Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. (2024) 118:120
https://doi.org/10.1007/s13398-024-01620-0

ORIGINAL PAPER

Parametrizing W-weighted BT inverse to obtain the W-weighted q-BT inverse

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Received: 23 March 2024 / Accepted: 22 May 2024

Abstract
The core-EP and BT inverses for rectangular matrices were studied recently in the literature. The main aim of this paper is to unify both concepts by means of a new kind of generalized inverse called W-weighted q-BT inverse. We analyze its existence and uniqueness by considering an adequate matrix system. Basic properties and some interesting characterizations are proved for this new weighted generalized inverse. Also, we give a canonical form of the W-weighted q-BT inverse by means of the weighted core-EP decomposition.

Keywords Weighted generalized inverses · q-BT inverse · W-weighted core-EP inverse · W-weighted Drazin inverse

Mathematics Subject Classification 15A09 · 15A24

1 Introduction and preliminaries

We denote by $\mathbb{C}^{m\times n}$ the set of all $m \times n$ complex matrices. Let $A \in \mathbb{C}^{m\times n}$. The conjugate transpose, rank, null space and column space of A are denoted by A^*, $\text{rank}(A)$, $\mathscr{N}(A)$, and $\mathscr{R}(A)$, respectively. The index of $A \in \mathbb{C}^{n\times n}$, denoted by $\text{Ind}(A)$, is the smallest nonnegative integer k such that $\text{rank}(A^k) = \text{rank}(A^{k+1})$. Moreover, $A^0 = I_n$ will refer to the $n \times n$ identity matrix, and 0 will denote the null matrix of appropriate size. The standard notations P_S and $P_{S,T}$ stand for the orthogonal projector onto a subspace S and a projector onto S along T, respectively, when $\mathbb{C}^n$ is equal to the direct sum of subspaces S and T.

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The Drazin inverse of a matrix $A \in \mathbb{C}^{n\times n}$ is the unique matrix $X = A^d \in \mathbb{C}^{n\times n}$ that satisfies

$$XA^{k+1} = A^k, \quad XAX = X, \quad AX = XA, \text{ where } k = \text{Ind}(A).$$

When $\text{Ind}(A) = 1$, the Drazin inverse is called the group inverse of A and is denoted by $A^{\#}$.

The Moore–Penrose inverse of a matrix $A \in \mathbb{C}^{m\times n}$ is the unique matrix $X = A^{\dagger} \in \mathbb{C}^{n\times m}$ that satisfies the Penrose equations

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA.$$

We will denote by P_A the orthogonal projector $AA^{\dagger}$ onto the subspace $\mathscr{R}(A)$.

In 2014, Manjunatha Prassad and Mohana [13] introduced the core-EP inverse of a matrix $A \in \mathbb{C}^{n\times n}$ of index k as the unique matrix $X = A^{\textcircled{\dagger}} \in \mathbb{C}^{n\times n}$ that satisfies the conditions $XAX = X$ and $\mathscr{R}(X) = \mathscr{R}(X^*) = \mathscr{R}(A^k)$. That same year, Baksalary and Trenkler [2] defined the BT inverse of A as the matrix $A^{\diamond} = (AP_A)^{\dagger}$. When the matrix A has index 1, both inverses are reduced to the well-known core inverse $A^{\textcircled{\#}} = A^{\#}AA^{\dagger}$ of A [1].

In 1980, Cline and Greville [4] extended the Drazin inverse to rectangular matrices and it was called the W-weighted Drazin inverse. Let $W \in \mathbb{C}^{n\times m}$ be a fixed nonzero matrix. We recall that the W-weighted Drazin inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^{d,W}$, is the unique matrix $X \in \mathbb{C}^{m\times n}$ satisfying the three equations

$$XWAWX = X, \quad AWX = XWA, \quad XW(AW)^{k+1} = (AW)^k,$$

where $k = \max\{\text{Ind}(AW), \text{Ind}(WA)\}$. If $k = 1$, the W-weighted Drazin inverse of A is called the W-weighted group inverse of A and is denoted by $A^{\#,W}$. When $m = n$ and $W = I_n$, we recover the Drazin inverse, that is, $A^{d,I_n} = A^d$.

The W-weighted Drazin inverse satisfies the following two dual representations

$$A^{d,W} = A[(WA)^d]^2 = [(AW)^d]^2A, \quad \text{whence} \quad A^{d,W}W = (AW)^d, \quad WA^{d,W} = (WA)^d. \tag{1.1}$$

Interesting representations and properties of the W-weighted Drazin inverse were studied in [17].

Similarly, the core-EP inverse was extended to rectangular matrices in [5]. It was named W-weighted core-EP inverse, and defined as $A^{\textcircled{\dagger},W} = (WAWP_{(AW)^k})^{\dagger}$, which is the unique solution of

$$WAWX = P_{(WA)^k}, \quad \mathscr{R}(X) \subseteq \mathscr{R}((AW)^k). \tag{1.2}$$

For the particular case $k = 1$, the W-weighted core-EP inverse of A is known as the W-weighted core inverse of A and denoted by $A^{\textcircled{\#},W}$. Clearly, when $m = n$ and $W = I_n$, we recover the core-EP inverse, that is, $A^{\textcircled{\dagger},I_n} = A^{\textcircled{\dagger}}$.

The W-weighted core-EP inverse satisfies the following interesting properties [5, 12]

$$A^{\textcircled{\dagger},W} = A[(WA)^{\textcircled{\dagger}}]^2, \quad A^{\textcircled{\dagger},W}WP_{(AW)^k} = (AW)^{\textcircled{\dagger}}, \quad P_{(WA)^k}WA^{\textcircled{\dagger},W} = (WA)^{\textcircled{\dagger}}. \tag{1.3}$$

Recently, the W-weighted BT inverse of A was defined in [10] as the unique matrix $X = A^{\diamond,W} \in \mathbb{C}^{m\times n}$ satisfying the following equations

$$XWAWX = X, \quad XWA = [W(AW)^2(AW)^{\dagger}]^{\dagger}WA, \quad AWX = AW[(WA)^2W(AW)^{\dagger}]^{\dagger}. \tag{1.4}$$

It was also established that $A^{\diamond,W} = (WAWP_{AW})^{\dagger}$.

Interesting results including different kinds of weighted generalized inverses can be found in [14–16].

In this paper we unify the definitions given in (1.2) and (1.4) given rise a new kind of generalized inverse called W-weighted q-BT inverse. We analyze its existence and uniqueness by considering an adequate matrix system.

This paper is organized as follows. In Sect. 2, we present results of existence and uniqueness of the W-weighted q-BT inverse. More precisely, the existence will be characterized as the unique solution of three matrix equations. In Sect. 3, we obtain some characterizations of the W-weighted q-BT inverse. As an interesting consequence, we present new characterizations of the W-weighted core-EP and W-weighted BT inverses. In Sect. 4, we obtain a canonical form of the W-weighted q-BT inverse by using a simultaneous decomposition of the matrices A and W called the weighted core-EP decomposition. Finally, some more properties of this new generalized inverse are investigated.

2 Existence and uniqueness

In this section, we define and investigate the W-weighted q-BT inverse for rectangular matrices $A \in \mathbb{C}^{m\times n}$ by considering a non-null weight $W \in \mathbb{C}^{n\times m}$.

We start with a result of existence and uniqueness. Before that, we need the following auxiliary lemma.

Lemma 2.1 *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times s}$. Then $P_B(AP_B)^\dagger = (AP_B)^\dagger$.*

Proof Since $(I_n - P_B)P_B A^* = 0$ trivially holds, we have that $\mathscr{R}((AP_B)^\dagger) \subseteq \mathscr{N}((I_n - P_B))$ is always valid, which in turn is equivalent to $P_B(AP_B)^\dagger = (AP_B)^\dagger$. □

Theorem 2.2 *Let $A \in \mathbb{C}^{m\times n}$, $0 \neq W \in \mathbb{C}^{n\times m}$ and $q \in \mathbb{N}\cup\{0\}$. The system of equations*

$$XWAWX = X, \quad XWA = (WAWP_{(AW)^q})^\dagger WA, \quad AWX = AW(WAWP_{(AW)^q})^\dagger, \tag{2.1}$$

is consistent and has a unique solution $X = (WAWP_{(AW)^q})^\dagger$.

Proof *Existence.* Let $X := (WAWP_{(AW)^q})^\dagger$. Clearly, X satisfies the two last equations in (2.1). Moreover, from Lemma 2.1 we have

$$\begin{aligned} XWAWX &= (WAWP_{(AW)^q})^\dagger WAW(WAWP_{(AW)^q})^\dagger \\ &= (WAWP_{(AW)^q})^\dagger WAWP_{(AW)^q}(WAWP_{(AW)^q})^\dagger \\ &= (WAWP_{(AW)^q})^\dagger \\ &= X. \end{aligned}$$

Thus, X is a solution to (2.1).

Uniqueness. Any arbitrary solution X to the system (2.1) satisfies

$$\begin{aligned} X &= (XWA)WX \\ &= (WAWP_{(AW)^q})^\dagger W(AWX) \\ &= (WAWP_{(AW)^q})^\dagger WAW(WAWP_{(AW)^q})^\dagger \\ &= (WAWP_{(AW)^q})^\dagger WAWP_{(AW)^q}(WAWP_{(AW)^q})^\dagger \\ &= (WAWP_{(AW)^q})^\dagger, \end{aligned}$$

which implies that the matrix $X = (WAWP_{(AW)^q})^\dagger$ is the unique solution to (2.1). □

The example below shows that the uniqueness of the solution of the system (2.1) cannot be guaranteed when the second condition is removed. Similar examples can be found by removing the first and the third conditions and mantaining the remaining two.

Example 2.3 Consider the system $XWAWX = X$ and $AWX = AW(WAWP_{AW})^\dagger$, where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is easy to see that $k = \max\{\text{Ind}(AW), \text{Ind}(WA)\} = \max\{3, 3\} = 3$. Let $X_0 := (WAWP_{AW})^\dagger$. By Theorem 2.2, it is clear that $X_0WAWX_0 = X_0$ and $AWX_0 = AW(WAWP_{AW})^\dagger$.

Now, we consider the matrix $X_1 := Q_{AW}X_0 + (I_m - Q_{AW})W^*$ where $Q_{AW} := (AW)^\dagger AW$. Then,

$$\begin{aligned} X_1WAWX_1 &= [Q_{AW}X_0 + (I_m - Q_{AW})W^*]WAW[Q_{AW}X_0 + (I_m - Q_{AW})W^*] \\ &= [Q_{AW}X_0 + (I_m - Q_{AW})W^*]WAWX_0 \\ &= Q_{AW}X_0WAWX_0 + (I_m - Q_{AW})W^*WAWX_0 \\ &= Q_{AW}X_0 + (I_m - Q_{AW})W^*WAWX_0 \\ &= Q_{AW}X_0 + (I_m - Q_{AW})W^* \\ &= X_1; \end{aligned}$$

$$\begin{aligned} AWX_1 &= AW[Q_{AW}X_0 + (I_m - Q_{AW})W^*] \\ &= AWQ_{AW}X_0 \\ &= AWX_0 = AW(WAWP_{AW})^\dagger. \end{aligned}$$

Thus, X_0 and X_1 both satisfy $XWAWX = X$ and $AWX = AW(WAWP_{AW})^\dagger$. Finally, we observe that, due to Theorem 2.2, the matrix X_0 is also a solution of the equation $XWA = (WAWP_{AW})^\dagger WA$. However, X_1 does not satisfy such an equation. In fact,

$$X_1WA = \begin{bmatrix} \frac{3}{5} & \frac{3}{5} & -\frac{1}{5} & -1 \\ 0 & 0 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & -\frac{5}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{7}{6} & -\frac{8}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & -\frac{2}{3} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = X_0WA = (WAWP_{AW})^\dagger WA.$$

Definition 2.4 Let $A \in \mathbb{C}^{m\times n}$, $0 \neq W \in \mathbb{C}^{n\times m}$, $k = \max\{\text{Ind}(AW), \text{Ind}(WA)\}$, and $q \in \mathbb{N}\cup\{0\}$. The unique matrix $X \in \mathbb{C}^{m\times n}$ that satisfies the system (2.1) is called the W-weighted q-BT inverse of A, and is denoted by $A^{\diamond_q, W}$.

Remark 2.5 Note that when $m = n$ and $W = I_n$, the W-weighted q-BT inverse of A gives rise a new generalized inverse for square matrices. For simplicity, it will be denoted as $A^{\diamond_q} := (AP_{A^q})^\dagger$ and will be called the q-BT inverse of A.

The motivation for the study of this new kind of generalized inverse is stated in the following result by showing that it extends certain inverses known in the literature.

Corollary 2.6 *Let* $A \in \mathbb{C}^{m\times n}$, $0 \neq W \in \mathbb{C}^{n\times m}$, $k = \max\{Ind(AW), Ind(WA)\}$, *and* $q \in \mathbb{N}\cup\{0\}$. *Then*

(i) $A^{\diamond_q,W} = (WAW)^\dagger$ *if* $q = 0$;
(ii) $A^{\diamond_q,W} = A^{\diamond,W}$ *if* $q = 1$;
(iii) $A^{\diamond_q,W} = A^{\textcircled{\dagger},W}$ *if either* $q = Ind(AW)$ *or* $q \geq k$.

Proof (i) Follows from Theorem 2.2 with $q = 0$.
(ii) It is a consequence from Theorem 2.2 and the expression of $A^{\diamond,W}$ recalled below (1.4).
(iii) It follows from Theorem 2.2, the definition of $A^{\textcircled{\dagger},W}$ and the fact that $P_{(AW)^q} = P_{(AW)^k}$ when either $q = \text{Ind}(AW)$ or $q \geq k$.

□

Remark 2.7 When $WAW = A$, from the above corollary it follows that the W-weighted q-BT inverse of A reduces to the Moore–Penrose inverse of A. Note that the condition $WAW = A$ is a Stein equation (in A). We recall that this equation has important applications in system theory, among them, the stability analysis of discrete-time systems [11].

Remark 2.8 If $m = n$ and $W = I_n$, from Corollary 2.6 we deduce that the W-weighted q-BT inverse concides with the BT inverse and core-EP inverse, when $q = 1$ and $q \geq k = \text{Ind}(A)$, respectively.

An interesting relationship between the products AW and WA is

$$(AW)^{\ell-1}A = A(WA)^{\ell-1}, \quad \ell \in \mathbb{N}. \tag{2.2}$$

Corollary 2.9 *Let* $A \in \mathbb{C}^{m\times n}$, $0 \neq W \in \mathbb{C}^{n\times m}$ *and* $q \in \mathbb{N}\cup\{0\}$. *Then*

$$A^{\diamond_q,W} = [W(AW)^{(q+1)}[(AW)^q]^\dagger]^\dagger = [(WA)^{q+1}W[(AW)^q]^\dagger]^\dagger.$$

Proof Follows from Theorem 2.2 and (2.2). □

In the following example we show that when $1 < q < k$ (eventually with $q \neq \text{Ind}(AW)$), this new inverse is different from other known ones.

Example 2.10 Consider the matrices

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since $\text{Ind}(AW) = 3$ and $\text{Ind}(WA) = 2$, we have $k = \max\{\text{Ind}(AW), \text{Ind}(WA)\} = 3$. Therefore, we must consider $q = 2$. Thus, the W-weighted core-EP inverse, the W-weighted BT inverse, and the W-weighted 2-BT inverse are given by

$$A^{\textcircled{\dagger},W} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^{\diamond,W} = \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ \frac{1}{6} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^{\diamond_2,W} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Some properties of the W-weighted q-BT inverse are established below. For example, the W-weighted q-BT inverse can be expressed in terms of the q-BT inverse. In particular, the q-BT inverse provides the range and null space of the W-weighted BT inverse.

Theorem 2.11 *Let $A \in \mathbb{C}^{m\times n}$, $0 \neq W \in \mathbb{C}^{n\times m}$ and $q \in \mathbb{N}\cup\{0\}$. Then the following statements hold:*

(i) $A^{\diamond_q,W} = (W[(AW)^{\diamond_q}]^{\dagger})^{\dagger}$.
(ii) $\mathscr{R}(A^{\diamond_q,W}) = \mathscr{R}(P_{(AW)^q}(WAW)^*)$ *and* $\mathscr{N}(A^{\diamond_q,W}) = \mathscr{N}(P_{(AW)^q}(WAW)^*)$.
(iii) $\mathscr{R}(A^{\diamond,W}) = \mathscr{R}([(AW)^{\diamond_q}]^{\dagger})^* W^*)$ *and* $\mathscr{N}(A^{\diamond,W}) = \mathscr{N}([(AW)^{\diamond_q}]^{\dagger})^* W^*)$.
(iv) $\mathscr{R}(A^{\diamond_q,W}) = \mathscr{R}([((AW)^q)^{\dagger}]^*[(AW)^{q+1}]^* W^*)$ *and* $\mathscr{N}(A^{\diamond_q,W}) = \mathscr{N}([(AW)^{q+1}]^* W^*)$.
(v) $\mathscr{R}(A^{\diamond_q,W}) \subseteq \mathscr{R}((AW)^q)$.
(vi) $P_{(AW)^q} A^{\diamond_q,W} = A^{\diamond_q,W}$.

Proof (i) By Theorem 2.2 we have $A^{\diamond_q,W} = (W[AWP_{(AW)^q}])^{\dagger}$. Now, by Remark 2.5 we deduce $AWP_{(AW)^q} = ((AW)^{\diamond_q})^{\dagger}$, whence the statement is clear.

(ii) By Theorem 2.2 we have $A^{\diamond_q,W} = (WAWP_{(AW)^q})^{\dagger}$. Now, the statement follows of the properties $\mathscr{R}(B^{\dagger}) = \mathscr{R}(B^*)$ and $\mathscr{N}(B^{\dagger}) = \mathscr{N}(B^*)$.

(iii) It follows immediately from part (i).

(iv) By Corollary 2.9 and the property $\mathscr{R}(B^{\dagger}) = \mathscr{R}(B^*)$ we get

$$\mathscr{R}(A^{\diamond_q,W}) = \mathscr{R}([W(AW)^{(q+1)}((AW)^q)^{\dagger}]^*) = \mathscr{R}([((AW)^q)^{\dagger}]^*[(AW)^{q+1}]^* W^*).$$

Similarly, Corollary 2.9 and the property $\mathscr{N}(B^{\dagger}) = \mathscr{N}(B^*)$ imply

$$\begin{aligned}\mathscr{N}(A^{\diamond_q,W}) &= \mathscr{N}([((AW)^q)^{\dagger}]^*[(AW)^{q+1}]^* W^*)\\ &= \mathscr{N}([((AW)^q)^{\dagger}]^*[(AW)^q]^*(AW)^* W^*)\\ &\subseteq \mathscr{N}([(AW)^q]^*[((AW)^q)^{\dagger}]^*[(AW)^q]^*(AW)^* W^*)\\ &= \mathscr{N}([(AW)^{q+1}]^* W^*)\\ &\subseteq \mathscr{N}([((AW)^q)^{\dagger}]^*[(AW)^{q+1}]^* W^*)\\ &= \mathscr{N}(A^{\diamond_q,W}).\end{aligned}$$

Thus, $\mathscr{N}(A^{\diamond,W}) = \mathscr{N}([(AW)^{q+1}]^* W^*)$.

(v) It directly follows from (ii) and the fact that $\mathscr{R}(P_{(AW)^k}) = \mathscr{R}((AW)^k)$.

(vi) It is sufficient to note that $P_{(AW)^q} A^{\diamond_q,W} = A^{\diamond_q,W}$ holds if and only if $\mathscr{R}(A^{\diamond_q,W}) \subseteq \mathscr{N}(I_m - P_{(AW)^q}) = \mathscr{R}(P_{(AW)^q}) = \mathscr{R}((AW)^q)$, which is true due to part (v).

□

We finish this section by showing that the W-weighted q-BT inverse can be written as a generalized inverse with prescribed range and null space. Moreover, some idempotent matrices related to the W-weighted q-BT inverse are found.

Proposition 2.12 *Let $A \in \mathbb{C}^{m\times n}$, $0 \neq W \in \mathbb{C}^{n\times m}$ and $q \in \mathbb{N}\cup\{0\}$. Then the following representations are valid:*

(i) $A^{\diamond_q,W} = (WAW)^{(2)}_{\mathscr{R}(P_{(AW)^q}(WAW)^*),\, \mathscr{N}([(AW)^{q+1}]^* W^*)}$;
(ii) $WAWA^{\diamond_q,W} = P_{\mathscr{R}(W[(AW)^{\diamond_q}]^{\dagger}(WAW)^*),\, \mathscr{N}([(AW)^{q+1}]^* W^*)}$;
(iii) $A^{\diamond_q,W}WAW = P_{\mathscr{R}(P_{(AW)^q}(WAW)^*),\, \mathscr{N}([(AW)^{q+1}]^* W^* WAW)}$.

Proof (i) By definition of the W-weighted q-BT inverse we know that $A^{\diamond,W}WAWA^{\diamond,W} = A^{\diamond,W}$. Now, parts (ii) and (iv) of Theorem 2.11 imply $\mathscr{R}(A^{\diamond_q,W}) = \mathscr{R}(P_{(AW)^q}(WAW)^*)$ and $\mathscr{N}(A^{\diamond,W}) = \mathscr{N}([(AW)^{q+1}]^* W^*)$, respectively. Thus, the statement follows by definition of an outer inverse with prescribed range and null space.

(ii) Since $A^{\diamond,W}WAWA^{\diamond,W} = A^{\diamond,W}$ by definition, we have that $WAWA^{\diamond_q,W}$ is idempotent. Also, from Theorem 2.11 (ii) we obtain

$$\begin{aligned}\mathscr{R}(WAWA^{\diamond_q,W}) &= WAW\mathscr{R}(A^{\diamond_q,W})\\ &= WAW\mathscr{R}(P_{(AW)^q}(WAW)^*)\\ &= W\mathscr{R}(AWP_{(AW)^q}(WAW)^*)\\ &= W\mathscr{R}([(AW)^{\diamond_q}]^\dagger(WAW)^*)\\ &= \mathscr{R}(W[(AW)^{\diamond_q}]^\dagger(WAW)^*).\end{aligned}$$

On the other hand, note that $\mathscr{N}(WAWA^{\diamond,W}) = \mathscr{N}(A^{\diamond,W})$ because $A^{\diamond,W}$ is an outer inverse of WAW. Thus, from Theorem 2.11 (iv) we have $\mathscr{N}(WAWA^{\diamond,W}) = \mathscr{N}([(AW)^{q+1}]^*W^*)$.

(iii) By Theorem 2.11 (ii) we know that $\mathscr{R}(A^{\diamond_q,W}) = \mathscr{R}(P_{(AW)^q}(WAW)^*)$. Thus, as $A^{\diamond,W}WAWA^{\diamond,W} = A^{\diamond,W}$, clearly $\mathscr{R}(A^{\diamond_q,W}WAW) = \mathscr{R}(A^{\diamond_q,W}) = \mathscr{R}(P_{(AW)^q}(WAW)^*)$.

Similarly, from Theorem 2.11 (iv) we know that $\mathscr{N}(A^{\diamond_q,W}) = \mathscr{N}([(AW)^{q+1}]^*W^*)$. On the other hand, it is easy to see that $\mathscr{N}(B) = \mathscr{N}(C)$ implies $\mathscr{N}(BD) = \mathscr{N}(CD)$, where B, C, and D are complex rectangular matrices of adequate sizes. Therefore, $\mathscr{N}(A^{\diamond,W}WAW) = \mathscr{N}([(AW)^{q+1}]^*W^*WAW)$. □

Recall that the Moore–Penroe inverse [3], the core-EP inverse [6, Theorem 3.2] and the BT inverse [7, Theorem 4.7] of a matrix $A \in \mathbb{C}^{n\times n}$ of index k, are outer inverses that can be represented as outer inverse with prescribed range and null spaces as:

$$A^\dagger = A^{(2)}_{\mathscr{R}(A^*),\mathscr{N}(A^*)}, \quad A^{\textcircled{\dagger}} = A^{(2)}_{\mathscr{R}(A^k),\mathscr{N}((A^k)^*)} \quad \text{and} \quad A^\diamond = A^{(2)}_{\mathscr{R}(P_AA^*),\mathscr{N}((A^2)^*)}. \tag{2.3}$$

Our next theorem shows that the representations given in (2.3) are particular cases of the following expression for the q-BT inverse.

Corollary 2.13 *Let $A \in \mathbb{C}^{n\times n}$ and $q \in \mathbb{N}\cup\{0\}$. Then the following statements hold:*

(i) $A^{\diamond_q} = A^{(2)}_{\mathscr{R}(P_{A^q}A^*),\mathscr{N}([A^{q+1}]^*)}$

(ii) $AA^{\diamond_q} = P_{\mathscr{R}([A^{\diamond_q}]^\dagger A^*),\mathscr{N}([A^{q+1}]^*)}$.

(iii) $A^{\diamond_q}A = P_{\mathscr{R}(P_{A^q}A^*),\mathscr{N}([A^{q+1}]^*A)}$.

Proof Items (i)–(iii) immediately follow from Proposition 2.12 by taking $m = n$ and $W = I_n$. □

Remark 2.14 From Corollary 2.13 (i), it is clear that when $q = 0$ and $q = 1$, we recover the expressions given in (2.3) for the Moore–Penrose inverse and the BT inverse, respectively. On the other hand, if $q \geq k = \mathrm{Ind}(A)$ we have that $\mathscr{R}(P_{A^q}A^*) = \mathscr{R}((AP_{A^q})^*) = \mathscr{R}((AP_{A^k})^\dagger) = \mathscr{R}(A^{\textcircled{\dagger}}) = \mathscr{R}(A^k)$. Also, by definition of index, we obtain $\mathscr{N}((A^{q+1})^*) = \mathscr{N}((A^{k+1})^*) = \mathscr{N}((A^k)^*)$. In consequence, $A^{\diamond_q} = A^{(2)}_{\mathscr{R}(A^k),\mathscr{N}((A^k)^*)} = A^{\textcircled{\dagger}}$.

3 Algebraic characterizations

In this section we give some algebraic characterizations of the W-weighted q-BT inverse.

Theorem 3.1 *Let $A \in \mathbb{C}^{m\times n}$, $0 \neq W \in \mathbb{C}^{n\times m}$, $k = \max\{Ind(AW), Ind(WA)\}$, and $q \in \mathbb{N}\cup\{0\}$. There exists a unique matrix X satisfying the conditions*

$$P_{(AW)^q}X = (WAWP_{(AW)^q})^\dagger \quad \text{and} \quad \mathscr{R}(X) \subseteq \mathscr{R}((AW)^q) \tag{3.1}$$

and is given by $X = A^{\diamond_q,W}$.

Proof *Existence*. Let $X := A^{\diamond_q, W}$. From parts (v) and (vi) of Theorem 2.11 it is clear that X is a solution to (3.1).

Uniqueness. Any matrix X satisfying conditions (3.1), in particular satisfies $\mathscr{R}(X) \subseteq \mathscr{R}((AW)^q)$ which is equivalent to $P_{(AW)^q} X = X$. Thus, from the condition $P_{(AW)^q} X = (WAWP_{(AW)^q})^\dagger$, we get $X = (WAWP_{(AW)^q})^\dagger$, which gives the conclusion. □

Theorem 3.2 *Let $A \in \mathbb{C}^{m\times n}$, $0 \neq W \in \mathbb{C}^{n\times m}$, $k = \max\{Ind(AW), Ind(WA)\}$, and $q \in \mathbb{N} \cup \{0\}$. The unique matrix X satisfying the conditions*

$$AWX = AW(WAWP_{(AW)^q})^\dagger \quad \text{and} \quad \mathscr{R}(X) \subseteq \mathscr{R}(P_{(AW)^q}(WAW)^*) \tag{3.2}$$

is given by $X = A^{\diamond_q, W}$.

Proof *Existence*. Let $X := A^{\diamond_q, W}$. By Definition 2.4 and Theorem 2.11 (ii) it is clear that X satisfies both conditions in (3.2).

Uniqueness. Let X be an arbitrary matrix satisfying both conditions in (3.2). Since $\mathscr{R}((WAWP_{(AW)^q})^\dagger) = \mathscr{R}(P_{(AW)^q}(WAW)^*)$, the second condition in (3.2) implies $X = (WAWP_{(AW)^q})^\dagger Z$ for some matrix Z. Now, from Lemma 2.1 and the first equation in (3.2) we obtain

$$\begin{aligned}
X &= (WAWP_{(AW)^q})^\dagger Z \\
&= (WAWP_{(AW)^q})^\dagger WAWP_{(AW)^q}(WAWP_{(AW)^q})^\dagger Z \\
&= (WAWP_{(AW)^q})^\dagger WAW[(WAWP_{(AW)^q})^\dagger Z] \\
&= (WAWP_{AW})^\dagger W(AWX) \\
&= (WAWP_{AW})^\dagger WAW(WAWP_{(AW)^q})^\dagger \\
&= (WAWP_{(AW)^q})^\dagger WAWP_{(AW)^q}(WAWP_{(AW)^q})^\dagger \\
&= (WAWP_{(AW)^q})^\dagger \\
&= A^{\diamond_q, W},
\end{aligned}$$

which gives the uniqueness. □

A similar result can be obtained using the null space.

Theorem 3.3 *Let $A \in \mathbb{C}^{m\times n}$, $0 \neq W \in \mathbb{C}^{n\times m}$, $k = \max\{Ind(AW), Ind(WA)\}$, and $q \in \mathbb{N} \cup \{0\}$. The unique matrix X that satisfies both conditions*

$$XWA = (WAWP_{(AW)^q})^\dagger WA \quad \text{and} \quad \mathscr{N}(P_{(AW)^q}(WAW)^*) \subseteq \mathscr{N}(X) \tag{3.3}$$

is given by $X = A^{\diamond, W}$.

As a consequence of above results we obtain some characterizations of the q-BT inverse of a square matrix.

Theorem 3.4 *Let $A \in \mathbb{C}^{n\times n}$. The following statements are equivalent:*

(i) *X is the q-BT inverse of A;*
(ii) *$XAX = X$, $AX = A(AP_{A^q})^\dagger$, and $XA = (AP_{A^q})^\dagger A$;*
(iii) *$P_{A^q}X = (AP_{A^q})^\dagger$ and $\mathscr{R}(X) \subseteq \mathscr{R}(A^q)$;*
(iv) *$AX = A(AP_{A^q})^\dagger$ and $\mathscr{R}(X) \subseteq \mathscr{R}(P_{A^q}A^*)$;*
(v) *$XA = (AP_{A^q})^\dagger A$ and $\mathscr{N}(P_{A^q}A^*) \subseteq \mathscr{N}(X)$.*

4 Canonical form of the *W*-weighted *q*-BT inverse

In [5] the authors introduced a simultaneous unitary block upper triangularization of a pair of rectangular matrices, called the weighted core-EP decomposition of the pair (A, W). More precisely, we have the following result:

Theorem 4.1 *Let $A \in \mathbb{C}^{m\times n}$ and $0 \neq W \in \mathbb{C}^{n\times m}$ with $k = \max\{Ind(AW), Ind(WA)\}$. Then there exist two unitary matrices $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$, two nonsingular matrices $A_1, W_1 \in \mathbb{C}^{t\times t}$, and two matrices $A_3 \in \mathbb{C}^{(m-t)\times(n-t)}$ and $W_3 \in \mathbb{C}^{(n-t)\times(m-t)}$ such that A_3W_3 and W_3A_3 are nilpotent of indices $Ind(AW)$ and $Ind(WA)$, respectively, with*

$$A = U\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} V^* \quad \text{and} \quad W = V\begin{bmatrix} W_1 & W_2 \\ 0 & W_3 \end{bmatrix} U^*. \tag{4.1}$$

The following lemma allows us to find the Moore–Penrose inverse of a partitioned matrix with some of its diagonal block nonsingular.

Lemma 4.2 [10] *Let $A = U\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} V^* \in \mathbb{C}^{m\times n}$ be such that $A_1 \in \mathbb{C}^{t\times t}$ is nonsingular and $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary. Then*

$$A^\dagger = V\begin{bmatrix} A_1^*\Omega & -A_1^*\Omega A_2 A_3^\dagger \\ (I_{n-t} - Q_{A_3})A_2^*\Omega & A_3^\dagger - (I_{n-t} - Q_{A_3})A_2^*\Omega A_2 A_3^\dagger \end{bmatrix} U^*, \tag{4.2}$$

where $\Omega = [A_1A_1^* + A_2(I_{n-t} - Q_{A_3})A_2^*]^{-1}$. In consequence,

$$P_A = U\begin{bmatrix} I_t & 0 \\ 0 & P_{A_3} \end{bmatrix} U^*. \tag{4.3}$$

Now, we present a representation for the W-weighted q-BT inverses by using the weighted core-EP decomposition.

Theorem 4.3 *Let $A \in \mathbb{C}^{m\times n}$, $0 \neq W \in \mathbb{C}^{n\times m}$, $k = \max\{Ind(AW), Ind(WA)\}$, and $q \in \mathbb{N} \cup \{0\}$. If A and W are written as in (4.1), then the W-weighted q-BT inverse of A is given by*

$$A^{\diamond_q, W} = U\begin{bmatrix} (W_1A_1W_1)^*\Omega_W & -(W_1A_1W_1)^*\Omega_W M A_3^{\diamond_q, W_3} \\ (P_{(A_3W_3)^q} - P_{A_3^{\diamond_q, W_3}})M^*\Omega_W & A_3^{\diamond_q, W_3} - (P_{(A_3W_3)^q} - P_{A_3^{\diamond_q, W_3}})M^*\Omega_W M A_3^{\diamond_q, W_3} \end{bmatrix} V^*, \tag{4.4}$$

where

$$M := W_1A_1W_2 + W_1A_2W_3 + W_2A_3W_3$$

and

$$\Omega_W := [W_1A_1W_1(W_1A_1W_1)^* + M(P_{(A_3W_3)^q} - P_{A_3^{\diamond_q, W_3}})M^*]^{-1}.$$

Proof We assume that A and W are written as in (4.1). Applying Theorem 2.2, we have $A^{\diamond_q, W} = (WAWP_{(AW)^q})^\dagger$. It can be easily obtained that

$$WAW = V\begin{bmatrix} W_1A_1W_1 & W_1A_1W_2 + (W_1A_2 + W_2A_3)W_3 \\ 0 & W_3A_3W_3 \end{bmatrix} U^* = V\begin{bmatrix} W_1A_1W_1 & M \\ 0 & W_3A_3W_3 \end{bmatrix} U^*,$$

where $M := W_1A_1W_2 + W_1A_2W_3 + W_2A_3W_3$, and

$$P_{(AW)^q} = U\begin{bmatrix} I_t & 0 \\ 0 & P_{(A_3W_3)^q}\end{bmatrix}U^*.$$

Thus, we have that

$$A^{\diamond_q,W} = (WAWP_{(AW)^q})^\dagger = U\begin{bmatrix} W_1A_1W_1 & MP_{(A_3W_3)^q} \\ 0 & W_3A_3W_3P_{(A_3W_3)^q}\end{bmatrix}^\dagger V^* = U\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4\end{bmatrix}V^*,$$

where we are considering the partition given by the blocks B_1, B_2, B_3 and B_4 having appropriate sizes induced by the central matrix in the previous step. By Theorem 4.1, $W_1A_1W_1$ is nonsingular. In order to determine the blocks B_1, B_2, B_3, and B_4 we will use Lemma 4.2. Taking $Z := P_{(A_3W_3)^q}(I_{n-t} - Q_{W_3A_3W_3P_{(A_3W_3)^q}})P_{(A_3W_3)^q}$, we get

$$\Omega_W = [W_1A_1W_1(W_1A_1W_1)^* + MZM^*]^{-1}. \tag{4.5}$$

Moreover, from Lemma 2.1 and Theorem 2.2, it follows

$$\begin{aligned} Z &= (P_{(A_3W_3)^q})^2 - [P_{(A_3W_3)^q}(W_3A_3W_3P_{(A_3W_3)^q})^\dagger]W_3A_3W_3(P_{(A_3W_3)^q})^2 \\ &= P_{(A_3W_3)^q} - [W_3A_3W_3P_{(A_3W_3)^q}]^\dagger W_3A_3W_3P_{(A_3W_3)^q} \\ &= P_{(A_3W_3)^q} - A_3^{\diamond_q,W_3}[A_3^{\diamond_q,W_3}]^\dagger \\ &= P_{(A_3W_3)^q} - P_{A_3^{\diamond_q,W_3}}. \end{aligned} \tag{4.6}$$

From (4.6) and (4.5) we have

$$\Omega_W = [W_1A_1W_1(W_1A_1W_1)^* + M(P_{(A_3W_3)^q} - P_{A_3^{\diamond_q,W_3}})M^*]^{-1}.$$

Finally,

$$\begin{aligned} B_1 &= (W_1A_1W_1)^*\Omega_W, \\ B_2 &= -(W_1A_1W_1)^*\Omega_W MP_{(A_3W_3)^q}(W_3A_3W_3P_{(A_3W_3)^q})^\dagger \\ &= -(W_1A_1W_1)^*\Omega_W MA_3^{\diamond_q,W_3}, \\ B_3 &= (I_{n-t} - Q_{W_3A_3W_3P_{(A_3W_3)^q}})(MP_{(A_3W_3)^q})^*\Omega_W \\ &= (P_{(A_3W_3)^q} - P_{A_3^{\diamond_q,W_3}})M^*\Omega_W, \\ B_4 &= A_3^{\diamond_q,W_3} - B_3MP_{(A_3W_3)^q}(W_3A_3W_3P_{(A_3W_3)^q})^\dagger \\ &= A_3^{\diamond_q,W_3} - (P_{(A_3W_3)^q} - P_{A_3^{\diamond_q,W_3}})M^*\Omega_W MA_3^{\diamond_q,W_3}, \end{aligned}$$

which completes the proof. □

Corollary 4.4 *Let $A \in \mathbb{C}^{m\times n}$, $0 \neq W \in \mathbb{C}^{n\times m}$, and $k = \max\{Ind(AW), Ind(WA)\}$. If A and W are written as in (4.1), then the W-weighted BT inverse of A is given by*

$$A^{\diamond_1,W} = A^{\diamond,W} = U\begin{bmatrix} (W_1A_1W_1)^*\Omega_W & -(W_1A_1W_1)^*\Omega_W MA_3^{\diamond,W_3} \\ (P_{A_3W_3} - P_{A_3^{\diamond,W_3}})M^*\Omega_W & A_3^{\diamond,W_3} - (P_{A_3W_3} - P_{A_3^{\diamond,W_3}})M^*\Omega_W MA_3^{\diamond,W_3}\end{bmatrix}V^*, \tag{4.7}$$

where

$$\begin{aligned} M &= W_1A_1W_2 + (W_1A_2 + W_2A_3)W_3, \quad \textit{and} \\ \Omega_W &= [W_1A_1W_1(W_1A_1W_1)^* + M(P_{A_3W_3} - P_{A_3^{\diamond,W_3}})M^*]^{-1}, \end{aligned}$$

and the W-weighted core-EP inverse of A is given by

$$A^{\diamond_q, W} = A^{\textcircled{\dagger}, W} = U \begin{bmatrix} (W_1 A_1 W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*, \quad \text{for } q \geq k. \tag{4.8}$$

Proof By Corollary 2.6 we know that $A^{\diamond_q, W} = A^{\diamond, W}$ if $q = 1$ and $A^{\diamond_q, W} = A^{\textcircled{\dagger}, W}$ if $q \geq k$. Clearly, if $q = 1$, (4.4) reduces to the expression given in (4.7). On the other hand, if $q \geq k$ we obtain $(A_3 W_3)^q = 0$. In fact, since $A_3 W_3$ is nilpotent of index at most k, we have $P_{(A_3 W_3)^q} = 0$. Hence, $A_3^{\diamond_q, W_3} = (W_3 A_3 W_3 P_{(A_3 W_3)^q})^\dagger = 0$. Now, from (4.5), it follows that $\Omega_W = [W_1 A_1 W_1 (W_1 A_1 W_1)^*]^{-1}$. In this way, (4.4) reduces to (4.8). □

Remark 4.5 When $k = \max\{\text{Ind}(AW), \text{Ind}(WA)\} = 1$, the above representations coincide with the W-weighted core inverse, that is, $A^{\diamond, W} = A^{\textcircled{\dagger}, W} = A^{\textcircled{\#}, W}$.

If $A \in \mathbb{C}^{n \times n}$ has index k, by applying Theorem 4.1 with $m = n$ and $W = I_n$, we obtain the following canonical form of A

$$A = U \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*, \tag{4.9}$$

where $U \in \mathbb{C}^{n \times n}$ is unitary, T is nonsingular, $\text{rank}(T) = \text{rank}(A^k)$, and N is nilpotent of index k. This representation of A is called the core-EP decomposition of A [18].

By using (4.9) we can give a canonical form for the q-BT inverse of a square matrix.

Corollary 4.6 *Let $A \in \mathbb{C}^{n \times n}$, $k = Ind(A)$, and $q \in \mathbb{N} \cup \{0\}$. If A is written as in (4.9), then the q-BT inverse of A is given by*

$$A^{\diamond_q} = U \begin{bmatrix} T^* \Delta & -T^* \Delta S N^{\diamond_q} \\ (P_N - P_{N^{\diamond_q}}) S^* \Delta & N^{\diamond_q} - (P_N - P_{N^{\diamond_q}}) S^* \Delta S N^{\diamond_q} \end{bmatrix} U^*, \tag{4.10}$$

where $\Delta = (T T^* + S(P_N - P_{N^{\diamond_q}}) S^*)^{-1}$.

Corollary 4.7 *Let $A \in \mathbb{C}^{m \times n}$, $0 \neq W \in \mathbb{C}^{n \times m}$, and $k = \max\{Ind(AW), Ind(WA)\}$. If A and W are written as in (4.1), then it results that*

$$(AW)^{\diamond_q} = U \begin{bmatrix} (A_1 W_1)^* \Delta & -(A_1 W_1)^* \Delta S (A_3 W_3)^{\diamond_q} \\ (P_{A_3 W_3} - P_{(A_3 W_3)^{\diamond_q}}) S^* \Delta & (A_3 W_3)^{\diamond_q} - (P_{A_3 W_3} - P_{(A_3 W_3)^{\diamond_q}}) S^* \Delta S (A_3 W_3)^{\diamond_q} \end{bmatrix} U^*,$$

with $\Delta = (A_1 W_1 (A_1 W_1)^* + S(P_{A_3 W_3} - P_{(A_3 W_3)^{\diamond_q}}) S^*)^{-1}$ and $S = A_1 W_2 + A_2 W_3$, and

$$(WA)^{\diamond_q} = U \begin{bmatrix} (W_1 A_1)^* \Delta & -(W_1 A_1)^* \Delta S (W_3 A_3)^{\diamond_q} \\ (P_{W_3 A_3} - P_{(W_3 A_3)^{\diamond_q}}) S^* \Delta & (W_3 A_3)^{\diamond_q} - (P_{W_3 A_3} - P_{(W_3 A_3)^{\diamond_q}}) S^* \Delta S (W_3 A_3)^{\diamond_q} \end{bmatrix} U^*,$$

with $\Delta = (W_1 A_1 (W_1 A_1)^* + S(P_{W_3 A_3} - P_{(W_3 A_3)^{\diamond_q}}) S^*)^{-1}$ and $S = W_1 A_2 + W_2 A_3$.

Proof From Theorem 4.1 we obtain

$$AW = U \begin{bmatrix} A_1 W_1 & A_1 W_2 + A_2 W_3 \\ 0 & A_3 W_3 \end{bmatrix} U^*, \tag{4.11}$$

where U is unitary, $A_1 W_1$ is nonsingular, and $A_3 W_3$ is nilpotent of index $\text{Ind}(AW)$.

Clearly, (4.11) is a core-EP decomposition of AW. Thus, the the expression for $(AW)^{\diamond_q}$ follows from Corollary 4.6.

The expression for $(WA)^{\diamond_q}$ can be found in a similar way. □

We recall that the W-weighted Drazin inverse and the W-weighted core-EP inverse of A satisfy the interesting identities $A^{d,W} = [(AW)^d]^2 A = A[(WA)^d]^2$ and $A^{\textcircled{\dagger},W} = A[(WA)^{\textcircled{\dagger}}]^2$, from (1.1) and (1.3), respectively.
However, these equalities do not remain valid for the W-weighted q-BT inverse whenever $1 \leq q < k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$, as we can check with the following example.

Example 4.8 Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\} = \max\{3, 2\} = 3$. For $1 \leq q < 3$ we obtain

$$A^{\diamond_1,W} = \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ \frac{1}{6} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [(AW)^{\diamond_1}]^2 A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A[(WA)^{\diamond_1}]^2 = \begin{bmatrix} \frac{3}{25} & 0 & 0 \\ \frac{2}{25} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A^{\diamond_2,W} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [(AW)^{\diamond_2}]^2 A = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A[(WA)^{\diamond_2}]^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Remark 4.9 If we take $q = 3$ in the above example (i.e., $q = k = 3$), from Corollary 4.4 we have that $A^{\diamond_3,W} = A^{\textcircled{\dagger},W}$. Thus, from (1.3) we obtain $A^{\diamond_3,W} = A[(WA)^{\diamond_3}]^2$, which can be verified in the example given above, that is,

$$A^{\diamond_3,W} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [(AW)^{\diamond_3}]^2 A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad A[(WA)^{\diamond_3}]^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Acknowledgements We would like to thank the anonymous referees for their careful reading of the paper and their valuable comments and suggestions that help us to improve the reading of the paper.

Author contributions All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature. The first and third author are partially supported by Universidad Nacional de Río Cuarto (Grant PPI 18/C559) and CONICET (Grant PIBAA 28720210100658CO). The second author was partially supported by Universidad Nacional de La Pampa, Facultad de Ingeniería (Grant Resol. Nro. 135/19) and Ministerio de Economía, Industria y Competitividad (Spain) [Grant Red de Excelencia RED2022-134176-T].

Data availibility No data was used.

Declarations

Conflict of interest The authors have no Conflict of interest.

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