ENTROPY NUMBERS AND BOX DIMENSION OF POLYNOMIALS AND HOLOMORPHIC FUNCTIONS

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Abstract. We study entropy numbers and box dimension of (the image of) homogeneous polynomials and holomorphic functions between Banach spaces. First, we see that entropy numbers and box dimensions of subsets of Banach spaces are related. We show that the box dimension of the image of a ball under a homogeneous polynomial is finite if and only if it spans a finite-dimensional subspace, but this is not true for holomorphic functions. Furthermore, we relate the entropy numbers of a holomorphic function to those of the polynomials of its Taylor series expansion. As a consequence, if the box dimension of the image of a ball by a holomorphic function f is finite, then the entropy numbers of the polynomials in the Taylor series expansion of f at any point of the ball belong to ℓ_p for every $p > 1$.

Introduction

In this note, we study the *compactness* of the image of homogeneous polynomials and holomorphic functions between Banach spaces. Let E and F be Banach spaces, $U \subset E$ an open set and $f\colon U \to F$ a holomorphic mapping. Whenever f maps a ball $B \subset U$ onto a relatively compact set, we use entropy numbers or box dimension (see the definitions in Section 2) to *measure* the compactness of $f(B)$. For $x_0 \in U$, let $P_m f(x_0)$ be the m-homogeneous polynomial of the Taylor series expansion of f at x_0. This article was originally motivated by the following question, posed by Richard Aron to the fourth author: given $\varepsilon > 0$ and some ball $B \subset U$, can we relate the *degree of compactness* (in terms of entropy numbers or box dimension) of $f(B)$ and that of $P_m f(x_0)(B)$? Similar questions were addressed in [1, 2, 13, 16] for other ways of measuring the compactness of a set. Also, entropy numbers and, in general, the theory of s-numbers and quasi s-numbers of multilinear operators was treated in [3, 8, 9, 10].

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Before dealing with this and other similar questions, we will see in Section 1 that entropy numbers and box dimension are closely related. Indeed, Proposition 1.2 essentially states that, for a connected set K, the box dimension of K is finite if and only if the entropy numbers of K decay exponentially.

Also, we show an example of a connected set K such that its box dimension is infinite and the sequence $(e_n(K))_n \in \ell_1$. In Section 2 we study m-homogeneous polynomials and holomorphic functions. In particular, we relate the box dimension of the image and the linear dimension of the subspace it spans. For example, we present a function f defined on the complex unit disk Δ such that, for every smaller disk $D \subset \Delta$, the (upper) box dimension of $f(D)$ is 2 while $f(D)$ spans an infinite dimensional subspace (Examples 2.4 and 2.5). On the other hand, we see that such an example cannot exist for m-homogeneous polynomials. In fact, if the image of an m-homogeneous polynomial spans an infinite dimensional subspace, then its box dimension must be infinite (Theorem 2.2).

Finally, in Section 4, for a given holomorphic function $f\colon U \to F$, $x_0 \in U$ and $\varepsilon > 0$ we obtain in Lemma 4.3 a relationship between the entropy numbers $e_n(f(x_0 + \varepsilon B_E))$ and $e_N(P_m f(x_0)(B_E))$, where n and N are related (here, B_E is the unit ball of E). As a consequence, we see in Proposition 4.2 that if the (upper) box dimension of $f(x_0 + \varepsilon B_E)$ is finite, then the sequence $(e_n(P_m f(x_0)(B_E)))_n$ belongs to ℓ_p for every $p > 1$. The problem of measuring $P_m f(B_E)$ (in some sense) in terms of the image of f is closely related to the problem of measuring the absolutely convex hull of a set K in terms of K (see the proof of [2, Proposition 3.4]). For entropy numbers, this geometric problem was studied by many authors (see, for example, [4, 5, 12, 10] and the references therein). Our Proposition 4.2 seems to provide sharper results than those obtained from the proof of [2, Proposition 3.4] together with the known relationships between the entropy numbers of a set and its absolutely convex hull (see the discussion after Proposition 4.2).

Throughout the article, we consider complex Banach spaces E and F and write B_E for the open unit ball of E and Δ for the open unit disc in $\mathbb{C}$. Also, ℓ_p^N stands for $\mathbb{C}^N$ endowed with the ℓ_p norm. We write ϵ_j for the vector/sequence which has 1 in the j-th coordinate and 0 elsewhere. This notation will be used both in $\mathbb{C}^N$ and in ℓ_p.

For a m-homogenoeus polynomial P, by $\overset{\vee}{P}(x_1, \ldots, x_m)$ we denote the unique symmetric m-linear operator such that $P(x) = \overset{\vee}{P}(x, \ldots, x) = \overset{\vee}{P}(x^m)$.

1. Entropy numbers, covering numbers and box dimension

A possible way to refine the concept of compactness in a Banach space is via the so-called entropy numbers. Recall that the *n-th entropy number* $\mathcal{E}_n(K)$ of a set K of a metric space (X,d) is defined as

$$\mathcal{E}_n(K)\colon = \inf\left\{\varepsilon>0\colon \exists\, x_1,\dots,x_n\in X\colon K\subset\bigcup_{i=1}^{n}B_X(x_i,\varepsilon)\right\},$$

where $B_X(x,\varepsilon)=\{\widetilde{x}\in X\colon d(x,\widetilde{x})<\varepsilon\}$. The *n-th dyadic entropy numbers* of K are given by $e_n(K)=\mathcal{E}_{2^{n-1}}(K)$. A subset K of a Banach space is relatively compact if and only if the sequence $(e_n(K))_{n\in\mathbb{N}}\in c_0$. Stronger conditions on the decay rate of $(e_n(K))_{n\in\mathbb{N}}\in c_0$ lead to stronger versions of compactness.

The concept of covering numbers is closely related to that of entropy numbers. For a bounded set $L\subset X$ and $\varepsilon>0$, the *ε-covering number* $N(L,\varepsilon)$ is given by

$$N(L,\varepsilon)\colon = \min\left\{n\in\mathbb{N}\colon \exists\, x_1,\dots,x_n\in X\colon L\subset\bigcup_{i=1}^{n}B_X(x_i,\varepsilon)\right\}.$$

Also, the *upper* and *lower box counting dimension of* L are given by

$$\overline{\dim}_B L=\limsup_{\varepsilon\to0^+}\frac{\log N(L,\varepsilon)}{-\log(\varepsilon)}\quad\text{and}\quad \underline{\dim}_B L=\liminf_{\varepsilon\to0^+}\frac{\log N(L,\varepsilon)}{-\log(\varepsilon)}.$$

In the case that $\lim_{\varepsilon\to0^+}\frac{\log N(L,\varepsilon)}{-\log(\varepsilon)}$ exists, we say that the *box counting dimension* of L is

$$\dim_B L=\lim_{\varepsilon\to0^+}\frac{\log N(L,\varepsilon)}{-\log(\varepsilon)}.$$

We remark that, if $L\subset X$ is not totally bounded, then $\underline{\dim}_B L=\infty$. Also, if $g\colon X\to Y$ is a Lipschitz function and $L\subset X$ is a bounded subset, then $\overline{\dim}_B g(L)\le\overline{\dim}_B L$ and $\underline{\dim}_B g(L)\le\underline{\dim}_B L$. We refer to [7] for the basics of the theory of fractal geometry.

The (linear) dimension of a subset U of a complex vector space $\mathbb{V}$ is the dimension of $\operatorname{span}\{U\}$ and is denoted by $\dim U$. Note that for an open and bounded set $U\subset\mathbb{C}$ we have $\underline{\dim}_B U=\overline{\dim}_B U=2$, while $\dim\ U=1$.

The following result can be found in [7, Chapter 3.1] for sets in $\mathbb{R}^N$, but the proof works line by line for general metric spaces?

Proposition 1.1. *Let (X,d) a metric space and $L\subset X$ a bounded set. If $(\delta_n)_n$ is a decreasing sequence of real numbers such that $\lim_{n\to\infty}\delta_n=0$ and $\liminf_{n\to\infty}\frac{\delta_{n+1}}{\delta_n}>0$, then*

$$\overline{\dim}_B L=\limsup_{n\to\infty}\frac{\log N(L,\delta_n)}{-\log(\delta_n)}\qquad \underline{\dim}_B L=\liminf_{n\to\infty}\frac{\log N(L,\delta_n)}{-\log(\delta_n)}.$$

The following proposition gives the connection between upper box dimension and the asymptotic behaviour of the entropy numbers.

Proposition 1.2. *Let (X,d) be a metric space and $L \subset X$ a connected and totally bounded set. Then, $\overline{\dim}_B L < \infty$ if and only if $\limsup_{n\to\infty} e_n(L)^{1/n} < 1$.*

Proof. Suppose that $\overline{\dim}_B = d < \infty$. By [11, Corollary 5] (which is stated for subsets of $\mathbb{R}^N$ but holds in general metric spaces), there exists $n_\varepsilon \in \mathbb{N}$ such that for every $n > n_\varepsilon$ we have

$$e_n(L) \leq 2(2^{n-1}+1)^{-\frac{1}{d+\varepsilon}} \leq 2^{1-\frac{n}{d+\varepsilon}}.$$

This gives

$$\limsup_{n\to\infty} e_n(L)^{1/n} \leq 2^{-\frac{1}{d+\varepsilon}} < 1.$$

Conversely, suppose that $\limsup_{n\to\infty} e_n(L)^{1/n} < 1$ and take $0 < \beta < 1$, $n_0 \in \mathbb{N}$ such that $e_n(L) < \beta^n$ for all $n \geq n_0$. For each n, the definiton of $e_n(L)$ gives us $x_1, x_2, \ldots, x_{2^n-1} \in X$ such that $L \subset \bigcup_{j=1}^{2^{n-1}} B(x_j, 2e_n(L))$. Therefore, $N(L, 2e_n(L)) \leq 2^{n-1}$ and for $n \geq n_0$ we have

$$\frac{\log\left(N(L,2e_n(L))\right)}{-\log(2e_n(L))} \leq \frac{(n-1)\log 2}{-\log(2e_n(L))} < \frac{(n-1)\log 2}{-\log(2\beta^n)}$$
$$= \frac{(n-1)\log 2}{-\log(2) - n\log(\beta)} \xrightarrow[n\to\infty]{} -\frac{\log(2)}{\log(\beta)} < +\infty.$$

In order to conclude that $\overline{\dim}_B L < \infty$, we want to use Proposition 1.1 with $\delta_n = 2e_n(L)$, which are clearly decreasing and convergent to zero (since L is totally bounded). The fact that $\liminf_{n\to\infty} \frac{\delta_{n+1}}{\delta_n} > 0$ follows from Lemma 1.3 below. □

Lemma 1.3. *Let (X,d) be a metric space and $L \subset X$ a connected set. Then,*

$$\liminf_{n\to\infty} \frac{e_{n+1}(L)}{e_n(L)} \geq \frac{1}{5}.$$

Proof. Fix $n \in \mathbb{N}$, take $\delta > 0$ and set $r = (1+\delta)e_{n+1}(L)$. There exists a subset $L_0 = \{x_1, x_2, \ldots, x_{2^n}\}$ of X such that $L \subset \bigcup_{x\in L_0} B_X(x,r)$.

First, we claim that for $x \in L_0$, there exists $\widetilde{x} \in L_0$, $x \neq \widetilde{x}$ such that $d(x,\widetilde{x}) < 2r$. Indeed, suppose that $B_X(x,r) \cap B_X(\widetilde{x},r) = \emptyset$ for every $\widetilde{x} \in L_0 \setminus \{x\}$. Now, the open set $V = \bigcup_{\widetilde{x}\in L_0\setminus\{x\}} B_X(\widetilde{x},r)$ satisfies $L \subset V \cup B_X(x,r)$ and $V \cap B_X(x,r) = \emptyset$, which is impossible since L connected.

Now, take $x_{j_1} \in L_0$ and let $M_1 = \{x \in L_0 \colon d(x_{j_1},x) < 2r\}$. Note that M_1 have at least 2 points of L_0: x_{j_1} and the one given by the previous claim. If $L_0 \subset B_X(x_{j_1}, 4r)$, then $L \subset B_X(x_{j_1}, 5r)$ and $5r \geq \mathcal{E}_1(L) \geq e_n(L)$. Thus

$$\frac{e_{n+1}(L)}{e_n(L)} \geq \frac{e_{n+1}(L)}{5r} = \frac{e_{n+1}(L)}{5e_{n+1}(L)(1+\delta)} = \frac{1}{5(1+\delta)}$$

and we are done, since δ is arbitrary. If, on the contrary, there exists $x_{j_2} \in L_0$ such that $d(x_{j_2}, x_{j_1}) \geq 4r$, we take $M_2 = \{x \in L_0 \colon d(x_{j_2},x) < 2r\}$. By the claim above, M_2

has at least 2 points and, clearly, $M_2 \cap M_1 = \emptyset$. Now, if $L_0 \subset B_X(x_{j_1}, 4r) \cup B_X(x_{j_2}, 4r)$, then $L \subset B_X(x_{j_1}, 5r) \cup B_X(x_{j_2}, 5r)$. This implies that $5r \geq \mathcal{E}_2(L) \geq e_n(L)$, which gives

$$\frac{e_{n+1}(L)}{e_n(L)} > \frac{1}{5(1+\delta)}$$

and we are done. If not, there exists $x_{j_3} \in L_0$ such that $x_{j_3} \notin B_X(x_{j_1}, 4r) \cup B_X(x_{j_2}, 4r)$ and we take $M_3 = \{x \in L_0 \colon d(x_{j_3}, x) < 2r\}$. Again by the claim, M_3 have at least 2 points and, also, $M_1 \cap M_3 = \emptyset$ and $M_2 \cap M_3 = \emptyset$. Then, if $L_0 \subset \bigcup_{i=1,2,3} B_X(x_{j_i}, 4r)$, reasoning as before we obtain that

$$\frac{e_{n+1}(L)}{e_n(L)} \geq \frac{1}{5(1+\delta)}.$$

If not, we continue with this procedure and, since L_0 have 2^n points, this procedure comes to an end. So for some $m \in \mathbb{N}$ we get subsets $M_i \subset L_0$ and points $x_{j_i} \in M_i$, $1 \leq i \leq m$, such that $L_0 \subset \bigcup_{i=1}^m B_X(x_{j_i}, 5r)$, which implies that that $5r \geq \mathcal{E}_m(L)$. Since the sets M_i are disjoint and each has at least 2 points, then $m \leq \frac{2^n}{2} = 2^{n-1}$. Thus $5r \geq \mathcal{E}_m(L) \geq e_n(L)$ and then

$$\frac{e_{n+1}(L)}{e_n(L)} \geq \frac{1}{5(1+\delta)},$$

which give the desired result. □

We finish this section with an example that will be used later. This example shows a connected set K in a Banach space E for which the sequence $(e_n(K))_n$ belongs to ℓ_1 while $\dim_B K = \infty$ which, thanks to the above proposition, is equivalent to $\limsup_{n\to\infty} e_n(K)^{1/n} = 1$.

Example 1.4. For $0 < \varepsilon < 1$, consider the set $K = \{(x_n)_n \subset \mathbb{C} \colon |x_n| \leq \varepsilon^n\} \subset c_0$. Then $\underline{\dim}_B K = \infty$ and $(e_n(K))_n \in \ell_1$.

Proof. Fix $N \in \mathbb{N}$ and denote by $\Pi_N \colon c_0 \to c_0$ the projection onto the first coordinates. Let $K^N = \Pi_N(K)$. Note that $K^N \subset K$ and that for any $(x_n)_n \in K$, $\|\Pi_N((x_n)_n) - (x_n)_n\| \leq \varepsilon^N$. Then we have the inequalities

$$e_n(K^N) \leq e_n(K) \leq e_n(K^N) + \varepsilon^N \quad \text{for all } n \in \mathbb{N} \tag{1}$$

We define the diagonal operator $D_N \colon c_0 \to c_0$ by $D((x_n)_n) = (x_1\varepsilon, x_2\varepsilon^2, \ldots, x_N\varepsilon^N, 0, \ldots)$. Since $D_N(B_{c_0}) = K^N$, we can apply [6, Proposition 1.3.2] to estimate the entropy numbers of K^N as

$$\sup_{1\leq k\leq N} 2^{-\frac{(n-1)}{2k}} \varepsilon^{\frac{k+1}{2}} \leq e_n(K^N) \leq 6 \sup_{1\leq k<N} 2^{-\frac{(n-1)}{2k}} \varepsilon^{\frac{k+1}{2}}. \tag{2}$$

Combining (1) and (2), we obtain

$$\sup_{1\leq k\leq N} 2^{-\frac{(n-1)}{2k}}\varepsilon^{\frac{k+1}{2}} \leq e_n(K) \leq 6 \sup_{1\leq k<N} 2^{-\frac{(n-1)}{2k}}\varepsilon^{\frac{k+1}{2}} + \varepsilon^N.$$

The above inequality holds for every $N \in \mathbb{N}$, so writing $s = \min\{\varepsilon, \frac{1}{2}\}$ and $S = \max\{\varepsilon, \frac{1}{2}\}$, and using simple calculations, we find positive constants C_1 and C_2 such that

$$C_1 s^{\sqrt{n-1}} \leq e_n(D) \leq C_2 S^{\sqrt{n-1}}.$$

Thus, $(e_n(D))_n \in \ell_1$.

To see that $\dim_B K = \infty$, fix again $N \in \mathbb{N}$ and define $T_N\colon c_0 \to \mathbb{C}^N$ by $T_N((x_n)_n) = (x_1, x_2, \ldots, x_N)$. Since $T_N(K) \subset \mathbb{C}^N$ has non-empty interior, we obtain that $2N = \underline{\dim}_B(T_N(K)) \leq \underline{\dim}_B(K)$. Since N was arbitrary, the result follows. □

2. On the dimension of the image

We begin this section with the following simple observation: if E and F are normed spaces, for a linear operator $T\colon E \to F$ there is clear relationship between the box dimension of $T(B_E)$ and its linear dimension. Indeed, if $\dim T(E) = N$ is finite, then $T(E)$ is an isomorphic copy of $\mathbb{C}^N$ and $T(B_E)$ corresponds (via such isomporphisim) to an open subset of $\mathbb{C}^N$. Then, we have $\dim_B(T(B_E)) = 2N$ (which means, in particular, that $T(B_E)$ has finite box dimension). On the other hand, if $T(E)$ has infinite linear dimension, we can take $\{x_n\}_n$ a sequence of linearly independent elements in $T(B_E)$. Now, for each N, $\text{span}\{x_1, \ldots, x_N\} \cap T(B_E)$ is homeomorphic (via the restriction of a linear isomorphism) to an open subset of $\mathbb{C}^N$. Since bi-Lipschitz mappings preserve box dimension, we have

$$N = \dim_B(\text{span}\{x_1, \ldots, x_N\} \cap T(B_E)) \leq \dim_B(T(B_E)))$$

for all N. Therefore, the box dimension of $T(B_E)$ is also infinite. In particular, we have the following.

Remark 2.1. If E and F are normed spaces and $T\colon E \to F$ is a linear operator, then $T(B_E)$ has infinite (linear) dimension if and only if it has infinite box dimension.

The aim of this section is to study possible analogous results for homogeneous polynomials and holomorphic functions between normed spaces. In the polynomial case, we obtain a result analogous to Remark 2.1, but the proof is much more involved. Our result in this direction is the following.

Theorem 2.2. *Let $P\colon E \to F$ be a homogeneous polynomial. Then,* $\dim P(B_E) = \infty$ *if and only if* $\underline{\dim}_B P(B_E) = \infty$.

We devote Section 3 to the proof of this theorem, which involves some results on polynomials of several complex variables that we think are interesting in their own. Regarding holomorphic mappings, the situation is completely different: Example 2.4 below shows that the box dimension of the image of f may be finite while its linear dimension is infinite. Let $f\colon U \to F$ where $U \subset E$ is an open set. It is clear that, if F is finite dimensional, then $f(U)$ has finite both linear and box dimensions. If E is finite dimensional, things do not go so smoothly (see again Example 2.4). We start with the following simple positive result.

Remark 2.3. Let E and F be Banach spaces, E finite dimensional, and $U \subset E$ be an open subset. If $f : U \to F$ is holomorphic and $K \subset U$ is compact, then $\overline{\dim}_B f(K)$ is finite.

Proof. By standard compactness arguments, it is enough to show the result for K a closed ball (in the finite dimensional Banach space E). Since f is holomorphic, it is continuously (Fréchet) differentiable. If $Df(z)$ denotes the differential of f at z, let M be the maximum of $\|Df(z)\|$ for $z \in K$. By [14, Theorem 13.8] we have

$$\|f(x) - f(y)\| \leq \|x - y\| \cdot \sup_{0 \leq t \leq 1} \|Df(x + t(y - x)\| \leq K\|x - y\|.$$

Therefore, f is Lipschitz on K and then $\overline{\dim}_B(f(K)) \leq \overline{\dim}_B(K) < +\infty$. □

The following examples show, on the one hand, that an analogue to Theorem 2.2 does not hold for holomorphic mappings. Also, we see in Example 2.4 that the restriction to a compact subset of U is necessary in the previous proposition.

Example 2.4. For $1 \leq p \leq \infty$, let $f\colon \Delta \to \ell_p$ (c_0 if $p = \infty$) be given by $f(z) = (z, z^2, z^3, \ldots)$. Then $f(\Delta)$ is not a relative compact set and, in particular, $\underline{\dim}_B f(\Delta) = \infty$. Also, for any $0 < r < 1$, the linear dimension of $f(r\Delta)$ is infinite while its box dimension is finite. In other words, $f(r\Delta)$ has finite box dimensional but it is not contained in any finite dimensional subspace of ℓ_p.

Proof. Take the sequence $(z_n)_n \subset \Delta$ given by $z_n = (\frac{1}{2})^{\frac{1}{2^{n-1}}}$. Note that, if $n < m$, then

$$\|f(z_n) - f(z_m)\|_{\ell_p} \geq |\left((f(z_n) - f(z_m)\right)_{2^{m-1}}| = \left|\left(\frac{1}{2}\right)^{\frac{2^{m-1}}{2^{n-1}}} - \frac{1}{2}\right| \geq \left|\frac{1}{4} - \frac{1}{2}\right| = \frac{1}{2}.$$

This shows that the sequence $(f(z_n))_n \subset f(\Delta)$ is uniformly separated, and hence $f(\Delta)$ cannot be totally bounded.

By Remark 2.3 we know that for $0 < r < 1$ we have $\overline{\dim}_B(f(r\Delta)) < \infty$. To see that $f(r\Delta)$ has infinite linear dimension, fix $\delta < r$ and define $w_n = \frac{\delta}{n}$ for $n \in \mathbb{N}$. Let us see that $\{(f(w_n))_n\}$ is a linearly independent set. If not, for some $N \in \mathbb{N}$ and some not all zero scalars $a_1, \dots, a_N$ we must have $\sum_{n=1}^N a_n f(z_n) = 0$, and so

$$\begin{pmatrix} \delta & \frac{\delta}{2} & \cdots & \frac{\delta}{N} \\ \delta^2 & \left(\frac{\delta}{2}\right)^2 & \cdots & \left(\frac{\delta}{N}\right)^2 \\ \vdots & \vdots & \vdots & \vdots \\ \delta^N & \left(\frac{\delta}{2}\right)^N & \cdots & \left(\frac{\delta}{N}\right)^N \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

This is impossible since the (Vandermonde) determinant of the matrix is nonzero. □

With the same ideas we can prove the following.

Example 2.5. Consider entire function $f : \mathbb{C} \to \ell_p$ given by $f(z) = (z, \frac{z^2}{2!}, \frac{z^3}{3!}, \dots)$. Then, for every $r > 0$ the linear dimension of $f(r\Delta)$ is infinite while its box dimension is finite.

3. The proof of Theorem 2.2

Let $P : E \to F$ be a m-homogenoeous polynomial. If dim $P(B_E) < \infty$, then it is clear that $\underline{\dim}_B P(B_E) < \infty$, since it is a subset of a finite dimensional vector space.

Suppose now that dim $P(B_E) = \infty$. Given $N \in \mathbb{N}$ we can take $x_1, \dots, x_N \in B_E$ such that $\{P(x_1), \dots, P(x_N)\}$ are lineary independent. We choose a bounded linear projection $R\colon F \to \operatorname{span}\{P(x_1), \dots, P(x_N)\}$. The multinomial formula gives us

$$\begin{aligned} R \circ P\left(\textstyle\sum_{i=1}^N a_i x_i\right) &= R\Big(\sum_{k_1+\ldots+k_N=n} \frac{n!}{k_1 \ldots k_N!} \prod_{1\le j\le N} a_j^{k_j} \overset{\vee}{P}(x_1^{k_1}, \dots, x_N^{k_N})\Big) \\ &= p_1(a_1, \dots, a_n)P(x_1) + \ldots + p_N(a_1, \dots, a_N)P(x_N) \end{aligned} \tag{3}$$

where, for each $1 \le j \le N$, $p_j : \mathbb{C}^N \to \mathbb{C}$ is an m-homogeneous polynomial. Note that $p_j(\epsilon_j) = 1$ and $p_j(\epsilon_i) = 0$ if $i \ne j$. This means that our polynomials $p_1, \dots, p_N$ are linearly independent. We define $\boldsymbol{f} : \mathbb{C}^N \to \mathbb{C}^N$ by

$$\boldsymbol{f}(z_1, \dots, z_N) = (p_1(z_1, \dots, z_N), \dots, p_N(z_1, \dots, z_N))$$

and $T\colon \mathbb{C}^N \to \operatorname{span}\{P(x_1), \dots P(x_N)\}$ by

$$T(z_1, \dots, z_N) = \sum_{j=1}^N z_j P(x_j),$$

which is a linear isomorphism.

Since $\{P(a_1x_1 + \ldots + a_Nx_N)\colon \sum_{j=1}^N |a_n| \le 1\} \subset P(B_E)$, we have

$$\dim_B(P(B_E)) \ge \dim_B \left\{ R \circ P(a_1x_1 + \ldots + a_Nx_N)\colon \sum_{j=1}^N |a_j| \le 1 \right\} = \dim_B(\boldsymbol{f}(B_{\ell_1^N})).$$

The proof of Theorem 2.2 is finished once we show that $\dim_B(\boldsymbol{f}(B_{\ell_1^N}))$ goes to infinity as N does. This is shown in Corollary 3.3 below, where we see that $\dim_B(\boldsymbol{f}(B_{\ell_1^N}))$ is at least $N^{\frac{1}{m}} - m$. In order to get Corollary 3.3 we need some preparation, which we develop in the following subsection.

3.1. **On the rank of the jacobian of homogeneous polynomials.** Consider $p_1, \dots, p_r$ m-homogeneous polynomials from $\mathbb{C}^N$ to $\mathbb{C}$ such that the $r \times N$ jacobian matrix $J_{p_1,\dots,p_N} := \left(\frac{\partial p_i}{\partial z_j}\right)_{1\le i\le r, 1\le j\le N}$ has maximal rank. Assume w.l.o.g. that the first principal minor

$$Q\colon = Q(z_1, \dots, z_N) = \det\left(\frac{\partial p_i}{\partial z_j}(z_1, \dots, z_N)\right)_{1\le i,j\le r}$$

is not zero. For $i = 1, \dots, r$, $j = r+1, \dots, N$, denote with $Q_{ij} = Q_{ij}(z_1, \dots, z_N)$ the determinant of the submatrix of the jacobian matrix whose columns are indexed by $\{1, \dots, r\} \cup \{j\} \setminus \{i\}$, multiplied by $(-1)^i$. All these polynomials Q and Q_{ij} are homogeneous of degree $r(m-1)$.

Is easy to check that any other m-homogeneous polynomial P such that the rank of the jacobian matrix of $p_1, \dots, p_r, P$ is equal to r must be a solution of the following linear system of partial differential equations:

$$Q\,\frac{\partial P}{\partial z_j} + \sum_{i=1}^{r} Q_{ij}\,\frac{\partial P}{\partial z_i} = 0,\ \ j = r+1, \dots, N. \tag{4}$$

We will exhibit a bound on the dimension of the $\mathbb{C}$-vector space of homogeneous polynomials of degree m satisfying (4) which does not depend on N.

Theorem 3.1. *The dimension of the $\mathbb{C}$-vector space of all m-homogeneous polynomials from $\mathbb{C}^N$ to $\mathbb{C}$ satisfying (4) is bounded from above by $\binom{r+m-2}{m-2}$.*

Proof. As $Q \neq 0$, after a linear change of variables in $\mathbb{C}^N$ if necessary, we may assume that the monomial $z_1^{r(m-1)}$ appears in the Taylor expansion of Q. For $\alpha = (\alpha_1, \dots, \alpha_N) \in (\mathbb{Z}_{\ge 0})^N$, we denote with z^α the product $z_1^{\alpha_1} \dots z_N^{\alpha_N}$, and we set $|\alpha| = \alpha_1 + \cdots + \alpha_N$. We can then write

$$\begin{aligned} Q &= \textstyle\sum_{|\alpha|=r(m-1)} Q_\alpha z^\alpha \\ Q_{ij} &= \textstyle\sum_{|\beta|=r(m-1)} Q_{ij\beta} z^\beta \\ P &= \textstyle\sum_{|\gamma|=m} P_\gamma z^\gamma. \end{aligned}$$

Let $\epsilon_1, \dots, \epsilon_n$ be the elements of the standard basis of $\mathbb{Z}^n$. By setting to zero all the coefficients in the polynomial (4), we get the following linear system of equations for

$(P_\gamma)_{|\gamma|=m}$:

(5)

$$\sum_{\alpha+\gamma=\delta+\epsilon_j} \gamma_j\, Q_\alpha\, P_\gamma + \sum_{i=1}^{r} \sum_{\beta+\gamma=\delta+\epsilon_i} \gamma_i\, Q_{ij\beta}\, P_\gamma = 0; \quad j = r+1, \ldots, n,\ |\delta| = r(m-1)+m-1.$$

This is a homogeneous system of $\binom{n+r(m-1)+m-2}{r(m-1)+m-1}$ linear equations in the $\binom{n+m-1}{m}$ variables $(P_\gamma)_{|\gamma|=m}$. We will show that the corank of this system is bounded from above by the number of monomials of degree m in r variables, which is equal to $\binom{r+m-1}{m-1}$.

To do so, we will show that for any $\tilde{\gamma} \in (\mathbb{Z}_{\geq 0})^N$ such that $|\tilde{\gamma}| = m$, and $\tilde{\gamma}_j > 0$ for some $j > r$, there is an equation from (5) from where we can express $P_{\tilde{\gamma}}$ as a function of all those P_γ with $z^\gamma \prec z^{\tilde{\gamma}}$ in the standard lexicographic order $z_1 \prec x_2 \prec \ldots \prec z_N$. This will amount to a triangular submatrix of the linear system (5) with the desired corank.

Write then $\tilde{\gamma} = (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_j, 0, \ldots, 0)$, with $\tilde{\gamma}_j \neq 0$, $j > r$, and set

$$\tilde{\delta} := r(m-1) + \tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_{j-1}, \tilde{\gamma}_j - 1, 0, \ldots, 0).$$

We extract the equation corresponding to $\delta = \tilde{\delta}$ in (5), and find that

- $P_{\tilde{\gamma}}$ appears in a nonzero term in the first sum (corresponding to $\tilde{\alpha} = (r(m-1), 0, \ldots, 0)$, recall that by hypothesis we have that $Q_{\tilde{\alpha}} \neq 0$ and also $\tilde{\gamma}_j \neq 0$);
- all the P_γ appearing in the second (double) sum have $\gamma_\ell = 0$ if $\ell > j$ and $\gamma_j \leq \tilde{\gamma}_j - 1$, so we have that $z^\gamma \prec z^{\tilde{\gamma}}$;
- the rest of the P_γ appearing in the first sum must satisfy

$$\begin{array}{rcl} \gamma_1 & \leq & r(m-1) + \tilde{\gamma}_1 \\ \gamma_2 & \leq & \tilde{\gamma}_2 \\ \vdots & & \vdots \\ \gamma_j & \leq & \tilde{\gamma}_j \\ \gamma_\ell & = & 0 \ \forall \ell > j. \end{array}$$

  As $\gamma \neq \tilde{\gamma}$ then there exists a unique $i > 1$ such that $\gamma_i < \tilde{\gamma}_i$ and $\gamma_\ell = \tilde{\gamma}_\ell$ for $\ell > i$. This implies that $z^\gamma \prec z^{\tilde{\gamma}}$ also in this case, which concludes with the proof of the claim.

□

Remark 3.2. The bound is sharp as it can be seen easily by choosing as $P_i = z_i^m$ for $i = 1, \ldots, r$. Then, the $\mathbb{C}$-vector subspace of all the polynomials of degree m in the variables $z_1, \ldots, z_r$ satisfy (4). On the other hand, it is a classical result (see [15]) that one can make a linear change of variables such that the polynomial system depends polynomially on $r < N$ variables if and only if the equations (4) can be defined with Q, $Q_{ij} \in \mathbb{C}$.

Corollary 3.3. *Let $p_1, \dots, p_N$ m-homogeneous polynomials from $\mathbb{C}^N$ to $\mathbb{C}$ which are linearly independent. Then the rank of the jacobian matrix of this family (i.e. the dimension of the image of the map $\boldsymbol{f} : \mathbb{C}^N \to \mathbb{C}^N$ defined by these polynomials) is at least $N^{\frac{1}{m}} - m$.*

Proof. Denote with r the rank of $J_{p_1,\dots,p_N}$, and suppose w.l.o.g. that $J_{p_1,\dots,p_N}$ has maximal rank. Then, we have that all p_j satisfies (4) for $j = 1, \dots, N$. As they are linearly independent, from Theorem 3.1, we deduce straightforwardly that

$$N \leq \binom{r+m-1}{m-1} \leq (r+m)^m. \quad \square$$

4. Entropy numbers for holomorphic functions and their Taylor coefficients

In this section we relate the entropy numbers of f with the entropy numbers of the polynomials of the Taylor series expansion of f and viceversa. We start with a simple example. Let $f\colon B_{c_0} \to c_0$ be defined as $f((x_n)_n) = (x_1, x_2^2, x_3^3, \dots)$. It is clear that $P_m f(0)((x_n)_n) = (0, \dots, x_m^m, 0, \dots)$, getting that $\overline{\dim}_B P_m f(0)(B_E) = 2$. On the other hand, given $\varepsilon > 0$ we have, for every $n \in \mathbb{N}$

$$e_n(f(\varepsilon B_{c_0})) = e_n(\{(x_n)_n \subset \mathbb{C}\colon |x_n| \leq \varepsilon^n\}),$$

which are the entropy numbers of the set from Example 1.4. Thus, we see that $\overline{\dim}_B f(\varepsilon B_{c_0}) = \infty$ for every $\varepsilon > 0$. This example shows the (expected) fact that the homogeneous polynomials in the Taylor expansion of f can have small dimensional images while f maps any ball in an infinite dimensional set. In particular, by Proposition 1.2 (and its proof) we have that

$$\limsup_{n\to+\infty} e_n(P_m f(0)(B_{c_0}))^{1/n} \leq \frac{\sqrt{2}}{2} \text{ for all } m$$

while, on the other hand,

$$\limsup_{n\to+\infty} e_n(f(\varepsilon B_{c_0}))^{1/n} = 1.$$

This shows that a (uniformly) fast decay of the entropy numbers of $P_m f(0)(B_E)$ does not imply a fast decay in the entropy numbers of $f(\varepsilon B_E)$ for $\varepsilon > 0$.

Recall that, by Example 1.4, the sequence $(e_n(f(\varepsilon B_{c_0})))_n$ belongs to ℓ_1 for every $\varepsilon > 0$. The following example shows a bit more.

Example 4.1. There is an holomorphic function $f\colon B_{c_0} \to c_0$ such that every polynomial of its Taylor expansion at 0 has finite rank (hence, for every $\varepsilon > 0$, $\dim_B P_m f(0)(\varepsilon B_{c_0})$

is finite) but for any $\varepsilon > 0$ the sequence $(e_n(f(\varepsilon B_{c_0}))_n$ does not belong to ℓ_p for any $1 \leq p < \infty$.

Proof. Consider $(\sigma_m)_m$ the partition of the natural numbers such that each σ_m is a finite set with $m!$ consecutive elements:

$$\sigma_1 = \{1\}; \quad \sigma_2 = \{\underbrace{2,3}_{2!}\}; \sigma_3 = \{\underbrace{4,5,6,7,8,9}_{3!}\}; \quad \sigma_4 = \{\underbrace{\dots}_{4!}\}; \quad \dots$$

and define $f\colon B_{c_0} \to c_0$ by

$$f((x_n)_n) = (x_1, \underbrace{x_2^2, x_3^2}_{2!}, \underbrace{x_4^3, x_5^3, \dots, x_9^3}_{3!}, \dots).$$

In other words, the j-th coordinate of $f((x_n)_n)$ is x_j^N if $j \in \sigma_N$. Note that for every $m \in \mathbb{N}$ we have

$$P_m f(0)((x_n)_n) = (0, 0, \dots, \underbrace{x_j^m, x_{j+1}^m, \dots,}_{m!}, 0 \dots),$$

where $j, j+1, \dots \in \sigma_m$. Now, denote by $\Pi_{\sigma_m}\colon c_0 \to c_0$ the projection onto the coordinates belonging to σ_m. To see that $(e_n(f(\varepsilon B_{c_0})))_n$ does not belong to ℓ_p for any $\varepsilon > 0$, we may suppose that $\varepsilon = 2^{-r}$ for some $r \in \mathbb{N}$. Note that a sequence $(x_n)_n$ belongs to $f(\Pi_N(2^{-r}B_{c_0}))$ if and only if $|x_j| \leq 2^{-rN}$ for $j \in \sigma_N$ and $x_j = 0$ otherwise. Then, applying for instance [6, 1.3.2], we have

$$e_{N!+1}\left(\Pi_{\sigma_N} f(2^{-r}B_{c_0})\right) = \sup_{1 \leq k \leq N!} 2^{-\frac{N!}{2k}} 2^{-rN} = \frac{2}{\sqrt{2}}\, 2^{-rN},$$

Now, for $1 \leq p < \infty$ we have

$$\begin{aligned}
\sum_{n=1}^{\infty} e_n(f(2^{-r}B_{c_0}))^p &= e_1(f(2^{-r}B_{c_0}))^p + \sum_{N=1}^{\infty} \sum_{n=N!+1}^{(N+1)!} e_n(f(2^{-r}B_{c_0}))^p \\
&\geq \sum_{N=2}^{\infty} \sum_{n=N!+1}^{(N+1)!} e_{N!+1}(f(2^{-r}B_{c_0}))^p \\
&\geq \sum_{N=2}^{\infty} \sum_{n=N!+1}^{(N+1)!} e_{N!+1}(\Pi_{\sigma_N}(f(2^{-r}B_{c_0})))^p \\
&\geq \sum_{N=2}^{\infty} ((N+1)! - N!) 2^{-p(\frac{1}{2}+rN)} \\
&\geq \sum_{N=2}^{\infty} 2^{-\frac{1}{2}} N! N (2^{-rp})^N.
\end{aligned}$$

Since the last term diverges for every r and p, we are done. ☐

Note that in the above example, $f(\Pi_N(2^{-r}B_{c_0}))$ coincides with $P_N f(0)(2^{-r}B_{c_0})$ and, also, that the dimension of $P_m f(0)(2^{-r}B_{c_0})$ is finite for every m but goes to infinity as m does. It would be interesting to know if the sequence $(e_n(f(\varepsilon B_E)))_n$ must decay fast if the dimensions of $P_m f(0)(2^{-r}B_{c_0})$ are uniformly bounded.

The last goal of this note is to find some bounds for the entropy numbers of $P_m f(x_0)(\varepsilon B_E)$ in terms of properties of f. Let us start by taking a standard approach. Following (the proof of) [2, Proposition 3.4], for an holomorphic mapping $f\colon U \to F$ and $x_0 \in U$, there is $\varepsilon > 0$ such that

$$P_m f(x_0)(\varepsilon B_E) \subset \overline{\mathrm{coe}\,(f(x_0+\varepsilon B_E))}, \tag{6}$$

where coe denotes the absolutely convex hull of a set. Thus, a good natural starting point to estimate the entropy numbers of $P_m f(x_0)(\varepsilon B_E)$ in terms of those of $f(x_0 + \varepsilon B_E)$ is to estimate $e_n(\mathrm{coe}\,(f(x_0+\varepsilon B_E))$ in terms of $e_n(f(x_0+\varepsilon B_E))$. Now, suppose that $\overline{\dim}_B f(x_0+\varepsilon B_E) = N < \infty$. By [11, Corollary 5], we get that for every $\varepsilon > 0$,

$$\mathcal{E}_n(f(x_0+\varepsilon B_E)) \leq 2(n+1)^{-1/(N+\varepsilon)}.$$

From this estimate and [5, Proposition 4.5], we get

$$e_n(\mathrm{coe}\,(f(x_0+\varepsilon B_E))) \leq C(n+1)^{-1/N}. \tag{7}$$

for some $C > 0$ (note that, when we take coe, we *pay* the price of changing $\mathcal{E}_n$ to e_n). Finally, (6) and (7) give the following estimate for the entropy numbers of $P_m f(x_0)(\varepsilon B_E)$.

$$e_n(P_m f(x_0)(\varepsilon B_E)) \leq e_n(\mathrm{coe}\,(f(x_0+\varepsilon B_E))) \leq C(n+1)^{-1/N}.$$

This bound allows us to deduce for example, that $(e_n(P_m f(x_0)(\varepsilon B_E)))_n \in \ell_p$ for every $p > N$. Our last theorem improves this claim, showing that under the same assumptions we actually have $(e_n(P_m f(x_0)(\varepsilon B_E)))_n \in \ell_p$ for every $p > 1$.

Theorem 4.2. *Let E and F be Banach spaces, $U \subset E$ be an open set, $x_0 \in U$ and $\varepsilon > 0$ be such that $x_0 + \varepsilon B_E \subset U$. Let $f\colon U \to F$ be a holomorphic function such that $\overline{\dim}_B f(x_0+\varepsilon B_E) < \infty$. Then, $(e_n(P_m f(x_0)(B_E))_{n\in\mathbb{N}} \in \ell_p$ for every $p > 1$ and every $m \in \mathbb{N}$.*

Before proving the theorem, we need a technical lemma. In what follows, for $a \in \mathbb{R}$, we write $\lceil a \rceil = \min\{k \in \mathbb{Z} | k \geq a\}$.

Lemma 4.3. *Let E and F be Banach spaces, $U \subset E$ be an open set, $x_0 \in U$ and $\varepsilon > 0$ be such that $x_0 + \varepsilon B_E \subset U$. For an holomorphic function $f\colon U \to F$ and $n, m \in \mathbb{N}$, the following inequality holds*

$$e_{(n-1)C_n+1}(P_m f(x_0)(\varepsilon B_E)) \leq 2e_n(f(x_0+\varepsilon B_E))$$

where $C_n = \left\lceil \dfrac{C}{e_n(f(x_0+\varepsilon B_E))} \right\rceil$ for some positive constant $C = C(f)$ which depends only on f.

Proof. We may suppose that $x_0 = 0$ and $\varepsilon = 1$. Fix $n, m \in \mathbb{N}$. Given $\delta > 0$, there exists $M = \{y_1, y_2, \dots, y_{2^{n-1}}\} \subset Y$ such that $f(B_E) \subset \bigcup_{k=1}^{2^{n-1}} y_k + (e_n(f(B_E)) + \delta)\, B_F$.

Write $C = \sup_{x\in B_E} \|P_1 f(x)\|$ and let $C_n = \left\lceil \dfrac{2\pi C}{e_n(f(B_E))} \right\rceil$. We split the interval $[0, 2\pi]$ into C_n disjoint intervals $J_1, \dots J_{C_n}$ of length $\frac{2\pi}{C_n}$. Note that, for $x \in B_E$, and t_0, t_1 in one of this intervals, we have

$$\|f(e^{it_0}x) - f(e^{it_1}x)\| \leq C|e^{it_0} - e^{it_1}| \leq C\frac{2\pi}{C_n} \leq e_n(f(B_E)).$$

As a consequence, if some t_0 and some $y_k \in M$ satisfy $\|y_k - f(e^{it_0}x)\| \leq e_n(f(B_E))$ then, for any other t in the same interval as t_0, we have $\|y_k - f(e^{it}x)\| \leq 2e_n(f(B_E))$. We define the set

$$L = \{y \in F \colon y = \frac{1}{2\pi}\sum_{j=1}^{C_n} \int_{J_j} z_j e^{-itm} dt, \text{ for some } z_1, \dots, z_{C_n} \in M\}.$$

Note that L has $(2^{n-1})^{C_n} = 2^{C_n(n-1)}$ elements. The proof is complete if we show that for $x \in B_E$, there exists $y \in L$ such that $\|y - P_m f(0)(x)\| \leq 2e_n(f(B_E))$. By the Cauchy integral formula (see for instance [14, Corollary 7.3]) we have

$$P_m f(0)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it}x)e^{-itm} dt = \frac{1}{2\pi}\sum_{j=1}^{C_n}\int_{J_j} f(e^{it}x)e^{-itm} dt.$$

For each j, take $z_j \in M$ such that $\|z_j - f(e^{it}x)\| \leq 2e_n(f(B_E))$ for all $t \in J_j$. Then, $y = \frac{1}{2\pi}\sum_{j=1}^{C_n}\int_{J_j} z_j e^{-itm} dt \in L$ and

$$\begin{aligned}\|y - P_n f(0)(x)\| &= \left\|\frac{1}{2\pi}\sum_{j=1}^{C_n}\int_{J_j} \left(f(e^{it}x) - z_j\right) e^{-itm} dt\right\| \\ &\leq \frac{1}{2\pi}\sum_{j=1}^{C_n}\int_{J_j} 2e_n(f(B_E)) dt \\ &= 2e_n(f(B_E)).\end{aligned}$$

□

Proof of Theorem 4.2. By Lemma 4.3, given $p > 1$ we have

$$\begin{aligned}\sum_{n=1}^{\infty} e_n(P_m f(0)(B_E))^p &= \sum_{j=1}^{\infty}\sum_{n=(j-1)C_j+1}^{j\,C_{j+1}} e_n(P_m f(0)(B_E))^p \\ &\leq \sum_{n=1}^{\infty}(nC_{n+1} - (n-1)C_n) e_{(n-1)C_n+1}(P_m f(0)(B_E))^p \\ &\leq 2^p\sum_{n=1}^{\infty}(nC_{n+1} - (n-1)C_n) e_n(f(B_E))^p,\end{aligned}$$

We finish the proof by showing that

$$\limsup_{n\to\infty} \sqrt[n]{(nC_{n+1} - (n-1)C_n) e_n(f(B_E))^p} < 1.$$

Since $f(B_E)$ is a connected set, by Proposition 1.3 we have

$$\liminf_{n\to\infty} \frac{e_{n+1}(f(B_E))}{e_n(f(B_E))} \geq \frac{1}{5}.$$

Thus, there exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$,

$$\frac{n}{n^2 + (n-1)C} \left(\frac{C}{\frac{e_{n+1}(f(B_E))}{e_n(f(B_E)}} + e_n(f(B_E)) \right) < 1.$$

This implies that for every $n > n_0$

$$n \left(\frac{C}{e_{n+1}(f(B_E))} + 1 \right) - \frac{(n-1)C}{e_n(f(B_E))} < \frac{n^2}{e_n(f(B_E))},$$

and since $\frac{C}{e_n(f(B_E))} \leq C_n \leq \frac{C}{e_n(f(B_E))} + 1$, we get the inequality

$$nC_{n+1} - (n-1)C_n \leq \frac{n^2}{e_n(f(B_E))}.$$

From this last inequality we obtain that,

$$\sqrt[n]{(nC_{n+1} - (n-1)C_n)e_n(f(B_E))^p} \leq \left(\sqrt[n]{e_n(f(B_E))} \right)^{p-1} \sqrt[n]{n^2}. \tag{8}$$

Since we are assuming that $\overline{\dim}_B f(B_E) < \infty$, by Proposition 1.2 we have that $\limsup_{n\to\infty} \sqrt[n]{e_n(f(B_e))} < 1$. This and (8) complete the proof. □

We do not know if Proposition 4.2 holds for $p = 1$. More precisely, we have the following question

Question 4.4. Let E and F be Banach spaces, $U \subset E$ an open set, and $x_0 \in U$. Take $f \colon U \to E$ an holomorphic function and suppose that there exists $\varepsilon > 0$ such that $\dim_B f(x_0 + \varepsilon B_E) < \infty$. Is it true that $(e_n(P_m f(x_0)(B_E))_n$ belongs to ℓ_1 for all $m \in \mathbb{N}$?

A probably more natural question, whose positive answer would clearly imply a positive answer to Question 4.4, is the following.

Question 4.5. Let E and F be Banach spaces, $U \subset E$ an open set, and $x_0 \in U$. Take $f \colon U \to E$ an holomorphic function and suppose that there exists $\varepsilon > 0$ such that $\dim_B f(x_0 + \varepsilon B_E) < \infty$. Is it true that for every $m \in \mathbb{N}$, $\overline{\dim}_B P_m f(x_0)(B_E) < \infty$?

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