to\infty}\frac{1}{N}\,\mathbb{E}\Big(\Big|\sum_{1\le i<j\le N} x_i x_j\Big|\Big) = \lim_{N\to\infty}\mathbb{E}\Big(\frac{|Z_N^2-1|}{2}\Big) = \mathbb{E}\Big(\frac{|Z^2-1|}{2}\Big).$$

Computing the latter integral, we finally arrive at

$$\lim_{N\to\infty}\frac{\boldsymbol{\lambda}(\mathscr{B}^N_{=2})}{N} = \mathbb{E}\Big(\frac{|Z^2-1|}{2}\Big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{|t^2-1|}{2}e^{\frac{-t^2}{2}}\,dt = \sqrt{\frac{2}{\pi e}}. \qquad \square$$

The general case of arbitrary degrees $d\in\mathbb{N}$ is technically more involved. In the previous proof for $d=2$, the key step is to rewrite

$$\frac{1}{N}\sum_{1\le i<j\le N} x_i x_j$$

in terms of a polynomial in one variable.

For arbitrary d (as in Theorem 4.1), we require an adequate decomposition of the random variable

$$Y_N(x) = \frac{1}{N^{d/2}}\sum_{|S|=d} x^S = \frac{1}{N^{d/2}}\sum_{\alpha\in\Lambda_T(d,N)} x^\alpha\,. \tag{35}$$

In order to derive the expectation of these kernels, we offer two proofs with independently interesting features. Both approaches have two steps (see also the proof of Theorem 4.3): in a first step the kernels Y_N are reformulated in such a way that in a second step the central limit theorem may be applied properly. In both approaches the second steps are basically identical, whereas the arguments for the first ones are substantially different. The first approach (see Section 4.2) doesn't need any knowledge on Hermite polynomials. It is mainly based on a natural decomposition of multi indices α into their 'even part' α_E and their 'tetrahedral part' α_T. At the very end we arrive at the limit formula from (30) discovering 'posthum' that it may be written in terms of Hermite polynomials. Knowing this fact, one in a second approach may use a (somewhat classical) formula of Beckner from [3] to reach the same goal. Since we use this formula as a sort of black box, this second approach here appears to be shorter.

4.2. **First approach - decomposing indices.** For any index $\alpha \in \Lambda(d,N)$ there are a unique integer $0 \le k \le d/2$ and a unique decomposition

$$\alpha = \alpha_T + \alpha_E$$

of α into the sum of a tetrahedral index $\alpha_T \in \Lambda_T(d-2k,N)$ and an even index $\alpha_E \in \Lambda_E(2k,N)$ (so all coordinates of α_E are even and $|\alpha_E| = 2k$). This in particular implies that

$$x^\alpha = x^{\alpha_T} \cdot \underbrace{x^{\alpha_E}}_{=1} = x^{\alpha_T} \quad \text{for every} \quad x \in \{-1,+1\}^N.$$

The way to find such a decomposition of a given index $\alpha \in \Lambda(d,N)$ is rather simple: Given $1 \le j \le N$, the j-th coordinate of the tetrahedral part $\alpha_T \in \Lambda(d,N)$ is equal to 1, whenever α_j is odd, and 0 else. The even part is defined as $\alpha_E := \alpha - \alpha_T \in \Lambda(d,N)$. Defining

$$k := \frac{|\alpha_E|}{2},$$

we indeed see that $\alpha_T \in \Lambda_T(d-2k,N)$ and $\alpha_E \in \Lambda_E(2k,N)$. Moreover, since all coordinates of α_E are even, there exists a unique $\beta \in \Lambda(d,N)$ such that $\alpha_E = 2\beta$, that is, there is a canonical way to identify $\Lambda_E(2k,N)$ and $\Lambda(k,N)$.

All together, this leads to the following identification of index sets:

$$\Lambda(d,N) \leftrightsquigarrow \bigcup_{k=0}^{\lfloor d/2 \rfloor} \Lambda_T(d-2k,N) \times \Lambda_E(2k,N) \leftrightsquigarrow \bigcup_{k=0}^{\lfloor d/2 \rfloor} \Lambda_T(d-2k,N) \times \Lambda(k,N)\,, \tag{36}$$

where the second identification comes from the fact that there is a canonical correspondence between $\Lambda_E(2k,N)$ and $\Lambda(k,N)$.

We say that two indices $\alpha \in \Lambda(d_1,N)$ and $\beta \in \Lambda(d_2,N)$ do not share variables whenever they have disjoint support.

Lemma 4.4. *Let $\alpha \in \Lambda(d,N)$ and $k \le d/2$. Assume that the tetrahedral part $\alpha_T \in \Lambda_T(d-2k,N)$ and the even part $\alpha_E \in \Lambda(2k,N)$ of α do not share variables, and that $\alpha_E = 2\beta$ with $\beta \in \Lambda_T(k,N)$. Then*

$$|[\alpha]| = \frac{d!}{2^k}.$$

Proof. We deduce from (4) that

$$|[\alpha]| = \frac{d!}{\alpha!} = \frac{d!}{(\alpha_T+\alpha_E)!} = \frac{d!}{\alpha_T!\alpha_E!} = \frac{d!}{(2\beta)!} = \frac{d!}{2^k}. \qquad \square$$

Let us begin analyzing the idea to rewrite the random variable from (35) for arbitrary degrees $d \ge 2$. Taking $\sum_{i=1}^N x_i$ for $x \in \{-1,+1\}^N$ to the power d and writing $x_{\mathbf{i}} = x_{i_1} \dots x_{i_1}$ for $\mathbf{i} \in \mathcal{M}(d,N)$, we get

$$\left(\sum_{i=1}^N x_i\right)^d = \sum_{\mathbf{i}\in\mathcal{M}(d,N)} x_{\mathbf{i}} = \sum_{\mathbf{j}\in\mathcal{J}(d,N)} \sum_{\mathbf{i}\in[\mathbf{j}]} x_{\mathbf{i}} = \sum_{\mathbf{j}\in\mathcal{J}(d,N)} |[\mathbf{j}]| x_{\mathbf{j}} = \sum_{\alpha\in\Lambda(d,N)} |[\alpha]| x^\alpha\,.$$

Decomposing each $\alpha \in \Lambda(d,N)$, according to (the first identification in) (36), and using the fact that $x_i^2 = 1$, we have

$$\left(\sum_{i=1}^N x_i\right)^d = \sum_{k=0}^{\lfloor d/2\rfloor}\left(\sum_{\alpha_E\in\Lambda_E(2k,N)}\ \sum_{\alpha_T\in\Lambda_T(d-2k,N)} |[\alpha_T+\alpha_E]|x^{\alpha_T}\right).$$

Consequently, by (4)

$$\left(\sum_{i=1}^N x_i\right)^d = \sum_{\alpha_T\in\Lambda_T(d,N)} d!x^{\alpha_T} + \sum_{k=1}^{\lfloor d/2\rfloor}\left(\sum_{\alpha_E\in\Lambda_E(2k,N)}\ \sum_{\alpha_T\in\Lambda_T(d-2k,N)} |[\alpha_T+\alpha_E]|x^{\alpha_T}\right),$$

so that rearranging terms leads to

$$\sum_{\alpha_T\in\Lambda_T(d,N)} x^{\alpha_T} = \frac{1}{d!}\left[\left(\sum_{i=1}^N x_i\right)^d - \sum_{k=1}^{\lfloor d/2\rfloor}\left(\sum_{\alpha_T\in\Lambda_T(d-2k,N)}\ \sum_{\alpha_E\in\Lambda_E(2k,N)} |[\alpha_T+\alpha_E]|x^{\alpha_T}\right)\right]. \tag{37}$$

To illustrate this, note that for $d \in \{2,3\}$ we get

- $\displaystyle\sum_{1\le i<j\le N} x_ix_j = \frac{1}{2}\left[\left(\sum_{i=1}^N x_i\right)^2 - N\right],$
- $\displaystyle\sum_{1\le i<j<k\le N} x_ix_jx_k = \frac{1}{6}\left[\left(\sum_{i=1}^N x_i\right)^3 - (3N-2)\left(\sum_{i=1}^N x_i\right)\right].$

The following two technical lemma analyze (37) in more detail.

Lemma 4.5. *Given $d, N \in \mathbb{N}$, we for each $x \in \{-1,+1\}^N$ have*

$$\sum_{\alpha\in\Lambda_T(d,N)} x^\alpha = \frac{1}{d!}\left[\left(\sum_{i=1}^N x_i\right)^d - \sum_{k=1}^{\lfloor d/2\rfloor} C_{d,k,N}\sum_{\alpha_T\in\Lambda_T(d-2k,N)} x^{\alpha_T}\right],$$

where for $1 \le k \le \lfloor d/2\rfloor$

$$C_{d,k,N} = \binom{N-d+2k}{k}\frac{d!}{2^k} + D_{d,k,N} \quad \text{and} \quad 0 \le D_{d,k,N} \le N^{k-1}2dd!.$$

In particular,

$$\lim_{N\to\infty}\frac{C_{d,k,N}}{N^k} = \frac{d!}{k!\,2^k}. \tag{38}$$

Proof. We fix some $1 \le k \le \lfloor d/2\rfloor$, and note that (in view of equation (37)) we need to study

$$\sum_{\alpha_T\in\Lambda_T(d-2k,N)}\ \sum_{\alpha_E\in\Lambda_E(2k,N)} |[\alpha_T+\alpha_E]|x^{\alpha_T},$$

in order to be able to first define and second control the factor $C_{d,k,N}$. We fix some $\alpha_T \in \Lambda_T(d-2k,N)$, and start to decompose

$$\sum_{\alpha_E\in\Lambda_E(2k,N)} |[\alpha_T+\alpha_E]|x^{\alpha_T}.$$

Let us denote the set of even indices which do not share variables with α_T, by $\Lambda_E(\alpha_T) \subset \Lambda_E(2k,N)$, and use $\Lambda_E(\alpha_T)^c \subset \Lambda_E(2k,N)$ for its complement in $\Lambda_E(2k,N)$. Then

$$\sum_{\alpha_E \in \Lambda_E(2k,N)} |[\alpha_T + \alpha_E]| x^{\alpha_T} = \Big[\underbrace{\sum_{\alpha_E \in \Lambda_E(\alpha_T)^c} |[\alpha_T + \alpha_E]|}_{=:A} + \underbrace{\sum_{\alpha_E \in \Lambda_E(\alpha_T)} |[\alpha_T + \alpha_E]|}_{=:B} \Big] x^{\alpha_T},$$

and we handle both sums separately.

Before we start to estimate, note that $A+B$ does not depend on α_T, that is, for each α_T the sum

$$\sum_{\alpha_E \in \Lambda_E(2k,N)} |[\alpha_T + \alpha_E]|$$

is the same. Indeed, given two different tetrahedral multi indices $\alpha_T, \alpha'_T \in \Lambda_T(d-2k,N)$, the natural bijection between index sets that maps α_T to α'_T, given by a suitable permutation of coordinates, also lets the sums invariant. This allows us to define

$$C_{d,k,N} := A + B\,.$$

Estimating A : In order to estimate the cardinality of $\Lambda_E(\alpha_T)^c$, observe that any multi index in $\Lambda_E(\alpha_T)^c$ needs to share at least one of the $d-2k$ possible variables of α_T, and therefore

$$|\Lambda_E(\alpha_T)^c| \le (d-2k)|\Lambda_E(2k-2,N)| = (d-2k)|\Lambda(k-1,N)| \le (d-2k)N^{k-1}\,. \tag{39}$$

Clearly, $\alpha_T + \alpha_E \in \Lambda(d,N)$ for any $\alpha_E \in \Lambda_E(\alpha_T)^c$, and hence by (4)

$$A = \sum_{\alpha_E \in \Lambda_E(\alpha_T)^c} |[\alpha_T + \alpha_E]| = \sum_{\alpha_E \in \Lambda_E(\alpha_T)^c} \frac{d!}{(\alpha_T + \alpha_E)!} \le (d-2k)N^{k-1} d!.$$

Estimating B : We have

$$B = \sum_{\alpha_E \in \Lambda_E(\alpha_T)} |[\alpha_T + \alpha_E]| = \sum_{\alpha_E \in \Lambda_E(2k,N-d+2k)} |[\alpha_T + \alpha_E]|.$$

We then may decompose the index set $\Lambda_E(2k,N-d+2k)$ into the set of those indices which use k variables, denoted by $\Lambda_E(2k,N-d+2k)^k$, and the set that contains all even indices with less than k variables, denoted by $\Lambda_E(2k,N-d+2k)^{<k}$, so

$$B = \underbrace{\sum_{\alpha_E \in \Lambda_E(2k,N-d+2k)^{<k}} |[\alpha_T + \alpha_E]|}_{=:B^{<k}} + \underbrace{\sum_{\alpha_E \in \Lambda_E(2k,N-d+2k)^{k}} |[\alpha_T + \alpha_E]|}_{=:B^{=k}}\,.$$

Observe that given a multi index in $\Lambda_E(2k,N-d+2k)^{<k}$, it is mandatory that some variable appears to at least the 4th power (since all the indices in the set $\Lambda_E(2k,N-d+2k)^{<k}$ are even). Going through all the possible $N-d+2k$ variables, we get the bound

$$\begin{aligned} |\Lambda_E(2k,N-d+2k)^{<k}| &\le (N-d+2k)|\Lambda_E(2k-4,N-d+2k)| \\ &= (N-d+2k)|\Lambda(k-2,N-d+2k)| \le (N-d+2k)^{k-1}\,, \end{aligned}$$

and then (as above with (4))

$$B^{<k} = \sum_{\alpha_E \in \Lambda_E(2k, N-d+2k)^{<k}} |[\alpha_T + \alpha_E]| \leq \left|\Lambda_E(2k, N-d+2k)^{<k}\right| d! \leq N^{k-1} d!.$$

On the other hand, note that for each $\alpha \in \Lambda_E(2k, N-d+2k)^k$ there is $\beta \in \Lambda_T(k, N-d+2k)$ such that $\alpha = 2\beta$. Thus, this defines a one to one mapping between $\Lambda_E(2k, N-d+2k)^k$ and $\Lambda_T(k, N-d+2k)$, so we get

$$|\Lambda_E(2k, N-d+2k)^k| = |\Lambda_T(k, N-d+2k)| = \binom{N-d+2k}{k}.$$

Then by Lemma 4.4

$$B^{=k} = \sum_{\alpha_E \in \Lambda_E(2k, N-d+2k)^k} |[\alpha_T + \alpha_E]| = \binom{N-d+2k}{k} \frac{d!}{2^k}.$$

Combining step: We define $D := A + B^{<k}$ (note that $D = D_{d,k,N}$ in fact depends on d, k and N). Then $D \leq N^{k-1} 2d d!$, and all in all we obtain

$$C_{d,k,N} = \sum_{\alpha_E \in \Lambda_E(2k,N)} |[\alpha_T + \alpha_E]| = A + B = B^{=k} + D_{d,k,N} \leq \binom{N-d+2k}{k} \frac{d!}{2^k} + N^{k-1} 2d d!.$$

Finally, for a fixed k, a standard calculation gives (38). □

The following lemma goes one step further - namely, rewriting the random variable from (35) in a way that later allows us to calculate the limit of its mean. Notice that for $d = 0$ this random variable equals the constant function of value 1.

Lemma 4.6. *Given $d \in \mathbb{N}_0$ and $N \in \mathbb{N}$, there is $\varphi_{d,N} \in C(\mathbb{R})$ such that for all $x \in \{-1,+1\}^N$*

$$\frac{1}{N^{d/2}} \sum_{\alpha \in \Lambda_T(d,N)} x^\alpha = P_d\left(\frac{\sum_{i=1}^N x_i}{\sqrt{N}}\right) + \varphi_{d,N}\left(\frac{\sum_{i=1}^N x_i}{\sqrt{N}}\right), \tag{40}$$

where P_d is as in Theorem 4.1 and $\varphi_{0,N} = \varphi_{1,N} = 0$. Moreover, we have that

$$\varphi_{d,N}\left(\frac{\sum_{i=1}^N x_i}{\sqrt{N}}\right) \xrightarrow{P} 0, \quad \text{as } N \to \infty, \tag{41}$$

and

$$\frac{1}{N^{d/2}} \sum_{\alpha \in \Lambda_T(d,N)} x^\alpha \xrightarrow{D} P_d(Z), \quad \text{as } N \to \infty, \tag{42}$$

where Z is a normal distribution with mean 0 and variance 1.

Proof. The proof will be by induction on d. Recall from Theorem 4.1 that $P_0 = 1$ and $P_1 = t$. Then for $d = 0$ there is nothing to prove, and for $d = 1$ the proof by (32) is obvious as well.

Let us fix some $d \geq 2$, and assume that the result is true for all degrees $\leq d-1$. The aim is to prove the result for d. Dividing the equality from Lemma 4.5 by $N^{d/2} = N^k N^{(d-2k)/2}$ and using the inductive hypothesis, we have

$$\begin{aligned}\frac{1}{N^{d/2}} \sum_{\alpha \in \Lambda_T(d,N)} x^\alpha &= \frac{1}{d!}\left(\frac{\sum_{i=1}^N x_i}{\sqrt{N}}\right)^d - \sum_{k=1}^{\lfloor d/2 \rfloor} \frac{C_{d,k,N}}{d! N^k}\left(\frac{1}{N^{(d-2k)/2}} \sum_{\alpha_T \in \Lambda_T(d-2k,N)} x^{\alpha_T}\right)\\ &= \frac{1}{d!}\left(\frac{\sum_{i=1}^N x_i}{\sqrt{N}}\right)^d - \sum_{k=1}^{\lfloor d/2 \rfloor} \frac{C_{d,k,N}}{d! N^k}\left(P_{d-2k}\left(\frac{\sum_{i=1}^N x_i}{\sqrt{N}}\right) + \varphi_{d-2k,N}\left(\frac{\sum_{i=1}^N x_i}{\sqrt{N}}\right)\right).\end{aligned}$$

Defining for $t \in \mathbb{R}$

$$\varphi_{d,N}(t) := \sum_{k=1}^{\lfloor d/2 \rfloor}\left(\frac{C_{d,k,n}}{d! N^k} - \frac{1}{k! 2^k}\right)\left(P_{d-2k}(t) + \varphi_{d-2k,N}(t)\right)$$

and recalling the definition of P_d, we see that (40) holds. It remains to show the two limit formulas from (41) and (42). By the inductive hypothesis for each $1 \leq k \leq d/2$

$$P_{d-2k}\left(\frac{\sum_{i=1}^N x_i}{\sqrt{N}}\right) \xrightarrow{D} P_{d-2k}(Z) \quad \text{and} \quad \varphi_{d-2k,N}\left(\frac{\sum_{i=1}^N x_i}{\sqrt{N}}\right) \xrightarrow{P} 0, \quad \text{as} \quad N \to \infty,$$

Moreover, by Lemma 4.5, (38) we have that $\lim_{N\to\infty} \dfrac{C_{d,k,n}}{d! N^k} - \dfrac{1}{k! 2^k} = 0$, so that (41) follows by Slutsky's theorem. Since convergence in distribution is preserved under continuous functions, we conclude that

$$P_d\left(\frac{\sum_{i=1}^N x_i}{\sqrt{N}}\right) \xrightarrow{D} P_d(Z), \quad \text{as} \quad N \to \infty,$$

and then we obtain (42) from another application of Slutsky's theorem, using (40) and (41). □

Finally, we come to the proof of the main contribution of this section, Theorem 4.1, which extends Theorem 4.3 to all possible degrees.

Proof of Theorem 4.1. We for $N \in \mathbb{N}$ define the random variable $Y_N(x)$ as in (35). Applying Lemma 4.6, the central limit theorem as in (32), the fact that convergence in distribution is preserved under continuous functions, and Slutsky's theorem, we get

$$Y_N(x) = P_d\left(\frac{\sum_{i=1}^N x_i}{\sqrt{N}}\right) + \varphi_{d,N}\left(\frac{\sum_{i=1}^N x_i}{\sqrt{N}}\right) \xrightarrow{D} P_d(Z), \quad \text{as} \quad N \to \infty.$$

Now orthogonality of the Fourier–Walsh basis assures that for all N

$$\mathbb{E}|Y_N|^2 = \frac{|\Lambda_T(d,N)|}{N^d} \leq 1,$$

which gives the uniform integrability of all Y_N (see the remark from (33)). Using [4, Theorem 3.5], this implies

$$\lim_{N\to\infty}\mathbb{E}|Y_N| = \mathbb{E}[|P_d(Z)|] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|P_d(t)|e^{-\frac{t^2}{2}}dt,$$

which by Theorem 3.1 finishes the proof of (30). The second claim follows with similar arguments. Observe first that by another application of Theorem 3.1 we have

$$\boldsymbol{\lambda}(\mathscr{B}^N_{\le d}) = \mathbb{E}\left[\left|\sum_{\alpha\in\Lambda_T(\le d,N)} x^\alpha\right|\right].$$

Also by Lemma 4.6, (42), for every $0\le k\le d$

$$\frac{1}{N^{k/2}}\sum_{\alpha\in\Lambda_T(k,N)} x^\alpha \xrightarrow{D} P_k(Z), \quad \text{as} \quad N\to\infty,$$

where Z is as above. Hence we as before use Slutsky's theorem and the fact that a sequence of random variables converges in probability whenever it converges in distribution to a constant, to see that for every $0\le k<d$

$$\frac{1}{N^{d/2}}\sum_{\alpha\in\Lambda_T(k,N)} x^\alpha \xrightarrow{P} 0, \quad \text{as} \quad N\to\infty.$$

Now one more application of Slutsky's theorem shows that

$$\lim_{N\to\infty}\frac{1}{N^{d/2}}\sum_{\alpha\in\Lambda_T(\le d,N)} x^\alpha = \lim_{N\to\infty}\frac{1}{N^{d/2}}\sum_{k=1}^{d}\sum_{\alpha\in\Lambda_T(k,N)} x^\alpha = \lim_{N\to\infty}\frac{1}{N^{d/2}}\sum_{\alpha\in\Lambda_T(d,N)} x^\alpha = P_d(Z),$$

where the limit is taken in the sense of distribution. Again by orthogonality

$$\mathbb{E}\left[\left|\frac{1}{N^{d/2}}\sum_{\alpha\in\Lambda_T(\le d,N)} x^\alpha\right|^2\right] = \frac{|\Lambda_T(\le d,N)|}{N^d} < \infty,$$

implying that the random variables $\frac{1}{N^{d/2}}\sum_{\alpha\in\Lambda_T(\le d,N)} x^\alpha$, $N\in\mathbb{N}$ are uniformly integrable, and this is enough to conclude that

$$\lim_{N\to\infty}\frac{\boldsymbol{\lambda}(\mathscr{B}^N_{\le d})}{N^{d/2}} = \mathbb{E}(|P_d(Z)|)$$

(see again (34)). Together with the first claim this finishes the proof. □

4.3. **Second approach - Beckner's formula.** The following identity of Beckner from [3, Equation (5)] rephrases $\sum_{|S|=d} x_S$ directly in terms of Hermite polynomials, and may hence serve as a substitute of Lemma 4.5.

Lemma 4.7. *For each $d,N\in\mathbb{N}$ and $x\in\{-1,+1\}^N$*

$$\frac{1}{N^{\frac{d}{2}}}\sum_{|S|=d} x_S = \frac{1}{d!}\left(h_d\left(\frac{x_1+\ldots+x_N}{\sqrt{N}}\right) + \frac{1}{N}\sum_{k=1}^{\lfloor d/2\rfloor} a_{d,k,N}h_{d-2k}\left(\frac{x_1+\ldots+x_N}{\sqrt{N}}\right)\right),$$

where the coefficients $a_{d,k,N}$ are bounded in N and h_ℓ is the ℓ-th Hermite polynomial.

With this the proof of Theorem 4.1 follows similarly as above replacing the polynomials P_d by $h_d/d!$. Indeed, applying Lemma 4.7, the fact that convergence in distribution is preserved under continuous functions together with (32) gives that for all $\ell \in \mathbb{N}_0$ we again get

$$h_\ell\left(\frac{\sum_{i=1}^N x_i}{\sqrt{N}}\right) \xrightarrow{D} h_\ell(Z), \quad \text{as} \quad N \to \infty.$$

Also, as the constants $a_{d,k,N}$ are bounded in N, we have $\lim_{N\to\infty} \frac{a_{d,k,N}}{N} = 0$ for every $1 \le k \le \lfloor d/2 \rfloor$, so that by Slutky's theorem (Y_N again as in (35))

$$Y_N(x) = \frac{1}{d!}\left(h_d\left(\frac{x_1 + \ldots + x_N}{\sqrt{N}}\right) + \frac{1}{N}\sum_{k=1}^{\lfloor d/2 \rfloor} a_{d,k} h_{d-2k}\left(\frac{x_1 + \ldots + x_N}{\sqrt{N}}\right)\right) \xrightarrow{D} \frac{1}{d!}h_d(Z), \quad \text{as} \quad N \to \infty.$$

The rest of the proof proceeds in a similar way as before.

While it may seem that this approach is more concise, it actually conceals the use of a deep specific result as if it were a 'hidden toolkit'. Furthermore, we would like to underscore that the first approach, founded on the contemporary combinatorial technique known as 'monomial decomposition' (see e.g., [16, 31]), reinforces this fresh perspective. We firmly believe that this approach holds substantial potential for wider applications.

5. Sidon vs projection constant

Given a set $\mathscr{S} \subset \{S \subset [N]\}$, we now establish an intimate link of the Sidon constant of $\mathscr{B}^N_{\mathscr{S}}$ with its projection constant. To do so, we define

$$\mathscr{S}^\flat = \{S \setminus \{i\} \colon S \in \mathscr{S}, \, i \in S\}.$$

Moreover, the following constant

$$\kappa := \left(\prod_{k=1}^{\infty} \operatorname{sinc} \frac{\pi}{\mathfrak{p}_k}\right)^{-1} = 2.209\ldots, \tag{43}$$

is going to appear, where $(\mathfrak{p}_k)_{k\ge1}$ is the sequence of prime numbers and $\operatorname{sinc} x := (\sin x)/x$.

The main result is as follows.

Theorem 5.1. *Let $2 \le d \le N$. Then*

$$\mathbf{Sid}\big(\mathscr{B}^N_{=d}\big) \;\le\; C(d)\, \boldsymbol{\lambda}\big(\mathscr{B}^N_{=d-1}\big), \tag{44}$$

where $C(d) \le e^d (2d)\kappa^d 2^{d-1}$. Additionally,

$$\mathbf{Sid}\big(\mathscr{B}^N_{\le d}\big) \;\le\; \widetilde{C}(d) \max_{1\le k\le d} \boldsymbol{\lambda}\big(\mathscr{B}^N_{=k-1}\big), \tag{45}$$

where $\widetilde{C}(d) \le (2.69076)^{2d}(d+1)(1+\sqrt{2})^d (2d)\kappa^d 2^d$.

Note that the estimates of $\mathbf{Sid}\big(\mathscr{B}^N_{=d}\big)$ and $\mathbf{Sid}\big(\mathscr{B}^N_{\le d}\big)$ are anyway trivial if $0 \le d \le 1$, so there is no need to compare them with projection constants. In fact, the previous the statement is an obvious consequence of the following more general result.

Theorem 5.2. *Let $2 \le d \le N$ and $\mathscr{S} \subset \big\{S \subset [N] : |S| = d\big\}$. Then*

$$\mathbf{Sid}\big(\mathscr{B}^N_{\mathscr{S}}\big) \le C(d)\, \|\mathbf{Q} : \mathscr{B}^N_{=d} \to \mathscr{B}^N_{\mathscr{S}}\|\, \boldsymbol{\lambda}\big(\mathscr{B}^N_{\mathscr{S}^\flat}\big),$$

where the constant $C(d)$ is as in (44) and $\mathbf{Q}$ is the projection annihilating Fourier coefficients with indices S not in $\mathscr{S}$.

More generally, if $\mathscr{S} \subset \big\{S \subset [N] : |S| \le d\big\}$, then

$$\mathbf{Sid}\big(\mathscr{B}^N_{\mathscr{S}}\big) \le \widetilde{C}(d) \max_{1\le k\le d} \|\mathbf{Q}_k : \mathscr{B}^N_{=k} \to \mathscr{B}^N_{\mathscr{S}_{=k}}\| \max_{1\le k\le d} \boldsymbol{\lambda}(\mathscr{B}^N_{(\mathscr{S}_{=k})^\flat}),$$

where $\mathscr{S}_{=k} = \{S \in \mathscr{S} : |S| = k\}$, the constant $\widetilde{C}(d)$ is as in (45) and $\mathbf{Q}_k$ is the projection annihilating Fourier coefficients with indices S not in $\mathscr{S}_{=k}$.

The proof of Theorem 5.2 is given in Section 5.3. In the coming two sections we prepare it collecting a few independently interesting facts.

5.1. **Annihilating coefficients.**

In what follows we need a lemma, which is a discrete variant of a result for polynomials on the N-dimensional torus due to Ortega-Cerdà, Ounaïes and Seip in [35]. We include a proof for the sake of completeness.

Lemma 5.3. *Let $2 \le d \le N$. Then*

$$\big\|\mathbf{Q} \colon \mathscr{P}_d(\ell^N_\infty) \to \mathscr{B}^N_{=d}\big\| \le \kappa^d 2^{d-1},$$

where $\mathbf{Q}$ is the projection annihilating coefficients with non-tetrahedral indices. In particular,

$$\boldsymbol{\lambda}\big(\mathscr{B}^N_{=d}\big) \le \kappa^d 2^{d-1} \boldsymbol{\lambda}\big(\mathscr{P}_d(\ell^N_\infty)\big).$$

Proof. As usual, we write $\pi(x)$ for the counting function of the prime numbers. Now, given

$$t = (t_1, \dots, t_{\pi(d)}) \in \mathscr{R} := [0,1]^{\pi(d)},$$

define

$$r_d(t) = c_d \exp\left(2\pi i\Big(\frac{t_1}{2} + \frac{t_2}{3} + \cdots + \frac{t_{\pi(d)}}{\mathfrak{p}_{\pi(d)}}\Big)\right),$$

where

$$c_d = \prod_{k=1}^{\pi(d)} \left(\frac{\mathfrak{p}_k}{2\pi i}\Big(e^{\frac{2\pi i}{\mathfrak{p}_k}} - 1\Big)\right)^{-1}.$$

Note that the function $r_d \colon \mathscr{R} \to \mathbb{C}$ has the following properties:

(i) $\int_{\mathscr{R}} r_d(t)\, d\mu(t) = 1$,

(ii) $\int_{\mathscr{R}} r_d^k(t)\,d\mu(t) = 0$ for each $2 \le k \le d$,

(iii) $|r_d(t)| \le \kappa$ for all $t \in \mathscr{R}$;

here $d\mu$ denotes the Lebesgue measure on $\mathscr{R}$. Indeed, (i) and (ii) are trivial and follow by the definition of the function, and (iii) holds because $|r_d(t)| = |c_d|$ and

$$|c_d|^{-2} = \prod_{k=1}^{\pi(d)} \frac{\mathfrak{p}_k^2}{(2\pi)^2}\left|e^{\frac{2\pi i}{\mathfrak{p}_k}} - 1\right|^2 = \prod_{k=1}^{\pi(d)} \operatorname{sinc}^2 \frac{\pi}{\mathfrak{p}_k}.$$

Take now some $P \in \mathscr{P}_d(\ell_\infty^N)$ and a representation $Px = \sum_{|\alpha|=d} c_\alpha x^\alpha$, $x \in \mathbb{R}^N$. Then by the properties (i) and (ii) we have

$$\mathbf{Q}P(x) = \int_{\mathscr{R}^N} P_{\mathbb{C}}(x_1 r_d(t^1), \dots, x_n r_d(t^N))\,d\mu(t^1)\cdots d\mu(t^n)\,,\ \ x \in \ell_\infty^N.$$

where $P_{\mathbb{C}}(z) = \sum_{|\alpha|=d} c_\alpha z^\alpha$, $z \in \mathbb{C}^N$. By (iii) we deduce that

$$|P_{\mathbb{C}}(x_1 r_d(t^1), \dots, x_N r_d(t^N))| \le \kappa^d \|P_{\mathbb{C}}\|_{\mathscr{P}_d(\ell_\infty^N(\mathbb{C}))}$$

for every $x \in B_{\ell_\infty^N}$, and therefore

$$\|\mathbf{Q}(P)\|_{\mathscr{B}_{=d}^N} \le \kappa^d \,\|P_{\mathbb{C}}\|_{\mathscr{P}_d(\ell_\infty^N(\mathbb{C}))}.$$

But by a result of Visser [46] we know that

$$\|P_{\mathbb{C}}\|_{\mathscr{P}_d(\ell_\infty^N(\mathbb{C}))} \le 2^{d-1}\|P\|_{\mathscr{P}_d(\ell_\infty^N)}\,.$$

All together this proves the first statement; the second one is then an immediate consequence. □

5.2. **Sidon vs Gordon–Lewis constant.** The following fact is the first of two major steps towards the proof of Theorem 5.2.

Theorem 5.4. *Let* $2 \le d \le N$ *and* $\mathscr{S} \subset \{S \subset [N] : |S| = d\}$. *Then*

$$\boldsymbol{gl}\big(\mathscr{B}_{\mathscr{S}}^N\big) \ \le\ \boldsymbol{\chi}\big(\mathscr{B}_{\mathscr{S}}^N\big) \ \le\ \mathbf{Sid}\big(\mathscr{B}_{\mathscr{S}}^N\big) \ \le\ e^d\,\boldsymbol{gl}\big(\mathscr{B}_{\mathscr{S}}^N\big)\,. \tag{46}$$

Additionally, if $\mathscr{S} \subset \{S \subset [N] : |S| \le d\}$ *then*

$$\boldsymbol{gl}\big(\mathscr{B}_{\mathscr{S}}^N\big) \ \le\ \boldsymbol{\chi}\big(\mathscr{B}_{\mathscr{S}}^N\big) \ \le\ \mathbf{Sid}\big(\mathscr{B}_{\mathscr{S}}^N\big) \ \le\ (2.69076)^{2d}\,\boldsymbol{gl}\big(\mathscr{B}_{\mathscr{S}}^N\big)\,. \tag{47}$$

Observe that the first estimate in Theorem 5.4 is immediate from the Gordon–Lewis inequality as formulated in (7), and the second one is trivial.

The proof of the third estimate is more involved and needs preparation. The first lemma needed is taken from [8, Proposition 21.14] (its roots have to be traced back to the works [36] and [41]).

Lemma 5.5. *Let X_n be an n-dimensional Banach space with a basis $(x_k)_{k=1}^n$, and denote the coefficient functionals of this basis by (x_k^*). Suppose that there exist constants $K_1, K_2 \geq 1$ such that for every choice of $\lambda, \mu \in \mathbb{C}^n$ the two diagonal operators*

$$D_\lambda \colon X_n \to \ell_2^n, \quad x_k \mapsto \lambda_k e_k$$
$$D_\mu^* \colon X_n^* \to \ell_2^n, \quad x_k^* \mapsto \mu_k e_k$$

satisfy $\pi_1(D_\lambda) \leq K_1 \left\| \sum_{k=1}^n \lambda_k x_k^* \right\|_{X_n^*}$ and $\pi_1(D_\mu^*) \leq K_2 \left\| \sum_{k=1}^n \mu_k x_k \right\|_{X_n}$. Then

$$\boldsymbol{\chi}\big((x_k), X_n\big) \leq K_1 K_2 \, \boldsymbol{gl}(X_n).$$

The second lemma is an almost immediate consequence of the so-called hypercontractivity of functions on the Boolean cube.

Lemma 5.6. *Let $2 \leq d \leq N$ and $\mathscr{S} \subset \big\{S \subset [N] \colon |S| = d\big\}$. Then*

$$\pi_1\big(I \colon \mathscr{B}_{\mathscr{S}}^N \to \ell_2(\mathscr{S})\big) \leq e^{\frac{d}{2}}, \tag{48}$$

where $I(f) = (\hat{f}(S))_{S \in \mathscr{S}}$ for $f \in \mathscr{B}_{\mathscr{S}}^N$. More generally, if $\mathscr{S} \subset \big\{S \subset [N] \colon |S| \leq d\big\}$, then

$$\pi_1\big(I \colon \mathscr{B}_{\mathscr{S}}^N \to \ell_2(\mathscr{S})\big) \leq (2.69076)^d. \tag{49}$$

Proof. Suppose that $\mathscr{S} \subset \big\{S \subset [N] \colon |S| = d\big\}$ and take finitely many $f_1, \ldots, f_M \in \mathscr{B}_{\mathscr{S}}^N$. Then by (18) we have

$$\begin{aligned}
\sum_{k=1}^M \|I f_k\|_{\ell_2(\mathscr{S})} &= \sum_{k=1}^M \|f_k\|_{L_2(\{-1,+1\}^N)} \\
&\leq e^{\frac{d}{2}} \sum_{k=1}^M \|f_k\|_{L_1(\{-1,+1\}^N)} = e^{\frac{d}{2}} \sum_{k=1}^M \mathbb{E}[|f_k|] \\
&= e^{\frac{d}{2}} \mathbb{E}[\sum_{k=1}^M |f_k|] \leq e^{\frac{d}{2}} \sup_{x \in \{-1,+1\}^N} \sum_{k=1}^M |f_k(x)| \leq e^{\frac{d}{2}} \sup_{\varphi \in B_{(\mathscr{B}_{\mathscr{S}}^N)^*}} \sum_{k=1}^M |\varphi(f_k)|,
\end{aligned}$$

which gives (48). With the same argument and the use of (19) instead of (18) we obtain (49). ☐

Proof of Theorem 5.4. As explained above, we may concentrate on the third estimate. For $A \in \mathscr{S}$ we write $\psi_A \colon \mathscr{B}_{\mathscr{S}}^N \to \mathbb{C}$ for the coefficient functionals of the canonical basis $(\chi_A)_{A \in \mathscr{S}}$ in $\mathscr{B}_{\mathscr{S}}^N$. They form the orthogonal basis of the dual $(\mathscr{B}_{\mathscr{S}}^N)^*$ in the sense that

$$\langle \chi_A, \psi_B \rangle_{\mathscr{B}_{\mathscr{S}}^N, (\mathscr{B}_{\mathscr{S}}^N)^*} = \delta_{A,B}.$$

Given two real sequences $\lambda = (\lambda_A)_{A \in \mathscr{S}}$ and $(\mu_A)_{A \in \mathscr{S}}$, we consider the two diagonal operators

$$D_\lambda \colon \mathscr{B}_{\mathscr{S}}^N \longrightarrow \mathscr{B}_{\mathscr{S}}^N, \quad D_\lambda(\chi_A) = \lambda_A \chi_A$$
$$D_\mu^* \colon (\mathscr{B}_{\mathscr{S}}^N)^* \longrightarrow \mathscr{B}_{\mathscr{S}}^N, \quad D_\mu^*(\psi_A) = \mu_A \chi_A,$$

and show that

$$\|D_\lambda\| \le \Big\| \sum_{A\in\mathscr{S}} \lambda_A \psi_A \Big\|_{(\mathscr{B}_{\mathscr{S}}^N)^*} \tag{50}$$

$$\|D_\mu^*\| \le \Big\| \sum_{A\in\mathscr{S}} \mu_A \chi_A \Big\|_{\mathscr{B}_{\mathscr{S}}^N}. \tag{51}$$

If we combine these two estimates with Lemma 5.6, together with the ideal property of the π_1-norm and also Lemma 5.5, then the conclusion follows. Let us prove (50). Define for $x \in \{-1,+1\}^N$ the diagonal map

$$D_x\colon \{-1,+1\}^N \to \{-1,+1\}^N,\ y \mapsto xy,$$

and note that $\chi_A \circ D_x = x^A \chi_A$ for every $A \in \mathscr{S}$. Then for $x \in \{-1,+1\}^N$ and $f = \sum_{A\in\mathscr{S}} \widehat{f}(A)\chi_A \in \mathscr{B}_{\mathscr{S}}^N$ we get

$$\begin{aligned}
\Big|\Big[D_\lambda\Big(\sum_{\mathscr{S}} \widehat{f}(A)\chi_A\Big)\Big](x)\Big| &= \Big|\sum_{\mathscr{S}} \lambda_A \widehat{f}(A)\chi_A(x)\Big| = \Big|\sum_{\mathscr{S}} \lambda_A \widehat{f}(A) x^A\Big| \\
&= \Big|\langle \sum_{\mathscr{S}} \widehat{f}(A) x^A \chi_A, \sum_{\mathscr{S}} \lambda_A \psi_A \rangle_{\mathscr{B}_{\mathscr{S}}^N,(\mathscr{B}_{\mathscr{S}}^N)^*}\Big| \\
&= \Big|\langle \sum_{\mathscr{S}} \widehat{f}(A) \chi_A \circ D_x, \sum_{\mathscr{S}} \lambda_A \psi_A \rangle_{\mathscr{B}_{\mathscr{S}}^N,(\mathscr{B}_{\mathscr{S}}^N)^*}\Big| \\
&\le \Big\| \sum_{\mathscr{S}} \widehat{f}(A)\chi_A \circ D_x \Big\|_{\mathscr{B}_{\mathscr{S}}^N} \Big\| \sum_{\mathscr{S}} \lambda_A \psi_A \Big\|_{(\mathscr{B}_{\mathscr{S}}^N)^*} \\
&= \Big\| \sum_{\mathscr{S}} \widehat{f}(A)\chi_A \Big\|_{\mathscr{B}_{\mathscr{S}}^N} \Big\| \sum_{\mathscr{S}} \lambda_A \psi_A \Big\|_{(\mathscr{B}_{\mathscr{S}}^N)^*}.
\end{aligned}$$

Obviously, this leads to the estimate from (50), and to see (51) we simply repeat the argument. This completes the proof. □

5.3. **Gordon–Lewis vs projection constant.** With the following theorem we establish the second step for the proof of Theorem 5.2. If one combines it with Theorem 5.4, the proof of Theorem 5.2 is immediate.

Theorem 5.7. *Let $2 \le d \le N$ and $\mathscr{S} \subset \{S \subset [N]\colon |S| = d\}$. Then*

$$\boldsymbol{gl}\big(\mathscr{B}_{\mathscr{S}}^N\big) \le c(d)\, \big\|\mathbf{Q}\colon \mathscr{B}_{=d}^N \to \mathscr{B}_{\mathscr{S}}^N\big\|\, \boldsymbol{\lambda}\big(\mathscr{B}_{\mathscr{S}^\flat}^N\big),$$

where $c(d) \le (2d)\,\kappa^d 2^{d-1}$ and $\mathbf{Q}$ denotes the projection annihilating Fourier coefficients with indices S not in $\mathscr{S}$. More generally, if $\mathscr{S} \subset \{S \subset [N]\colon |S| \le d\}$, then

$$\boldsymbol{gl}\big(\mathscr{B}_{\mathscr{S}}^N\big) \le \widetilde{c}(d) \max_{1\le k\le m} \big\|\mathbf{Q}_k\colon \mathscr{B}_{=k}^N \to \mathscr{B}_{\mathscr{S}_{=k}}^N\big\| \max_{1\le k\le d} \boldsymbol{\lambda}\big(\mathscr{B}_{(\mathscr{S}_{=k})^\flat}^N\big),$$

where $\widetilde{c}(d) \le (d+1)(1+\sqrt{2})^d (2d)\kappa^d 2^d$ and $\mathbf{Q}_k$ denotes the projection annihilating Fourier coefficients with indices S not in $\mathscr{S}_{=k} = \{S \in \mathscr{S}\colon |S| = k\}$.

Again we need preparation for the proof, and start with two elementary observations. We start with a result taken from [8, Lemma 22.2].

Lemma 5.8. *For every finite dimensional Banach lattice X, and every finite dimensional Banach space Y one has*

$$\boldsymbol{gl}\big(\mathscr{L}(X,Y)\big) \le \boldsymbol{\lambda}(Y)\,.$$

The second tool is an elementary piece of multilinear algebra.

Lemma 5.9. *For $f \in \mathscr{B}^N_{=d}$ let P_f be the associated d-homogeneous polynomial on $\mathbb{R}^N$ given by*

$$P_f(x) = \sum_{1\le j_1<\dots<j_d\le N} \hat{f}(\{j_1,\dots,j_d\})\, x_{j_1}\dots x_{j_d}\,.$$

Then the unique d-linear symmetrization $\check{P}_f\colon (\mathbb{R}^N)^d \to \mathbb{R}$ of P_f is given by

$$\check{P}_f(u^1,\dots,u^d) = \sum_{\substack{\mathbf{i}\in[N]^d\\ i_k\neq i_\ell \text{ for } k\neq\ell}} \frac{\hat{f}(\{i_1,\dots,i_d\})}{d!}\, u^1_{i_1}\dots u^d_{i_d}\,.$$

Proof. Let $(a_{\mathbf{i}})_{\mathbf{i}\in\mathscr{M}(d,N)}$ be the symmetric matrix defining $\check{P}_f$. Then for all $x\in\mathbb{R}^N$ we have

$$\sum_{1\le j_1<\dots<j_d\le N} \hat{f}(\{j_1,\dots,j_d\})x_{\mathbf{j}} = \check{P}_f(x,\dots,x) = \sum_{\mathbf{i}\in\mathscr{M}(d,N)} a_{\mathbf{i}}x_{\mathbf{i}} = \sum_{\mathbf{j}\in\mathscr{J}(d,N)}\sum_{\mathbf{i}\in[\mathbf{j}]} a_{\mathbf{j}}x_{\mathbf{j}} = \sum_{\mathbf{j}\in\mathscr{J}(d,N)} |[\mathbf{j}]|a_{\mathbf{j}}x_{\mathbf{j}}\,,$$

and hence $\hat{f}(\{j_1,\dots,j_d\}) = d!a_{\mathbf{j}}$ whenever $1\le j_1<\dots<j_d\le N$, and $=0$ else. □

Proof of Theorem 5.7. To see the first statement, we consider the following commutative diagram:

(52)
$$\begin{array}{ccccc}
\mathscr{B}^N_{\mathscr{S}} & & \xrightarrow{\quad id \quad} & & \mathscr{B}^N_{\mathscr{S}} \\
\downarrow U_d & & & & \uparrow \mathbf{R} \\
\mathscr{L}\big(\ell^N_\infty,\mathscr{B}^N_{\mathscr{S}^\flat}\big) & \xrightarrow[I_d]{} & \mathscr{L}\big(\ell^N_\infty,\mathscr{P}_{d-1}(\ell^N_\infty)\big) & \xrightarrow[V_d]{} & \mathscr{P}_d(\ell^N_\infty),
\end{array}$$

where $\mathbf{R}$ is the canonical projection annihilating coefficients with multi indices not generated by sets in $\mathscr{S}$, I_d is the canonical isometric embedding and

$$\big(U_d(f)x\big)(u) := \check{P}_f(u,\dots,u,x) \;\text{ for }\; f\in\mathscr{B}^N_{\mathscr{S}} \;\text{ and }\; x\in\ell^N_\infty, u\in\{-1,+1\}^N\,,$$

$$V_d(T)(y) := (Ty)(y) \;\text{ for }\; T\in\mathscr{L}\big(\ell^N_\infty,\mathscr{B}^N_{\mathscr{S}^\flat}\big) \;\text{ and }\; y\in\ell^N_\infty.$$

We show that U_d, as an operator from $\mathscr{B}_{\mathscr{S}}^N$ into $\mathscr{L}\big(\ell_\infty^N, \mathscr{B}_{\mathscr{S}^\flat}^N\big)$, is well-defined. Indeed, by Lemma 5.9 for $x \in \ell_\infty^N$ and $u \in \{-1,+1\}^N$

$$\begin{aligned}
\check{P}_f(u,\dots,u,x) &= \sum_{\substack{\mathbf{i}\in[N]^d \\ i_k \neq i_\ell \text{ for } k\neq \ell}} \frac{\hat{f}(\{i_1,\dots,i_d\})}{d!} u_{i_1}\dots u_{i_{d-1}} x_{i_d} \\
&= \sum_{\substack{\mathbf{i}\in[N]^{d-1} \\ i_k \neq i_\ell \text{ for } k\neq \ell}} \Big(\sum_{\substack{1\le \ell \le N \\ i_k \neq \ell}} \frac{\hat{f}(\{i_1,\dots,i_d\})}{d!} x_\ell \Big) u_{\mathbf{i}} \\
&= \sum_{1\le j_1<\dots<j_d\le N} (d-1)! \Big(\sum_{\substack{1\le \ell \le N \\ j_k \neq \ell}} \frac{\hat{f}(\{j_1,\dots,j_d\})}{d!} x_\ell \Big) u_{\mathbf{j}} \\
&= \sum_{S\in\mathscr{S}^\flat} (d-1)! \Big(\sum_{\substack{1\le \ell \le N \\ j_k \neq \ell}} \frac{\hat{f}(S)}{d!} x_\ell \Big)\, u^S .
\end{aligned}$$

By the polarization formula from [10, Proposition 4] we have $\|U_d\| \le 2d$, and moreover trivially $\|V_d\| \le 1$. Hence by the ideal property (8) of the Gordon–Lewis constant

$$\boldsymbol{gl}\big(\mathscr{B}_{\mathscr{S}}^N\big) \le 2d \,\big\|\mathbf{R}\colon \mathscr{P}_d(\ell_\infty^N) \to \mathscr{B}_{\mathscr{S}}^N\big\|\, \boldsymbol{gl}\big(\mathscr{L}\big(\ell_\infty^N, \mathscr{B}_{\mathscr{S}^\flat}^N\big)\big) \le 2d\, \|\mathbf{R}\colon \mathscr{P}_d(\ell_\infty^N) \to \mathscr{B}_{\mathscr{S}}^N\|\, \boldsymbol{\lambda}\big(\mathscr{B}_{\mathscr{S}^\flat}^N\big),$$

where for the last estimate we use Lemma 5.8. Finally, since by Lemma 5.3

$$\big\|\mathbf{R}\colon \mathscr{P}_d(\ell_\infty^N) \to \mathscr{B}_{\mathscr{S}}^N\big\| \le \kappa^d 2^{d-1} \big\|\mathbf{Q}\colon \mathscr{B}_{=d}^N \to \mathscr{B}_{\mathscr{S}}^N\big\|,$$

the proof of the first claim is complete.

For the second claim we assume that $\mathscr{S} \subset \big\{S \subset [N] : |S| \le d\big\}$, and consider the following commutative diagram

(53)

$$\begin{array}{ccccc}
\mathscr{B}_{\mathscr{S}}^N & \xrightarrow{\quad \mathrm{id}_{\mathscr{B}_{\mathscr{S}}^N} \quad} & \mathscr{B}_{\mathscr{S}}^N & & \\
\Big\downarrow {\scriptstyle \mathbf{O}\oplus\bigoplus \mathbf{P}_k} & & \Big\uparrow {\scriptstyle \Sigma} & & \\
\mathbb{C}\oplus_\infty \bigoplus_\infty \mathscr{B}_{\mathscr{S}_{=k}}^N & \xrightarrow{\mathrm{id}_{\mathbb{C}}\oplus\bigoplus \mathrm{id}_{\mathscr{B}_{\mathscr{S}_{=k}}^N}} & \mathbb{C}\oplus_1 \bigoplus_1 \mathscr{B}_{\mathscr{S}_{=k}}^N & \xleftarrow{\mathrm{id}_{\mathbb{C}}\oplus\bigoplus \mathbf{R}_k} & \mathbb{C}\oplus_1 \bigoplus_1 \mathscr{P}_k(\ell_\infty^N) \\
\Big\downarrow {\scriptstyle \mathrm{id}_{\mathbb{C}}\oplus\bigoplus U_k} & & & & \Big\uparrow {\scriptstyle \mathrm{id}_{\mathbb{C}}\oplus\bigoplus V_k} \\
\mathbb{C}\oplus_\infty \bigoplus_\infty \mathscr{L}\big(\ell_\infty^N, \mathscr{B}_{(\mathscr{S}_{=k})^\flat}^N\big) & \xrightarrow{\mathrm{id}_{\mathbb{C}}\oplus\bigoplus I_k} & \mathbb{C}\oplus_\infty \bigoplus_\infty \mathscr{L}\big(\ell_\infty^N, \mathscr{P}_{k-1}(\ell_\infty^N)\big) & \xrightarrow{\Phi} & \mathbb{C}\oplus_1 \bigoplus_1 \mathscr{L}\big(\ell_\infty^N, \mathscr{P}_{k-1}(\ell_\infty^N)\big).
\end{array}$$

Let us explain our notation in this diagram: Note first that $\mathbf{P}_k$ and $\mathbf{R}_k$ stand for the canonical projections annihilating coefficients. By Klimek's result already used in (27), for each k, we have

$$\|\mathbf{P}_k \colon \mathscr{B}_{\mathscr{S}}^N \to \mathscr{B}_{\mathscr{S}=k}^N\| \le (1+\sqrt{2})^d,$$

and from Lemma 5.3, we conclude that

$$\left\|\mathbf{R}_k \colon \mathscr{P}_k(\ell_\infty^N) \to \mathscr{B}^N_{\mathscr{S}_{=k}}\right\| \le \kappa^d 2^{d-1} \left\|\mathbf{Q}_k \colon \mathscr{B}^N_{=k} \to \mathscr{B}^N_{\mathscr{S}_{=k}}\right\|.$$

Note that U_k, V_k and I_k for each $1 \le k \le d$ are the operators from (52). If $f = a_0 + \sum_{k=1}^d f_k$ is a decomposition of $f \in \mathscr{B}^N_{\mathscr{S}}$, then $\mathbf{O}(f) = a_0$, and hence $\big(\mathbf{O} \oplus \bigoplus \mathbf{P}_k\big)(f) = \big(a_0, (f_k)_{k=1}^d\big)$. Additionally, $\sum$ is the mapping which assigns to every $\big(a_0, (f_k)_{k=1}^m\big)$ the polynomial $f = a_0 + \sum_{k=1}^d f_k$, and Φ stands for the identity map. The notation for the rest of the maps is self-explaining.

Applying the ideal property (8), we all together arrive at

$$\boldsymbol{gl}\big(\mathscr{B}^N_{\mathscr{S}}\big) \le 2d(d+1)(1+\sqrt{2})^d \max_{1\le k\le d} \left\|\mathbf{R}_k \colon \mathscr{P}_k(\ell_\infty^N) \to \mathscr{B}^N_{\mathscr{S}=k}\right\| \, \boldsymbol{gl}\Big(\mathbb{C} \oplus_\infty \bigoplus_\infty \mathscr{L}\big(\ell_\infty^N, \mathscr{B}^N_{(\mathscr{S}_{=k})^\flat}\big)\Big).$$

It is easy to check that for any Banach spaces X and Y one has $\boldsymbol{gl}\big(X \oplus_\infty Y\big) \le 2\max\{\boldsymbol{gl}(X), \boldsymbol{gl}(Y)\}$. Thus, to complete the proof, it suffices to show that

$$\boldsymbol{gl}\Big(\bigoplus_\infty \mathscr{L}\big(\ell_\infty^N, \mathscr{B}^N_{(\mathscr{S}_{=k})^\flat}\big)\Big) \le \max_{1\le k\le m} \boldsymbol{\lambda}\big(\mathscr{B}^N_{(\mathscr{S}_{=k})^\flat}\big).$$

Indeed, using standard properties of ε- and π-tensor products (see e.g., [6]), we have

$$\begin{aligned}
\bigoplus_\infty \mathscr{L}\big(\ell_\infty^N, \mathscr{B}^N_{(\mathscr{S}_{=k})^\flat}\big) &\hookrightarrow \bigoplus_\infty \mathscr{L}\big(\ell_\infty^N, \bigoplus_\infty \mathscr{B}^N_{(\mathscr{S}_{=k})^\flat}\big) \\
&= \ell_\infty^d \otimes_\varepsilon \big[\ell_1^N \otimes_\varepsilon \bigoplus_\infty \mathscr{B}^N_{(\mathscr{S}_{=k})^\flat}\big] \\
&= \big[\ell_\infty^d \otimes_\varepsilon \ell_1^N\big] \otimes_\varepsilon \bigoplus_\infty \mathscr{B}^N_{(\mathscr{S}_{=k})^\flat} \\
&= \big(\ell_1^d \otimes_\pi \ell_\infty^N\big)^* \otimes_\varepsilon \bigoplus_\infty \mathscr{B}^N_{(\mathscr{S}_{=k})^\flat} = \mathscr{L}\big(\ell_1^d(\ell_\infty^N), \bigoplus_\infty \mathscr{B}^N_{(\mathscr{S}_{=k})^\flat}\big),
\end{aligned}$$

where the first space in fact is 1-complemented in the second one, and all other identifications are isometries. Then we deduce from Lemma 5.8 that

$$\boldsymbol{gl}\Big(\bigoplus_\infty \mathscr{L}\big(\ell_\infty^N, \mathscr{B}^N_{(\mathscr{S}_{=k})^\flat}\big)\Big) \le \boldsymbol{gl}\Big(\mathscr{L}\big(\ell_1^d(\ell_\infty^N), \bigoplus_\infty \mathscr{B}^N_{(\mathscr{S}_{=k})^\flat}\big)\Big) \le \boldsymbol{\lambda}\Big(\bigoplus_\infty \mathscr{B}^N_{(\mathscr{S}_{=k})^\flat}\Big).$$

Since

$$\boldsymbol{\lambda}\Big(\bigoplus_\infty \mathscr{B}^N_{(\mathscr{S}_{=k})^\flat}\Big) = \gamma_\infty\big(\mathrm{id}_{\bigoplus_\infty \mathscr{B}^N_{(\mathscr{S}_{=k})^\flat}}\big) \le \max_{1\le k\le m} \gamma_\infty\big(\mathrm{id}_{\mathscr{B}^N_{(\mathscr{S}_{=k})^\flat}}\big) = \max_{1\le k\le m} \boldsymbol{\lambda}\big(\mathscr{B}^N_{(\mathscr{S}_{=k})^\flat}\big),$$

the proof is complete. □

5.4. **Sidon constants and the Bohnenblust-Hille inequality.** In Theorem 5.1 we prove that the projection constant of $\mathscr{B}^N_{\mathscr{S}}$ and its Sidon constant (see again (2)) are closely related. In the following result we describe the asymptotic behaviour of $\mathrm{Sid}(\mathscr{B}^N_{\mathscr{S}})$ in the case that $\mathscr{S}$ is 'big'.

Proposition 5.10. *Given integers* $1 \le d \le N$*, let* $\mathscr{S} \subset [N]$ *be such that* $(N/d)^{d/2} \le |\mathscr{S}|$ *and* $|S| \le d$ *for all* $S \in \mathscr{S}$*. Then there are constants* $C_1, C_2 \ge 1$ *(independent of* $N, d, \mathscr{S}$*) such that*

$$C_1 \frac{1}{\sqrt{N}} |\mathscr{S}|^{\frac{1}{2}} \le \mathbf{Sid}(\mathscr{B}^N_{\mathscr{S}}) \le C_2^{\sqrt{d \log d}} \frac{1}{\sqrt{N}} |\mathscr{S}|^{\frac{1}{2}}.$$

For the proof of the upper bound we need the so-called subexponential Bohnenblust-Hille inequality for functions on Boolean cubes from [11, Theorem 1]: There is a constant $C \ge 1$ such that for each $1 \le d \le N$ and every $f \in \mathscr{B}^N_{\le d}$ one has

$$\Big(\sum_{\substack{S \subset [N] \\ |S| \le d}} |\widehat{f}(S)|^{\frac{2d}{d+1}} \Big)^{\frac{d+1}{2d}} \le C^{\sqrt{d \log d}} \|f\|_\infty. \tag{54}$$

Proof of Proposition 5.10. The upper bound follows from Hölder's inequality and (54). That is, for all functions $f \in \mathscr{B}^N_{\mathscr{S}}$,

$$\sum_{|S| \le d} |\hat{f}(S)| \le \Big(\sum_{|S| \le d} |\hat{f}(S)|^{\frac{2d}{d+1}} \Big)^{\frac{2d}{d+1}} |\mathscr{S}|^{\frac{d-1}{2d}} \le C^{\sqrt{d \log d}} \frac{1}{|\mathscr{S}|^{\frac{1}{d}}} |\mathscr{S}|^{\frac{1}{2}} \|f\|_\infty.$$

But by assumption $\sqrt{N} \le \sqrt{d} |\mathscr{S}|^{\frac{1}{d}}$, and hence the claim follows from the definition of Sidon constants given in (2). The proof of the lower estimate is probabilistic. Indeed, by the Kahane-Salem-Zygmund inequality for the Boolean cube (see, e.g., [11, Lemma 3.1]) there is a family $(\varepsilon_S)_{S \in \mathscr{S}}$ of signs such that for $f = \sum_{S \in \mathscr{S}} \varepsilon_S \chi_S$ we have

$$\|f\|_\infty \le 6 \sqrt{\log 2}\, \sqrt{N} \Big(\sum_{S \in \mathscr{S}} |\varepsilon_S|^2 \Big)^{\frac{1}{2}},$$

and hence

$$|\mathscr{S}| = \sum_{S \in \mathscr{S}} |\widehat{f}(S)| \le \mathbf{Sid}(\mathscr{B}^N_{\mathscr{S}})\, 6 \sqrt{\log 2}\, \sqrt{N} |\mathscr{S}|^{\frac{1}{2}}.$$

This completes the argument. □

Corollary 5.11. *There are constants* $C_1, C_2 > 0$ *such that for each integer* $1 \le d \le N$ *one has*

$$C_1 \frac{1}{\sqrt{N}} \binom{N}{d}^{\frac{1}{2}} \le \mathbf{Sid}\big(\mathscr{B}^N_{=d}\big) \le C_2^{\sqrt{d \log d}} \frac{1}{\sqrt{N}} \binom{N}{d}^{\frac{1}{2}}$$

and

$$C_1 \frac{1}{\sqrt{N}} \left(\sum_{k=0}^{d} \binom{N}{k} \right)^{\frac{1}{2}} \le \mathbf{Sid}\big(\mathscr{B}^N_{\le d}\big) \le C_2^{\sqrt{d \log d}} \frac{1}{\sqrt{N}} \left(\sum_{k=0}^{d} \binom{N}{k} \right)^{\frac{1}{2}}.$$

Proof. Since $\left(\frac{N}{d}\right)^d \le \binom{N}{d} \le \sum_{k=0}^d \binom{N}{k}$ (see again (24)), both sets $\mathscr{S} = \{S \colon |S| = d\}$ and $\mathscr{S} = \{S \colon |S| \le d\}$ satisfy the assumptions of Proposition 5.10. □

Corollary 5.12. *There are constants* $C_1, C_2 > 0$ *such that for all* $1 \le d \le N$,

$$C_1 \frac{1}{\sqrt{d}} \left(\frac{N}{d}\right)^{\frac{d-1}{2}} \le \mathbf{Sid}\big(\mathscr{B}^N_{=d}\big) \le \mathbf{Sid}\big(\mathscr{B}^N_{\le d}\big) \le C_2^{\sqrt{d \log d}} e^{\frac{d}{2}} \frac{1}{\sqrt{d}} \left(\frac{N}{d}\right)^{\frac{d-1}{2}}.$$

In particular, we have the following hypercontractive comparison:

$$\mathbf{Sid}\big(\mathscr{B}^N_{=d}\big) \sim_{C^d} \left(\frac{N}{d}\right)^{\frac{d-1}{2}} \quad \textit{and} \quad \mathbf{Sid}\big(\mathscr{B}^N_{\le d}\big) \sim_{C^d} \left(\frac{N}{d}\right)^{\frac{d-1}{2}}.$$

Proof. Both first and the third estimate follow from the preceding corollary. For the lower one use again (24), and for the upper note that it suffices to check that

$$\frac{1}{\sqrt{N}} \Big(\sum_{k=0}^{d} \binom{N}{k} \Big)^{\frac{1}{2}} \le e^{\frac{d}{2}} \frac{1}{\sqrt{d}} \left(\frac{N}{d}\right)^{\frac{d-1}{2}};$$

indeed, this is another consequence of (25). □

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