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Spectral flow and the conformal block expansion for strings in AdS_3

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Abstract: We present a detailed study of spectrally flowed four-point functions in the SL(2,$\mathbb{R}$) WZW model, focusing on their conformal block decomposition. Dei and Eberhardt conjectured a general formula relating these observables to their unflowed counterparts. Although the latter are not known in closed form, their conformal block expansion has been formally established. By combining this information with the integral transform that encodes the effect of spectral flow, we show how to describe a considerable number of s-channel exchanges, including cases with both flowed and unflowed intermediate states. For all such processes, we compute the normalization of the corresponding conformal blocks in terms of products of the recently derived flowed three-point functions with arbitrary spectral flow charges. Our results constitute a highly non-trivial consistency check, thus strongly supporting the aforementioned conjecture, and establishing its computational power.

Keywords: AdS-CFT Correspondence, Conformal Field Models in String Theory, Long Strings

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1 Introduction

String propagation in AdS_3 is one of the most studied realizations of the AdS/CFT correspondence, partly because it allows for a level of computational control that is perhaps unprecedented beyond the purely topological instances of the holographic duality in more than two bulk dimensions. Indeed, this applies to the boundary theory, which is a two-dimensional CFT, believed to be closely related to a symmetric orbifold model, but also to the worldsheet theory, which, in the pure NS-NS case, is given by a potentially solvable Wess-Zumino-Witten (WZW) model, thus giving access to the description of string dynamics well beyond the supergravity limit.

The bosonic sector of the worldsheet CFT is described by the WZW model based on the universal cover of SL(2,$\mathbb{R}$). Its distinctive feature is that, in addition to the *standard*

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representations, which are constructed as in any rational WZW model, one must also incorporate their images under the so-called spectral flow automorphisms of the affine symmetry algebra. For instance, these are necessary to describe the classical configurations known as long strings in AdS_3. Spectral flow thus plays a central role in generating a consistent physical spectrum in the SL(2,$\mathbb{R}$) WZW model [1]. In the supersymmetric case, it also allows for the construction of the tower of short string states forming the spacetime chiral ring [2–5].

From the computational point of view, including vertex operators with non-zero spectral flow charges entails considerable technical difficulties for the calculation of the corresponding correlation functions [6]. Correlators involving only unflowed states can be obtained simply by analytical extension from those of the Euclidean counterpart, that is, the H_3^+ coset model [7, 8]. This includes two- and three-point functions, which were computed exactly in analogy to the Liouville case [9, 10], the OPE structure, and also four-point functions. The latter are not known in closed form, but they can be determined recursively from the leading order solution to the Knizhnik-Zamolodchikov (KZ) equation. This establishes their factorization properties at the primary level, by which we mean their description as a sum over the exchange of all intermediate states allowed by the fusion rules of the SL(2,$\mathbb{R}$) model. On the other hand, correlation functions with flowed insertions require much more intricate techniques. These originally involved an increase in the effective number of insertions due to the inclusion of the so-called spectral flow operators [6, 11], rendering the analysis substantially more challenging.

At the level of the flowed four-point functions, which will be our main focus, this approach only led to concrete results for a handful of particular cases where all spectral flow charges are quite low. Let us stress that we are mostly interested in x-basis correlators, namely those written in the appropriate language for holographic considerations. This is because the complex variable x parametrizes the asymptotic boundary of AdS_3. The only four-point functions which were analyzed in detail by using the techniques of [6, 11] were the singly-flowed case [12, 13], and, under some assumptions, the spectral flow conserving correlators [14]. Some additional m-basis results were provided in [15–18]. Roughly speaking, the latter applies only to correlation functions for which all flowed insertions are located either at $x = 0$ or at $x = \infty$. Determining the OPE structure for vertex operators belonging to the spectrally flowed sectors of the theory and the factorization properties of the corresponding four-point functions remains an important open problem. Solving it will likely provide crucial insights to understand the SL(2,$\mathbb{R}$) WZW model and, perhaps, even prove the $\mathrm{AdS}_3/\mathrm{CFT}_2$ holographic duality beyond the tensionless limit.[1]

Recently, an alternative technique based on the *local* Ward identities was introduced in [20, 31]. This allowed the authors to propose a closed formula for the three-point functions with arbitrary spectral flow charges. This was then proven in [32, 33] by making use of the ramified holomorphic covering maps from the string worldsheet to the AdS_3 boundary. These allow one to bypass some of the main complications stemming from the OPEs between flowed vertex operators and the currents of the model. Covering maps are crucial in the tensionless limit, and remain quite useful beyond it. Moreover, a general integral expression relating four-point correlation functions in non-trivial spectral flow sectors to their unflowed cousins

[1]The tensionless case was established in [19–23]. A concrete proposal for the holographic CFT beyond the tensionless limit was put forward in [24–26] and further tested in [27–30] using free-field methods.

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was conjectured in [34] by means of a comprehensive case-by-case analysis. This formula has not been proven in general, although it has been explicitly verified in several cases. It is thus natural to wonder if, and how, it can be used as a computational technique, for instance in order to study how these four-points functions factorize.

Let us stress that this question is a highly non-trivial one. On the one hand, the integral transform involved in defining the x-basis correlators from the perspective of [31, 34] is quite challenging. It cannot be carried out in full generality even in the considerably simpler case of three-point functions. On the other hand, in the four-point case, the usefulness of the conjecture of [34] for the task at hand relies on our knowledge of the corresponding *unflowed* four-point functions. As discussed above, the only explicit expression for this kind of correlator is the leading order in the conformal block expansion in powers of the worldsheet cross-ratio obtained by Teschner in [8], together with the recursion relations that, in principle, define the following orders. Given that the expression conjectured in [34] mixes the spacetime and worldsheet cross-ratios with multiple integration variables in a complicated way, whether or not it is possible to recover a factorized expansion for the spectrally flowed four-point function at leading order, together with a consistent set of recursion relations for computing the subleading contributions to the conformal blocks, is yet to be understood.

In this paper, we address precisely these questions. Although the inherent complexity of four-point functions in the spectrally flowed sectors of the SL(2,$\mathbb{R}$) WZW model prevents us from carrying out the analysis in full generality, we work with arbitrary spectral flow charges for the external states and isolate a considerable number of non-trivial channels. We explore channels involving both flowed and unflowed intermediate states, and prove that, in all cases, one obtains a consistent contribution to the factorized expansion of the full correlator. More precisely, we describe how the insertion of the factorized expression for the unflowed correlator in the integrand of the conjectured transform accounting for the effect of spectral flow leads to (1) the expected overall dependence on both the worldsheet and the boundary cross-ratios, (2) the leading order expression for the flowed conformal blocks, and (3) their appropriate normalization in terms of flowed structure constants. This involves the full complicated form of the latter observables, as derived in [31, 33]. Our results constitute an involved consistency check based on the fundamental properties of the worldsheet CFT, which provides strong support for the conjecture of [34], and further establishes its computational power.

Regarding the derivation of the subleading contributions to the conformal blocks, we also expect the method based on the local Ward identities to allow for the determination of the corresponding recursions relations. We leave this calculation and that of the remaining factorization channels for future work.

The paper is organized as follows. In section 2 we review the SL(2,$\mathbb{R}$) WZW model, focusing on the role played by spectral flow in the construction of its spectrum, and its incidence while computing two- and three-point functions. We also review the known facts about the unflowed four-point functions [6, 8] and introduce the conjecture of [34] relating them to their spectrally flowed counterparts.

We then initiate our discussion of the factorization properties of flowed four-point functions in section 3 by analyzing in detail a number of sample cases, involving only vertex operators with either zero or small spectral flow charges. The advantage of studying these

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specific situations is two-fold. First, it allows us to reproduce the results that were previously derived in [12–14, 18] employing different methods, and compare their analysis with the one based on the local Ward identities. As we will show in the following sections, the latter is best suited for an extension to the general case. Moreover, the sample cases we consider here provide a simplified context for the detailed description of certain technical aspects of our approach, which will also play an important role in the rest of the paper. At the end of this section we further present a roadmap that will guide us through our study of the more general spectrally flowed four-point functions.

In section 4 we move past specific cases to study the factorization of four-point correlators with generic spectral flow assignments, first focusing on those admitting a well-defined m-basis limit. These are the so-called edge cases, in the language of [31, 33, 34], because the spectral flow charges nearly saturate the range allowed by the selection rules of the model, derived originally in [6]. For the maximally spectral flow violating case, we also extend our analysis to the x-basis correlator, for which the integral transform associated with spectral flow can be carried out explicitly.

Section 5 contains the main results of this paper. We discuss x-basis four-point functions with arbitrary spectral flow charges. We first show that the formula conjectured in [34] correctly accounts for the exchange of unflowed states along the s-channel, whenever they are allowed by the SL(2,$\mathbb{R}$) fusion rules. In other words, we show that using our knowledge of the unflowed conformal blocks, we can derive those associated with the exchange of such intermediate states between the different spectrally flowed vertex operators. We also obtain the corresponding normalizations, given by products of the relevant three-point functions, thus rendering such contributions consistent with the usual factorization ansatz. We then consider the exchange of flowed states and show that, for all channels isolated in the limit of small (worldsheet) cross-ratio, combining the formula of [34] with the results of [8] for unflowed correlators leads to contributions to the factorized four-point function that are again consistent with the conformal symmetry constraints. Finally, in section 6 we summarize our results, and discuss their impact for future investigations. Several useful identities are relegated to appendix A.

2 Review of the SL(2,$\mathbb{R}$) WZW model

2.1 Currents, spectrum and vertex operators

Let us start by introducing the basic ingredients of the SL(2,$\mathbb{R}$) WZW model at level $k > 3$ [1, 6, 35, 36]. The symmetry algebra is characterized by the affine currents $J^a(z)$ with $a = 3, \pm$, satisfying the following operator product expansions (OPEs),

$$J^a(z)J^b(w) \sim \frac{k\eta^{ab}/2}{(z-w)^2} + \frac{if^{ab}{}_cJ^c(w)}{z-w}, \tag{2.1}$$

where $\eta^{+-} = -2\eta^{33} = 2$, $f^{+-}{}_3 = -2$ and $f^{3+}{}_+ = -f^{3-}{}_- = 1$.

Physical states come in three families. The long string states belong to the (flowed) continuous representations $C_j^{\alpha,\omega}$ and have

$$j \in \frac{1}{2} + i\mathbb{R}, \quad \alpha \in [0,1), \quad \omega \in \mathbb{Z}. \tag{2.2}$$

Here j is the unflowed $\mathrm{SL}(2,\mathbb{R})$ spin and ω is the spectral flow charge. The short string states belong to the (flowed) discrete representations $\mathcal{D}_j^{\pm,\omega}$, which are of the highest/lowest-weight type and have

$$\frac{1}{2} < j < \frac{k-1}{2}\,, \quad \omega \in \mathbb{Z}\,. \tag{2.3}$$

Supergravity states belong to the unflowed discrete representations $\mathcal{D}_j^{\pm} \equiv \mathcal{D}_j^{\pm,0}$. While such states can be understood by analytic continuation from the H_3^+ model studied in [7, 8], the presence of spectrally flowed states is crucial for the consistency of the worldsheet spectrum.

In the so-called x-basis, the vertex operators are denoted as $V_{jh}^{\omega}(x,z)$. Here, x and z are the boundary and worldsheet complex coordinates, respectively, while h is the spacetime conformal weight. Incidentally, here we can restrict to $\omega > 0$ since states with negative ω are encoded in the same vertex operators [6]. Then, h is defined by the J_0^3 eigenvalue m before spectral flow, i.e. $h = m + \frac{k}{2}\omega$. For $\omega = 0$ we have $h = j$, hence we simply write $V_j(x,z)$. Here and in what follows we mostly omit the antiholomorphic variables, namely $\bar{x}, \bar{z}$ and $\bar{h}$.

For strings propagating in an $\mathrm{AdS}_3 \times \mathcal{M}_{\mathrm{int}}$ geometry supported by pure NS-NS fluxes, the zero modes J_0^a are identified with the global modes of the spacetime Virasoro algebra under the holographic duality, while the (AdS_3 part of the) worldsheet Virasoro algebra is obtained through the Sugawara construction, the corresponding central charge being $c = \frac{3k}{k-2}$. In contrast to the unflowed states, vertex operators $V_{jh}^{\omega}(x,z)$ with $\omega > 0$ are not affine primaries, but they are Virasoro primaries of weight

$$\Delta_j = -\frac{j(j-1)}{k-2} - h\omega + \frac{k}{4}\omega^2\,. \tag{2.4}$$

It should be noted that not all representations are actually independent. Indeed, in the continuous sector one has a reflection symmetry under $j \to 1-j$. For $\omega = 0$ this takes the form

$$V_j(x,z) = B(j) \int d^2x\, |x-x'|^{-4j} V_{1-j}(x',z)\,, \tag{2.5}$$

where the integral is performed over the full complex plane, and

$$B(j) = \frac{2j-1}{\pi} \frac{\Gamma[1-b^2(2j-1)]}{\Gamma[1+b^2(2j-1)]}\, \nu^{1-2j}\,, \quad b^2 = (k-2)^{-1} \tag{2.6}$$

with ν a free parameter of the theory which will be unimportant for us, while for $\omega > 0$ we have

$$V_{jh}^{\omega}(x,z) = R(j,h,\omega) V_{1-j,h}^{\omega}(x,z)\,, \tag{2.7}$$

with

$$R(j,h,\omega) = \frac{\pi\gamma\left(h - \frac{k}{2}\omega + j\right) B(j)}{\gamma(2j)\gamma\left(h - \frac{k}{2}\omega + 1 - j\right)}\,, \qquad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-\bar{x})}\,. \tag{2.8}$$

Moreover, in the discrete sector one has the so-called series identifications

$$V_{j,h=j+\frac{k}{2}\omega}^{\omega}(x,z) = \mathcal{N}(j) V_{\frac{k}{2}-j,h}^{\omega+1}(x,z)\,, \qquad \mathcal{N}(j) = \sqrt{\frac{B(j)}{B\left(\frac{k}{2}-j\right)}}\,, \tag{2.9}$$

relating lowest-weight states of spin j and spectral flow charge ω and highest-weight states of spin $\frac{k}{2}-j$ and spectral flow charge $\omega+1$.

A set of linear combinations of flowed vertex operators which are particularly useful for the study of correlation functions is given by the so-called y-basis operators, related to the ones introduced above by the Mellin-type transform[2] [31, 32]

$$V^{\omega}_{jh}(x,z) = \int d^2y\, |y^{j+\frac{k}{2}\omega-h-1}|^2 V^{\omega}_{j}(x,y,z)\,. \tag{2.10}$$

In the y-basis, the defining OPEs are as follows. For the $J^+(z)$ current we have

$$J^+(w)V^{\omega}_{j}(x,y,z) = \frac{\partial_y V^{\omega}_{j}(x,y,z)}{(w-z)^{\omega+1}} + \sum_{n=2}^{\omega} \frac{\left(J^{+}_{n-1}V^{\omega}_{j}\right)(x,y,z)}{(w-z)^n} + \frac{\partial_x V^{\omega}_{j}(x,y,z)}{(w-z)} + \cdots\,, \tag{2.11}$$

which includes a series of intermediate poles whose residues have no simple expression in terms of differential operators acting on the primaries (except for the $\omega=0$ OPEs, obtained by taking $y\to x$). Their presence complicates the computation of correlation functions considerably. On the other hand, for

$$J^3(x,z) = J^3(z) - xJ^+(z)\,, \qquad J^-(x,z) = J^-(z) - 2xJ^3(z) + x^2J^+(z)\,, \tag{2.12}$$

we have

$$J^3(x,w)V^{\omega}_{j}(x,y,z) = \frac{\left(y\partial_y + j + \frac{k}{2}\omega\right)V^{\omega}_{j}(x,y,z)}{(w-z)} + \cdots\,, \tag{2.13}$$

and

$$J^-(x,w)V^{\omega}_{j}(x,y,z) = (w-z)^{\omega-1}\left(y^2\partial_y + 2jy\right)V^{\omega}_{j}(x,y,z) + \cdots\,. \tag{2.14}$$

The vanishing of the first $\omega-1$ terms in the expansion of the product $J^-(x,w)V^{\omega}_{j}(x,y,z)$ around $z=w$ imposes strong constraints on the correlation functions, known as *local* Ward identities [20]. For y-basis operators the reflection symmetry reads

$$V^{\omega}_{j}(x,y,z) = B(j)\int d^2y'\, |y-y'|^{-4j} V^{\omega}_{1-j}(x,y',z)\,. \tag{2.15}$$

2.2 Two- and three-point functions

The study of exact correlation functions in the bosonic $\mathrm{SL}(2,\mathbb{R})$ model was initiated in [6], where the authors first derived the unflowed two- and three-point functions by analytic continuation from the H_3^+ results obtained in [8]. For the flowed cases, important advances were achieved in recent years by studying the local Ward identities and making use of the y-basis representation [5, 20, 31–34].

As the auxiliary complex variable y is somewhat unconventional from the CFT point of view, it will be useful to review the consequence of the global Ward identities for y-basis correlators [31]. For two-point functions, one has

$$\langle V^{\omega_1}_{j_1}(x_1,y_1,z_1)V^{\omega_2}_{j_2}(x_2,y_2,z_2)\rangle = \left|\frac{x_{12}^{-h_1^0-h_2^0}}{z_{12}^{\Delta_1^0+\Delta_2^0}}\right|^2 \langle V^{\omega_1}_{j_1}\left(0, y_1\frac{z_{12}^{\omega_1}}{x_{12}}, 0\right) V^{\omega_2}_{j_2}\left(\infty, y_2\frac{z_{12}^{\omega_2}}{x_{12}}, \infty\right)\rangle\,, \tag{2.16}$$

[2]Strictly speaking $\bar{h}$ is not necessarily the complex conjugate of h. In a slight abuse of notation, we will keep writing absolute values squared throughout the paper.

with

$$h_i^0 = j_i + \frac{k}{2}\omega_i\,, \qquad \Delta_i^0 = -\frac{j_i(j_i-1)}{k-2} - j_i w_i - \frac{k}{2}\omega_i^2\,, \tag{2.17}$$

and

$$V_j^\omega(\infty, y, \infty) = \lim_{x,z\to\infty} |x|^{4h^0}|z|^{4\Delta_j^0} V_j^\omega\left(x, y\frac{x^2}{z^{2\omega}}, z\right)\,. \tag{2.18}$$

In this language, the two-point functions read [31]

$$\begin{aligned}\langle V_{j_1}^{\omega_1}(0,y_1,0)V_{j_2}^{\omega_2}(\infty,y_1,\infty)\rangle =& \delta_{\omega_1,\omega_2}B(j_1)\delta(j_1-j_2)|1-y_1y_2|^{-4j_1} \\ &+ \delta_{\omega_1,\omega_2}\delta(j_1+j_2-1)|y_1|^{-2j_1}|y_2|^{-2j_2}\delta^{(2)}(1-y_1y_2)\,.\end{aligned} \tag{2.19}$$

By carrying out the integral transform in eq. (2.10) one recovers the original x-basis results of [6], namely

$$\langle V_{j_1h_1}^{\omega_1}(0,0)V_{j_2h_2}^{\omega_2}(\infty,\infty)\rangle = \delta_{\omega_1,\omega_2}\left(\delta(j_1-j_2)R(j_1,h_1,\omega_1)+\delta(j_1+j_2-1)\right)\,. \tag{2.20}$$

Three-point functions are more involved. The spectral flow charge is not a conserved quantity, although, as shown in [6], all non-zero n-point functions must satisfy

$$2\max(\omega_i) \leq \sum_{i=1}^n \omega_i + 1\,. \tag{2.21}$$

An exact expression for the y-basis three-point functions was first proposed in [31] and later proven in [32, 33]. The result depends on the parity of the total spectral flow charge $\omega = \omega_1+\omega_2+\omega_3$. Denoting

$$\ddot{\mathcal{F}}(y_i) \equiv \left\langle V_{j_1}^{\omega_1}(0,y_1,0)\, V_{j_2}^{\omega_2}(1,y_2,1)\, V_{j_3}^{\omega_3}(\infty,y_3,\infty)\right\rangle\,, \tag{2.22}$$

one gets

$$\ddot{\mathcal{F}}(y_i) = \begin{cases} C_-(j_1,j_2,j_3)\left|Z_{123}^{\frac{k}{2}-J}\prod_{i=1}^3 Z_i^{J-\frac{k}{2}-2j_i}\right|^2, & \text{for } \omega \text{ odd} \\ C_+(j_1,j_2,j_3)\left|Z_\emptyset^{J-k}\prod_{i<\ell}^3 Z_{i\ell}^{J-2j_i-2j_\ell}\right|^2, & \text{for } \omega \text{ even}\end{cases} \tag{2.23}$$

where $J = j_1+j_2+j_3$. Here, for any subset $I\subset\{1,2,3\}$ we have introduced[3]

$$Z_I(y_1,y_2,y_3) \equiv \sum_{i\in I:\ \varepsilon_i=\pm1} Q_{\boldsymbol{\omega}+\sum_{i\in I}\varepsilon_i e_i}\prod_{i\in I} y_i^{\frac{1-\varepsilon_i}{2}}\,, \tag{2.24}$$

with $\boldsymbol{\omega} = (\omega_1,\omega_2,\omega_3)$, $e_1=(1,0,0)$, $e_2=(0,1,0)$, and $e_3=(0,0,1)$. The Z_I play the role of *generalized differences*, and the numbers $Q_{\boldsymbol{\omega}}$ are defined as

$$Q_{\boldsymbol{\omega}} = 0 \qquad \text{if} \qquad \sum_j \omega_j < 2\max_{i=1,2,3}\omega_i \quad \text{or} \quad \sum_i \omega_i \in 2\mathbb{Z}+1\,, \tag{2.25}$$

while otherwise

$$Q_{\boldsymbol{\omega}} = S_{\boldsymbol{\omega}}\frac{G\left(\frac{-\omega_1+\omega_2+\omega_3}{2}+1\right)G\left(\frac{\omega_1-\omega_2+\omega_3}{2}+1\right)G\left(\frac{\omega_1+\omega_2-\omega_3}{2}+1\right)G\left(\frac{\omega_1+\omega_2+\omega_3}{2}+1\right)}{G(\omega_1+1)G(\omega_2+1)G(\omega_3+1)}\,, \tag{2.26}$$

[3] We reserve the original notation $(Z_I, Q_{\boldsymbol{\omega}}) \to (X_I, P_{\boldsymbol{\omega}})$ for the four-point functions, discussed below.

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with $G(n) = \prod_{i=1}^{n-1} \Gamma(i)$ the Barnes G function and $S_{\boldsymbol{\omega}}$ a simple phase, see [31]. Finally, the overall normalizations are set by the unflowed structure constants $C(j_1, j_2, j_3)$ as

$$C_+(j_1, j_2, j_3) = C(j_1, j_2, j_3) \,, \qquad C_-(j_1, j_2, j_3) = \mathcal{N}(j_3) C\left(j_1, j_2, \frac{k}{2} - j_3\right) . \tag{2.27}$$

Note that, although it is not obvious in this notation, $C_-(j_1, j_2, j_3)$ is invariant under the exchanges $1 \leftrightarrow 3$ and $2 \leftrightarrow 3$, as it should. This follows from the explicit form of $C(j_1, j_2, j_3)$, which was analyed in detail in [6–8]. For more general insertion points one again makes use of the global Ward identities, which in this case imply

$$\begin{aligned}\langle V_{j_1}^{\omega_1}(x_1,y_1,z_1) V_{j_2}^{\omega_2}(x_2,y_2,z_2) V_{j_3}^{\omega_3}(x_3,y_3,z_3)\rangle &= \left| \frac{x_{21}^{h_3^0-h_1^0-h_2^0} x_{31}^{h_2^0-h_1^0-h_3^0} x_{32}^{h_1^0-h_2^0-h_3^0}}{z_{21}^{\Delta_1^0+\Delta_2^0-\Delta_3^0} z_{31}^{\Delta_1^0+\Delta_3^0-\Delta_2^0} z_{32}^{\Delta_2^0+\Delta_3^0-\Delta_1^0}} \right|^2 \\ \times \left\langle V_{j_1}^{\omega_1}\left(0, y_1 \frac{x_{32} z_{21}^{\omega_1} z_{31}^{\omega_1}}{x_{21} x_{31} z_{32}^{\omega_1}}, 0\right) V_{j_2}^{\omega_2}\left(1, y_2 \frac{x_{31} z_{21}^{\omega_2} z_{32}^{\omega_2}}{x_{21} x_{32} z_{31}^{\omega_2}}, 1\right) \right. & \left. V_{j_3}^{\omega_3}\left(\infty, y_3 \frac{x_{21} z_{31}^{\omega_3} z_{32}^{\omega_3}}{x_{31} x_{32} z_{21}^{\omega_3}}, \infty\right)\right\rangle .\end{aligned} \tag{2.28}$$

It then follows from eq. (2.10) that the x-basis three-point function is given by

$$\begin{aligned}&\langle V_{j_1 h_1}^{\omega_1}(x_1, z_1) V_{j_2 h_2}^{\omega_2}(x_2, z_2) V_{j_3 h_3}^{\omega_3}(x_3, z_3)\rangle \\ &\qquad = C_{\boldsymbol{\omega}}(j_i, h_i) \left| \frac{x_{21}^{h_3-h_1-h_2} x_{31}^{h_2-h_1-h_3} x_{32}^{h_1-h_2-h_3}}{z_{21}^{\Delta_1+\Delta_2-\Delta_3^{\omega_3}} z_{31}^{\Delta_1^{\omega_1}+\Delta_3^{\omega_3}-\Delta_2^{\omega_2}} z_{32}^{\Delta_2^{\omega_2}+\Delta_3^{\omega_3}-\Delta_1^{\omega_1}}} \right|^2 ,\end{aligned} \tag{2.29}$$

with flowed structure constants

$$C_{\boldsymbol{\omega}}(j_i, h_i) \equiv \int \prod_{i=1}^{3} d^2 y_i \, |y_i^{j_i - h_i + \frac{k}{2}\omega_i - 1}|^2 \dddot{\mathcal{F}}_{\boldsymbol{\omega}}(y_i) \,. \tag{2.30}$$

Finally, let us recall that something special happens in the so-called collision limit $x_2 \to x_1$ [31, 33]. This corresponds to the limit [31]

$$\begin{aligned}&\langle V_{j_1}^{\omega_1}(0, y_1, 0) V_{j_2}^{\omega_2}(x, y_2, 1) V_{j_3}^{\omega_3}(\infty, y_3, \infty)\rangle \\ &= \lim_{x \to 0} |x^{j_3 - j_1 - j_2 + \frac{k}{2}(\omega_3 - \omega_1 - \omega_2)}|^2 \, \langle V_{j_1}^{\omega_1}\left(0, \frac{y_1}{x}, 0\right) V_{j_2}^{\omega_2}\left(1, \frac{y_2}{x}, 1\right) V_{j_3}^{\omega_3}\left(\infty, y_3 x, \infty\right)\rangle \, .\end{aligned} \tag{2.31}$$

Assuming for simplicity that $\omega_3 \geq \omega_{1,2}$, it turns out that only three-point functions satisfying $|\omega_3 - \omega_1 - \omega_2| \leq 1$ remain non-vanishing. These y-basis correlators simplify considerably, giving a y_i-dependence of the form

$$w_1 + w_2 - w_3 = 1 : \quad y_1^{-j_1+j_2+j_3-\frac{k}{2}} y_2^{j_1-j_2+j_3-\frac{k}{2}} (y_1 + y_2 + y_1 y_2 y_3)^{\frac{k}{2}-j_1-j_2-j_3} \,, \tag{2.32a}$$

$$w_1 + w_2 - w_3 = 0 : \quad (y_1 - y_2)^{-j_1-j_2+j_3} (1 + y_1 y_3)^{-j_1+j_2-j_3} (1 + y_2 y_3)^{j_1-j_2-j_3} \,, \tag{2.32b}$$

$$w_1 + w_2 - w_3 = -1 : \quad y_3^{j_1+j_2-j_3-\frac{k}{2}} (1 + y_1 y_3 + y_2 y_3)^{\frac{k}{2}-j_1-j_2-j_3} \,. \tag{2.32c}$$

Upon integrating over the y_i as usual, one finds that these are the only three-point functions for which the charge conservation condition $h_3 = h_1 + h_2$ holds.

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The details of the derivation leading to eqs. (2.23) are beyond the scope of this brief review. However, let us provide some intuition for interpreting these formulas. As mentioned above, the local Ward identities derived from including additional current insertions in the flowed correlators and using the OPE (2.14) give a series of linear constraints involving not only primary correlators and their x- and y-derivatives, but also descendant correlators. In principle, the total number of constraints allows to solve for these unknowns and, when the dust settles, one is left with three differential equations that must be satisfied by the Moebius-fixed y-basis primary correlators. In practice, however, this procedure quickly becomes rather cumbersome as the spectral flow charges are increased.

Luckily, at least for three-point functions there is a workaround to this problem. The motivation comes from the relation with the holographic CFT. It is known that, at $k = 3$, the AdS_3 becomes string-size and the strings become tensionless [37–39]. This leads to drastic simplifications of the worldsheet WZW model, which can be shown to be exactly dual to a symmetric orbifold CFT living on the boundary [19–21]. In this setup the holomorphic covering maps routinely employed in the computation of two-dimensional symmetric orbifold models [40–43] play a key role in computing the worldsheet correlators. It was shown in [20, 22, 23] that the latter localize on configurations where the worldsheet itself *is* the appropriate covering surface of the AdS_3 boundary. Importantly, the spectral flow charge ω of a given vertex operator is identified with the twist of its boundary avatar.

Although the story is more complicated for $k > 3$, where the dual theory is expected to be a suitable twist-two deformation of a somewhat different symmetric orbifold, [24–26] the covering maps remain very useful tools. For instance, the authors of [27, 29] were recently able to establish the holographic matching for a series of residues of n-point functions in the perturbative regime associated with poles in the space of complex (unflowed) spins by using the free-field description of the SL(2,$\mathbb{R}$) WZW model, valid at large radial distances. For the exact theory, which is the focus of the present work, the localisation property no longer holds. Nevertheless, the full solution for the case of $n = 3$ presented above can be derived by combining the properties of certain covering maps with the symmetries of the model, including in particular the series identifications (2.9) [31–33]. The relevant covering map data is encoded in the ratios of the Q_ω numbers defined in eq. (2.26).

2.3 Four-point functions and the KZ equation in the unflowed sector

The main focus of this paper is on four-point functions and their factorization properties. Let us first review some basic facts about the unflowed case, namely

$$\langle V_{j_1}(x_1, z_1) V_{j_2}(x_2, z_2) V_{j_3}(x_3, z_3) V_{j_4}(x_4, z_4)\rangle. \tag{2.33}$$

As usual, by using both worldsheet and spacetime conformal invariance one can fix the insertion points at $(z_1, z_2, z_3, z_4) = (0, 1, \infty, z)$ and $(x_1, x_2, x_3, x_4) = (0, 1, \infty, x)$ where

$$z = \frac{z_{32} z_{14}}{z_{12} z_{34}}\,, \qquad x = \frac{x_{32} x_{14}}{x_{12} x_{34}}\,, \tag{2.34}$$

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with $z_{ij} = z_i - z_j$ and $x_{ij} = x_i - x_j$. In other words, the four-point function (2.33) can be expressed as

$$
\begin{aligned}
&\langle V_{j_1}(x_1,z_1)V_{j_2}(x_2,z_2)V_{j_3}(x_3,z_3)V_{j_4}(x_4,z_4)\rangle = \\
&\qquad \left| \frac{x_{12}^{-j_1-j_2+j_3-j_4} x_{13}^{-j_1+j_2-j_3+j_4} x_{23}^{j_1-j_2-j_3+j_4} x_{34}^{-2j_4}}{z_{12}^{\Delta_1+\Delta_2-\Delta_3+\Delta_4} z_{13}^{\Delta_1-\Delta_2+\Delta_3-\Delta_4} z_{23}^{-\Delta_1+\Delta_2+\Delta_3-\Delta_4} z_{34}^{2\Delta_4}} \right|^2 \ddddot{\mathcal{F}}_0(x,z)\,,
\end{aligned} \tag{2.35}
$$

where we have defined

$$\ddddot{\mathcal{F}}_0(x,z) \equiv \langle V_{j_1}(0,0)V_{j_2}(1,1)V_{j_3}(\infty,\infty)V_{j_4}(x,z)\rangle\,, \tag{2.36}$$

leaving its j_i-dependence implicit. The function $\ddddot{\mathcal{F}}_0(x,z)$ must correspond to a (monodromy invariant) solution of the KZ equation, which follows from the fact that

$$L_{-1} - \frac{1}{2(k-2)}\left(J^+_{-1}J^-_0 + J^-_{-1}J^+_0 - 2J^3_{-1}J^3_0\right) \tag{2.37}$$

vanishes when acting on affine primaries. More precisely, this reads

$$\left[\partial_z - \frac{1}{k-2}\left(\frac{\mathcal{P}}{z} + \frac{\mathcal{Q}}{z-1}\right)\right]\ddddot{\mathcal{F}}_0(x,z) = 0\,, \tag{2.38}$$

where

$$\mathcal{P} = x^2(x-1)\partial_x^2 - \left[(\kappa-1)x + 2j_1 - 2j_4(x-1)\right]x\partial_x - 2j_4(j_1+\kappa x)\,, \tag{2.39a}$$

$$\mathcal{Q} = -x(1-x)^2\partial_x^2 + \left[(\kappa-1)(1-x) + 2j_2 + 2j_4 x\right](1-x)\partial_x - 2j_4[j_2+\kappa(1-x)]\,, \tag{2.39b}$$

with $\kappa = j_3 - j_1 - j_2 - j_3$. Beyond the usual singularities at $z = 0, 1, \infty$ and $x = 0, 1, \infty$, unflowed four-point functions have an additional one at $z = x$ [6],

$$\ddddot{\mathcal{F}}_0(x,z) \sim |x-z|^{2(k-j_1-j_2-j_3-j_4)}\,. \tag{2.40}$$

For unflowed vertex operators the OPE expansion can be derived by analytic continuation from that of the H_3^+ model [6, 8]. For states in the continuous representations (which are tachyonic for $\omega = 0$), one has

$$V_{j_1}(x_1,z_1)V_{j_4}(x_4,z_4) = \int_{\frac{1}{2}+i\mathbb{R}} dj \int_{\mathbb{C}} d^2x \frac{\left|z_{14}^{\Delta_j-\Delta_1-\Delta_4}\right|^2 C(j_1,j_4,j)V_{1-j}(x,z_1)}{\left|x_{14}^{j_1+j_4-j}(x_4-x)^{j_4+j-j_1}(x-x_1)^{j+j_1-j_4}\right|^2} + \cdots\,, \tag{2.41}$$

up to descendant contributions weighted by larger powers of z_{14}. The appearance of the operator V_{1-j} is related to the reflection symmetry (2.5). The analytic continuation of this OPE is required for exploring the short string sector, where j_1 or j_4 would take real values. Note that as j_{14} is moved around in the complex plane (2.41) can pick up additional contributions coming from the poles of $C(j_1,j_4,j)$ and the powers of x_{ij} that may cross the integration contour. The main aspects of this procedure were discussed in [6].

By inserting the OPE in eq. (2.33), one obtains the decomposition of the unflowed four-point function in terms of conformal blocks. More explicitly, one gets

$$\ddddot{\mathcal{F}}_0(x,z) = \int_{\frac{1}{2}+i\mathbb{R}} dj\, \mathcal{C}(j)|F_j(x,z)|^2, \tag{2.42}$$

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where

$$\mathcal{C}(j) = \frac{C(j_1, j_4, j)C(j, j_2, j_3)}{B(j)}\,, \tag{2.43}$$

and $F_j(x,z)$ has the z expansion

$$F_j(x,z) = z^{\Delta_j - \Delta_1 - \Delta_4} \sum_{n=0}^{\infty} f_{j,n}(x) z^n\,, \tag{2.44}$$

with

$$f_{j,0}(x) = x^{j-j_1-j_4} {}_2F_1\left[\begin{matrix} j-j_1+j_4,\ j-j_3+j_2 \\ 2j \end{matrix}\ \middle|\ x\right]. \tag{2.45}$$

This is consistent with the fact that the theory has a Virasoro symmetry acting on the variable x since this is the coordinate on the conformal boundary of AdS_3 where the holographic CFT is defined. From the worldsheet perspective, eq. (2.45) can be derived directly by using (2.5) to write $V_{1-j}(x,z)$ in terms of an integral involving $V_j(x')$ and using the identity

$$\begin{aligned} &\int_{\frac{1}{2}+i\mathbb{R}} dj\, \mathcal{C}(j) \left| \frac{\pi x^{j-j_1-j_4}}{2j-1} {}_2F_1\left[\begin{matrix} j-j_1+j_4,\ j-j_3+j_2 \\ 2j \end{matrix}\ \middle|\ x\right]\right|^2 = \\ &\int_{\frac{1}{2}+i\mathbb{R}} dj\, \mathcal{C}(j) \int d^2x' d^2x'' |x^{j-j_1-j_4} \left(x'-x''\right)^{2j-2} x'^{j_4-j_1-j}(x-x')^{j_1-j_4-j}(1-x'')^{j_3-j_2-j}|^2\,. \end{aligned} \tag{2.46}$$

Of course, $f_{j,0}(x)$ is a leading order solution of the KZ equation (2.38). Although, in principle, higher orders $f_{j,n>0}(x)$ can be obtained recursively, no closed-form expression is known for $F_j(x,z)$. It is important to keep in mind that the expressions in eqs. (2.41) and (2.42) are only valid if the external spins satisfy

$$\mathrm{Max}\left[|\mathrm{Re}(j_1-j_4)|, |\mathrm{Re}(1-j_1-j_4)|, |\mathrm{Re}(j_2-j_3)|, |\mathrm{Re}(1-j_2-j_3)|\right] < \frac{1}{2}\,. \tag{2.47}$$

Moving away from this range implies that the correlator will pick up additional contributions coming from the exchange of unflowed discrete states and, as shown in [6], in some cases also singly-flowed states.

It will also be necessary for our proposes to use the conformal block expansion of the unflowed four-point function in the limit $x, z \to 0$, with x/z fixed. This can be extrapolated from (2.42) by using the identity [26]

$$\ddot{\ddot{\mathcal{F}}}_0\left(x,z\right) = \mathcal{N}(j_1)\mathcal{N}(j_3)|x|^{-4j_4}|z|^{2j_4}\ddot{\ddot{\mathcal{F}}}_0\Big|_{\frac{k}{2}-j_1, \frac{k}{2}-j_3}\left(\frac{z}{x}, z\right). \tag{2.48}$$

This implies that

$$\ddot{\ddot{\mathcal{F}}}_0(zx,z) = \int_{\frac{1}{2}+i\mathbb{R}} dj \mathcal{N}(j_1)\mathcal{N}(j_3) \frac{C\left(\frac{k}{2}-j_1, j_4, j\right) C\left(j, j_2, \frac{k}{2}-j_3\right)}{B(j)} |G_j(x,z)|^2\,, \tag{2.49}$$

where the function $G_j(x,z)$ can be expanded as

$$G_j(x,z) = z^{\Delta_j - \Delta_1 - \Delta_4 - j_4 - j_1 + \frac{k}{4}} x^{-2j_4} \sum_{n=0}^{\infty} g_{j,n}(x) z^n\,, \tag{2.50}$$

the leading order being

$$G_j(x,z) = z^{\Delta_j-\Delta_1-\Delta_4-j_4-j_1+\frac{k}{4}} x^{\frac{k}{2}-j-j_1-j_4} {}_2F_1\left[\begin{matrix} j-\frac{k}{2}+j_1+j_4,\ j-\frac{k}{2}+j_3+j_2 \\ 2j \end{matrix}\,\middle|\, \frac{1}{x}\right]. \tag{2.51}$$

By virtue of eq. (2.48), these expressions are valid as long as

$$\text{Max}\left[|\text{Re}\left(j_1+j_4-\frac{k}{2}\right)|, |\text{Re}\left(1+j_{14}-\frac{k}{2}\right)|, |\text{Re}\left(j_2+j_3-\frac{k}{2}\right)|, |\text{Re}\left(1-j_{23}-\frac{k}{2}\right)|\right] < \frac{1}{2}, \tag{2.52}$$

where $j_{i\ell} = j_i - j_\ell$.

2.4 Conjecture for the flowed four-point functions

Spectrally flowed four-point functions are notoriously complicated. Recently, an integral expression relating them with the corresponding unflowed correlators was proposed in [34]. One can divide this conjecture into two parts. First, using the global Ward identities, which in this case imply

$$\begin{aligned} &\langle V_{j_1}^{\omega_1}(x_1,y_1,z_1) V_{j_2}^{\omega_2}(x_2,y_2,z_2) V_{j_3}^{\omega_3}(x_3,y_3,z_3) V_{j_4}^{\omega_4}(x_4,y_4,z_4)\rangle \\ &= \left| \frac{x_{21}^{-h_1^0-h_2^0+h_3^0-h_4^0} x_{31}^{-h_1^0+h_2^0-h_3^0+h_4^0} x_{32}^{h_1^0-h_2^0-h_3^0+h_4^0} x_{34}^{-2h_4^0}}{z_{21}^{-\Delta_1^0-\Delta_2^0+\Delta_3^0-\Delta_4^0} z_{31}^{-\Delta_1^0+\Delta_2^0-\Delta_3^0+\Delta_4^0} z_{32}^{\Delta_1^0-\Delta_2^0-\Delta_3^0+\Delta_4^0} z_{34}^{-2\Delta_4^0}} \right|^2 \\ &\quad \times \left\langle V_{j_1}^{\omega_1}\left(0, \frac{y_1\, x_{32}\, z_{21}^{\omega_1} z_{31}^{\omega_1}}{x_{21}\, x_{31}\, z_{32}^{\omega_1}}, 0\right) V_{j_2}^{\omega_2}\left(1, \frac{y_2\, x_{31}\, z_{21}^{\omega_2} z_{32}^{\omega_2}}{x_{21}\, x_{32}\, z_{31}^{\omega_2}}, 1\right) V_{j_3}^{\omega_3}\left(\infty, \frac{y_3\, x_{21}\, z_{31}^{\omega_3} z_{32}^{\omega_3}}{x_{31}\, x_{32}\, z_{21}^{\omega_3}}, \infty\right)\right. \\ &\qquad \left. V_{j_4}^{\omega_4}\left(\frac{x_{32}\, x_{14}}{x_{12}\, x_{34}}, \frac{y_4\, x_{31}\, x_{32}\, z_{21}^{\omega_4} z_{34}^{2\omega_4}}{x_{21}\, x_{34}^2\, z_{31}^{\omega_4} z_{32}^{\omega_4}}, \frac{z_{32}\, z_{14}}{z_{12}\, z_{34}}\right)\right\rangle, \end{aligned} \tag{2.53}$$

in order to focus on

$$\dddot{\mathcal{F}}_{\boldsymbol{\omega}}(x,y_i,z) \equiv \langle V_{j_1}^{\omega_1}(0,y_1,0) V_{j_2}^{\omega_2}(1,y_2,1) V_{j_3}^{\omega_3}(\infty,y_3,\infty) V_{j_4}^{\omega_4}(x,y_4,z)\rangle, \tag{2.54}$$

it was proposed that the y-basis differential equations for four-point functions with arbitrary spectral flow charges are solved by

$$\begin{aligned} \dddot{\mathcal{F}}_{\boldsymbol{\omega}}(x,y_i,z) = &|X_\emptyset^{j_1+j_2+j_3+j_4-k} X_{12}^{-j_1-j_2+j_3-j_4} X_{13}^{-j_1+j_2-j_3+j_4} \\ &\times X_{23}^{j_1-j_2-j_3+j_4} X_{34}^{-2j_4}|^2 \dddot{\mathcal{F}}_+\left(\frac{X_{14}X_{23}}{X_{12}X_{34}}, z\right) \end{aligned} \tag{2.55}$$

when the total spectral flow charge is even, and by

$$\begin{aligned} \dddot{\mathcal{F}}_{\boldsymbol{\omega}}(x,y_i,z) = \mathcal{N}(j_3)&|X_{123}^{\frac{k}{2}-j_1-j_2-j_3-j_4} X_1^{-j_1+j_2+j_3+j_4-\frac{k}{2}} X_2^{j_1-j_2+j_3+j_4-\frac{k}{2}} \\ &\times X_3^{j_1+j_2-j_3+j_4-\frac{k}{2}} X_4^{-2j_4}|^2 \dddot{\mathcal{F}}_-\left(\frac{X_2 X_{134}}{X_{123}X_4}, z\right) \end{aligned} \tag{2.56}$$

when it is odd. The corresponding x-basis correlators

$$\dddot{\mathcal{F}}_{\boldsymbol{\omega}}(x,z) \equiv \langle V_{j_1h_1}^{\omega_1}(0,0) V_{j_2h_2}^{\omega_2}(1,1) V_{j_3h_3}^{\omega_3}(\infty,\infty) V_{j_4h_4}^{\omega_4}(x,z)\rangle \tag{2.57}$$

are then obtained as

$$\dddot{\mathcal{F}}_{\boldsymbol{\omega}}(x,z)=\int\prod_{i=1}^{4}d^2y_i\,|y_i^{j_i+\frac{k}{2}\omega_i-h_i-1}|^2\,\dddot{\mathcal{F}}_{\boldsymbol{\omega}}(x,y_i,z). \tag{2.58}$$

In these expressions, the X_I are the appropriate generalizations of the factors appearing in the three-point functions discussed above, namely the Z_I defined in (2.24), while $\dddot{\mathcal{F}}_{\pm}(X,z)$ are arbitrary functions of the worldsheet cross-ratio z and of the relevant generalized cross-ratio, i.e. either $X=\frac{X_{14}X_{23}}{X_{12}X_{34}}$ or $X=\frac{X_2X_{134}}{X_{123}X_4}$. The structure of this conjecture is thus similar in spirit to the three-point case. More precisely, the X_I are still linear functions in the corresponding y_i,

$$X_I=z^{\frac{1}{2}\delta_{\{1,4\}\in I}}\left[(z-1)(-1)^{\omega_1(\omega_2+\omega_3)+\omega_4(\omega_2+\omega_3)}\right]^{\frac{1}{2}\delta_{\{2,4\}\in I}}\sum_{i\in I:\epsilon_i=\pm}P_{\boldsymbol{\omega}+\sum_{i\in I}\epsilon_i\hat{e}_i}\prod_{i\in I}y_i^{\frac{1-\epsilon_i}{2}}\,, \tag{2.59}$$

although now the coefficients $P_{\boldsymbol{\omega}}=P_{\boldsymbol{\omega}}(x,z)$ are polynomials that depend on both x and z. They are defined as

$$\begin{aligned}P_{\boldsymbol{\omega}}=\,&f(\boldsymbol{\omega})\,(1-x)^{\frac{1}{2}s(\omega_2+\omega_4-\omega_1-\omega_3)}\,(1-z)^{\frac{1}{4}s((\omega_1+\omega_2-\omega_3-\omega_4)(\omega_1+\omega_4-\omega_2-\omega_3))-\frac{1}{2}\omega_2\omega_4}\\&\times x^{\frac{1}{2}s(\omega_1+\omega_4-\omega_2-\omega_3)}z^{\frac{1}{4}s((\omega_1+\omega_2-\omega_3-\omega_4)(\omega_2+\omega_4-\omega_1-\omega_3))-\frac{1}{2}\omega_1\omega_4}\tilde{P}_{\boldsymbol{\omega}}(x,z)\,,\end{aligned} \tag{2.60}$$

where $s(\omega)=\omega\Theta(\omega)$, Θ being the Heaviside step function. On the other hand, the $\tilde{P}_{\boldsymbol{\omega}}(x,z)$ come from the different covering maps relevant for a given four-point function, see [34, 42]. A necessary condition for the existence of a rational function $\Gamma(z)$ with the appropriate branch points is

$$\sum_{i=1}^{4}\omega_i\in\mathbb{Z}\,,\quad\text{and}\quad\sum_{i=1}^{4}\omega_i>2\max\omega_i\,. \tag{2.61}$$

However, as opposed to the three-point case, this is not a sufficient condition. Indeed, once we further impose $\Gamma(z_i)=x_i$ for $i=1,2,3$ there is no freedom left, hence a covering map can only exist if a certain relation between the worldsheet and boundary insertion points holds, namely $\Gamma(z_4)=x_4$. The $\tilde{P}_{\boldsymbol{\omega}}(x,z)$ are irreducible polynomials which vanish whenever this condition holds, namely $\tilde{P}_{\boldsymbol{\omega}}(x,z)=\prod(z-\Gamma^{-1}(x))$, where the product runs over all possible covering maps.[4] As it turns out, their degree in z is given by the Hurwitz number

$$H_{\boldsymbol{\omega}}=\frac{1}{2}\min_{i=1,2,3,4}\left[\omega_i\left(\sum_{j=1}^{4}\omega_j-2\omega_i\right)\right]\,, \tag{2.62}$$

while their degree in x is

$$\Lambda_{\boldsymbol{\omega}}=\frac{1}{2}\left[\min(\omega_1+\omega_2,w_3+\omega_4)-\max(|\omega_1-\omega_2|,|\omega_3-\omega_4|)\right]\,. \tag{2.63}$$

Moreover, they are normalized such that if $H_\omega=0$ then $\tilde{P}_{\boldsymbol{\omega}}(x,z)=1$, and if $H_\omega<0$ then $\tilde{P}_{\boldsymbol{\omega}}(x,z)=0$.

[4]These are direct generalizations of the $Q_{\boldsymbol{\omega}}$ functions appearing in the three-point functions. Although they are not known in closed form, an algorithmic procedure for computing them was given in [42].

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The second part of the conjecture concerns the nature of the functions $\dddot{\mathcal{F}}_\pm(X,z)$. These are unconstrained by the local Ward identities. However, one still has to impose the KZ equation for the flowed correlator. The derivation of the corresponding differential equation is of course more involved than in the unflowed case because the vertex operators under consideration are not affine primaries. The authors of [34] argued that the structure of the prefactors in eqs. (2.55) and (2.56) is such that, as a function of z and of the generalized cross-ratio X, $\dddot{\mathcal{F}}_+$ satisfies precisely the KZ equation as the unflowed correlator with the same external spins, see (2.38). The same goes for $\dddot{\mathcal{F}}_-$ provided we flip $j_3 \to \frac{k}{2} - j_3$. It is therefore natural to identify these functions with the corresponding unflowed correlators,

$$\dddot{\mathcal{F}}_+ = \dddot{\mathcal{F}}_0 \,, \qquad \dddot{\mathcal{F}}_- = \dddot{\mathcal{F}}_0|_{j_3 \to \frac{k}{2} - j_3} \,. \tag{2.64}$$

Note that the unflowed structure constants play an analogous role in the context of flowed three-point functions, see eqs. (2.23) and (2.27). In this way, the flowed four-point functions can be understood as integral transforms of the unflowed ones.

This proposal is based on a detailed case-by-case analysis of the local Ward identities, combined with a series of non-trivial consistency checks. More explicitly, it was shown that 1) it reduces to the known three-point functions in the $\omega_4 = j_4 = 0$ limit, 2) it satisfies certain null vector constraints when they are present, 3) it is invariant under the pairwise exchange of the different insertions, 4) it is consistent with the reflection symmetry under $j_i \to 1 - j_i$, and 5) it reproduces the relatively small number of previously known results [6, 11–13, 15, 18]. However, it remains to be proven.

One of the main goals of this paper is to provide further support for this conjecture. We will do so by studying its factorization properties, i.e. by considering several non-trivial exchange channels and showing that, in all cases, one obtains perfect agreement with the expected OPE decomposition for the SL(2,$\mathbb{R}$) WZW model, including spectrally flowed sectors.

3 Sample cases for low spectral flow charges

In this section we focus on a few low-lying cases in terms of the external spectral flow charges ω_i, with $i = 1,2,3,4$, leaving a more general analysis for the following ones. These examples, namely the correlators with $\boldsymbol{\omega} = (0,0,1,0)$, $\boldsymbol{\omega} = (1,0,1,0)$, and $\boldsymbol{\omega} = (0,0,2,0)$, are interesting for several reasons. First, the cases with a single flowed operator with either $\omega_3 = 1$ or $\omega_3 = 2$ are the relevant ones for the comparison with previous results from [12, 13, 18], which were obtained by using different techniques. Second, in these cases, the expressions in eqs. (2.55) and (2.56) can easily be established explicitly, as will be shown below, and they are simple enough to allow for a tractable and detailed analysis which will provide some useful intuition for the general case.

3.1 The $\boldsymbol{\omega} = (0,0,1,0)$ case

We consider the four-point function

$$\langle V_{j_1}(x_1,z_1) V_{j_2}(x_2,z_2) V^1_{j_3}(x_3,y_3,z_3) V_{j_4}(x_4,z_4) \rangle \,. \tag{3.1}$$

In this case, there is a single constraint equation [20], namely that

$$\begin{aligned}&\langle J^{-}(x_3,z)V_{j_1}(x_1,z_1)V_{j_2}(x_2,z_2)V^{1}_{j_3}(x_3,y_3,z_3)V_{j_4}(x_4,z_4)\rangle\\&\quad=(y_3^2\partial_{y_3}+2j_3y_3)\langle V_{j_1}(x_1,z_1)V_{j_2}(x_2,z_2)V^{1}_{j_3}(x_3,y_3,z_3)V_{j_4}(x_4,z_4)\rangle+\mathcal{O}(z-z_3)\,.\end{aligned}\tag{3.2}$$

By inserting the OPEs (2.11)–(2.14) in eq. (3.2), and fixing $x_1=z_1=0$, $x_2=z_2=1$ and $x_3=z_3\to\infty$, this translates into the following differential equation for the Moebius-fixed y-basis correlator:

$$\left\{y_3(y_3-1)\partial_{y_3}+(x-z)\partial_x+j_1+j_2+j_3(2y_3-1)+j_4-\frac{k}{2}\right\}\dddot{\mathcal{F}}_{(0010)}(x,y_3,z)=0\,.\tag{3.3}$$

The general solution takes the form

$$\dddot{\mathcal{F}}_{(0010)}(x,y_3,z)=|y_3^{j_1+j_2-j_3+j_4+\frac{k}{2}}(1-y_3)^{\frac{k}{2}-j_1-j_2-j_3-j_4}|^2F(X,z)\,,\tag{3.4}$$

where

$$X=\frac{xy_3-z}{y_3-1}\tag{3.5}$$

is the appropriate generalized cross-ratio for this particular case, and $F(X,z)$ is an arbitrary function. On the other hand, as $\omega_4=0$, the KZ equation for the vertex V_{j_4} is derived analogously to the unflowed case. However, one picks up an additional contribution from the double pole in the OPE of the current with $V^1_{j_3}$. Making use of (3.4), one finds that the resulting constraint on $F(X,z)$ takes exactly the same form as in eqs. (2.38) and (2.39), with the replacements $x\to X$ and $j_3\to\frac{k}{2}-j_3$. Transforming back to the x-basis, and using the series identifications to fix the overall normalization as in [32, 33], we conclude that

$$\dddot{\mathcal{F}}_{(0010)}(x,z)=\mathcal{N}(j_3)\int d^2y_3|y_3^{-h_3+j_1+j_2+j_4-1}(1-y_3)^{\frac{k}{2}-j_1-j_2-j_3-j_4}|^2\dddot{\mathcal{F}}_{-}(X,z)\,,\tag{3.6}$$

$\dddot{\mathcal{F}}_{-}(X,z)$ being the unflowed four-point function with $j_3\to\frac{k}{2}-j_3$ as in (2.64). This is the expression conjectured in [34].

We now discuss the factorization properties of the correlator (3.6). The spectral flow selection rules for three-point functions in eq. (2.21) imply that this should involve two different intermediate channels for processes of the form $14\to32$: the exchanged state can only have $\omega=0$ or $\omega=1$, see figure 1. We expect to be able to study these processes by combining our knowledge of the factorized expansion of the *unflowed* four-point function appearing in the integrand of (3.6) with the integral transform that captures the effect of spectral flow. In other words, we aim for the conformal block decomposition of the spectrally flowed correlator by means of the corresponding decomposition of the unflowed one.

Now, the scanning of exchange channels presents several subtleties. For instance, it strongly depends on the values of the external spins j_i. We shall bypass this problem by restricting our study to spin configurations reducing to Teschner strip as in the Euclidean model [8] (up to the appropriate flips $j_i\to\frac{k}{2}-j$), for which a closed analytical expression for the four-point function is actually available. These domains only allow for the propagation of intermediate channels lying in the continuous series. The emergence of intermediate discrete states could be obtained after a proper analytical continuation [6, 8], which is beyond the

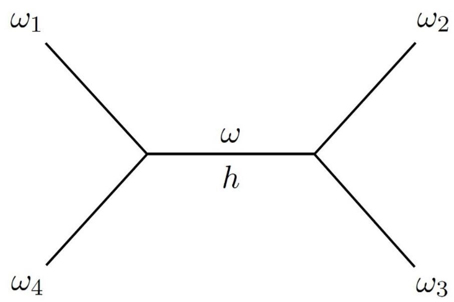

Figure 1. Schematic description of the exchange of states with spectral flow charge ω and chage h along the $14 \to 23$ channel.

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scope of this paper. Another factor that allows tuning new exchange states, in particular, channels in non-trivial spectral flow sectors, is related to the behaviour of the generalized cross-ratio X as z approaches zero and, indirectly, with the interchange of the y-integrals and the $z \to 0$ limit while determining conformal blocks. Indeed, we shall see that different rescalings of y_i depending on x and z while integrating (2.58) will lead us to recognize different intermediate channels, even when the external spins remain fixed.

Let us be more precise. When the external spins j_i satisfy the constraint

$$\text{Max}\left[|\text{Re}(j_1 - j_4)|, |\text{Re}(1 - j_1 - j_4)|, |\text{Re}\left(j_2 + j_3 - \frac{k}{2}\right)|, |\text{Re}\left(1 - j_2 + j_3 - \frac{k}{2}\right)|\right] < \frac{1}{2}, \tag{3.7}$$

the unflowed four-point function appearing in the integrand of (3.6) admits a factorized expression of the form

$$\dddot{\mathcal{F}}_{-}(X, z) = \int_{\frac{1}{2}+i\mathbb{R}} dj \mathcal{C}(j) F_j(X, z), \tag{3.8}$$

where

$$\mathcal{C}(j) \equiv \frac{C(j_1, j_4, j) C\left(j, j_2, \frac{k}{2} - j_3\right)}{B(j)}. \tag{3.9}$$

If we consider the limit $z \to 0$, while keeping x and y_3 fixed, the generalized cross-ratio goes as

$$X \to \frac{x y_3}{y_3 - 1}. \tag{3.10}$$

Eq. (2.44) then shows that, at leading order in z, we can replace

$$F_j(X, z) \to z^{\Delta_j^{(0)} - \Delta_1 - \Delta_4} \left(\frac{x y_3}{y_3 - 1}\right)^{j - j_1 - j_4} {}_2F_1\left[\begin{matrix} j - j_1 + j_4,\ j + j_3 + j_2 - \frac{k}{2} \\ 2j \end{matrix} \middle| \frac{x y_3}{y_3 - 1}\right]. \tag{3.11}$$

Here, we have introduced the notation $\Delta_j^{(0)}$ for the conformal dimension of the intermediate state at $\omega = 0$, which depends only on j. We obtain a contribution of the form

$$\begin{aligned} \dddot{\mathcal{F}}_{(0010)}(x, z) \sim \mathcal{N}(j_3) \int d^2 y_3 \int_{\frac{1}{2}+i\mathbb{R}} dj \mathcal{C}(j) \Bigg| & y_3^{-h_3 + j_1 + j_2 + j_4 - 1} (1 - y_3)^{\frac{k}{2} - j_1 - j_2 - j_3 - j_4} \\ & \times z^{\Delta_j^{(0)} - \Delta_1 - \Delta_4} \left(\frac{x y_3}{y_3 - 1}\right)^{j - j_1 - j_4} {}_2F_1\left[\begin{matrix} j - j_1 + j_4,\ j + j_3 + j_2 - \frac{k}{2} \\ 2j \end{matrix} \middle| \frac{x y_3}{y_3 - 1}\right] \Bigg|^2, \end{aligned} \tag{3.12}$$

which indeed corresponds to the exchange of an unflowed state, as can be read off from the overall powers of x and z.

We would now like to compute the integral over y_3. The subtleties related to the exchange of the j and y_3 integrals in (3.12) can be analyzed along the lines of [6]. Although this could lead to extra contributions associated to specific values of j, we expect these to be associated to either descendant contributions or two-particle states. Since we are interested in primary exchanges, we will ignore such contributions from now on. Using the complex integrals provided in appendix A.1, we get

$$\ddddot{\mathcal{F}}_{(0010)}(x,z) \sim \int_{\frac{1}{2}+i\mathbb{R}} dj\, \mathcal{C}(j) \frac{\mathcal{N}(j_3)\gamma\left(h_3+j_3-\frac{k}{2}\right)\gamma(j+j_2-h_3)}{\gamma\left(j+j_2+j_3-\frac{k}{2}\right)} \times \left| z^{\Delta_j^{(0)}-\Delta_1-\Delta_4} x^{j-j_1-j_4} {}_2F_1\left[\begin{matrix} j-j_1+j_4,\ j-h_3+j_2 \\ 2j \end{matrix}\middle| x\right]\right|^2 . \tag{3.13}$$

We find that the factors $\mathcal{C}(j)$ and $\mathcal{N}(j_3)$ combine with the γ-functions precisely in the right way to give the product of structure constants associated to the relevant flowed three-point functions, i.e. [6]

$$\mathcal{N}(j_3)\mathcal{C}(j) \frac{\gamma\left(h_3+j_3-\frac{k}{2}\right)\gamma(j+j_2-h_3)}{\gamma\left(j+j_2+j_3-\frac{k}{2}\right)} = \frac{C(j_1,j_4,j)C_{(001)}(j,j_2,j_3,h_3)}{B(j)} . \tag{3.14}$$

This shows that the structure of the factorized expression is analogous to that of the unflowed case. Indeed, on top of the structure constants we have obtained the *flowed* conformal block at leading order in z, that is, the hypergeometric function appearing in (2.44) but with j_3 replaced by h_3, which is the spacetime spin (or boundary weight) of the flowed vertex.

This kind of flowed four-point function was studied previously in [12, 13]. Their approach was quite different, as it was based on the spectral flow operator introduced in [11] and the subsequent definition of singly-flowed x-basis operators introduced in [6].[5] Nevertheless, the authors obtained a KZ equation and recursion relations which are, of course, equivalent to those we have just derived in the y-basis language. Our result, including the region of validity given in eq. (3.7), matches the combined results of [12, 13].

Now, according to the spectral flow selection rules, the four-point function under consideration can also receive contributions from the exchange of intermediate states with $\omega = 1$. In order to pick up a contribution of this type we turn back to eq. (3.6) and perform the change of variables $y_3 \to z y_3/x$. This choice will be justified in section (3.4) below. Keeping the new y_3 fixed, the generalized cross-ratio now behaves at small z as

$$X \to z(1-y_3) . \tag{3.15}$$

As a consequence, the leading order expression of the unflowed correlator must be read off from eq. (2.51). As long the external spins lie in the range

$$\text{Max}\left[\left|\text{Re}\left(j_1+j_4-\frac{k}{2}\right)\right|, \left|\text{Re}\left(1-j_4+j_1-\frac{k}{2}\right)\right|, |\text{Re}(j_2-j_3)|, |\text{Re}(1-j_2-j_3)|\right] < \frac{1}{2}, \tag{3.16}$$

[5] This was recently generalized to arbitrary ω in [32].

this leads to

$$\tilde{\tilde{\tilde{\mathcal{F}}}}_{(0010)}(x,z) \sim \mathcal{N}(j_1) \int d^2y_3 \int_{\frac{1}{2}+i\mathbb{R}} dj\, \mathcal{C}(j) \Big| z^{\Delta_j^{(0)}-h_3+j_2+\frac{k}{4}-\Delta_1-\Delta_4} x^{h_3-j_2-j_1-j_4} \\ \times y_3^{-h_3+j_1+j_2+j_4-1}(1-y_3)^{\frac{k}{2}-j-j_1-j_4}\, {}_2F_1\left[\begin{matrix} j_1+j_4+j-\frac{k}{2},\ j-j_3+j_2 \\ 2j \end{matrix}\ \middle|\ \frac{1}{1-y_3}\right]\Big|^2, \tag{3.17}$$

now with

$$\mathcal{C}(j) = \frac{C\left(\frac{k}{2}-j_1,j_4,j\right)C(j,j_2,j_3)}{B(j)}. \tag{3.18}$$

We thus find that the x-dependence simplifies considerably as we only get an overall power. For this to describe a consistent contribution to the factorization expansion associated with a singly-flowed exchange, this should be of the form $x^{h-j_1-j_4}$, with h the spacetime spin of the intermediate state. Hence, in this strict $z \to 0$ limit, we appear to only be capturing the process corresponding to $h = h_3 - j_2$. This interpretation is further supported by the power of z in (3.17) since

$$\Delta_j\Big|_{h=h_3-j_2}^{\omega=1} = \Delta_j^{(0)} - h_3 + j_2 + \frac{k}{4} - \Delta_1 - \Delta_4\,, \tag{3.19}$$

as can be seen from eq. (2.4). We will discuss why this specific channel has been isolated later on, when analyzing the general case in section (3.4). For now, we focus on showing that, at least for this particular contribution, the four-point function factorizes correctly, in the sense that the relevant flowed structure constants emerge directly from (3.17).

For this, we first note that the hypergeometric function appearing in (3.17) must have a different origin from the one that appeared so far, since it is not evaluated at x and is independent of the spacetime spins h_i. In order to show how it comes about, we proceed by reverse engineering. In analogy with the unflowed case, we expect to find certain factors corresponding to

$$\frac{C_{(001)}(j_1,j_4,j,h)C_{(101)}(j,j_2,j_3,h,h_3)}{R(j,h,1)}, \tag{3.20}$$

where the denominator stands for the relevant $\omega = 1$ propagator in eq. (2.8). Since $h = h_3 - j_2$, we can combine the explicit integral expressions given in section 2.2 for the flowed structure constants with the identity (A.26) to get

$$\begin{aligned}(3.20) &= \int d^2y\, d^2y_3\, |y^{-2j} y_3^{j_3-h_3+\frac{k}{2}-1}|^2 \\ &\quad\times \langle V_{j_1}(0,0)V_{j_4}(1,1)V^1_{j,h_3-j_2}(\infty,y^{-1},\infty)\rangle \langle V^1_{1-j}(0,y,0)V_{j_2}(0,1)V^1_{j_3}(\infty,y_3,\infty)\rangle \\ &= \mathcal{N}(j_1)C\left(\frac{k}{2}-j_1,j_4,j\right)C(1-j,j_2,j_3) \\ &\quad\times \int d^2y d^2y_3\, |y_3^{j_3-h_3+\frac{k}{2}-1}(1-y)^{\frac{k}{2}-j_1-j_4-j}y^{j_3+j-j_2-1}(1-yy_3)^{j_2+j-j_3-1}|^2\,. \end{aligned} \tag{3.21}$$

After the change of variable $y \to y_3^{-1}(1-y)$, the y-integral becomes of the hypergeometric type, giving

$$
\begin{aligned}
(3.20) = \mathcal{N}(j_1) C\left(\frac{k}{2}-j_1, j_4, j\right) \frac{\pi\gamma(j-j_3+j_2)\gamma(j+j_3-j_2)C(1-j,j_2,j_3)}{\gamma(2j)} \\
\times \int d^2y_3 \left| y_3^{j_1+j_4-h-1}(1-y_3)^{\frac{k}{2}-j-j_1-j_4} {}_2F_1\left[\begin{matrix} j+j_1+j_4-\frac{k}{2},\ j-j_3+j_2 \\ 2j \end{matrix} \middle| \frac{1}{1-y_3}\right]\right|^2,
\end{aligned}
\tag{3.22}
$$

This exactly reproduces the contribution to the four-point function under consideration given in eq. (3.17), by means of the identity

$$
\frac{\pi\gamma(j-j_3+j_2)\gamma(j+j_3-j_2)C(1-j,j_2,j_3)}{\gamma(2j)} = \frac{C(j,j_2,j_3)}{B(j)}\,, \tag{3.23}
$$

which follows directly from the reflection symmetry in the unflowed sector (2.5).

3.2 The $\boldsymbol{\omega} = (1,0,1,0)$ case

In this case the four point function is given by

$$
\begin{aligned}
\dddot{\mathcal{F}}_{(1010)}(x,z) = z^{j_4} \int d^2y_1\, d^2y_3\, |y_1^{\alpha_1-1} y_3^{\alpha_3-1} (1-y_1)^{-j_1-j_2+j_3-j_4}(1-y_3)^{j_1-j_2-j_3+j_4} \\
\times (z-xy_3)^{-2j_4}(1-y_1y_3)^{-j_1+j_2-j_3+j_4}|^2 \dddot{\mathcal{F}}_+\left[\frac{x-zy_1}{1-y_1}\frac{1-y_3}{z-xy_3}, z\right],
\end{aligned}
\tag{3.24}
$$

where we have used the shorthand $\alpha_i = j_i + \frac{k}{2}\omega_i - h_i$. The structure of the integrand follows from the two independent local Ward identities, which imply that the y-basis correlator $\dddot{\mathcal{F}}_{(1010)}(x,y_1,y_3,z)$ is annihilated by the differential operators

$$
y_3(y_3-1)\partial_{y_3} + (y_1-1)\partial_{y_1} + (x-z)\partial_x + j_1 + j_2 + j_3(2y_3-1) + j_4\,, \tag{3.25}
$$

and

$$
y_1(y_1-1)\partial_{y_1} + (y_3-1)\partial_{y_3} + \frac{x(x-z)}{z}\partial_x + j_1(2y_1-1) + j_2 + j_3 + j_4\left(\frac{2x}{z}-1\right). \tag{3.26}
$$

Combined with the KZ equation, this implies that $\dddot{\mathcal{F}}_+(X,z)$ should be identified with the corresponding unflowed correlator, as it satisfies the same differential equation in terms of the generalized cross-ratio

$$
X = \frac{x-zy_1}{1-y_1}\frac{1-y_3}{z-xy_3}\,. \tag{3.27}
$$

We now show that, as in the $\boldsymbol{\omega} = (0,0,1,0)$ case, considering different regimes for the external spins and taking the $z \to 0$ limit accordingly helps us isolate the different possibilities for the spectral flow charge of the intermediate state, which in this case can be either $\omega = 0$, $\omega = 1$ or $\omega = 2$.

For the singly-flowed channel we simply take the small z limit while keeping $y_{1,3}$ (and also x) fixed, hence $X \to \frac{y_3-1}{(1-y_1)y_3}$. Assuming that the external spins are in the range (2.47) and inserting the leading order expression for the unflowed conformal block, we obtain

$$\ddddot{\mathcal{F}}_{(1010)}(x,z) \sim \int d^2y_1 d^2y_3 \int_{\frac{1}{2}+i\mathbb{R}} dj\mathcal{C}(j) \Big| z^{\Delta_j^{(0)}-\Delta_1^{(0)}-\Delta_4+j_4} x^{-2j_4} y_3^{\alpha_3+j_1-j_4-j-1}(1-y_1)^{j_3-j_2-j}$$
$$\times y_1^{\alpha_1-1}(1-y_3)^{j-j_2-j_3}(1-y_1y_3)^{-j_1+j_2-j_3+j_4} {}_2F_1\left[\begin{matrix} j-j_1+j_4,\ j-j_3+j_2 \\ 2j \end{matrix} \,\middle|\, \frac{y_3-1}{(1-y_1)y_3}\right]\Big|^2, \tag{3.28}$$

with $\mathcal{C}(j) = B(j)^{-1}C(j_1,j_4,j)C(j,j_2,j_3)$. We now find a power-law dependence in x that is factorized from the y_i-integrals, and is compatible with the exchange of a flowed state with $h = h_1 - j_4$. The power of z supports this conclusion since, for $\omega = 1$, we have

$$\Delta_j^{(0)} - \Delta_1^{(0)} - \Delta_4 + j_4 = \Delta_j^{(0)} - (h_1 - j_4) + \frac{k}{4} - \Delta_1 - \Delta_4 = \Delta_j - \Delta_1 - \Delta_4\,. \tag{3.29}$$

The factorization along this particular channel is then consistent iff the remaining factors, including the y_i-integrals, give precisely the normalization

$$\frac{C_{(101)}(j_1,j_4,j,h_1,h)C_{(101)}(j,j_2,j_3,h,h_3)}{R(j,h,1)}\,. \tag{3.30}$$

Proceeding as in the previous section, we use the identities of section A.3 to rewrite this as

$$(3.30) = C(j_1,j_4,j)C(1-j,j_2,j_3)\int d^2y\, d^2y_1\, d^2y_3\, |y_1^{\alpha_1-1}y_3^{\alpha_3-1}$$
$$\times (1-y_3)^{1-j-j_2-j_3}(y-y_1)^{j_4-j-j_1}(1-y)^{j+j_3-j_2-1}(1-y_3y)^{j_2+j-j_3-1}|^2\,, \tag{3.31}$$

with $\alpha_i = j_i + \frac{k}{2} - h_i$. The integral in the auxiliary variable y can be performed as in the previous section. It is simplified by changing variables $y \to y_3^{-1}[1-(1-y_3)y]$, and the result is proportional to a hypergeometric function evaluated at $\frac{y_3-1}{(1-y_1)y_3}$. After using again (3.23), we recover precisely the expression on the r.h.s. of eq. (3.28), as expected.

As it turns out, in this case yet another contribution corresponding to the exchange of a state with $\omega = 1$ can be obtained from (3.24). This can be seen by first taking $(y_1,y_3) \to (z^{-1}y_1, zy_3)$ and only then taking $z \to 0$ with the new $y_{1,3}$ fixed. A similar analysis shows that this corresponds to an intermediate spacetime spin of the form $h = h_3 - j_2$.

It is also possible to isolate the unflowed channel by changing variables to $y_1 \to xy_1/z$. After doing so, the generalized cross-ratio behaves at small z as

$$X \to \frac{z(1-y_1)(1-y_3)}{xy_1y_3}, \tag{3.32}$$

and therefore we must use the expansion (2.51) for the unflowed correlator. This leads to a contribution of the form

$$\ddddot{\mathcal{F}}_{(1010)}(x,z) \sim \int d^2y_1\, d^2y_3 \int_{\frac{1}{2}+i\mathbb{R}} dj\, \mathcal{C}(j) \Big| x^{j-h_1-j_4} z^{\Delta_j-\Delta_1-\Delta_4}$$
$$\times y_1^{j+j_4-h_1-1} y_3^{j+j_2-h_3-1}(1-y_1)^{\frac{k}{2}-j-j_1-j_4}(1-y_3)^{\frac{k}{2}-j-j_2-j_3}$$
$$\times {}_2F_1\left[\begin{matrix} j_1+j_4+j-\frac{k}{2},\ j_3+j_2+j-\frac{k}{2} \\ 2j \end{matrix} \,\middle|\, \frac{xy_1y_3}{(1-y_1)(1-y_3)}\right]\Big|^2, \tag{3.33}$$

where now

$$\mathcal{C}(j)=\frac{\mathcal{N}(j_1)C\left(\frac{k}{2}-j_1,j_4,j\right)\mathcal{N}(j_3)C\left(j,j_2,\frac{k}{2}-j_3\right)}{B(j)}\,. \tag{3.34}$$

Taking $y_i\to y_i/(y_i+1)$ gives an integral that can be carried out in analogy to what was done around (3.12), leading to

$$\begin{aligned}\ddddot{\mathcal{F}}_{(1010)}(x,z)\sim\int_{\frac{1}{2}+i\mathbb{R}}dj\,\mathcal{C}(j)\frac{\gamma(j+j_2-h_3)\gamma\left(j_3+h_3-\frac{k}{2}\right)}{\gamma\left(j_3+j_2+j-\frac{k}{2}\right)}\frac{\gamma(j+j_4-h_1)\gamma\left(j_1+h_1-\frac{k}{2}\right)}{\gamma\left(j_1+j_4+j-\frac{k}{2}\right)}\\ \times\left|x^{j-h_1-j_4}z^{\Delta_j-\Delta_1-\Delta_4}{}_2F_1\left[\begin{matrix}j-h_1+j_4,\ j-h_3+j_2\\ 2j\end{matrix}\middle|\,x\right]\right|^2.\end{aligned} \tag{3.35}$$

The expression (3.33) is valid as long as the external spins are in the strip

$$\mathrm{Max}\left[|\mathrm{Re}\left(j_1+j_4-\frac{k}{2}\right)|,|\mathrm{Re}\left(1+j_{14}-\frac{k}{2}\right)|,|\mathrm{Re}\left(j_2+j_3-\frac{k}{2}\right),|\mathrm{Re}\left(1-j_{23}-\frac{k}{2}\right)|\right]<\frac{1}{2}\,. \tag{3.36}$$

Our result is thus consistent with the expected factorization structure for the process associated to the exchange of an unflowed state thanks to the identity

$$\begin{aligned}\mathcal{C}\left(j\right)\frac{\gamma(j+j_4-h_1)\gamma\left(j_1+h_1-\frac{k}{2}\right)\gamma(j+j_2-h_3)\gamma\left(j_3+h_3-\frac{k}{2}\right)}{\gamma\left(j_1+j_4+j-\frac{k}{2}\right)\gamma\left(j_3+j_2+j-\frac{k}{2}\right)}\\ =\frac{C_{(100)}(j_1,j_4,j,h_1)C_{(001)}(j,j_2,j_3,h_3)}{B(j)}\,.\end{aligned} \tag{3.37}$$

Finally, we note that the $\omega=2$ channel sits at the edge of the selection rules for three-point functions, since $\omega=\omega_1+\omega_4+1=\omega_3+\omega_2+1$. For reasons that will become clear below, we were only able to isolate this flowed channel in the spacetime collision limit $x\to 0$, which corresponds to taking $h=h_1+j_4$. Evaluating (3.24) at $x=0$, rescaling $y_1\to zy_1$, and using (2.51), we get a leading order behavior given by

$$\begin{aligned}\ddddot{\mathcal{F}}_{(1010)}(0,z)\sim\int_{\frac{1}{2}+i\mathbb{R}}dj\mathcal{N}(j_1)\mathcal{N}(j_3)\frac{C\left(\frac{k}{2}-j_1,j_4,j\right)C\left(j,j_2,\frac{k}{2}-j_3\right)}{B(j)}|z^{\Delta_j-\Delta_1-\Delta_4}|^2\\ \times\int d^2y_1d^2y_3\left|y_1^{k-j-h_1-j_4-1}y_3^{\alpha_3-1}(1-y_3)^{\frac{k}{2}-j-j_2-j_3}\right.\\ \left.\times\,{}_2F_1\left[\begin{matrix}j_1+j_4+j-\frac{k}{2},\ j_3+j_2+j-\frac{k}{2}\\ 2j\end{matrix}\middle|\,\frac{1}{y_1(y_3-1)}\right]\right|^2,\end{aligned} \tag{3.38}$$

where the intermediate conformal weight is given by

$$\Delta_j=\Delta_j^{(0)}-2(h_1+j_4)+k. \tag{3.39}$$

The overall power of z is compatible with a process where the intermediate channel has $\omega=2$ and the aforementioned spacetime weight. For the decomposition to be consistent, the remaining factors should match

$$\frac{C_{(102)}(j_1,j_4,j,h_1,h)C_{(201)}(j,j_2,j_3,h,h_3)}{R(j,h,2)}\,, \tag{3.40}$$

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which, thanks to the identity (A.26), can be written as

$$\mathcal{N}(j_1)\mathcal{N}(j_3)C\left(\frac{k}{2}-j_1,j_4,1-j\right)C\left(j,j_2,\frac{k}{2}-j_3\right)\int d^2y_1d^2y_3d^2y \tag{3.41}$$
$$\times\,|y_1^{j_1-h_1+\frac{k}{2}-1}y_3^{j_3-h_3+\frac{k}{2}-1}y^{j_1+j_4+j-\frac{k}{2}-1}\left(1-y_1y\right)^{\frac{k}{2}-j_1-j_4+j-1}\left(1+y-y_3\right)^{\frac{k}{2}-j-j_2-j_3}|^2\,.$$

This is seen to be true by taking $y\to y/y_1$, performing the integral over the new y, and using (3.23) once again.

3.3 The $\boldsymbol{\omega}=(0,0,2,0)$ case

For completeness we also consider the correlator with $\boldsymbol{\omega}=(0,0,2,0)$. This corresponds to the simplest configuration lying at the edge of the range allowed by the selection rules (2.21). The four-point function is given by

$$\dddot{\mathcal{F}}_{(0020)}(x,z)=f(z)\int d^2y_3|y_3^{j_1+j_2+j_4-h_3-1}\left[(1-z)z+y_3(z-x)\right]^{k-j_1-j_2-j_3-j_4}|^2\,. \tag{3.42}$$

Note that the derivation of this formula from the general expression in eq. (2.55) is a bit subtle [34]. This will be discussed more generally in section 4.2 below. In particular, the definition of the function $f(z)$, which is independent of y_i and x, is provided in eq. (4.17).

For the particular case at hand, the local Ward identities give two constraints. Although in principle one of them would be used to solve for the unknown correlator corresponding to the insertion of $(J_1^+V_{j_3}^2)(x_3,z_3)$, this actually drops out. One finds that the correlator is annihilated by the differential operator

$$y_3\partial_{y_3}+(z-x)\partial_x+k-j_1-j_2+j_3-j_4\,, \tag{3.43}$$

hence the general solution is given by the integrand in eq. (3.42) times an arbitrary function $F(y_3(x-z),z)$. Since we also know that at large y_3 this must scale as $y_3^{-2j_3}$ [33], we can fix $F(y_3(x-z),z)=f(z)$. The KZ equation then fixes $f(z)$ in terms of the unflowed correlator.

As it turns out, correlators saturating the bound on spectral flow are also simpler to deal with in terms of their factorization properties. Indeed, unlike in the previous cases, the integral in (3.42) can be carried out explicitly even before taking the small z limit. After rescaling $y_3\to z(1-z)y_3/(z-x)$, we obtain

$$\dddot{\mathcal{F}}_{(0020)}(x,z)=f(z)\left|\frac{[(1-z)z]^{k-h_3-j_3}}{(z-x)^{j_1+j_2+j_4-h_3}}\right|^2\frac{\pi\gamma(j_1+j_2+j_4-h_3)\gamma(j_3+h_3-k)}{\gamma(j_1+j_2+j_3+j_4-k)}. \tag{3.44}$$

As follows from eqs. (4.17) and (2.51), at small z the function $f(z)$ behaves as

$$f(z)\sim\int_{\frac{1}{2}+i\mathbb{R}}dj\mathcal{N}(j_1)\mathcal{N}(j_3)\frac{C\left(\frac{k}{2}-j_1,j_4,j\right)C\left(j,j_2,\frac{k}{2}-j_3\right)}{B(j)}$$
$$\times\,|z^{\Delta_j-\Delta_1-\Delta_4+j_2+j_3-\frac{3k}{4}}|^2\frac{\gamma(2j)\gamma(j_1+j_2+j_3+j_4-k)}{\gamma\left(j_1+j_4+j-\frac{k}{2}\right)\gamma\left(j_2+j_3+j-\frac{k}{2}\right)}\,. \tag{3.45}$$

Therefore, we get

$$\dddot{\tilde{\mathcal{F}}}_{(0020)}(x,z) \sim \int_{\frac{1}{2}+i\mathbb{R}} dj \mathcal{N}(j_1)\mathcal{N}(j_3) \frac{C\left(\frac{k}{2}-j_1,j_4,j\right) C\left(j,j_2,\frac{k}{2}-j_3\right)}{B(j)} \tag{3.46}$$
$$\times \left|z^{\Delta_j-\Delta_1-\Delta_4+j_2-h_3+\frac{k}{4}} x^{h_3-j_2-j_1-j_4}\right|^2 \frac{\pi\gamma(j_1+j_2+j_4-h_3)\gamma(j_3+h_3-k)\gamma(2j)}{\gamma\left(j_1+j_4+j-\frac{k}{2}\right)\gamma\left(j_2+j_3+j-\frac{k}{2}\right)}.$$

As usual, the overall powers of x and z allow us to recognize the intermediate channel in question. It describes the exchange of a state with $\omega = 1$, which is indeed the only channel allowed by the selection rules (2.21), and $h = h_3 - j_2$. The numerical constants then combine to give the relevant product of flowed structure constants,

$$\frac{C_{(001)}(j_1,j_4,j,h)C_{(102)}(j,j_2,j_3,h,h_3)}{R(j,h,1)}, \tag{3.47}$$

which can be found in [6, 44].

3.4 Lessons for the general case

Let us try and build on the results obtained in the previous sections and provide some intuition for the more general cases. On general grounds, worldsheet conformal invariance combined with the spectrum of the SL(2,$\mathbb{R}$) WZW model implies that four-point functions should factorize (in the $14 \to 23$ channel) schematically as follows:

$$\dddot{\tilde{\mathcal{F}}}_{\boldsymbol{\omega}}(x,z) = \sum_{\Delta} \frac{C_{1,4,\Delta}C_{\Delta,2,3}}{B_\Delta} |F_\Delta(x,z)|^2 \,. \tag{3.48}$$

Here we sum over all possible primary intermediate states of conformal dimension Δ, while $F_\Delta(x,z)$ represents the corresponding conformal block, which also accounts for the descendant contributions. Let us emphasize that we are discussing Virasoro primaries, as opposed to the affine ones, hence we must include the spectrally flowed operators. The z-independent factors $C_{ij\Delta}$ and B_Δ then capture the relevant structure constants and inverse propagator. In our context the sum over Δ is actually a sum over the spectral flow charge ω of the exchanged state, together with a double integral over its flowed and unflowed spins, namely h and j, respectively.

When the external ω_i vanish and the corresponding j_i are sufficiently closed to those of the continuous representations, only continuous unflowed states are exchanged, hence the integral over j is along the contour $\frac{1}{2}+i\mathbb{R}$ [6, 8]. The above examples, and, more generally, the conjecture of [34], suggest that, for some appropriately defined range of values for the external spins j_i, which does not necessarily coincide with Teschner's strip (2.47), one has

$$\dddot{\tilde{\mathcal{F}}}_{\boldsymbol{\omega}}(x,z) = \int_{\frac{1}{2}+i\mathbb{R}} dj \left[\frac{C_{(\omega_1,\omega_4,0)}(j_1,j_4,j,h_1,h_4)C_{(0,\omega_2,\omega_3)}(j,j_2,j_3,h_2,h_3)}{B(j)}|F^0_j(x,z)|^2\right. \tag{3.49}$$
$$\left. + \sum_{\omega>0}\sum_{h} \frac{C_{(\omega_1,\omega_4,\omega)}(j_1,j_4,j,h_1,h_4,h)C_{(\omega,\omega_2,\omega_3)}(j,j_2,j_3,h,h_2,h_3)}{R(j,h,\omega)}|F^{\omega}_{jh}(x,z)|^2\right],$$

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with $B(j)$, $R(j,h,\omega)$ and $C_\omega(j_i,h_i)$ defined in eqs. (2.8) and (2.30). Here we have separated the contributions from the exchange of unflowed states from the flowed ones because for the latter there is an additional sum over the allowed values of h (and $\bar{h}$). At leading order in z the conformal blocks must be of the form[6]

$$
\begin{aligned}
|F^{\omega}_{jh}(x,z)|^2 = & |z^{\Delta-\Delta_1-\Delta_4}x^{h-h_1-h_4}|^2 \\
& \times \frac{(2h-1)^2}{\pi^2}\int d^2x' d^2x'' |\,(x'-x'')^{2h-2}\,x'^{h_4-h_1-h}(x-x')^{h_1-h_4-h}(1-x'')^{h_3-h_2-h}|^2\,,
\end{aligned}
\tag{3.50}
$$

where for $\omega=0$ we take $h=j$. As in the unflowed case the integral eq. (3.50) is of the hypergeometric type, albeit with all j_i replaced by the corresponding h_i.

The main obstacle for showing that the proposal of [34] factorizes in this way is that the complicated expressions in (2.55) and (2.56) do not allow us to obtain the four-point functions in closed form. The reason is two-fold. On the one hand, they provide integral expressions for the former in terms of the y_i-variables, but the structure of the generalized differences X_I and cross-ratio X where these variables appear is quite involved, so much so that even in the three-point function case (where the Moebius-fixed unflowed correlator is simply a constant) it has not been possible to compute these integrals in closed form. On the other hand, the integrands involve the unflowed four-point functions which, as discussed in section 2.3, are not known in closed form. Nevertheless, below we will extend the discussion of the particular examples considered so far to four-point functions with arbitrary spectral flow charges, which leads to an important number of non-trivial results concerning the factorization properties of the expressions proposed in [34].

Let us now try to provide a heuristic intuition for why we have only been able to isolate exchanges with spacetime spins such as $h=h_1-h_4$ when dealing with flowed channels. (Here we assume for simplicity that $\omega_1 \geq \omega_4$). The main issue is that, due to the explicit form of the conformal dimension Δ given in eq. (2.4) the small z limit interacts non-trivially with the sum over h. In order to describe the consequences of this observation it will be useful to massage a bit the formal expansion presented in eqs. (3.49) and (3.50). Let us consider the contribution from the exchange of a state with a given spectral flow charge ω and write the flowed structure constants appearing in (3.49) in terms of their y-integrals. By using eqs. (2.7), (2.15) and (2.30), we get

$$
\begin{aligned}
\ddddot{\mathcal{F}}_\omega(x,z) \sim & \int_{\frac{1}{2}+i\mathbb{R}} dj \sum_h \frac{(2h-1)^2}{\pi^2}\int d^2y' d^2y'' d^2x' d^2x'' \prod_{i=1}^{4} d^2y_i\, |y_i^{h_i^0-h_i-1} z^{\Delta-\Delta_1-\Delta_4} \\
& \times y'^{h^0-h-1} y''^{\tilde{h}^0-h-1}|^2\, \ddddot{\mathcal{F}}_{(\omega_1,\omega_4,\omega)}(y_1,y_4,y')\, \ddddot{\mathcal{F}}^{(1-j)}_{(\omega,\omega_2,\omega_3)}(y'',y_2,y_3) \\
& \times |x^{h-h_1-h_4}\,(x'-x'')^{2h-2}\,x'^{h_4-h_1-h}(x-x')^{h_1-h_4-h}(1-x'')^{h_3-h_2-h}|^2\,,
\end{aligned}
\tag{3.51}
$$

with $h_i^0 = j_i + \frac{k}{2}\omega_i$, $h^0 = j+\frac{k}{2}\omega$ and $\tilde{h}^0 = 1-j+\frac{k}{2}\omega$. In the second three-point function the superscript indicates that one has to replace $j \to 1-j$. We can now remove the dependence of the overall power of z in h by rescaling

$$
y_1 \to y_1 z^{\omega_1}\,, \qquad y_4 \to y_4 z^{\omega_4}\,, \qquad y' \to y' z^{-\omega}\,,
\tag{3.52}
$$

[6]Strictly speaking, eq. (3.50) is valid only up to potential monodromy projections removing the contribution of the shadow operator [45]. In all cases considered in this paper this projection trivializes.

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which gives

$$
\begin{aligned}
\ddddot{\mathcal{F}}_{\boldsymbol{\omega}}(x,z) \sim \int_{\frac{1}{2}+i\mathbb{R}} dj \sum_h \frac{(2h-1)^2}{\pi^2} \int d^2y' d^2y'' d^2x' d^2x'' \prod_{i=1}^{4} d^2y_i \, |y_i^{h_i^0-h_i-1} z^{\Delta_j^0-\Delta_1^0-\Delta_4^0} \\
\times\, y'^{h^0-h-1} y''^{\tilde{h}^0-h-1}|^2 \dddot{\mathcal{F}}_{(\omega_1,\omega_4,\omega)}\left(y_1 z^{\omega_1}, y_4 z^{\omega_4}, \frac{y'}{z^{\omega}}\right) \dddot{\mathcal{F}}^{(1-j)}_{(\omega,\omega_2,\omega_3)}(y'', y_2, y_3) \\
\times\, |x^{h-h_1-h_4} \left(x'-x''\right)^{2h-2} x'^{h_4-h_1-h}(x-x')^{h_1-h_4-h}(1-x'')^{h_3-h_2-h}|^2 \,,
\end{aligned}
\tag{3.53}
$$

where $\Delta_i^0 = -\frac{j_i(j_i-1)}{k-2} - j_i\omega_i - \frac{k}{4}\omega_i^2$. This suggests that, at least in terms of scaling with z, the new y_i should be identified with those appearing in eqs. (2.55) and (2.56). The issue is that the arguments of $\dddot{\mathcal{F}}_{(\omega_1,\omega_4,\omega)}$ now scale either to zero or to infinity as we take $z \to 0$. This is only well defined for generic values of h_i when it reduces to a collision limit analogous to the one in eq. (2.31). As a consequence, we are able to select channels for which $h = h_1 - h_4$ and $\omega = \omega_1 - \omega_4 + \delta$ with $\delta = 0, \pm 1$ by implementing

$$
y_1 \to y_1 z^{-\omega_4+\delta} \,, \qquad y_4 \to y_4 z^{-\omega_1-\delta}
\tag{3.54}
$$

In such cases, the form of $\dddot{\mathcal{F}}_{(\omega_1,\omega_4,\omega)}$ reduces to that given in eq. (2.32). The integration over x' and x'' simplifies drastically for this value of h, which also allows us to make use of the identity (A.26) derived in the appendix. After the dust settles, we find that these particular contributions can be written as

$$
\begin{aligned}
\ddddot{\mathcal{F}}_{\boldsymbol{\omega}}(x,z) \sim \int_{\frac{1}{2}+i\mathbb{R}} dj \int \prod_{i=1}^{4} d^2y_i \, |y_i^{h_i^0-h_i-1} z^{\Delta-\Delta_1-\Delta_4} x^{-2h_4}|^2 \\
\times \int d^2y \, |y^{2j-2}|^2 \dddot{\mathcal{F}}_{(\omega_1,\omega_4,\omega_1-\omega_4+\delta)}\left(y_1, y_4, y\right) \dddot{\mathcal{F}}^{(1-j)}_{(\omega_1-\omega_4+\delta,\omega_2,\omega_3)}(y^{-1}, y_2, y_3) \,.
\end{aligned}
\tag{3.55}
$$

Recall that the dependence of three-point functions with respect to the y-variables is much more complicated than that in x or z. The latter is fixed in the usual way by conformal symmetry, while the former depends non-trivially on the choice of spectral flow charges, see eq. (2.23). Nevertheless, the crucial point is that even though the factors $Z_I(y_i)$ are not simple differences such as y_{ij}, they remain linear in all the y_i, and their powers depend on the unflowed spins j_i analogously to the unflowed case (up to shifts of the form $j_i \to \frac{k}{2} - j_i$ in the odd parity case). This implies that the integral appearing in the second line of eq. (3.55) is again of the hypergeometric type and can be carried out following appendix (A.1). The explicit result will be given below when discussing several cases in detail.

Finally, let us also state that, depending on the values of the spectral flow charges, a similar analysis involving rescalings of y_2 and y_3 with factors of z^{ω} where $\omega = \pm(\omega_3 - \omega_2) + \delta$ can be used to compute the contribution from channels where the three-point function on the r.h.s. becomes extremal, i.e. with either $h = h_3 - h_2$ or $h = h_2 - h_3$. The same goes for channels with $\omega = \omega_1 + \omega_4$, etc.

4 Edge cases and m-basis limits

In this section we move past particular cases and start testing the factorization properties of the SL(2,$\mathbb{R}$) four-point functions proposed in [34] for more general spectral flow configurations. As a stepping stone, we consider correlators which admit a well-defined m-basis limit. The latter were dubbed as edge cases in [31, 33, 34].

4.1 Factorization in the m-basis

In the m-basis approach [6, 13] one usually works with the so-called flowed primary states, denoted as $V^{\omega}_{jm}(z)$ [32]. They are obtained by considering the affine primary states and applying the spectral flow operation along the direction of the Cartan current $J^3(z)$. One can obtain them as limits of the x-basis operators we have been using so far:

$$\lim_{x\to 0} V^{w}_{jh}(x,z) = V^{\omega}_{jm}(z)\,, \qquad \lim_{x\to\infty} x^{2h} V^{w}_{jh}(x,z) = V^{-\omega}_{j,-m}(z)\,, \qquad m = h - \frac{k}{2}\omega\,. \tag{4.1}$$

The m-basis four-point function

$$\langle V^{w_1}_{j_1 m_1}(0) V^{w_2}_{j_2 m_2}(1) V^{-w_3}_{j_3,-m_3}(\infty) V^{w_4}_{j_4 m_4}(z)\rangle \tag{4.2}$$

thus corresponds to the collision limit[7] $x_2, x_4 \to 0$ of

$$\langle V^{w_1}_{j_1}(0,y_1,z_1) V^{w_2}_{j_2}(x_2,y_2,1) V^{w_3}_{j_3}(\infty,y_3,\infty) V^{w_4}_{j_4}(x_4,y_4,z)\rangle\,. \tag{4.3}$$

Since the global Ward identities (2.53) imply that for generic x_2 and x_4 this correlator takes the form

$$|x_2|^{2(-h_1^0-h_2^0+h_3^0-h_4^0)} \left\langle V^{w_1}_{j_1}\left(0,\frac{y_1}{x_2},0\right) V^{w_2}_{j_2}\left(1,\frac{y_2}{x_2},1\right) V^{w_3}_{j_3}\left(\infty, y_3\, x_2, \infty\right) V^{w_4}_{j_4}\left(\frac{x_4}{x_2},\frac{y_4}{x_2},z\right)\right\rangle,$$

one finds that the limit in (4.3) is only well-defined for spectral flow charges satisfying [34]

$$|w_3 - w_1 - w_2 - w_4| \leq 2\,. \tag{4.4}$$

The resulting formulae for the y-basis four-point functions are drastically simplified in the above regime. We now study the factorization properties for the cases near the edge of the range allowed by the selection rules (2.21) separately. For simplicity, we will focus on a single flowed intermediate channel in each case.

Spectral flow conservation. We first consider the case $\omega_3 = \omega_1 + \omega_2 + \omega_4$, where the total spectral flow is conserved. After rescaling the y_i variables by the appropriate factors of z and $z-1$ and changing $y_3 \to -y_3^{-1}$ one gets [34]

$$\begin{aligned}
\langle V^{w_1}_{j_1 m_1}(0) V^{w_2}_{j_2 m_2}(1) V^{-w_3}_{j_3,-m_3}(\infty) V^{w_4}_{j_4 m_4}(z)\rangle &= \left|z^{-\frac{k w_1 w_4}{2} - w_1 m_4 - w_4 m_1}(1-z)^{\frac{-k w_2 w_4}{2} - w_2 m_4 - w_4 m_2}\right|^2 \\
&\times \int \prod_{i=1}^{4} d^2 y_i \prod_{i\neq 3} \left|y_i^{j_i - m_i - 1} y_3^{j_3+m_3-1} y_{12}^{-j_1-j_2+j_3-j_4} y_{13}^{-j_1+j_2-j_3+j_4} y_{23}^{j_1-j_2-j_3+j_4} y_{34}^{-2j_4}\right|^2 \\
&\times \left\langle V^0_{j_1}(0;0) V^0_{j_2}(1;1) V^0_{j_3}(\infty;\infty) V^0_{j_4}\left(\frac{y_{32}\, y_{41}}{y_{21}\, y_{34}}; z\right)\right\rangle\,.
\end{aligned} \tag{4.5}$$

[7] One could also consider taking $x_4 \to 0$ but $x_2 \to \infty$. We will consider such configurations later on.

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We see that, in this limit, the y_i variables effectively play the role of the traditional x_i for the unflowed correlator. The result is then simply the usual m-basis unflowed expression.[8]

The factorization analysis thus parallels the unflowed one. The selection rules for the relevant three-point functions allow for a channel with $\omega = \omega_1 + \omega_4 = \omega_3 - \omega_2$. Further taking $x_2, x_4 \to 0$, the relevant expressions for the factorization limit take the form

$$\langle V_{j_1}^{w_1}(0,y_1,0) V_{j_4}^{w_4}(0,y_4,1) V_j^{w}(\infty,y,\infty)\rangle = \frac{C(j_1,j_4,j)}{|y_{14}^{j_1+j_4-j}(1+y_1 y)^{j+j_1-j_4}(1+y_4 y)^{j+j_4-j_1}|^2}\,, \tag{4.6}$$

$$\langle V_{j}^{w}(0,y',0) V_{j_2}^{w_2}(0,y_2,1) V_{j_3}^{w_3}(\infty,y_3,\infty)\rangle = \frac{C(j,j_2,j_3)|(y'-y_2)^{j_3-j_2-j}|^2}{|(1+y_2 y_3)^{j_2+j_3-j}(1+y_3 y')^{j+j_3-j_2}|^2}, \tag{4.7}$$

again up to a few unimportant signs, and where j is the intermediate spin. By taking $y \to -y^{-1}$ and $y_3 \to -y_3^{-1}$ one finds that, in this regime, the spectral flow preserving three-point functions take precisely the same form as the unflowed x-basis ones, but with $x_i \to y_i$.

It follows that the integration over the intermediate y thus parallels the derivation of [8] leading to the hypergeometric function appearing in eqs. (2.42)–(2.45). More explicitly, proceeding in analogy to what was done in the previous section and using (2.15) and (A.26) shows that we need to compute

$$\int d^2y |(y_1-y)^{j_4-j_1-j}(y_4-y)^{j_1-j_4-j}(yy_2-1)^{j_3-j_2+j-1}(yy_3-1)^{j_2-j_3+j-1}|^2\,,$$

which is of the hypergeometric type. Combined with the overall factor of $y_{14}^{j-j_1-j_4} y_{23}^{j-j_2-j_3}$, this precisely matches the factorization derived by inserting the small z expansion (at fixed y_i) of the unflowed correlator in eq. (4.5). Note that in this case the sum over the intermediate values of h in eq. (3.49) trivializes as a consequence of charge conservation [33]. In other words, given the values of the external m_i, the intermediate state must have $m = m_1 + m_4 = m_3 - m_2$. Of course, the overall power of z is consistent with our analysis: the factors coming from the expansion of the unflowed correlator add up to those in eq. (4.5), giving

$$\Delta_j^{(0)} - \Delta_1^{(0)} - \Delta_4^{(0)} - \frac{k w_1 w_4}{2} - w_1 m_4 - w_4 m_1 = \Delta - \Delta_1 - \Delta_4\,. \tag{4.8}$$

Spectral flow violation in one unit. We now consider the case $\omega_3 = \omega_1 + \omega_2 + \omega_4 + 1$ (the analysis for the case with $\omega_3 = \omega_1 + \omega_2 + \omega_4 - 1$ is analogous). Even though spectral flow is not a conserved quantity, non-vanishing m-basis n-point functions for which the total spectral flow is non-zero are usually referred to as spectral flow *violating* correlators. Manipulations similar to those of the previous section give

$$\begin{aligned}\langle V^{w_1}_{j_1 m_1}(0) V^{w_2}_{j_2 m_2}(1) V^{-w_3}_{j_3,-m_3}(\infty) V^{w_4}_{j_4 m_4}(z)\rangle &= \left| z^{-\frac{kw_1w_4}{2}-w_1m_4-w_4m_1}(1-z)^{\frac{-kw_2w_4}{2}-w_2m_4-w_4m_2}\right|^2 \\ &\times \int \prod_{i=1}^{4} d^2y_i \prod_{i\neq 3} |y_i^{j_i-m_i-1} y_3^{j_3+m_3-1}(y_1+y_2+y_3)^{\frac{k}{2}-j_1-j_2-j_3-j_4}|^2 \\ &\times \left\langle V^0_{j_1}(0;0) V^0_{j_2}(1;1) V^0_{\frac{k}{2}-j_3}(\infty;\infty) V^0_{j_4}\left(\frac{y_1+zy_3-y_4}{y_1+y_2+y_3};z\right)\right\rangle\,. \end{aligned} \tag{4.9}$$

[8]This can be understood from parafermionic decomposition [1], which implies that in the m-basis one only cares about the total spectral flow [11], which vanishes in this particular case.

Taking the small z limit and using the expansion for the unflowed correlator, we find that the y-integral becomes

$$\int_{\frac{1}{2}+i\mathbb{R}} dj \prod_{i=1}^{4} d^2y_i \prod_{i\neq 3} \Bigg| y_i^{j_i-m_i-1} y_3^{j_3+m_3-1} y_{14}^{j-j_1-j_4} (y_1+y_2+y_3)^{\frac{k}{2}-j-j_2-j_3} \\ \times {}_2F_1\left[\begin{matrix} j-j_1+j_4,\ j+j_2+j_3-\frac{k}{2} \\ 2j \end{matrix} \,\middle|\, \frac{y_{14}}{y_1+y_2+y_3}\right]\Bigg|^2 . \tag{4.10}$$

The latter can be obtained directly by integrating over the product of y-basis three-point functions involved in the exchange of a state with $\omega = \omega_1+\omega_4 = \omega_3-\omega_2-1$ (times the usual factor related to the corresponding reflection $j \to 1-j$) which, as before, is constrained by charge conservation. The only element we have not discussed so far is the three-point function on the r.h.s., which, up to signs, takes the form

$$\langle V_j^w(0,y',0) V_{j_2}^{w_2}(0,y_2,1) V_{j_3}^{w_3}(\infty,y_3,\infty)\rangle = \frac{\mathcal{N}(j_3) C\left(j,j_2,\frac{k}{2}-j_3\right)}{|y_3^{\frac{k}{2}+j_3-j_2-j}(1+y_2y_3+y'y_3)^{j+j_2+j_3-\frac{k}{2}}|^2} . \tag{4.11}$$

Indeed, the integral

$$\int d^2y |(y_1y-1)^{j_4-j_1+j-1}(y_4y-1)^{j_1-j_4+j-1}(1+y_2y_3+yy_3)^{\frac{k}{2}-j-j_2-j_3}|^2 ,$$

and the overall factors of $y_{14}^{1-j-j_1-j_4} y_3^{j+j_2-j_3-\frac{k}{2}}$ combine to give the integrand in (4.10). The product of the unflowed structure constants $C(j_1,j_4,j)$ and $\mathcal{N}(j_3)C\left(j,j_2,\frac{k}{2}-j_3\right)$, and the overall powers of z then lead to the expected factorization.

Spectral flow violation in two units. Finally, we consider the case $\omega_3 = \omega_1+\omega_2+\omega_4+2$. Correlators with $\omega_3 = \omega_1+\omega_2+\omega_4-2$ can be treated similarly. This corresponds to the situation where spectral flow violation is maximal [11, 18]. As discussed above, in this case, the conjecture of [34] looks slightly different. We get

$$\langle V_{j_1m_1}^{w_1}(0) V_{j_2m_2}^{w_2}(1) V_{j_3,-m_3}^{-w_3}(\infty) V_{j_4m_4}^{w_4}(z)\rangle = |z^{m_2-j_2-m_3-j_3}(1-z)^{m_1-j_1-m_3-j_3}|^2 f(z) \\ \times \int d^2y_i \prod_{i=1,2,4} |y_i^{j_i-m_i-1} y_3^{j_3+m_3-1}(y_1+y_2+y_3+y_4)^{k-j_1-j_2-j_3-j_4}|^2 , \tag{4.12}$$

where $f(z)$ is given in eq. (4.17) below.

Only a single channel is allowed, for which the exchanged state has spectral flow charge $\omega = \omega_1+\omega_4+1 = \omega_3-\omega_2-1$. After changing variables to $y_3 \to -y_3^{-1}$, the relevant integral can be written as

$$\int d^2y |(y+y_1+y_4)^{\frac{k}{2}-j_1-j_4+j-1} y_3^{2j_3-2}(y+y_2+y_3)^{\frac{k}{2}-j_2-j_3-j}|^2 \tag{4.13} \\ = \pi \frac{\gamma\left(\frac{k}{2}-j_1-j_4+j\right)\gamma(j_1+j_2+j_3+j_4-k)}{\gamma\left(j+j_2+j_3-\frac{k}{2}\right)} |(y_1+y_2+y_3+y_4)^{k-j_1-j_2-j_3-j_4}|^2 ,$$

where j is the spin of the intermediate state. Comparing with (4.12) at leading order in z, by means of (3.45) and (3.23), we obtain a consistent factorization limit.

4.2 Edge cases beyond the collision limit

Of course, the analysis performed above is quite restrictive, as the m-basis limit only captures a subset of four-point functions for which all spectral flows are, so to speak, along the same direction in x-space [19]. We would now like to relax this constraint, and work directly in the x-basis, i.e. for generic values of x_2 and x_4.

As it turns out, from the three choices of spectral flow charges considered in the previous section, the first two, namely those we have called spectral flow conserving and spectral flow violating by one unit, can be analyzed along the same lines as those with generic values of the ω_i, hence we will discuss them in the next section. Here we focus on the edge case

$$\omega_3 = \omega_1 + \omega_2 + \omega_4 + 2, \tag{4.14}$$

and set $(x_1, x_2, x_3, x_4) = (0, 1, \infty, x)$ as usual. For such configurations, the quantity $X_\emptyset$ appearing in the conjectured general formula (2.55) vanishes. However, one also finds that the generalized cross-ratio behaves as $X \to z$. It is therefore necessary to make use of the identity

$$z - X = \frac{X_\emptyset X_{1234}}{X_{12}X_{34}} \tag{4.15}$$

to rewrite the y-basis correlator as

$$\ddot{\bar{\mathcal{F}}}_{\text{edge}}(x, y_i, z) = |X_{1234}^{k-j_1-j_2-j_3-j_4} X_{12}^{2j_3-k} X_{13}^{-j_1+j_2-j_3+j_4} X_{23}^{j_1-j_2-j_3+j_4} X_{34}^{j_1+j_2+j_3-j_4-k}|^2 f(z), \tag{4.16}$$

where the function $f(z)$ is defined as a limit of the unflowed correlator, namely

$$f(z) = \lim_{x\to z} |x - z|^{2(j_1+j_2+j_3+j_4-k)} \ddot{\bar{\mathcal{F}}}_0(x, z)\,. \tag{4.17}$$

As in the example considered in section 3.3, we will be able to carry out the integration over the y_i variables explicitly. In order to show this we need to specify the relevant X_I factors. For this choice of spectral flow charges, many of the polynomials $P_{\boldsymbol{\omega}}(x, z)$ involved in the definition of the X_I either vanish or trivialize. We have

$$X_{12} = P_{\boldsymbol{\omega}+e_1+e_2}\,, \quad X_{13} = P_{\boldsymbol{\omega}+e_1-e_3} y_3\,, \quad X_{23} = P_{\boldsymbol{\omega}+e_2-e_3} y_3\,, \quad X_{34} = P_{\boldsymbol{\omega}-e_3+e_4} y_3\,, \tag{4.18}$$

and

$$\begin{aligned} \frac{X_{1234}}{\sqrt{z(1-z)}} = P_{\boldsymbol{\omega}+e_1+e_2+e_3+e_4} + y_3 (P_{\boldsymbol{\omega}+e_1+e_2-e_3+e_4} + P_{\boldsymbol{\omega}-e_1+e_2-e_3+e_4} y_1 \\ + P_{\boldsymbol{\omega}+e_1-e_2-e_3+e_4} y_2 + P_{\boldsymbol{\omega}+e_1+e_2-e_3-e_4} y_4)\,. \end{aligned} \tag{4.19}$$

Here and in what follows we leave the dependence of the $P_{\boldsymbol{\omega}}(x, z)$ in x and z implicit. By rescaling the y_i-variables appropriately we can express the exact x-basis four-point function

$$\ddot{\bar{\mathcal{F}}}_{\text{edge}}(x, z) = \int d^2 y_i \prod_{i=1}^{4} |y_i^{j_i + \frac{k}{2}\omega_i - h_i - 1}|^2 \ddot{\bar{\mathcal{F}}}_{\text{edge}}(x, y_i, z), \tag{4.20}$$

as

$$\dddot{\bar{\mathcal{F}}}_{\text{edge}}(x,z) = h(x,z) \int d^2 y_i \prod_{i\neq 3} \left| y_i^{j_i+\frac{k}{2}\omega_i - h_i - 1} y_3^{h_1+h_2+h_4-h_3-1} \left(1+\sum_{i=1}^{4} y_i\right)^{k-j_1-j_2-j_3-j_4} \right|^2$$
$$= \frac{\pi^4 h(x,z)\gamma(h_1+h_2+h_4-h_3)\gamma\left(j_3+h_3-\frac{k}{2}\omega_3\right)\prod_{i\neq 3}\gamma\left(j_i+\frac{k}{2}\omega_i-h_i\right)}{\gamma(j_1+j_2+j_3+j_4-k)}, \tag{4.21}$$

with

$$h(x,z) = f(z) \left| \frac{P_{\boldsymbol{\omega}+e_1-e_3}^{-j_3-j_1+j_2+j_4} P_{\boldsymbol{\omega}+e_2-e_3}^{j_1-j_2-j_3+j_4}}{[z(1-z)]^{\frac{1}{2}(j_1+j_2+j_3+j_4-k)} P_{\boldsymbol{\omega}+e_1+e_2}^{k-2j_3} P_{\boldsymbol{\omega}-e_1+e_2-e_3+e_4}^{\alpha_1}} \right.$$
$$\left. \times \frac{P_{\boldsymbol{\omega}-e_3+e_4}^{j_1+j_2+j_3-j_4-k} P_{\boldsymbol{\omega}+e_1+e_2+e_3+e_4}^{j_1+j_2+j_4-h_3+\frac{k}{2}(\omega_3-2)}}{P_{\boldsymbol{\omega}+e_1-e_2-e_3+e_4}^{\alpha_2} P_{\boldsymbol{\omega}+e_1+e_2-e_3-e_4}^{\alpha_4} P_{\boldsymbol{\omega}+e_1+e_2-e_3+e_4}^{h_1+h_2+h_4-h_3}} \right|^2. \tag{4.22}$$

In order to analyze this result from the factorization perspective we consider the behaviour of the relevant $P_{\boldsymbol{\omega}}(x,z)$ in the $z\to 0$ limit. This is done in appendix A.2, where we show that they reduce to a numerical prefactor of the form $N_{\boldsymbol{\omega}} = n_{\boldsymbol{\omega}} \tilde{P}_{\boldsymbol{\omega}}(1,0)$, together with a number of powers of x and $(1-x)$. Further using eq. (3.45) leads to the following small z behavior of the function $h(x,z)$:

$$h(x,z) \sim \left| z^{\Delta_j - \Delta_1 - \Delta_4 + \frac{k\omega_1\omega_4}{2} + \frac{k}{2}(\omega_3-\omega_2) - \frac{3k}{4} - (h_3-h_2)(\omega_3-\omega_2-1) + \omega_1 h_1 + \omega_4 h_4} \right.$$
$$\left. \times \left(N_{++-+} x\right)^{h_3-h_1-h_2-h_4} \right|^2 \frac{\gamma(2j)\gamma(j_1+j_2+j_3+j_4-k)}{\gamma\left(j_1+j_4+j-\frac{k}{2}\right)\gamma\left(j_2+j_3+j-\frac{k}{2}\right)}. \tag{4.23}$$

The overall powers of x and z we have obtained show that we have picked up the contribution coming from the exchange of an intermediate state with quantum numbers

$$\omega = \omega_3 - \omega_2 - 1 = \omega_1 + \omega_4 + 1\,, \qquad h = h_3 - h_2\,. \tag{4.24}$$

Recall that this value of ω is the only one allowed by the selection rules (2.21). We conclude that, as long as the external spins j_i are in the range (3.36), the small z limit of four-point functions saturating the bound on spectral flow can be written as

$$\dddot{\bar{\mathcal{F}}}_{\text{edge}}(x,z) \sim \int_{\frac{1}{2}+i\mathbb{R}} dj \left| z^{\Delta_j-\Delta_1-\Delta_4} \left(N_{++-+}x\right)^{h-h_1-h_4} \right|^2 C\left(j_1,j_4,\frac{k}{2}-j\right) C\left(\frac{k}{2}-j,j_2,j_3\right)$$
$$\times \frac{\mathcal{N}(j)^2 \pi^4 \gamma(2j)\gamma\left(j_3+h_3-\frac{k}{2}\omega_3\right)\prod_{i\neq 3}\gamma\left(j_i+\frac{k}{2}\omega_i-h_i\right)}{B(j)\gamma\left(j_1+j_4+j-\frac{k}{2}\right)\gamma\left(j_2+j_3+j-\frac{k}{2}\right)}. \tag{4.25}$$

This is consistent with the expected factorization structure. Indeed, for these values of ω and h we find that the constants in this expression precisely match the product of three-point functions appearing in eq. (3.49), namely

$$\frac{C_{(\omega_1,\omega_4,\omega_1+\omega_4+1)}(j_i,h_i) C_{(\omega_3-\omega_2-1,\omega_2,\omega_3)}(j_i,h_i)}{R(j,h,\omega)}\,, \tag{4.26}$$

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thanks to the identity

$$N_{++-+} = n_{(\omega_1+1,\omega_2+1,\omega_3-1,\omega_4+1)}\tilde{P}_{(\omega_1+1,\omega_2+1,\omega_3-1,\omega_4+1)}(1,0) = Q_{(\omega_3-\omega_2,\omega_2+1,w_3-1)}. \quad (4.27)$$

This is a particular case of eq. (A.14). It corresponds to a non-trivial relation between the (small z limit of the) polynomials $P_{\boldsymbol{\omega}}(x,z)$ involved in the conjecture of [34] for flowed four-point functions and the numerical factors $Q_{\boldsymbol{\omega}}$ appearing in the flowed three-point functions of [31, 33].

5 Correlators with arbitrary spectral flow charges

We now discuss the general case, namely, four-point correlators with arbitrary assignments of spectral flow charges. We first show that, whenever it is allowed by the AdS$_3$ selection rules, the formula put forward in [34] for flowed four-point functions in terms of their unflowed counterparts consistently accounts for the exchange of unflowed states. We then move to arbitrary insertions and show that, in the small z limit, the properties of correlation functions in the unflowed sector of the model lead to consistent contributions coming from the exchange of states with a non-zero spectral flow charge. The precise agreement with the structure anticipated in eq. (3.49) we derive involves the full set of highly non-trivial spectrally flowed three-point functions obtained in [31, 33].

5.1 Contributions from the exchange of unflowed states

As it follows from (2.21), only four-point functions for which the external spectral flow charges satisfy

$$|\omega_1 - \omega_4| \leq 1\,, \qquad |\omega_3 - \omega_2| \leq 1\,, \quad (5.1)$$

can contain contributions from the exchange of unflowed states along the $14 \to 23$ channel. For concreteness, we first focus on the cases where $\omega_1 = \omega_4 = \omega_L$ and $\omega_2 = \omega_3 = \omega_R$, and then briefly discuss other situations. Consequently, the original four-point function and both three-point functions involved in the following discussion will be of the even-parity type.

The relevant expression for the four-point function under consideration is given in eq. (2.55). The analysis of section A.2 implies that, after the following rescaling

$$y_1 \to (-1)^{\omega_L}\frac{x}{z^{\omega_L}}y_1, \qquad y_4 \to \frac{x}{z^{\omega_L}}y_4, \qquad y_2 \to (-1)^{\omega_R}y_2, \quad (5.2)$$

the relevant factors X_I, at leading order in z, take the form

$$\begin{aligned}
X_{14} &\to z^{-\frac{(\omega_L+1)^2}{2}+\frac{1}{2}}x^{\omega_L+1}(1+y_1y_4),\\
X_{23} &\to z^{-\frac{\omega_L^2}{2}}x^{\omega_L}(1+y_2y_3),\\
X_{12} &\to z^{-\frac{\omega_L(\omega_L+1)}{2}}x^{\omega_L+1}(1+y_2+y_1(1-x+y_2)),\\
X_{13} &\to z^{-\frac{\omega_L(\omega_L+1)}{2}}x^{\omega_L+1}(1-y_3+y_1(1+(x-1)y_3)),\\
X_{34} &\to z^{-\frac{\omega_L(\omega_L+1)}{2}}x^{\omega_L+1}(1-y_4+y_3(x-1+y_4)).
\end{aligned} \quad (5.3)$$

In particular, this shows that for the generalized cross-ratio we have

$$\frac{X_{14}X_{23}}{X_{12}X_{34}} \to \frac{x(1+y_1y_4)(1+y_2y_3)}{(1+y_2+y_1(1-x+y_2))(1-y_4+y_3(x-1+y_4))}\,. \tag{5.4}$$

Therefore, after using (2.45) under the assumption that the j_i are in the range (2.47), we obtain a contribution to the factorized expansion of the form

$$\begin{aligned}\dddot{\mathcal{F}}_{\boldsymbol{\omega}}(x,z) \sim \int_{\frac{1}{2}+i\mathbb{R}} dj\mathcal{C}(j)|x^{j-h_1-h_4}z^{\Delta_j^{(0)}-\Delta_1-\Delta_4}|^2 \int d^2y_i \prod_{i=1}^{4}\Big|y_i^{j_i+\frac{k}{2}\omega_i-h_i-1} \\ \times (1+y_2+y_1(1-x+y_2))^{-j-j_2+j_3}(1-y_4+y_3(x-1+y_4))^{j_1-j-j_4} \\ \times (1-y_3+y_1(1+(x-1)y_3))^{-j_1+j_2-j_3+j_4}(1+y_1y_4)^{j-j_1-j_4}(1+y_2y_3)^{j-j_2-j_3} \\ \times {}_2F_1\left[\begin{matrix}j-j_1+j_4,\ j-j_3+j_2\\ 2j\end{matrix}\,\middle|\,\frac{x(1+y_1y_4)(1+y_2y_3)}{(1+y_2+y_1(1-x+y_2))(1-y_4+y_3(x-1+y_4))}\right]\Big|^2\,.\end{aligned} \tag{5.5}$$

Here we have written the leading order expression for the unflowed correlator as in eq. (2.44) and used the shorthand $\mathcal{C}(j) = C(j_1,j_4,j)C(j,j_2,j_3)/B(j)$. The overall powers of x and z we have obtained in (5.5) show that we have indeed isolated the contribution associated with the exchange of an unflowed state of spin j.

In order to show that this expression is consistent with the expectation set up in section 3.4 we proceed by reverse engineering. For the case at hand, the relevant contribution to the r.h.s. of (3.49) reads

$$\dddot{\mathcal{F}}_{\boldsymbol{\omega}}(x,z) \sim \int_{\frac{1}{2}+i\mathbb{R}} dj\,\frac{C_{(\omega_L,\omega_L,0)}(j_i,h_i)C_{(0,\omega_R,\omega_R)}(j_i,h_i)}{B(j)}|F^0_{jh}(x,z)|^2\,, \tag{5.6}$$

where, at leading order in z,

$$F^0_{jh}(x,z) = z^{\Delta_j^{(0)}-\Delta_1-\Delta_4}x^{j-h_1-h_4}{}_2F_1\left[\begin{matrix}j-h_1+h_4,\ j-h_3+h_2\\ 2j\end{matrix}\,\middle|\,x\right]. \tag{5.7}$$

This can be re-written in terms of an integral over two auxiliary variables x' and x'', interpreted as the locations of the intermediate vertex operators. This is analogous to (2.46), but with the replacements $j_i \to h_i$ for the external spins. On the other hand, inserting the integral expressions for the flowed structure constants obtained from the y-basis analysis, we have

$$\begin{aligned}\frac{C_{(\omega_L,\omega_L,0)}(j_i,h_i)C_{(0,\omega_R,\omega_R)}(j_i,h_i)}{B(j)} = \mathcal{C}(j)\int d^2y_i \prod_{i=1}^{4}|y_i^{h_i^0-h_i-1}(1+y_1y_4)^{j-j_1-j_4} \\ \times (1+y_2y_3)^{j-j_2-j_3}(y_4+1)^{j_1-j_4-j}(y_1-1)^{j_4-j_1-j}(y_3-1)^{j_2-j_3-j}(y_2+1)^{j_3-j_2-j}|^2\,.\end{aligned} \tag{5.8}$$

with $h_i^0 = j_i + \frac{k}{2}\omega_i$. Plugging this back into eq. (5.6), we see that the full dependence on the h_i quantum numbers coming from the hypergeometric integral can be eliminated by rescaling

$$y_1 \to \frac{x-x'}{x'}y_1\,, \qquad y_2 \to \frac{y_2}{1-x''}\,, \qquad y_3 \to (1-x'')y_3\,, \qquad y_4 \to \frac{x'}{x-x'}y_4\,. \tag{5.9}$$

Indeed, as could have been anticipated from the global Ward identity (2.28), this effectively replaces the h_i appearing in the powers of x', $(x-x')$ and $(1-x'')$ by the corresponding h_i^0.

Moreover, for the spectral flow charges under consideration we have that $h_1^0 - h_4^0 = j_1 - j_4$ and $h_2^0 - h_3^0 = j_2 - j_3$, hence the resulting factors nicely combine with those involving the y_i variables, thus leading to

$$\bar{\mathcal{F}}_\omega(x,z) \sim \int_{\frac{1}{2}+i\mathbb{R}} dj\mathcal{C}(j) \int d^2y_i d^2x' d^2x'' \prod_{i=1}^{4} |y_i^{h_i^0-h_i-1} (xy_1 + x'(1-y_1))^{j_4-j_1-j} \times (x - x'(-y_4+1))^{j_1-j_4-j} (y_3 - 1 - y_3x'')^{j_2-j_3-j} (y_2 + 1 - x'')^{j_3-j_2-j} (x'-x'')^{2j-2} |^2 . \tag{5.10}$$

We have landed on an integral over x' and x'' which is again of the hypergeometric type, with all powers written in terms of the unflowed spins j_i. This showcases how the presence of the integrals over the y_i associated to the external states allows one to perform changes of variables that, roughly speaking, end up rewriting flowed quantities in terms of the unflowed ones. After integrating out x' and x'', we finally re-obtain the expression derived from the conjectured formula for the flowed four-point function, namely eq. (5.5).

For completeness, let us provide an additional example involving an odd three-point function. We set $\omega_1 = \omega_4 + 1 = \omega_L + 1$ and $\omega_2 = \omega_3 = \omega_R$. After taking $y_1 \to (-z)^{-\omega_1} xy_1$ and $y_4 \to z^{-\omega_4} xy_4$, the corresponding generalized cross-ratio behaves at small z as

$$\frac{X_2 X_{134}}{X_4 X_{123}} \to \frac{z(1+y_2)(1-y_3+y_1(1-y_4+y_3(x-1+y_4)))}{xy_1(1+y_2y_3)} . \tag{5.11}$$

Therefore, after using the expansion (2.51), the relevant contribution to the four-point becomes

$$\begin{aligned}\bar{\mathcal{F}}_\omega(x,z) \sim & \int_{\frac{1}{2}+i\mathbb{R}} dj\mathcal{C}(j) |z^{\Delta_j^{(0)}-\Delta_1-\Delta_4} x^{j-h_1-h_4}|^2 \int \prod_{i=1}^{4} d^2y_i \Big| y_i^{h_i^0-h_i-1} y_1^{j+j_4-j_1-\frac{k}{2}} \\ & \times (1+y_2y_3)^{j-j_2-j_3} (1+y_2)^{j_3-j_2-j} (1-y_3)^{j_1+j_2-j_3+j_4-\frac{k}{2}} \\ & \times (1-y_3+y_1(1-y_4+y_3(x-1+y_4)))^{\frac{k}{2}-j-j_1-j_4} \\ & \times {}_2F_1\left[\begin{matrix} j+j_1+j_4-\frac{k}{2},\ j-j_3+j_2 \\ 2j \end{matrix} \,\middle|\, \frac{xy_1(1+y_2y_3)}{(1+y_2)(1-y_3+y_1(1-y_4+y_3(x-1+y_4)))}\right] \Big|^2 ,\end{aligned} \tag{5.12}$$

where $\mathcal{C}(j)$ now stands for

$$\mathcal{C}(j) = \frac{\mathcal{N}(j_1) C\left(\frac{k}{2} - j_1, j_4, j\right) C(j, j_2, j_3)}{B(j)} . \tag{5.13}$$

As before, the overall dependence in z and x matches with the desired unflowed intermediate channel.

In order to re-derive this expression from the expected factorized form (3.49) we proceed analogously to the previous case. In other words, we combine the hypergeometric integral (2.46) with all external j_i replaced by h_i with the y-basis expression

$$\frac{C_{(\omega_L+1,\omega_L,0)}(j_i,h_i) C_{(0,\omega_R,\omega_R)}(j_i,h_i)}{B(j)} = \mathcal{C}(j) \int \prod_{i=1}^{4} d^2y_i |y_i^{h_i^0-h_i-1} y_1^{j+j_4-j_1-\frac{k}{2}} (1+y_2)^{j_3-j_2-j} (1-y_1(1+y_4))^{\frac{k}{2}-j_1-j_4-j} (1-y_3)^{j_2-j_3-j} (1+y_2y_3)^{j-j_2-j_3} |^2 , \tag{5.14}$$

perform the appropriate rescalings, use that $h_1^0 - h_4^0 = j_1 - j_4 + \frac{k}{2}$, and finally compute the integral over x' and x''. The rest of the combinations allowed by (5.1) can be analyzed similarly.

5.2 Contributions from the exchange of spectrally flowed states

We now move to the exchange of states with non-zero spectral flow charges, which allows us to work with arbitrary values of the external ω_i. For concreteness, we first focus on four-point functions of odd-parity type, i.e. those described by eq. (2.56), and assume that the ω_i satisfy the following inequalities:

$$\omega_1 - \omega_4 > |\omega_2 - w_3| \,. \tag{5.15}$$

This is the first option considered in eq. (A.13), which showcases how the corresponding polynomials $P_{\boldsymbol{\omega}}(x,z)$ behave at small z. The identity (A.14) then allows us to derive the leading order expressions for each of the terms in the relevant X_I, see eq. (2.59). Roughly speaking, we find that all X_I reduce to one of the Z_I factors appearing in the flowed three-point functions (2.23) that are expected to appear in a given contribution to the factorization limit. Indeed, at small z, we get

$$\begin{aligned}
X_1 &\sim z^{-\frac{(\omega_1+1)\omega_4}{2}} x^{\omega_4} \left[Q_{(\omega_1-\omega_4+1,\omega_2,\omega_3)} + Q_{(\omega_1-\omega_4-1,\omega_2,\omega_3)}(-z)^{\omega_4} y_1\right] , \\
X_2 &\sim z^{-\frac{\omega_1\omega_4}{2}} x^{\omega_4} \left[Q_{(\omega_1-\omega_4,\omega_2+1,w_3)} + Q_{(\omega_1-\omega_4,\omega_2-1,w_3)} y_2\right] , \\
X_3 &\sim z^{-\frac{\omega_1\omega_4}{2}} x^{\omega_4} \left[Q_{(\omega_1-\omega_4,\omega_2,w_3+1)} + Q_{(\omega_1-\omega_4,\omega_2,w_3-1)} y_3\right] , \\
X_4 &\sim z^{-\frac{\omega_1(\omega_4+1)}{2}} x^{\omega_4+1} \left[Q_{(\omega_1-\omega_4-1,\omega_2,\omega_3)} - Q_{(\omega_1-\omega_4+1,\omega_2,\omega_3)} z^{\omega_1} x^{-2} y_4\right] ,
\end{aligned} \tag{5.16}$$

with the $Q_{\boldsymbol{\omega}}$ defined in (2.26), while

$$\begin{aligned}
X_{123} \sim z^{-\frac{(\omega_1+1)\omega_4}{2}} x^{\omega_4} \Big[& Q_{(\omega_1-\omega_4+1,\omega_2+1,\omega_3+1)} + Q_{(\omega_1-\omega_4-1,\omega_2+1,\omega_3+1)}(-z)^{\omega_4} y_1 \\
& + Q_{(\omega_1-\omega_4+1,\omega_2-1,\omega_3+1)} y_2 + Q_{(\omega_1-\omega_4+1,\omega_2+1,\omega_3-1)} y_3 + Q_{(\omega_1-\omega_4-1,\omega_2-1,\omega_3+1)}(-z)^{\omega_4} y_1 y_2 \\
& + Q_{(\omega_1-\omega_4+1,\omega_2-1,\omega_3-1)} y_2 y_3 + Q_{(\omega_1-\omega_4-1,\omega_2+1,\omega_3-1)}(-z)^{\omega_4} y_1 y_3 \\
& + Q_{(\omega_1-\omega_4+1,\omega_2+1,\omega_3+1)}(-z)^{\omega_4} y_1 y_2 y_3 \Big] ,
\end{aligned} \tag{5.17}$$

and

$$\begin{aligned}
X_{134} \sim z^{-\frac{(\omega_1+1)(\omega_4+1)}{2}} x^{\omega_4+1} \Big[& \left(1 + (-z)^{\omega_4} y_1 z^{\omega_1} x^{-2} y_4\right) \Big(Q_{(\omega_1-\omega_4,\omega_2,\omega_3+1)} \\
& + Q_{(\omega_1-\omega_4,\omega_2,\omega_3-1)} y_3 \Big) + z^{\omega_1+1} x^{-2} y_4 \left(Q_{(\omega_1-\omega_4+2,\omega_2,\omega_3+1)} + Q_{(\omega_1-\omega_4+2,\omega_2,\omega_3-1)} y_3 \right) \\
& - (-z)^{\omega_4+1} y_1 \left(Q_{(\omega_1-\omega_4-2,\omega_2,\omega_3+1)} + Q_{(\omega_1-\omega_4-2,\omega_2,\omega_3-1)} y_3 \right) \Big] .
\end{aligned} \tag{5.18}$$

In what follows we will show that, as anticipated in section 3.4, from this expansion we can read off contributions to the factorized expansion of the flowed four-point functions associated with the exchange of flowed states with charges $\omega = \omega_1 - \omega_4 + \delta$ with $\delta = -1, 0, +1$.

Let us start with the case where the three-point function on the left of figure 1 is, in the language of [31, 33], of the even edge type, namely when the intermediate state has $\omega = \omega_1 - \omega_4$. Importantly, the three-point function on the r.h.s. remains unconstrained, except for the fact that it must be an odd-parity one. We will make use of the notation in eq. (2.23) for the y_i-dependent factors involved in the latter three-point function. By rescaling

$$y_1 \to (-z)^{-\omega_4} y_1 , \qquad y_4 \to z^{-\omega_1} x^2 y_4 \,, \tag{5.19}$$

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and working at first non-trivial order in z, we find that the above expressions give

$$[X_1, X_2, X_3, X_4] \to z^{-\frac{\omega_1\omega_4}{2}} x^{\omega_4} \left[z^{-\frac{\omega_4}{2}} Z_1(y_1), Z_2(y_2), Z_3(y_3), -xz^{-\frac{\omega_1}{2}} y_4 Z_1(-y_4^{-1}) \right] , \tag{5.20}$$

while

$$X_{123} \to z^{-\frac{(\omega_1+1)\omega_4}{2}} x^{\omega_4+1} Z_{123}(y_1, y_2, y_3) , \tag{5.21}$$

and

$$X_{134} \to z^{-\frac{(\omega_1+1)(\omega_4+1)}{2}} x^{\omega_4+1} (1 + y_1 y_4) Z_3(y_3) . \tag{5.22}$$

The overall powers of x and z cancel out in the generalized cross-ratio, giving

$$\frac{X_2 X_{134}}{X_4 X_{123}} \to \frac{(-1)^{\omega+\omega_2+1} (1 + y_1 y_4) Z_2(y_2) Z_3(y_3)}{y_4 Z_1(-y_4^{-1}) Z_{123}(y_1, y_2, y_3)} . \tag{5.23}$$

Here, the Z_I functions are those corresponding to a spectrally flowed three-point function with charges $(\omega, \omega_2, \omega_3)$. Note that, somewhat surprisingly, either y_1 or y_4^{-1} seem to play a role analogous to a putative y-variable associated to the intermediate state. Finally, combining these results with the leading order expression for the unflowed correlator appearing in (2.56), we get

$$\ddddot{\mathcal{F}}_{\boldsymbol{\omega}}(x, z) \sim |x^{-2h_4} z^{h_1\omega_4 + h_4\omega_1 - \frac{k}{2}\omega_1\omega_4}|^2 \int \prod_i d^2 y_i |y_i^{j_i + \frac{k}{2}\omega_i - h_i - 1}|^2 \mathcal{G}(y_i, z) , \tag{5.24}$$

where, at small z and for external spins in the range (3.7), we have

$$\mathcal{G}(y_i, z) \sim \int_{\frac{1}{2}+i\mathbb{R}} dj \mathcal{C}(j) |z^{\Delta_j^{(0)} - \Delta_1^{(0)} - \Delta_4^{(0)}} \hat{\mathcal{G}}(y_i)|^2 , \tag{5.25}$$

with

$$\begin{aligned} \hat{\mathcal{G}}(y_i) &= Z_1(y_1)^{-j_1+j_2+j_3+j_4-\frac{k}{2}} Z_2(y_2)^{j-j_2+j_3-\frac{k}{2}} Z_3(y_3)^{j+j_2-j_3-\frac{k}{2}} \\ &\quad \times Z_{123}(y_1, y_2, y_3)^{\frac{k}{2}-j-j_2-j_3} \left[y_4 Z_1(-y_4^{-1}) \right]^{j_1-j-j_4} (1 + y_1 y_4)^{j-j_1-j_4} \\ &\quad \times {}_2F_1 \left[\begin{matrix} j - j_1 + j_4, \ j + j_2 + j_3 - \frac{k}{2} \\ 2j \end{matrix} \middle| \frac{(-1)^s (1 + y_1 y_4)}{y_4 Z_1(-y_4^{-1})} \frac{Z_2(y_2) Z_3(y_3)}{Z_{123}(y_1, y_2, y_3)} \right] , \end{aligned} \tag{5.26}$$

and

$$\mathcal{C}(j) = \frac{C(j_1, j_4, j) \mathcal{N}(j_3) C\left(j, j_2, \frac{k}{2} - j_3\right)}{B(j)} . \tag{5.27}$$

As it happened in the sample cases studied in section 3, the overall power of x shows that the $z \to 0$ limit has fixed the spacetime spin of the exchanged state to be

$$h = h_1 - h_4 . \tag{5.28}$$

Moreover, the remaining integral is independent of x, which is consistent with the fact that the hypergeometric function in (3.49) trivializes for this particular value of h. This is also consistent with the power of z in eq. (5.24) since,

$$\Delta_j^{(0)} - \Delta_1^{(0)} - \Delta_4^{(0)} + h_1\omega_4 + h_4\omega_1 - \frac{k}{2}\omega_1\omega_4 = \Delta_j \Big|_{h=h_1-h_4}^{\omega=\omega_1-\omega_4} - \Delta_1 - \Delta_4 . \tag{5.29}$$

In order to argue that this is consistent with the expected factorization structure in eq. (3.49) we need to show that it matches the relevant product of three-point functions. As before, we proceed by reverse engineering. In the present context, the starting point corresponds to

$$\frac{C_{(\omega_1,\omega_4,\omega_1-\omega_4)}(j_i,h_i)C_{(\omega_1-\omega_4,\omega_2,\omega_3)}(j_i,h_i)}{R(j,h_1-h_4,\omega_1-\omega_4)}\,. \tag{5.30}$$

Following the analysis of section 3.4, and by means of the identity given in eq. (A.26), it suffices to show that

$$\begin{aligned}\mathcal{I}_{\text{odd}} \equiv \int d^2y \Big|&(1+yy_4)^{j_1+j-j_4-1}(y-y_1)^{j_4+j-j_1-1}(1+y_1y_4)^{1-j_1-j_4-j} \\ &\times Z_{123}^{\frac{k}{2}-j-j_2-j_3}(y,y_2,y_3)\, Z_1^{j_3+j_2-j-\frac{k}{2}}(y)\, Z_2^{j+j_3-j_2-\frac{k}{2}}(y_2) Z_3^{j+j_2-j_3-\frac{k}{2}}(y_3)\Big|^2\end{aligned} \tag{5.31}$$

integrates to $\hat{\mathcal{G}}(y_i)$ up to the appropriate multiplicative factor depending only on the unflowed spins. Roughly speaking, in eq. (5.31) we have included the y-variable of the intermediate state. As discussed in section (3.4), the Z_I do not take the form of the usual differences y_{ij} and contain complicated numerical factors coming from the relevant holomorphic covering maps [31, 33], but they remain at most linear functions of each of the y_i. Since their powers involve the usual combinations of the external spins j_i and of the intermediate spin j (up to shifts of the form $j \to \frac{k}{2}-j$), this allows us to carry out the integral over y as in appendix A.1, leading once again to a result that can be expressed in terms of hypergeometric functions of the ${}_2F_1$ type. More precisely, denoting

$$Z_{123}(y,y_2,y_3) \equiv A^+(y_2,y_3) + A^-(y_2,y_3)y\,, \qquad Z_1(y) \equiv B^+ + B^-y, \tag{5.32}$$

we find that the resulting hypergeometric function is evaluated at

$$\frac{(1+y_1y_4)}{y_4Z_1\left(-y_4^{-1}\right)}\frac{\left[A^-(y_2,y_3)B^+ - A^+(y_2,y_3)B^-\right]}{Z_{123}\left(y_1,y_2,y_3\right)}\,. \tag{5.33}$$

Finally, from the explicit expressions for $A^\pm(y_2,y_3)$ and $B^\pm$ one can check that

$$A^-(y_2,y_3)B^+ - A^+(y_2,y_3)B^- = (-1)^{\omega+\omega_2}Z_2\left(y_2\right)Z_3\left(y_3\right)\,, \tag{5.34}$$

hence reproducing the (small z limit of the) generalized cross-ratio appearing in (5.26). The precise matching is given by

$$C\left(j,j_2,\frac{k}{2}-j_3\right)\mathcal{N}(j_3)C(j_1,j_4,1-j)\mathcal{I}_{\text{odd}} = \mathcal{C}(j)|\hat{\mathcal{G}}(y_i)|^2\,. \tag{5.35}$$

We have thus recovered the expected factorization structure. Moreover, one of the relevant three-point functions, namely that on the r.h.s. of figure 1, involves the full complexity of the exact odd-parity spectrally flowed three-point functions of the SL(2,$\mathbb{R}$) WZW model, derived recently in [31–33].

For the four-point function under consideration, we can also isolate the contribution coming from the exchange of a flowed state with $\omega = \omega_1 - \omega_4 - 1$. Indeed, by rescaling

$$y_1 \to (-z)^{-\omega_4-1}y_1, \qquad y_4 \to z^{-\omega_1+1}x^2y_4\,, \tag{5.36}$$

we find that generalized cross-ratio goes to

$$\frac{X_2X_{134}}{X_4X_{123}} \to z\frac{A^+(y_2)\left[(1+y_1y_4)B^+(y_3)-y_1B^-(y_3)\right]}{(-1)^{\omega+\omega_2+1}y_1Z_\emptyset Z_{23}(y_2,y_3)}, \tag{5.37}$$

where now the Z_I are those corresponding to an even spectrally flowed three-point function with charges $(\omega,\omega_2,\omega_3)$. The factors $A^\pm$ and $B^\pm$ are defined accordingly by

$$Z_{12}(y,y_2) = A^+(y_2)+A^-(y_2)y, \qquad Z_{13}(y,y_3) = B^+(y_3)+B^-(y_3)y\,. \tag{5.38}$$

This time the leading order cross-ratio is proportional to z, hence the factorization limit is governed by eq. (2.51) whenever the external spins are in the range (3.16). Therefore the corresponding contribution to the factorized form of the four-point function is given by

$$\ddot{\mathcal{F}}_{\boldsymbol{\omega}}(x,z) \sim |x^{-2h_4}z^{h_1(\omega_4-1)+h_4(\omega_1+1)-\frac{k}{2}(\omega_1\omega_4+\omega_1-\omega_4)-j_1-j_4}|^2\int\prod_i d^2y_i|y^{j_i+\frac{k}{2}\omega_i-h_i-1}|^2\mathcal{G}(y_i,z), \tag{5.39}$$

where

$$\mathcal{G}(y_i,z) \sim \int_{\frac{1}{2}+i\mathbb{R}} dj\mathcal{C}(j)|z^{\Delta_j^{(0)}-\Delta_1^{(0)}-\Delta_4^{(0)}-j_1-j_4+\frac{k}{4}}\hat{\mathcal{G}}(y_i)|^2\,, \tag{5.40}$$

with

$$\begin{aligned}\hat{\mathcal{G}}(y_i) &= Z_{23}(y_2,y_3)^{j-j_2-j_3}A^+(y_2)^{j_3-j_2-j}B^+(y_3)^{j_1+j_4+j_2-j_3-\frac{k}{2}}Z_\emptyset^{j+j_2+j_3-k}\\ &\times\left[(1+y_1y_4)B^+(y_3)-y_1B^-(y_3)\right]^{\frac{k}{2}-j-j_1-j_4}y_1^{j+j_4-j_1-\frac{k}{2}}\\ &\times{}_2F_1\left[\begin{matrix}j+j_1+j_4-\frac{k}{2},\; j-j_3+j_2\\ 2j\end{matrix}\;\middle|\;\frac{(-1)^{\omega+\omega_2+1}y_1Z_\emptyset Z_{23}(y_2,y_3)}{A^+(y_2)\left[(1+y_1y_4)B^+(y_3)-y_1B^-(y_3)\right]}\right].\end{aligned} \tag{5.41}$$

and

$$\mathcal{C}(j) = \frac{\mathcal{N}(j_1)C\left(\frac{k}{2}-j_1,j_4,j\right)C(j,j_2,j_3)}{B(j)}\,. \tag{5.42}$$

The overall x-power again shows that we have picked up an intermediate spacetime spin $h = h_1-h_4$, and the remaining z-dependence is also consistent since

$$\Delta_j^{(0)}-\Delta_1^{(0)}-\Delta_4^{(0)}+h_1(\omega_4-1)+h_4(\omega_1+1)-\frac{k}{2}(\omega_1\omega_4+\omega_1-\omega_4)+\frac{k}{4} = \Delta_j\Big|_{h=h_1-h_4}^{\omega=\omega_1-\omega_4-1}-\Delta_1-\Delta_4\,. \tag{5.43}$$

As in the previous case, it is possible to show that the entire function matches the product

$$\frac{C_{(\omega_1,\omega_4,\omega_1-\omega_4-1)}(j_i,h_i)C_{(\omega_1-\omega_4-1,\omega_2,\omega_3)}(j_i,h_i)}{R(j,h_1-h_4,\omega_1-\omega_4-1)} \tag{5.44}$$

by integrating an expression analogous to eq. (5.31). Since the four-point function spectral flow parity has not changed, the three-point function on the r.h.s. must now be of the even-parity type. The corresponding integral takes the form

$$\begin{aligned}\mathcal{I}'_{\text{odd}} = \int d^2y\Big|&y_1^{j+j_4-j-\frac{k}{2}}(1+y_1y_4+y_1y)^{\frac{k}{2}-j-j_1-j_4}Z_\emptyset^{1-j+j_2+j_3-k}\\ &\times y^{2j-2}Z_{23}(y_2,y_3)^{1-j-j_2-j_3}Z_{13}(y^{-1},y_3)^{j_2+j-j_3-1}Z_{12}(y^{-1},y_2)^{j_3+j-j_2-1}\Big|^2\,.\end{aligned} \tag{5.45}$$

Indeed, this leads exactly to the $\hat{\mathcal{G}}(y_i)$ in eq. (5.41) by means of the identity

$$(-1)^{\omega+\omega_2+1} Z_\emptyset Z_{23}(y_2,y_3) = A^+(y_2)B^-(y_3) - A^-(y_2)B^+(y_3)\,. \tag{5.46}$$

The necessary steps to isolate the $\omega = \omega_1 - \omega_4 + 1$ intermediate channel also follow analogously by rescaling

$$y_1 \to (-z)^{-\omega_4+1} y_1\,, \qquad y_4 \to z^{-\omega_1-1} x^2 y_4\,. \tag{5.47}$$

In the small z limit, the generalized cross-ratio now gives

$$\frac{X_2 X_{134}}{X_4 X_{123}} \to z \frac{A^-(y_2)\left[(1+y_1y_4)B^-(y_3) + y_4 B^+(y_3)\right]}{y_4 Z_\emptyset Z_{23}(y_2,y_3)}\,, \tag{5.48}$$

where the Z_I corresponds to the factors involved in the three-point function with charges $\boldsymbol{\omega} = (\omega_1 - \omega_4 + 1, \omega_2, \omega_3)$, and with

$$Z_{12}(y,y_2) A^+(y_2) + A^-(y_2) y\,, \qquad Z_{13}(y,y_3) = B^+(y_3) + B^-(y_3) y\,. \tag{5.49}$$

The four-point function limit then becomes

$$\begin{aligned}
\dddot{\mathcal{F}}_{\boldsymbol{\omega}}(x,z) = &\int_{\frac{1}{2}+i\mathbb{R}} dj \mathcal{C}(j) |z^{\Delta_j - \Delta_1 - \Delta_4} x^{-2h_4}|^2 \int \prod_{i=1}^4 d^2 y_i \bigg| y_i^{h_i^0 - h_i - 1} \\
&\times Z_\emptyset^{j+j_2+j_3-k} Z_{23}^{j-j_2-j_3} A^-(y_2)^{j_3-j-j_2} B^-(y_3)^{j_1+j_2-j_3+j_4-\frac{k}{2}} \\
&\times y_4^{j_1+j-j_4-\frac{k}{2}} \Big((1+y_1y_4)B^-(y_3) + y_4 B^+(y_3)\Big)^{\frac{k}{2}-j_1-j_4-j} \\
&\times {}_2F_1\left[\begin{matrix} j+j_1+j_4-\frac{k}{2},\ j-j_3+j_2 \\ 2j \end{matrix} \,\middle|\, \frac{y_4 Z_\emptyset Z_{23}(y_2,y_3)}{A^-(y_2)\left((1+y_1y_4)B^-(y_3)+y_4B^+(y_3)\right)}\right] \bigg|^2,
\end{aligned} \tag{5.50}$$

where $\Delta_j = \Delta_j\Big|_{h=h_1-h_4}^{\omega=\omega_1-\omega_4+1}$. Again, the normalization matches the product

$$\frac{C_{(\omega_1,\omega_4,\omega_1-\omega_4+1)}(j_i,h_i) C_{(\omega_1-\omega_4+1,\omega_2,\omega_3)}(j_i,h_i)}{R(j,h_1-h_4,\omega_1-\omega_4+1)}\,, \tag{5.51}$$

which is now computed from the integral

$$\begin{aligned}
\mathcal{I}_{\text{odd}} = \int d^2 y \Big| &y_4^{j+j_1-j_4-\frac{k}{2}} y^{j_1+j_4-j-\frac{k}{2}} (y_4 + (1+y_1y_4)y)^{\frac{k}{2}-j-j_1-j_4} Z_\emptyset^{1-j+j_2+j_3-k} \\
&\times y^{2j-2} Z_{23}(y_2,y_3)^{1-j-j_2-j_3} Z_{13}(y^{-1},y_3)^{j_2+j-j_3-1} Z_{12}(y^{-1},y_2)^{j_3+j-j_2-1}\Big|^2.
\end{aligned} \tag{5.52}$$

Let us stress that the matchings we have obtained for $\omega = \omega_1 - \omega_4 \pm 1$ involves all the even-parity spectrally flowed three-point functions. This ends our analysis of four-point functions satisfying (5.15).

For completeness, let us now consider spectral flow charges satisfying

$$\omega_2 - \omega_3 > |\omega_1 - \omega_4|\,. \tag{5.53}$$

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Following a similar path as in the previous cases and using the rescalings

$$
\begin{aligned}
y_1 &\to \frac{xy_1}{z^{\omega_1}(1-x)}, \qquad & y_2 &\to (-1)^{\omega_3}\frac{(1-x)z^{\omega}y_2}{x}, \\
y_3 &\to -\frac{xy_3}{(1-x)z^{\omega}}, \qquad & y_4 &\to \frac{x(1-x)y_4}{z^{\omega_4}},
\end{aligned}
\tag{5.54}
$$

now leads to a small z expression of the form

$$
\begin{aligned}
\ddddot{\mathcal{F}}_{\boldsymbol{\omega}}(x,z) \sim & |x^{h-h_1-h_4}(1-x)^{h_1-h_4-h} z^{j_1+j_4-\frac{k}{4}-h\omega+h_1\omega_1+h_4\omega_4+\frac{k}{4}\left(\omega^2-\omega_1^2-\omega_4^2\right)}|^2 \\
& \times \int \prod_i d^2y_i |y_i^{j_i+\frac{k}{2}\omega_i-h_i-1}|^2 \mathcal{G}(y_i,z)\,.
\end{aligned}
\tag{5.55}
$$

The overall powers of x and z now correspond to the exchange of an intermediate state with

$$
\omega = \omega_2-\omega_3\,, \qquad h = h_2-h_3\,. \tag{5.56}
$$

Note that one important difference is that there is also a factor of $(1-x)^{h_1-h_4-h}$. This is consistent with the fact that for $h = h_2 - h_3$ the hypergeometric function analogous to (2.45) but with the external j_i replaced by h_i simplifies, but it does not trivialize completely. Moreover, in this case the generalized cross-ratio becomes

$$
\frac{X_2X_{134}}{X_4X_{123}} \to z(-1)^{\omega+\omega_1}\frac{y_3Z_3(y_2)Z_{123}(y_1,y_4,y_3^{-1})}{Z_2(y_4)Z_1(y_1)(1-y_2y_3)}\,, \tag{5.57}
$$

with $Z_I(y_i)$ the factors involved in the l.h.s. three-point function with $\boldsymbol{\omega} = (\omega_1,\omega_4,\omega)$. For spins in the range (3.16) we can thus use eq. (2.51) to provide the leading order expression for the relevant unflowed correlator, leading to

$$
\mathcal{G}(y_i,z) \sim \int_{\frac{1}{2}+i\mathbb{R}} dj\,\mathcal{C}(j)|z^{\Delta_j^{(0)}-\Delta_1^{(0)}-\Delta_4^{(0)}-j_1-j_4+\frac{k}{4}}\hat{\mathcal{G}}(y_i)|^2\,, \tag{5.58}
$$

which gives the expected power of z, where

$$
\begin{aligned}
\hat{\mathcal{G}}(y_i) = & (1-y_2y_3)^{j-j_2-j_3} Z_1(y_1)^{j_4+j-j_1-\frac{k}{2}} \left(y_3Z_{123}(y_1,y_4,y_3^{-1})\right)^{\frac{k}{2}-j-j_1-j_4} \\
& \times Z_2(y_4)^{j_1+j-j_4-\frac{k}{2}} Z_3(y_2)^{j_3-j_2-j}\left(y_3Z_3\left(y_3^{-1}\right)\right)^{j_1+j_2-j_3+j_4-\frac{k}{2}} \\
& \times {}_2F_1\left[\begin{matrix} j_1+j_4+j-\frac{k}{2},\ j-j_3+j_2 \\ 2j \end{matrix}\,\middle|\, (-1)^{\omega+\omega_1}\frac{Z_2(y_4)Z_1(y_1)(1-y_2y_3)}{y_3Z_3(y_2)Z_{123}(y_1,y_4,y_3^{-1})}\right].
\end{aligned}
\tag{5.59}
$$

and

$$
\mathcal{C}(j) = \frac{\mathcal{N}(j_1)C\left(\frac{k}{2}-j_1,j_4,j\right)C(j,j_2,j_3)}{B(j)}\,. \tag{5.60}
$$

After taking $y_2 \to -y_2^{-1}$ and $y_3 \to y_3^{-1}$, the $\hat{\mathcal{G}}(y_i)$ we have obtained takes exactly the same form as in eq. (5.26), hence it can be obtained from the product of the y-basis three-point functions in an analogous way. As before, one can also derive the contributions associated with exchanged states of winding $\omega = \omega_2-\omega_3 \pm 1$ (and $h = h_2 - h_3$) in a similar manner.

This completes our study of the small z behavior and factorization properties of spectrally flowed four-point functions of the odd-parity type.

The even-parity cases, i.e. the four-point functions described by (2.55), can be studied in the same way. Let us briefly discuss one example for completeness. We assume that

$$\omega_3 - \omega_2 > |\omega_1 - \omega_4| \,. \tag{5.61}$$

After taking

$$y_1 \to \frac{x}{z^{\omega_1}} y_1 \,, \qquad y_2 \to (-1)^{\omega} \frac{x}{z^{\omega}} y_2 \,, \qquad y_3 \to \frac{z^{\omega}}{x} y_3 \,, \qquad y_4 \to \frac{x}{z^{\omega_4}} y_4 \,, \tag{5.62}$$

the relevant small z contribution describes the exchange of a state with

$$\omega = \omega_3 - \omega_2 \,, \qquad h = h_3 - h_2 \,. \tag{5.63}$$

More explicitly, this takes the form

$$\bar{\dddot{\mathcal{F}}}_{\omega}(x,z) \sim \int_{\frac{1}{2}+i\mathbb{R}} dj \left| x^{h-h_1-h_4} z^{\Delta_j - \Delta_{j_1} - \Delta_4} \right|^2 \mathcal{C}(j) \int \prod_i d^2 y_i \left| y_i^{j_i + \frac{k}{2}\omega_i - h_i - 1} \hat{\mathcal{G}}(y_i) \right|^2 , \tag{5.64}$$

with

$$\begin{aligned}
\hat{\mathcal{G}}(y_i) = {} & Z_{\emptyset}^{j+j_1+j_4-k} (1+y_2 y_3)^{j-j_2-j_3} \left[y_2 Z_{13}\left(y_1, -y_2^{-1}\right)\right]^{j_3-j_2-j} \\
& \times Z_{13}^{j_4-j_1-j_3+j_2}(y_1,y_3) Z_{23}(y_4,y_3)^{j_1-j-j_4} Z_{12}(y_1,y_4)^{j-j_1-j_4} \\
& \times {}_2F_1 \left[\begin{matrix} j-j_1+j_4, \; j-j_3+j_2 \\ 2j \end{matrix} \, \middle| \, \frac{(-1)^{\omega+\omega_1}(1+y_2y_3) Z_{\emptyset} Z_{12}(y_1,y_4)}{y_2 Z_{13}(y_1,-y_2^{-1}) Z_{23}(y_1,y_3)} \right] .
\end{aligned} \tag{5.65}$$

and

$$\mathcal{C}(j) = \frac{C(j_1,j_4,j) C(j,j_2,j_3)}{B(j)} \,. \tag{5.66}$$

As usual, this can be obtained from the integral associated to the relevant product of three-point functions, namely

$$\begin{aligned}
\mathcal{I}_{\text{even}} = \int d^2 y \Big| & (1+yy_2)^{j_3+j-j_2-1} (y-y_3)^{j_2+j-j_3-1} (1+y_2y_3)^{1-j_2-j_3-j} \\
& \times Z_{23}^{j_1-j-j_4}(y_4,y)\, Z_{13}^{j_4-j_1-j}(y_1,y)\, Z_{12}^{j-j_1-j_4}(y_1,y_4) Z_{\emptyset}^{j+j_1+j_4-k} \Big|^2 .
\end{aligned} \tag{5.67}$$

The matching holds thanks to the identity

$$A^-(y_4)B^+(y_1) - A^+(y_4)B^-(y_1) = (-1)^{\omega+\omega_1} Z_{\emptyset} Z_{12}(y_1,y_4) \,, \tag{5.68}$$

where

$$Z_{23}(y_4,y) = A^+(y_4) + A^-(y_4) y \,, \qquad Z_{13}(y_1,y) = B^+(y_1) + B^-(y_1) y \,. \tag{5.69}$$

The remaining contributions for these spectral flow charges and the rest of the cases in eq. (A.13) can be studied similarly.

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6 Concluding remarks and outlook

Let us briefly recapitulate what we have achieved in this paper. The WZW model based on the universal cover of SL(2,$\mathbb{R}$) at level k describes string propagation in an AdS$_3$ background of radius $\sqrt{k}$ in string units. The importance of spectral flow in this context was established a long time ago in [1] by solving a number of puzzles regarding the spectrum of the theory. Despite the fact that its role at the level of correlation functions was initially discussed shortly after in [6], structure constants involving states in the flowed sectors were derived in full generality only recently [31–33]. Four-point functions are even harder to study due to the intricate dependence on the worldsheet and boundary cross-ratios, usually denoted as z and x, respectively. In [34] the authors conjectured a formula relating flowed four-point functions with arbitrary spectral flow charges with their unflowed counterparts [6, 8] by means of a complicated integral transform, see eqs. (2.55) and (2.56).

Unflowed four-point functions are not known in closed form, but only in a formal expansion in powers of z. This is the so-called factorization expansion, which accounts for the exchange of unflowed states along the $14 \to 23$ channel in the language of figure 1. It thus makes sense to ask whether this can be used to show that the proposal of [34] leads to a consistent factorization picture in the flowed sectors of the theory, where the structure of the OPE has not been explored so far. This paper constitutes a first approach to this important question, which, if answered positively, would lend conclusive support to the above conjecture.

On general grounds, the factorization structure is expected to be of the form discussed in section (3.4), see eqs. (3.49) and (3.50). The difficulties we have encountered so far can be understood as follows. Flowed vertex operators have worldsheet conformal dimensions Δ that depend not only on their unflowed spin j but also on the spacetime spin h and the spectral flow charge. It follows that, for a given value of the intermediate ω, the small z limit we have used to study the flowed four-point functions interacts non-trivially with the sum over the quantum number h associated to the exchanged state. In terms of the integral transform from the y-basis to the x-basis which captures the effect of spectral flow, the consequence is that different scalings of the integration variables y_i with respect to z correspond to different channels in the conformal block decomposition.

Nevertheless, we have been able to consider four-point functions with arbitrary values of the external ω_i and study in detail all exchanges of unflowed states, together with an important number of flowed channels. We have shown that, at small z, in all cases one can isolate several contributions to the factorization expansion that are precisely consistent with the structure anticipated in eq. (3.49). Let us stress that this matching involves the full complexity of the spectrally flowed three-point functions of the model. We see this as providing substantial evidence for the general formulae in eqs. (2.55) and (2.56).

Of course, this is not the full story. We leave for the near future the possibility of computing the total sum over all allowed intermediate states. Further integrating over z, at least at small x, would allow us to study the problem at hand from the point of view of the OPE structure and conformal block decomposition of the holographic CFT. This could help establishing the duality proposed in [24, 25] beyond the perturbative analysis developed recently in [26, 27, 29].It would also be interesting to extend the present analysis to the supersymmetric AdS$_3 \times S^3 \times T^4$ model [2–5, 42, 46, 47]. Finally, let us highlight the fact that the ability to compute the worldsheet correlators in the SL(2,$\mathbb{R}$) model also leads to applications for black hole phenomenology [48–50], most notably in the context of the Fuzzball program and holography beyond AdS [51–69].

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A Useful identities

A.1 Complex integrals

Here we record some identities related to complex integrals. We start with

$$\int d^2y y^{a-1}\bar{y}^{\bar{a}-1}(1-y)^{b-1}(1-\bar{y})^{\bar{b}-1} = \frac{\pi\gamma(a)\gamma(b)}{\gamma(a+b)}\,, \tag{A.1}$$

where

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-\bar{x})}. \tag{A.2}$$

Along the paper, we also make use of the following generalized version of this formula:

$$\int \prod_{i=1}^{n} d^2y_i y_i^{a_i-1}\bar{y}_i^{\bar{a}_i-1}\left(1-\sum_{j=1}^{n} y_j\right)^{b-1}\left(1-\sum_{j=1}^{n}\bar{y}_j\right)^{\bar{b}-1} = \frac{\pi^n\gamma(b)\prod_{i=1}^{n}\gamma(a_i)}{\gamma\left(b+\sum_{i=1}^{n}a_i\right)}\,. \tag{A.3}$$

Another useful identity is given by [6]

$$\begin{aligned}\left|{}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix}\,\middle|\, x\right]\right|^2 &+ \lambda\left|x^{1-c}{}_2F_1\left[\begin{matrix}1-c+a,\ 1-c+b\\ 2-c\end{matrix}\,\middle|\, x\right]\right|^2\\ &= \frac{\pi\gamma(c)}{\gamma(b)\gamma(c-b)}\int d^2t|t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}|^2,\end{aligned} \tag{A.4}$$

where

$$\lambda = -\frac{\gamma(c)^2\gamma(a-c+1)\gamma(b-c+1)}{(1-c)^2\gamma(a)\gamma(b)}. \tag{A.5}$$

For $a = j+\alpha$, $b = j+\beta$ and $c = 2j$, the above expression becomes symmetric under $j \to 1-j$, hence for any function $g(j)$ invariant under this reflection we can write

$$\begin{aligned}&\int_{\frac{1}{2}+i\mathbb{R}} djg(j)\left|{}_2F_1\left[\begin{matrix}j+\alpha,\ j+\beta\\ 2j\end{matrix}\,\middle|\, x\right]\right|^2\\ &\quad= \int_{\frac{1}{2}+i\mathbb{R}} djg(j)\frac{\pi\gamma(2j)}{\gamma(j+\beta)\gamma(j-\beta)}\int d^2t|t^{j+\beta-1}(1-t)^{j-\beta-1}(1-xt)^{-j-\alpha}|^2.\end{aligned} \tag{A.6}$$

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A related identity is the Appell function-type integral, given by

$$\int d^2u|u^{\alpha-1}(1-u)^{\beta+\beta'-\alpha-1}(1-qu)^{-\beta}(1-pu)^{-\beta'}|^2$$
$$= |(1-p)|^{-2\alpha}\int d^2t\left|t^{\alpha-1}(1-t)^{\beta+\beta'-\alpha-1}\left(1-\frac{q-p}{1-p}t\right)^{-\beta}\right|^2, \tag{A.7}$$

which follows by taking

$$u = \frac{t}{1+p(t-1)}. \tag{A.8}$$

Assuming $g(j) = g(1-j)$ as before, this implies

$$\int_{\frac{1}{2}+i\mathbb{R}} dj g(j)\left|{}_2F_1\left[\begin{matrix} j+\alpha,\ j+\beta \\ 2j\end{matrix}\,\middle|\,\frac{q-p}{1-p}\right]\right|^2 \tag{A.9}$$
$$= \int_{\frac{1}{2}+i\mathbb{R}} dj \frac{\pi\gamma(2j)|(1-p)|^{2(j+\beta)}}{\gamma(j+\beta)\gamma(j-\beta)}\int d^2u|u^{j+\beta-1}(1-u)^{j-\beta-1}(1-qu)^{-(j+\alpha)}(1-pu)^{-(j-\alpha)}|^2.$$

A.2 The behaviour of $P_{\boldsymbol{\omega}}(x,z)$ near $z=0$

Here we discuss the small z behaviour of the polynomials $P_{\boldsymbol{\omega}}(x,z)$ involved in the integral formulas for spectrally flowed four-point functions (2.55) and (2.56). The corresponding definitions, provided in eq. (2.60), show that, at small z, there is an overall power-law dependence, accompanied by a factor $\tilde{P}_{\boldsymbol{\omega}}(x,z)$ which is regular at $z=0$,

$$\tilde{P}_{\boldsymbol{\omega}}(x,0) = x^{\Lambda_{\boldsymbol{\omega}}}\tilde{P}_{\boldsymbol{\omega}}(1,0), \tag{A.10}$$

with Λ_ω as in (2.63). More explicitly, we find

$$P_{\boldsymbol{\omega}}(x,z) \sim n(\boldsymbol{\omega})\tilde{P}_{\boldsymbol{\omega}}(1,0)\,(1-x)^{\frac{1}{2}s(\omega_2+\omega_4-\omega_1-\omega_3)}\,x^{\frac{1}{2}s(\omega_1+\omega_4-\omega_2-\omega_3)+\Lambda_{\boldsymbol{\omega}}}$$
$$\times\, z^{\frac{1}{4}s((\omega_1+\omega_2-\omega_3-\omega_4)(\omega_2+\omega_4-\omega_1-\omega_3))-\frac{1}{2}\omega_1\omega_4}. \tag{A.11}$$

The overall z power is completely determined in terms of $\max\left[|\omega_1-\omega_4|,|\omega_3-\omega_2|\right]$ since

$$s((\omega_1+\omega_2-\omega_3-\omega_4)(\omega_2+\omega_4-\omega_1-\omega_3)) = s\left((\omega_2-\omega_3)^2-(\omega_1-\omega_4)^2\right). \tag{A.12}$$

Once this is fixed, one further condition is necessary in order to fix the x-dependence. This can be written as follows:

$$\frac{P_{\boldsymbol{\omega}}(x,z)}{n(\boldsymbol{\omega})\tilde{P}_{\boldsymbol{\omega}}(1,0)} \sim \begin{cases} z^{-\frac{\omega_1\omega_4}{2}}x^{\omega_4} & \text{if } \omega_1-\omega_4 > |\omega_3-\omega_2|\,, \\ z^{-\frac{\omega_1\omega_4}{2}}x^{\omega_1}(1-x)^{\frac{\omega_2-\omega_3-\omega_1+\omega_4}{2}} & \text{if } \omega_4-\omega_1 > |\omega_3-\omega_2|\,, \\ z^{\frac{(\omega_2-\omega_3)^2}{4}-\frac{\omega_1^2}{4}-\frac{\omega_4^2}{4}}x^{\frac{\omega_2-\omega_3+\omega_1+\omega_4}{2}} & \text{if } |\omega_1-\omega_4| < \omega_3-\omega_2\,, \\ z^{\frac{(\omega_2-\omega_3)^2}{4}-\frac{\omega_1^2}{4}-\frac{\omega_4^2}{4}}x^{\frac{\omega_3-\omega_2+\omega_1+\omega_4}{2}}(1-x)^{\frac{\omega_2-\omega_3-\omega_1+\omega_4}{2}} & \text{if } |\omega_1-\omega_4| < \omega_2-\omega_3\,. \end{cases} \tag{A.13}$$

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Moreover, we have checked numerically with the help of the ancillary `Mathematica` file provided in [34] that, up to an unimportant overall sign, the normalization $n(\boldsymbol{\omega})\tilde{P}_{\boldsymbol{\omega}}(1,0)$ can be expressed as

$$|n(\boldsymbol{\omega})\tilde{P}_{\boldsymbol{\omega}}(1,0)| = |Q_{\max[|\omega_1-\omega_4|,|\omega_3-\omega_2|],\omega_2,\omega_3}Q_{\max[|\omega_1-\omega_4|,|\omega_3-\omega_2|],\omega_1,\omega_4}| \,, \tag{A.14}$$

where $Q_{\boldsymbol{\omega}}$ are the numbers involved in the y-basis flowed three-point functions [31, 33], see eq. (2.26). This relation is crucial when considering the factorization properties of the spectrally flowed four-point functions.

A.3 Useful properties of flowed three-point functions

Here we will derive an integral representation of the product of three-point functions given by

$$\frac{\langle V^{\omega_1}_{j_1,h_1}(0,0)V^{\omega_4}_{j_4,h_4}(x_4,1)V^{\omega}_{j,h}(\infty,\infty)\rangle\,\langle V^{\omega}_{j,h}(0,0)V^{\omega_2}_{j_2,h_2}(1,1)V^{\omega_3}_{j_3,h_3}(\infty,\infty)\rangle}{\langle V^{\omega}_{j,h}(0,0)V^{\omega}_{j,h}(\infty,\infty)\rangle}\,, \tag{A.15}$$

when one of the involved correlators is in the so-called collision limit, namely $x_4 \to 0$.

Let us consider three-point functions satisfying either $\omega_3 = \omega_1 + \omega_2$ or $\omega_3 = \omega_1 + \omega_2 \pm 1$. For such correlators taking $x_2 \to x_1$ leads to the conservation equation $h_3 = h_1 + h_2$. In the even-parity case one then gets structure constants of the form [31, 33]

$$\begin{aligned}
&\langle V^{\omega_1}_{j_1,h_1}(0,0)V^{\omega_2}_{j_2,h_2}(0,1)V^{\omega_3}_{j_3,h_3}(\infty,\infty)\rangle \\
&= C(j_1,j_2,j_3)\int\prod_{i=1}^{3} d^2y_i\,|y_i^{\alpha_i-1}(y_1-y_2)^{j_3-j_1-j_2}(1-y_2y_3)^{j_1-j_2-j_3}(1-y_1y_3)^{j_2-j_1-j_3}|^2\,,
\end{aligned} \tag{A.16}$$

up to an overall sign, and where we have introduced the shorthand $\alpha_i = j_i - h_i + \frac{k}{2}\omega_i$. By defining the function

$$\varphi_0(y_1) \equiv \int d^2y_2d^2y_3|y_2^{\alpha_2-1}y_3^{\alpha_3-1}(y_1-y_2)^{j_3-j_1-j_2}(1-y_2y_3)^{j_1-j_2-j_3}(1-y_1y_3)^{j_2-j_1-j_3}|^2\,, \tag{A.17}$$

one immediately notices that

$$\varphi_0(y_1) = |y_1^{-j_1+h_3-h_2-\frac{k}{2}\omega_1}|^2\varphi_0(1)\,. \tag{A.18}$$

It follows that the structure constant can be expressed as

$$\langle V^{\omega_1}_{j_1h_1}(0,0)V^{\omega_2}_{j_2h_2}(0,1)V^{\omega_3}_{j_3h_3}(\infty,\infty)\rangle_{\text{even}} = i(2\pi)^2C(j_1,j_2,j_3)\delta^{(2)}(h_3-h_1-h_2)\varphi_0(1). \tag{A.19}$$

Similarly, in the odd-parity case we have

$$\langle V^{\omega_1}_{j_1,h_1}(0,0)V^{\omega_2}_{j_2,h_2}(0,1)V^{\omega_3}_{j_3,h_3}(\infty,\infty)\rangle_{\text{odd}} = \mathcal{N}(j_1)C\left(\frac{k}{2}-j_1,j_2,j_3\right)\int d^2y_1\,|y_1^{\alpha_1-1}|^2\varphi_\pm(y_1)\,,$$

with

$$\varphi_+(y_1) = \int d^2y_2d^2y_3\,|y_2^{\alpha_2-1}y_3^{\alpha_3-1}\,y_3^{j_1+j_2-j_3-\frac{k}{2}}(1-y_1y_3-y_2y_3)^{\frac{k}{2}-j_1-j_2-j_3}|^2\,, \tag{A.20}$$

$$\varphi_-(y_1) = \int d^2y_2d^2y_3\,|y_2^{\alpha_2-1}y_3^{\alpha_3-1}\,y_2^{j_1+j_3-j_2-\frac{k}{2}}y_1^{j_2+j_3-j_1-\frac{k}{2}}(y_2+(1+y_3y_2)y_1)^{\frac{k}{2}-j_1-j_2-j_3}|^2. \tag{A.21}$$

Since $\varphi_\pm(y_1)$ satisfy (A.18) as well, we get

$$\begin{aligned}&\langle V^{\omega_1}_{j_1h_1}(0,0)V^{\omega_2}_{j_2h_2}(0,1)V^{\omega_3}_{j_3h_3}(\infty,\infty)\rangle_{\text{odd}} \\ &\qquad\qquad = i(2\pi)^2\mathcal{N}(j_1)C\left(\frac{k}{2}-j_1,j_2,j_3\right)\delta^{(2)}(h_3-h_1-h_2)\varphi_\pm(1).\end{aligned} \tag{A.22}$$

From this we derive a useful identity involving the normalization of certain x-basis spectrally flowed conformal blocks. This reads

$$\begin{aligned}&\frac{\langle V^{\omega_1}_{j_1,h_1}V^{\omega_4}_{j_4,h_4}V^{\omega}_{j,h_3-h_2}\rangle\,\langle V^{\omega}_{j,h_3-h_2}V^{\omega_2}_{j_2,h_2}V^{\omega_3}_{j_3,h_3}\rangle}{\langle V^{\omega}_{j,h_3-h_2}V^{\omega}_{j,h_3-h_2}\rangle} \\ &\qquad = B(1-j)\int d^2yd^2y'|(1-yy')^{2j-2}|^2\,\langle V^{\omega_1}_{j_1,h_1}V^{\omega_4}_{j_4,h_4}V^{\omega}_{j}(y)\rangle\,\langle V^{\omega}_{j}(y')V^{\omega_2}_{j_2,h_2}V^{\omega_3}_{j_3,h_3}\rangle\,,\end{aligned} \tag{A.23}$$

where on both sides the three-point function on the left is evaluated at $(x_1,x_4,x)=(z_1,z_4,z)=(0,0,\infty)$ while the one on the right one is evaluated at $(x,x_2,x_3)=(0,1,\infty)$ and $(z,z_2,z_3)=(0,1,\infty)$. Indeed, by using the reflection properties in eqs. (2.7) and (2.15) we get

$$\begin{aligned}&\frac{\langle V^{\omega_1}_{j_1,h_1}V^{\omega_4}_{j_4,h_4}V^{\omega}_{j,h_3-h_2}\rangle\,\langle V^{\omega}_{j,h_3-h_2}V^{\omega_2}_{j_2,h_2}V^{\omega_3}_{j_3,h_3}\rangle}{\langle V^{\omega}_{j,h_3-h_2}V^{\omega}_{j,h_3-h_2}\rangle} \\ &= \langle V^{\omega_1}_{j_1,h_1}V^{\omega_4}_{j_4,h_4}V^{\omega}_{j,h_3-h_2}\rangle\,\langle V^{\omega}_{1-j,h_3-h_2}V^{\omega_2}_{j_2,h_2}V^{\omega_3}_{j_3,h_3}\rangle \\ &= \int d^2y|y^{\alpha-2j}|^2\,\langle V^{\omega_1}_{j_1,h_1}V^{\omega_4}_{j_4,h_4}V^{\omega}_{j,h_3-h_2}\rangle\,\langle V^{\omega}_{1-j}(y)V^{\omega_2}_{j_2,h_2}V^{\omega_3}_{j_3,h_3}\rangle \\ &= B(1-j)\int d^2yd^2y'|y^{\alpha-2j}(y-y')^{2j-2}|^2\,\langle V^{\omega_1}_{j_1,h_1}V^{\omega_4}_{j_4,h_4}V^{\omega}_{j,h_3-h_2}\rangle\,\langle V^{\omega}_{j}(y')V^{\omega_2}_{j_2,h_2}V^{\omega_3}_{j_3,h_3}\rangle \\ &= B(1-j)\int d^2yd^2y'|y^{-\alpha}(1-yy')^{2j-2}|^2\,\langle V^{\omega_1}_{j_1,h_1}V^{\omega_4}_{j_4,h_4}V^{\omega}_{j,h_3-h_2}\rangle\,\langle V^{\omega}_{j}(y')V^{\omega_2}_{j_2,h_2}V^{\omega_3}_{j_3,h_3}\rangle\,,\end{aligned} \tag{A.24}$$

where $\alpha=j+\frac{k}{2}\omega-h$, and in the last step we have inverted $y\to y^{-1}$. The final expression in (A.24) reduces to the r.h.s. of eq. (A.23) by virtue of (A.18). Eq. (A.23) can also be expressed as

$$\begin{aligned}&\int d^2y|y^{2j-2}|^2\,\langle V^{\omega_1}_{j_1,h_1}V^{\omega_4}_{j_4,h_4}V^{\omega}_{1-j}\left(\frac{1}{y}\right)\rangle\,\langle V^{\omega}_{j}(y)V^{\omega_2}_{j_2,h_2}V^{\omega_3}_{j_3,h_3}\rangle \\ &\qquad\qquad = \frac{\langle V^{\omega_1}_{j_1,h_1}V^{\omega_4}_{j_4,h_4}V^{\omega}_{j,h_3-h_2}\rangle\,\langle V^{\omega}_{j,h_3-h_2}V^{\omega_2}_{j_2,h_2}V^{\omega_3}_{j_3,h_3}\rangle}{\langle V^{\omega}_{j,h_3-h_2}V^{\omega}_{j,h_3-h_2}\rangle}\,,\end{aligned} \tag{A.25}$$

or, equivalently, as

$$\begin{aligned}&\int d^2y|y^{-2j}|^2\,\langle V^{\omega_1}_{j_1,h_1}V^{\omega_4}_{j_4,h_4}V^{\omega}_{j}\left(\frac{1}{y}\right)\rangle\,\langle V^{\omega}_{1-j}(y)V^{\omega_2}_{j_2,h_2}V^{\omega_3}_{j_3,h_3}\rangle \\ &\qquad\qquad = \frac{\langle V^{\omega_1}_{j_1,h_1}V^{\omega_4}_{j_4,h_4}V^{\omega}_{j,h_3-h_2}\rangle\,\langle V^{\omega}_{j,h_3-h_2}V^{\omega_2}_{j_2,h_2}V^{\omega_3}_{j_3,h_3}\rangle}{\langle V^{\omega}_{j,h_3-h_2}V^{\omega}_{j,h_3-h_2}\rangle}\,.\end{aligned} \tag{A.26}$$

These identities are used multiple times along the paper while studying the factorization properties of different examples of spectrally flowed four-point functions.

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