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Holographic thermal propagator for arbitrary scale dimensions

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Abstract: Using the AdS/CFT correspondence we model the behaviour of the two-point correlator of an operator with arbitrary scale dimension Δ in arbitrary spacetime dimension d for small but non-zero temperature. The obtained propagator coincides in the low temperature regime with the known result for $d = 4$ for large Δ at the order T^d as well as with the T^d and T^{2d} terms of the exact all order result for $d = 2$. Furthermore, for arbitrary d we explicitly write down the expression for the order T^d of the propagator for arbitrary Δ, and present a conjecture for the order T^{2d} in the large Δ limit.

Keywords: AdS-CFT Correspondence, Thermal Field Theory

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Contents

1 Introduction

Correlation functions of operators in conformal field theories are constrained by symmetry. For example, the two-point correlation function in d-dimensional Euclidean momentum space of a spin zero primary operator with scale dimension $\Delta = d/2 + \nu$ is given by

$$G_2^{(0)}(k) \propto k^{2\nu}\,, \quad k^2 = \omega^2 + \vec{q}^{\,2}\,, \tag{1.1}$$

where we split the time and space part of the d-dimensional vector $k^\mu = (\omega, \vec{q})$ for later use.

A bit less obvious is what happens when in such a theory we introduce a non-zero temperature, i.e. in the spacetime $S^1_{1/T} \times \mathbb{R}^{d-1}$. Conformal symmetry is now broken by $T \neq 0$ and correlators get, as we will see, corrections of order $\mathcal{O}((T/k)^d)$. There have been recently two important developments in this direction. First, the problem has been attacked in the limit of $\vec{q} = 0$ for the absorption cross section by black branes [1] or using the AdS/CFT correspondence [2–4] for large scale dimensions $\Delta \gg 1$ [5–7]. This last limit was needed to be able to apply the geodesic approximation in the background geometry of a black brane in AdS, while the leading order has been given for general Δ recently in terms of the ambient space formalism [8]. Second, by using the solution to the connection problem of the Heun equation [9, 10] a complete analytic result for the thermal two-point correlation function has been found recently in [11] (with the generalisation to non-zero chemical potential in [12]).

These results are however not completely satisfactory. First, large scale dimensions are not necessarily realistic, so results for smaller values are welcome. Second, although the solution found in [11] is analytic, it is very hard to make it work properly in the small temperature expansion of the black brane case, mainly because an infinite number of terms contribute and the resummation is far from obvious.

In this work we want to improve the state of the art calculating the first two corrections (the $(T/k)^d$ and the $(T/k)^{2d}$ ones) to the zero temperature propagator for a completely generic scale dimension Δ and arbitrary spacetime dimension d. In some sense this paper is a continuation of [13], although it uses a different approach. The method for calculating the propagator we use in this paper is straightforward: we rewrite the usual second order ordinary differential equation (ODE) for a perturbation in the black brane AdS background as a matrix linear first order ODE, expand it in powers of the temperature and solve it by explicit integration. When we will not be able to explicitly and analytically compute the integrals involved, we will evaluate them numerically. As we will show, our method turns out to be tougher and tougher as the scale dimension increases, so it is particularly useful especially for low Δ and so it is complementary to the large Δ calculation of [5–7].

2 The propagator from the AdS/CFT approach

As we mentioned in the introduction we will model our CFT primary operator with scale dimension Δ by a scalar field in the AdS black brane background. The equation to solve in momentum space for the scalar field perturbation

$$\eta(z) = \int d^d x\, e^{-ix_\mu k^\mu} \phi(x,z) \tag{2.1}$$

is thus (see for example [14] and references therein)

$$\left(f(z/z_h)\, z \frac{d}{dz}\, z \frac{d}{dz} - d\, z \frac{d}{dz} - z^2 \left(k^2 + \omega^2\, \frac{1-f(z/z_h)}{f(z/z_h)} \right) - \Delta(\Delta - d) \right) \eta(z) = 0\,, \tag{2.2}$$

where

$$f(x) \equiv 1 - x^d \tag{2.3}$$

with the AdS radius $L = 1$ and the horizon radius (black brane position) expressed in terms of the spacetime dimension d and temperature T through

$$z_h = \frac{d}{4\pi T}\,. \tag{2.4}$$

Our choice of (Euclidean) boundary conditions for the two linearly independent solutions $\eta_\pm(z)$ are

$$\begin{aligned} z \to 0^+ : \eta_\pm(z) \quad &\longrightarrow z^{\frac{d}{2} \pm \nu}\,, \\ z \to z_h : |\eta_-(z)| &< \infty\,, \end{aligned} \tag{2.5}$$

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where

$$\nu = \Delta - d/2 \tag{2.6}$$

is taken to be positive with non integer 2ν all along the paper.

If we write

$$\eta(z) = \frac{z^{\frac{d-1}{2}}}{f(z/z_h)^{\frac{1}{2}}}\, h(z)\,, \tag{2.7}$$

then (2.2) implies that $h(z)$ obeys

$$\left(\frac{d^2}{dz^2} - U(z)\right) h(z) = 0\,, \tag{2.8}$$

where

$$U(z) = -\frac{1}{4\,z^2} + \frac{d^2}{4}\frac{2\,f(z/z_h)-1}{z^2\,f(z/z_h)^2} + \frac{\Delta(\Delta-d)}{z^2\,f(z/z_h)} + \frac{k^2}{f(z/z_h)} + \omega^2\,\frac{1-f(z/z_h)}{f(z/z_h)^2}\,. \tag{2.9}$$

Let $h_\pm(z)$ be a basis of solutions to (2.8). The general solution can be written as

$$h(z) = C_+\,h_+(z) + C_-\,h_-(z)\,, \tag{2.10}$$

where $C_\pm$ are arbitrary constants. Taking into account the derivative of (2.10), we can write,

$$\begin{pmatrix} h(z) \\ h'(z) \end{pmatrix} = \mathbf{w}\,(h_+, h_-; z)\,\begin{pmatrix} C_+ \\ C_- \end{pmatrix}\,, \tag{2.11}$$

where

$$\mathbf{w}\,(f, g; z) \equiv \begin{pmatrix} f(z) & g(z) \\ f'(z) & g'(z) \end{pmatrix} \tag{2.12}$$

is the wronskian matrix of f and g at the point z. For any basis of solutions of (2.8) its determinant, i.e. the wronskian, is a non-zero constant. From (2.11), our problem reduces to the computation of the wronskian matrix of a given basis. According to (2.5) the (Euclidean) boundary conditions are

$$\begin{aligned} z \to 0^+ &: h_\pm(z) \quad \longrightarrow z^{\frac{1}{2}\pm\nu}\,, \\ z \to z_h &: |h_-(z)| = 0\,. \end{aligned} \tag{2.13}$$

In this way, the wronskian results: $\det(\mathbf{w}\,(h_+, h_-; z)) = -2\,\nu$.

We can recast the second order equation (2.8) as the following first order system,

$$\frac{d}{dz}\begin{pmatrix} h(z) \\ h'(z) \end{pmatrix} = \mathbf{A}(z)\,\begin{pmatrix} h(z) \\ h'(z) \end{pmatrix}\,, \qquad \mathbf{A}(z) \equiv \begin{pmatrix} 0 & 1 \\ U(z) & 0 \end{pmatrix}\,, \tag{2.14}$$

or, for the wronskian matrix

$$\frac{d}{dz}\mathbf{w}\,(h_+, h_-; z) = \mathbf{A}(z)\;\mathbf{w}\,(h_+, h_-; z)\;. \tag{2.15}$$

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The potential $\mathbf{A}(z)$ can be split into a zero-temperature and a temperature-dependent part as

$$\mathbf{A}(z) = \mathbf{A}^{(0)}(z) + \mathbf{A}_T(z)\,, \tag{2.16}$$

where the $T = 0$ matrix is

$$\mathbf{A}^{(0)}(z) = k^2 \begin{pmatrix} 0 & 1 \\ 1 + \frac{\nu^2 - \frac{1}{4}}{q^2} & 0 \end{pmatrix}, \tag{2.17}$$

with

$$q \equiv kz\,, \tag{2.18}$$

while that the temperature-dependent part can be expanded in powers of $(T/k)^d$ as

$$\mathbf{A_T(z)} = k^2\, \sigma_- u_T(q)\,, \quad u_T(q) = \sum_{m \in \mathbb{N}^+} \frac{u^{(m)}(kz)}{(kz_h)^{dm}}\,, \tag{2.19}$$

where

$$\begin{aligned} \sigma_- &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ u^{(m)}(q) &= A_m\, q^{dm-2} + B_m\, q^{dm}\,, \\ A_m &\equiv \nu^2 - m\, \frac{d^2}{4}, \qquad B_m \equiv 1 + m\,(\omega/k)^2\,. \end{aligned} \tag{2.20}$$

2.1 The $T = 0$ case

At zero temperature (2.15) reads,

$$\frac{d}{dz}\mathbf{w}\left(h_+^{(0)}, h_-^{(0)}; z\right) = \mathbf{A}^{(0)}(z)\; \mathbf{w}\left(h_+^{(0)}, h_-^{(0)}; z\right)\,. \tag{2.21}$$

The solutions are $z^{\frac{1}{2}}$ times Bessel functions of order ν. A summary of their properties and useful expansions is given in appendix A. The basis (2.13) is given by

$$h_+^{(0)}(z) = \frac{2^\nu\, \Gamma(1+\nu)}{k^\nu}\; z^{\frac{1}{2}}\; I_\nu(q)\,, \tag{2.22}$$

$$h_-^{(0)}(z) = \frac{2^{1-\nu}\, k^\nu}{\Gamma(\nu)}\; z^{\frac{1}{2}}\; K_\nu(q)\,. \tag{2.23}$$

According to the AdS/CFT recipe the propagator is defined through the limit

$$h_-(z) \quad \overset{z\to 0^+}{\longrightarrow} \quad z^{\frac{1}{2}-\nu}\,(1 + \dots) + G_2(k)\; z^{\frac{1}{2}+\nu}\,(1 + \dots) \tag{2.24}$$

to be in the leading order

$$G_2^{(0)}(k) = \frac{\Gamma(-\nu)}{\Gamma(\nu)}\; \left(\frac{k}{2}\right)^{2\nu}\,. \tag{2.25}$$

2.2 The $T \neq 0$ case

Going to the general case, if we make the ansatz for any T,

$$\mathbf{w}\,(h_+, h_-; z) = \mathbf{w}\left(h_+^{(0)}, h_-^{(0)}; z\right)\,\left(\mathbf{1} + \mathbf{w}^{(T)}(q)\right)\,, \tag{2.26}$$

then from (2.15) and (2.21) we get for $\mathbf{w}^{(T)}(q)$

$$\frac{d}{dq}\mathbf{w}^{(T)}(q) = q\; u_T(q)\;\mathbf{b}(q)\;\left(\mathbf{1} + \mathbf{w}^{(T)}(q)\right)\,, \tag{2.27}$$

where $u_T(q)$ is given in (2.19) and the matrix $\mathbf{b}(q)$ is

$$\begin{aligned}\mathbf{b}(q) &\equiv z^{-1}\;\mathbf{w}\left(h_+^{(0)}, h_-^{(0)}; z\right)^{-1}\;\sigma_-\;\mathbf{w}\left(h_+^{(0)}, h_-^{(0)}; z\right)\\ &= \begin{pmatrix} I_\nu(q)\,K_\nu(q) & \alpha\,K_\nu(q)^2 \\ -\frac{1}{\alpha}\,I_\nu(q)^2 & -I_\nu(q)\,K_\nu(q)\end{pmatrix}\,,\end{aligned} \tag{2.28}$$

where

$$\alpha \equiv \frac{2^{1-2\nu}\,k^{2\nu}}{\Gamma(\nu)\,\Gamma(1+\nu)} = -2\frac{\sin(\pi\nu)}{\pi}\,G_2^{(0)}(k)\,. \tag{2.29}$$

In order to solve (2.27) we expand

$$\mathbf{w}^{(T)}(q) = \sum_{m\in\mathbb{N}^+}\frac{\mathbf{w}^{(m)}(q)}{(kz_h)^{m\,d}} \tag{2.30}$$

and get the following system of differential equations for the $\mathbf{w}^{(m)}(q)$'s, $m > 0$ (we define $\mathbf{w}^{(0)} = \mathbb{I}$)

$$\frac{d}{dq}\mathbf{w}^{(m)}(q) = \sum_{k=0}^{m-1} q\; u^{(m-k)}(q)\;\mathbf{b}(q)\;\mathbf{w}^{(k)}(q)\,, \tag{2.31}$$

to be solved iteratively, with the $u^{(m)}(q)$'s given in (2.20). The solution is

$$\mathbf{w}^{(m)}(q) = \mathbf{w}_0^{(m)} + \sum_{k=0}^{m-1}\int^q dz\; z\; u^{(m-k)}(z)\;\mathbf{b}(z)\;\mathbf{w}^{(k)}(z)\,. \tag{2.32}$$

From (2.26) and (2.12) we have

$$\begin{aligned} h_+(z) &= \left(1 + w_{11}^{(T)}(q)\right)\;h_+^{(0)}(z) + w_{21}^{(T)}(q)\;h_-^{(0)}(z)\,,\\ h_-(z) &= w_{12}^{(T)}(q)\;h_+^{(0)}(z) + \left(1 + w_{22}^{(T)}(q)\right)\;h_-^{(0)}(z)\,.\end{aligned} \tag{2.33}$$

The integration constants present in the computation of the matrix elements $w_{ij}^{(T)}(q)$ are fixed by (2.13), i.e. for $z \to 0$

$$z \to 0^+ : \quad h_\pm(z) \quad \longrightarrow z^{\frac{1}{2}\pm\nu}\,. \tag{2.34}$$

Since expanding everything around $T = 0$ is equivalent to expanding everything around $z_h = \infty$, we replace the last constraint of (2.13) with

$$z \to \infty : \quad h_-(z) \longrightarrow 0\,. \tag{2.35}$$

Near $z = 0$,

$$\begin{aligned} h_+(z) &= \left(1 + w_{11}^{(T)}(q)\right)\, z^{\frac{1}{2}+\nu}\,(1+\mathcal{O}(q)) \\ &\quad + w_{21}^{(T)}(q)\, z^{\frac{1}{2}-\nu}\, \left(1 + \mathcal{O}(q) + G_2^{(0)}(k)\, z^{2\nu}\,(1+\mathcal{O}(q))\right)\,, \\ h_-(z) &= w_{12}^{(T)}(q)\, z^{\frac{1}{2}+\nu}\,(1+\mathcal{O}(q)) \\ &\quad + \left(1 + w_{22}^{(T)}(q)\right)\, z^{\frac{1}{2}-\nu}\, \left(1 + \mathcal{O}(q) + G_2^{(0)}(k)\, z^{2\nu}\,(1+\mathcal{O}(q))\right)\,. \end{aligned} \tag{2.36}$$

Conditions (2.34) are fulfilled if

$$0 = w_{11}^{(T)}(0) = w_{22}^{(T)}(0) = q^{-2\nu}\, w_{21}^{(T)}(q)\Big|_{q\to 0} = q^{2\nu}\, w_{12}^{(T)}(q)\Big|_{q\to 0}\,. \tag{2.37}$$

The first three conditions imply that $w_{11,21,22}^{(T)}(0) = 0$, so they determine the respective integration constants $w_{0\,ij}^{(T)}$ in (2.32). Since in general up to a constant

$$z \to 0 : w_{ij}^{(m)}(z) \sim z^{md+(i-j)2\nu}\,, \tag{2.38}$$

$$z \to \infty : w_{ij}^{(m)}(z) \sim z^{md+1-|i-j|}e^{(i-j)2z}\,, \tag{2.39}$$

all (ij) except eventually (12) are well behaved close to $z = 0$. On the other side, close to $z = \infty$ it is exactly the (12) component which is well behaved. We can thus rewrite (2.32) incorporating (2.34) as (let define $w_{ij}^{(0)} = \delta_{ij}$)

$$w_{ij}^{(m)}(q) = \sum_{k=0}^{m-1} \int_0^q dz\, z\, u^{(m-k)}(z)\, \left(\mathbf{b}(z)\, \mathbf{w}^{(k)}(z)\right)_{ij} \tag{2.40}$$

for $(ij) = (11), (21), (22)$, while (2.35) implies

$$w_{12}^{(m)}(q) = -\sum_{k=0}^{m-1} \int_q^\infty dz\, z\, u^{(m-k)}(z)\, \left(\mathbf{b}(z)\, \mathbf{w}^{(k)}(z)\right)_{12}\,. \tag{2.41}$$

Taking into account that the behavior of a solution is always of the form: $a_- z^{\frac{1}{2}-\nu}(1 + \mathcal{O}(z)) + a_+ z^{\frac{1}{2}+\nu}(1+\mathcal{O}(z))$, we can write the following behaviour for $w_{12}^{(T)}(q)$ near $q = 0$,

$$w_{12}^{(T)}(q) \quad \overset{q\to 0}{\longrightarrow} \quad G_2^{(0)}(k)\, g^{(T)}(k) + a\, q^{-2\nu+d}\,(1+\mathcal{O}(q))\,, \tag{2.42}$$

where $g^{(T)}(k)$ is some free constant. Then the AdS/CFT recipe (2.24) gives the propagator in the form

$$G_2(k) = G_2^{(0)}(k)\, \left(1 + g^{(T)}(k)\right)\,. \tag{2.43}$$

Similarly as $\mathbf{w}^{(T)}(k)$ in (2.30) also $g^{(T)}(k)$ gets expanded:

$$g^{(T)}(k) = \sum_{m=1}^{\infty} \frac{g_m(\omega/k)}{(kz_h)^{md}}\,. \tag{2.44}$$

Thus, to get the finite temperature correction to the propagator one needs to extract the constant part of the expansion of $w_{12}^{(T)}(q)$ for $q \to 0$. If the integral (2.41) converges, then the constant part of the expansion (2.42) is simply

$$G_2^{(0)}(k) g_m(\omega/k) = -\sum_{k=0}^{m-1} \int_0^\infty dz\, z\, u^{(m-k)}(z) \, \left(\mathbf{b}(z)\, \mathbf{w}^{(k)}(z)\right)_{12} . \tag{2.45}$$

However, for $2\nu - md > 0$ the integral (2.41) diverges. In this case, to extract the constant part, one needs first to split the integrand

$$F(z) \equiv -\sum_{k=0}^{m-1} z\, u^{(m-k)}(z) \, \left(\mathbf{b}(z)\, \mathbf{w}^{(k)}(z)\right)_{12} \to \sum_{\alpha>0} \frac{F_{-\alpha}}{z^{\alpha+1}} + \sum_{\beta>0} F_\beta z^{\beta-1} \tag{2.46}$$

as $z \to 0$, where α, β are in general non integer real numbers. Notice that for finite ν there is always only a finite number of divergent terms (α). In this case (2.45) becomes

$$G_2^{(0)}(k) g_m(\omega/k) = \int_0^\infty dz \left(F(z) - \sum_{\alpha>0} \frac{F_{-\alpha}}{z^{\alpha+1}} \right) , \tag{2.47}$$

which renders the integration at $z = 0$ finite without changing the convergence at $z = \infty$.

Of course, if one is able to compute analytically (2.41), then this is not needed, and one just takes the constant term in the expansion around $q \to 0$ of the result as indicated in eq. (2.42). But apart from the $m = 1$ case we were unable to perform such an analytic integration, in which case the formula (2.47) (which reduces to (2.45) for $2\nu - md < 0$) is the one we will use in the numerical evaluation.

3 First correction: $(T/k)^d$

At first order we obtain

$$\mathbf{w}^{(1)}(q) = A_1 \begin{pmatrix} I_d[I_\nu\, K_\nu; q, 0] & \alpha\, I_d[{K_\nu}^2; q, \infty] \\ -\frac{1}{\alpha}\, I_d[{I_\nu}^2; q, 0] & -I_d[I_\nu\, K_\nu; q, 0] \end{pmatrix} + ((A_1, d) \to (B_1, d+2)) , \tag{3.1}$$

where we defined the integral

$$I_s[f; q, q_0] \equiv \int_{q_0}^q dz\, z^{s-1}\, f(z) . \tag{3.2}$$

Although we will not use them, these integrals are explicitly computed for completeness in appendix A.

From (2.33) we have in our case

$$\begin{aligned} h_+(z) &= \left(1 + \frac{1}{(kz_h)^d}\, w_{11}^{(1)}(q)\right) h_+^{(0)}(z) + \frac{1}{(kz_h)^d}\, w_{21}^{(1)}(q)\, h_-^{(0)}(z) + \mathcal{O}\left(\frac{1}{(kz_h)^{2d}}\right) , \\ h_-(z) &= \frac{1}{(kz_h)^d}\, w_{12}^{(1)}(q)\, h_+^{(0)}(z) + \left(1 + \frac{1}{(kz_h)^d}\, w_{22}^{(1)}(q)\right) h_-^{(0)}(z) + \mathcal{O}\left(\frac{1}{(kz_h)^{2d}}\right) . \end{aligned} \tag{3.3}$$

Using the definitions (3.1) we find the leading low q expansion as

$$w_{11}^{(1)}(q) \sim q^d\,, \quad w_{22}^{(1)}(q) \sim q^d\,, \quad w_{21}^{(1)}(q) \sim q^{2\nu+d}\,, \tag{3.4}$$

while the only constant piece is in

$$w_{12}^{(1)}(q) \sim G_2^{(0)}(k)\,(A_1\alpha_d + B_1\alpha_{d+2}) + \mathcal{O}\left(q^{-2\nu+d}\right)\,, \tag{3.5}$$

where we have defined

$$\alpha_s = \frac{2\sin(\pi\nu)}{\pi}\int_0^\infty dz z^{s-1} K_\nu^2(z)\,, \tag{3.6}$$

which is taken at face value for $s-2\nu>0$, while for $s-2\nu<0$ it must be understood in the sense of (2.47), i.e. with proper subtractions. However, in both cases (we are interested in positive integer s and $\nu > d/2$) the final result is

$$\alpha_s = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}\,\nu\frac{\Gamma\left(\frac{s}{2}+\nu\right)\Gamma\left(\frac{s}{2}-\nu\right)}{\Gamma(1+\nu)\Gamma(1-\nu)}\,. \tag{3.7}$$

The propagator (2.24) at this order is

$$G_2(k) = G_2^{(0)}(k)\left(1+\frac{g_1(\omega/k)}{(kz_h)^d}\right)\,, \tag{3.8}$$

with

$$\begin{aligned} g_1(\omega/k) &= A_1\alpha_d + B_1\alpha_{d+2} \\ &= \frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+3}{2}\right)}\frac{\Gamma\left(\frac{d}{2}+1+\nu\right)\Gamma\left(\frac{d}{2}+1-\nu\right)}{\Gamma(1+\nu)\Gamma(1-\nu)}\nu\left(d\left(\frac{\omega}{k}\right)^2-1\right)\,, \end{aligned} \tag{3.9}$$

which agrees with [8]. For $d=4$ we get

$$G_2(k)/G_2^{(0)}(k) - 1 = \left(\frac{\pi T}{k}\right)^4 \frac{2}{15}\,\nu\,(\nu^2-1)\,(\nu^2-4)\left(4\left(\frac{\omega}{k}\right)^2-1\right)\,, \tag{3.10}$$

which is the Fourier transform (B.9) of the result in [6] using $\Delta = 2+\nu$ (2.6). Notice that the result here is exactly the same as for large Δ. This peculiarity is not preserved at next order in the temperature expansion.

Since the $d=2$ solution is known [15], for even d there is an even easier way to compute this leading order correction to the propagator, without explicitly calculating the integral (3.6), which we now explicitly write as a function of ν:

$$\alpha_r(\nu) = \frac{2}{\pi}\sin(\pi\nu)\int_0^\infty dz z^{r-1}K_\nu^2(z)\,. \tag{3.11}$$

Using

$$K_\nu = -K'_{\nu+1} - \frac{\nu+1}{z}K_{\nu+1} \tag{3.12}$$

one can derive, applying repeatedly differentiation by parts

$$\alpha_{r+2}(\nu) = \alpha_{r+2}(\nu+1) - \frac{r(r-2\nu-2)}{2}\alpha_r(\nu+1)\,. \tag{3.13}$$

Eq. (3.13) means that there is a relation for the α_{d+2} once α_d is known. Since α_2 is known exactly, α_4 can be calculated.

Take the ansatz[1]

$$\alpha_4(\nu) = \sum_{n=1}^{d-1} c_n \nu^n \tag{3.14}$$

and plugging it into (3.13) we obtain ($\alpha_2(\nu) = \nu$)

$$c_3\left((\nu+1)^3 - \nu^3\right) + c_2\left((\nu+1)^2 - \nu^2\right) + c_1\left((\nu+1) - \nu\right) = -2\nu(\nu+1)\,. \tag{3.15}$$

Equating the same powers of ν we get

$$\alpha_4(\nu) = \frac{2}{3}\nu\left(1-\nu^2\right)\,, \tag{3.16}$$

which is the correct result got by direct integration of (2.29), which Mathematica is able to do easily. Notice that here the recursion relation is between coefficients in different dimensions but of the same order (coefficients of T^d vs. those of T^{d+2} but not of T^{2d} which are of next order except for $d=2$). Unfortunately we did not succeed in computing the next order term T^{2d} using the same idea.

4 Second correction: $(T/k)^{2d}$

Going to the second order correction, taking into account (2.37) and (2.35), we define

$$w_{ij}^{(2)}(q) = \begin{cases} \int_0^q dz\, z\, \left(u^{(2)}(z)\, b_{ij}(z) + u^{(1)}(z)\, b_{ik}(z)\, w_{kj}^{(1)}(z)\right) & , \quad (ij) \neq (12) \\ -\int_q^\infty dz\, z\, \left(u^{(2)}(z)\, b_{12}(z) + u^{(1)}(z)\, b_{1k}(z)\, w_{k2}^{(1)}(z)\right) & , \quad (ij) = (12) \end{cases} \tag{4.1}$$

With these definitions we assure that the conditions that define the basis $h_\pm(z)$,

$$\begin{aligned} w_{11}^{(2)}(0) = w_{21}^{(2)}(0) = w_{22}^{(2)}(0) = 0\,, \\ w_{12}^{(2)}(\infty) = 0 \end{aligned} \tag{4.2}$$

hold. To get the correction to the propagator, we focus on $w_{12}^{(2)}(q)$ and its limit

$$w_{12}^{(2)}(q) \quad \xrightarrow{q\to 0} \quad G_2^{(0)}(k)\, g_2(\omega/k) + a\, q^{-2\nu+2d}\,(1+\mathcal{O}(q))\,. \tag{4.3}$$

For $\nu < d$ the correction to the propagator is

$$g_2(\omega/k) = A_2\alpha_{2d} + B_2\alpha_{2d+2} + {A_1}^2\, g_{dd} + {B_1}^2\, g_{d+2,d+2} + 2A_1\, B_1\, g_{d,d+2}\,, \tag{4.4}$$

[1] The constant term is missing because g_1 is odd under $\nu \to -\nu$, see section 4.1, while higher orders c_n for $n \geq d$ turn out to be zero.

with

$$g_{rs} = \frac{1}{2}\int_0^\infty dz\,\left(F_{rs}(z) + F_{sr}(z)\right)\,, \tag{4.5}$$

whose integrand is explicitly given by

$$\frac{\pi}{2\sin(\pi\nu)}F_{rs}(z) = z^{r-1}\,I_\nu(z)\,K_\nu(z)\,I_s[K_\nu^2; z,\infty] - z^{r-1}K_\nu^2(z)\,I_s[I_\nu K_\nu; z, 0]\,. \tag{4.6}$$

The behaviour of the integral can be read from the behaviour of the Bessel functions, which main properties are summarised in appendix A. While for $z \to \infty$ both terms in (4.6) are suppressed by $e^{-2\,z}$, for $z \to 0$ we have

$$F_{rs}(z) \sim z^{-2\,\nu+r+s-1}\,. \tag{4.7}$$

So, the integral (4.5) converges if $2\,\nu < r+s$. Since we need $r+s = 2\,d, 2\,d+2, 2\,d+4$, for $\nu < d$ no substractions are needed, and the contribution to the propagator at order $(T/k)^{2d}$, i.e. $g_2(\omega/k)$, can be evaluated directly.

This procedure can be continued for larger ν using proper subtractions (2.47). For example, for $\nu < d+1$ one gets the same formula (4.4) but now with

$$g_{dd} = \int_0^\infty dz\left(F_{dd}(z) - \frac{2^{2\nu}}{2\nu d(d-2\nu)}\frac{\Gamma(\nu)}{\Gamma(-\nu)}z^{-2\nu+2d-1}\right)\,, \tag{4.8}$$

with all the rest the same.

More generally, for $2\nu > r+s$, the renormalised formula (4.5) looks like

$$\begin{aligned} g_{rs} = \frac{1}{2}\int_0^\infty dz\,\Bigg[F_{rs}(z) - \sum_{n=0}^{\lfloor 2\nu-r-s\rfloor} z^{-2\nu+r+s-1+n}\frac{\sqrt{\pi}\csc(\pi\nu)}{4\Gamma(n+1)\Gamma(n-\nu+1)} \\ \times\left(\frac{\pi 4^\nu\Gamma(\nu+1)(-1)^n\,{}_5F_4\left(-n,\frac{1}{2}-\nu,-n-\nu,\frac{s}{2}-\nu,\nu-n;\frac{1}{2}-n,1-2\nu,1-\nu,\frac{s}{2}-\nu+1;1\right)}{\Gamma(n+\nu+1)\Gamma\left(\frac{1}{2}-n\right)(-2\nu)\Gamma(1-\nu)\left(\frac{s}{2}-\nu\right)}\right. \\ \left.+\frac{\Gamma\left(n-\nu+\frac{1}{2}\right)\,{}_5F_4\left(\frac{1}{2},-n,\frac{s}{2},\nu-n,2\nu-n;\frac{s}{2}+1,1-\nu,\nu+1,-n+\nu+\frac{1}{2};1\right)}{\nu s\Gamma(n-2\nu+1)}\right)\Bigg] \\ +(r\leftrightarrow s)\,. \end{aligned} \tag{4.9}$$

Notice that both ${}_5F_4$ are actually finite sums.

4.1 $\nu \to -\nu$

A check of the results is the behaviour of the propagator for $\nu \to -\nu$ (or, equivalently, $\Delta \to d-\Delta$). In this case one should get the inverse propagator:

$$G_2(-\nu) = \frac{1}{G_2(\nu)}\,. \tag{4.10}$$

Since at this order

$$G_2(\nu) = G_2^{(0)}(\nu)\left(1 + \frac{g_1(\nu)}{(kz_h)^d} + \frac{g_2(\nu)}{(kz_h)^{2d}}\right) \tag{4.11}$$

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and since due to (2.25) indeed

$$G_2^{(0)}(-\nu) = \frac{1}{G_2^{(0)}(\nu)}\,, \tag{4.12}$$

eq. (4.10) implies

$$g_1(-\nu) = -g_1(\nu)\,, \tag{4.13}$$

which is easily satisfied by (3.9), and

$$g_2(-\nu) = g_1^2(\nu) - g_2(\nu)\,. \tag{4.14}$$

To check this one explicitly, let us first rewrite (4.5) in terms of the Wronskian

$$g_{rs} = \frac{1}{2}\alpha_r\alpha_s - \frac{1}{4\nu}\int_0^\infty dz\,\left(W(F_r, G_s; z) + W(F_s, G_r; z)\right) \tag{4.15}$$

and (see definitions in appendix A)

$$F_r(z) = \frac{z^r}{r}\left(2f_r(z) + g_r^{(+)}(z) + g_r^{(-)}(z)\right) + 2\nu\alpha_r\,, \tag{4.16}$$

$$G_r(z) = \frac{z^r}{4\nu r}\left(g_r^{(+)}(z) - g_r^{(-)}(z)\right)\,, \tag{4.17}$$

$$F_r'(z) = -\frac{4\nu}{\pi}\sin(\pi\nu) z^{r-1} K_\nu^2(z)\,, \tag{4.18}$$

$$G_r'(z) = \frac{z^{r-1}}{2}K_\nu(z)\left(I_\nu(z) + I_{-\nu}(z)\right)\,. \tag{4.19}$$

From (2.29), (A.8) and the above definitions it is straightforward to see that under $\nu \to -\nu$ the quantity α_r is odd, while A_m, B_m, F_r and G_r are even. Then in (4.14) the A_2, B_2 as well as the integrals of Wronskians exactly drop out, while the $\alpha_r\alpha_s/2$ terms in (4.15) take care of the g_1^2 term in (4.14). The relation (4.10) is indeed correct up to order T^{2d}.

5 Special d

So far the only check we did is the analytic comparison with the large ν solution of [6]. Now we will specialise to the $d = 2$ case which is exactly known. We will compare our numerical solution with this exactly known result and find agreement at second order T^4. Then we will pass on to $d = 4$ and predict the second order T^8 correction to the propagator.

5.1 $d = 2$

Equation (5.8) of [15] in our euclidean setting is

$$G(k) = C_\Delta \frac{\Gamma\left(z_+ + \frac{\Delta}{2}\right)}{\Gamma\left(z_+ + 1 - \frac{\Delta}{2}\right)} \frac{\Gamma\left(z_- + \frac{\Delta}{2}\right)}{\Gamma\left(z_- + 1 - \frac{\Delta}{2}\right)}\,, \tag{5.1}$$

where

$$C_\Delta = \frac{\Gamma(-\nu)}{\Gamma(\nu)2^{2\Delta-2}}\left(\frac{2}{z_h}\right)^{2\Delta-2} \tag{5.2}$$

is a normalisation constant, and

$$z_\pm \equiv \frac{z_h}{2}\,(\omega \pm i\,q)\,, \qquad z_+\,z_- = \frac{(kz_h)^2}{4}\,. \tag{5.3}$$

We need the expansion for large kz_h, i.e. large $|z_\pm|$. To get it, we use results in [16]. We have for large $|z|$

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b} \sum_{n\in\mathbb{N}} c_n(a,b)\,\frac{z^{-n}}{n!} \qquad \text{with} \qquad c_n(a,b) \equiv (-)^n\,(b-a)_n\,B_n^{(1+a-b)}(a)\,, \tag{5.4}$$

where $B_n^{(k)}(x)$ are generalized Bernoulli polynomials [16].

Applying to (5.1), taking into account that

$$c_{2n} = (1-\Delta)_{2n}\,B_{2n}^{(\Delta)}\,(\Delta/2)\,, \qquad c_{2n+1} = -(1-\Delta)_{2n+1}\,B_{2n+1}^{(\Delta)}\,(\Delta/2) = 0\,, \tag{5.5}$$

we get the expansion

$$\begin{aligned} G(k) &\sim C_\Delta\,(z_+\,z_-)^{\Delta-1}\left(1+\frac{c_2}{2!\,z_+{}^2}+\frac{c_4}{4!\,z_+{}^4}+\mathcal{O}(T^6)\right)\left(1+\frac{c_2}{2!\,z_-{}^2}+\frac{c_4}{4!\,z_-{}^4}+\mathcal{O}(T^6)\right) \\ &= G^{(0)}(k)\left(1+\frac{c_2}{2\,z_+{}^2}+\frac{c_2}{2\,z_-{}^2}+\frac{c_4}{24\,z_+{}^4}+\frac{c_2{}^2}{4\,z_+{}^2\,z_-{}^2}+\frac{c_4}{24\,z_-{}^4}+\mathcal{O}(T^6)\right)\,, \end{aligned} \tag{5.6}$$

where

$$c_2 = -\frac{1}{12}\prod_{l=0}^{2}(\Delta-l)\,, \qquad c_4 = \frac{1}{240}\,(5\,\Delta+2)\prod_{l=0}^{4}(\Delta-l)\,. \tag{5.7}$$

From (5.6) we find that the T^2-correction to the propagator

$$\frac{g_1(\omega/k)}{(kz_h)^2} = \left(\frac{2\pi T}{k}\right)^2\,\frac{1}{3}\,\Delta(\Delta-1)(\Delta-2)\,(1-2\,(\omega/k)^2) \tag{5.8}$$

coincides with our result (3.9) at $d=2$, while the T^4-correction is

$$\begin{aligned} \frac{g_2(\omega/k)}{(kz_h)^4} &= \left(\frac{2\pi T}{k}\right)^4\,\frac{1}{90}\,\Delta(\Delta-1)(\Delta-2) \\ &\times\left(9\,\Delta\,(\Delta-2)-12+(5\,\Delta+2)\,(\Delta-3)\,(\Delta-4)\,(1-2\,(\omega/k)^2)^2\right)\,. \end{aligned} \tag{5.9}$$

Writing

$$g_2(\omega/k) = \gamma_0 + \gamma_2(\omega/k)^2 + \gamma_4(\omega/k)^4 \tag{5.10}$$

and comparing γ_i from the numerical evalulation of the integrals (4.4) (with (4.8) instead of (4.5) for $2<\nu<5/2$) as functions of $\nu \in [1, 5/2]$ with the known analytic form (5.9) we get good agreement, see figures 1.

Since the expansion eq. (4.5) in [1] for the $D1/D5$-branes system agrees with the full result [15] by identifying $\Delta = 1+\nu$ and $q=0$, we obviously agree with [1] too.

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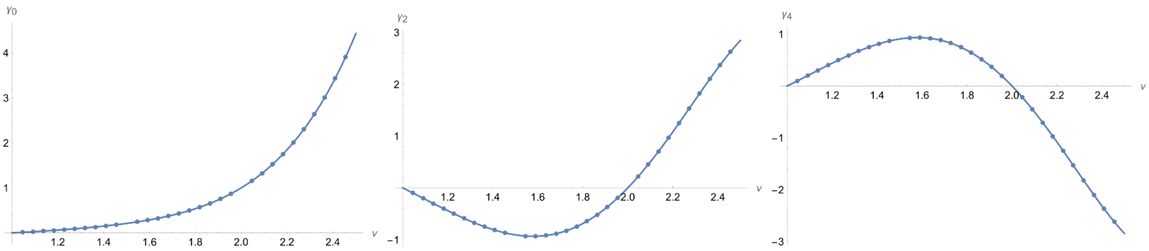

Figure 1. Comparison of γ_i defined in (5.10) between the exact result (5.9) (continuous curves) and our numerical evaluation of. eq. (4.4) (discrete points) as functions of $\nu \in [d/2, d+1/2]$ for the order T^{2d} correction to the Euclidean propagator in the $d = 2$ case.

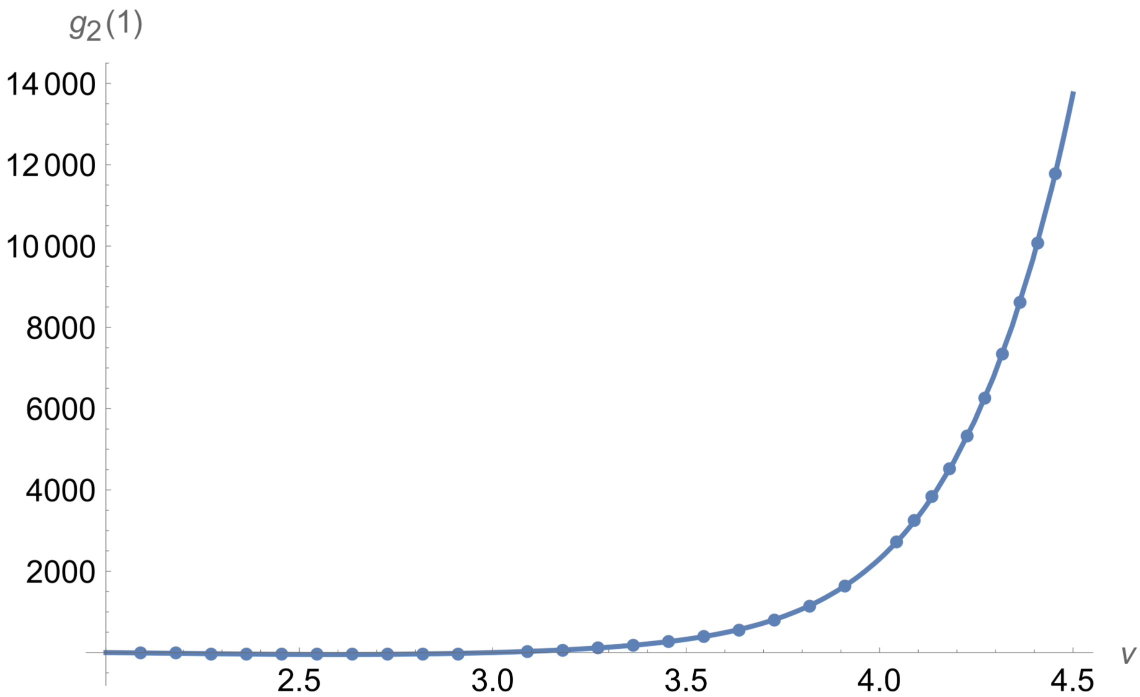

Figure 2. The function g_2 at $k = \omega$ for the $d = 4$ Euclidean propagator at order T^8 in our numerical evaluation of eq. (4.4) (discrete points) as functions of ν compared to the eq. (2.3) of [1] (full line).

5.2 $d = 4$

Here we first notice that our T^4 analytic result (3.10) completely agrees with the first nontrivial term in [1] by identifying $\Delta = 2 + \nu = l + 4$ and $\vec{q} = \vec{0}$. Next we compare our T^8 numerical result (4.4) for $g(1)$ with eq. (2.3) of [1] with the same identification as above, finding a perfect numerical agreement as shown in figure 2.

Reinvigorated by this matching at $d = 4$ with the case $\omega/k = 1$ and $\nu = l + 2$ of [1], and the matching in the previous section between the exactly known $d = 2$ result and our numerical evaluation of the second order contribution to the Euclidean propagator as a function of ν, which is summarised in figure 1, we go now to the arbitrary ω/k and $d = 4$ case. For the same γ_i, $i = 0, 2, 4$ functions of ν we find the T^8 contributions in figure 3.

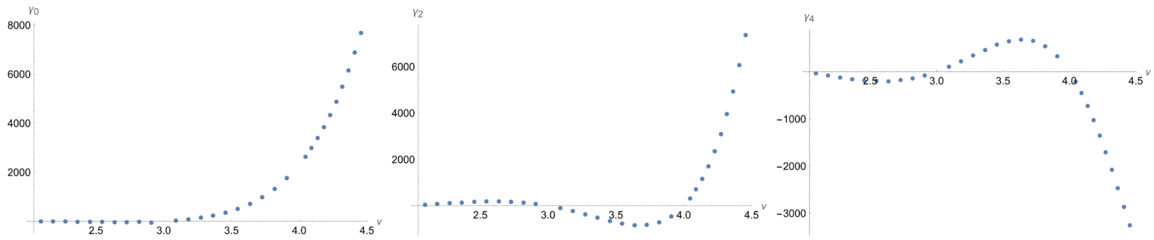

Figure 3. The functions γ_i defined in (5.10) for the $d = 4$ Euclidean propagator at order T^8 in our numerical evaluation of eq. (4.4) (discrete points) as functions of $\nu \in [d/2, d + 1/2]$.

Here the full ν dependent result is not known, but for large ν the approximate values can be found in [5, 6] and is here taken from (B.15):[2]

$$g_2^{\text{large }\nu}(\omega/k) = \frac{2}{225}\nu^{10}\left(4\frac{\omega^2}{k^2} - 1\right)^2 . \tag{5.11}$$

It is interesting to know how far is the approximate result (5.11) from our numerical evaluation. This is compared in figure 4, which shows that indeed the large ν approximation is not suitable for small ν, which makes our computation all the more important. On the other side, it should be stressed here that our method is not suitable for large ν calculations: in fact we need to explicitly subtract the divergent pieces and, as we have seen, their number increases linearly with ν. So the two methods, the geodesic approximation used in [6, 7] for large ν and our method here, should be considered as complementary rather than competitive.

Said that, we tried to analytically take the limit $\nu \to \infty$. Although we were unable to prove it definitely, we have some indications to conjecture that the large ν result for an arbitrary dimension d is

$$\begin{aligned} g_2^{\text{large }\nu}(\omega/k) &= \lim_{\nu\to\infty} \frac{1}{2}\left(A_1\alpha_d + B_1\alpha_{d+2}\right)^2 \\ &= \frac{2^{2d-5}\Gamma^4(d/2)}{(d+1)^2\Gamma^2(d)}\nu^{2d+2}\left(d\left(\frac{\omega}{k}\right)^2 - 1\right)^2 . \end{aligned} \tag{5.12}$$

This result coincides with (5.11) for $d = 4$ and with the large ν limit of (5.9) for $d = 2$. The reason why the extreme large limit is maybe doable is that one does not need to do it numerically, and analytical approximation of Bessel functions at large ν can be used [17].

Notice that (5.12) confirms the exponentiation property of the propagator at large Δ [6].

6 Conclusions

We showed in this paper the way to calculate the euclidean thermal propagator in a CFT using the AdS/CFT correspondence. Although the full formula for it has been recently

[2]Here there is an ambiguity of how to write the leading term, i.e. $\propto \nu^{10}$ or $\propto \Delta^{10}$: they are equivalent at large ν but make a difference for small ν. We choose a conservative approach and wrote this contribution as a function of ν, which is closer to our numerical result.

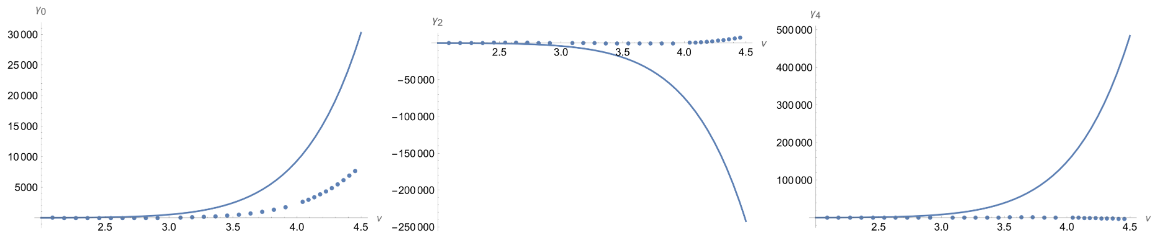

Figure 4. Comparison of γ_i defined in (5.10) for the $d = 4$ Euclidean propagator at order T^8 between the approximate result valid for large ν of [6] and our numerical evaluation of eq. (4.4) (discrete points) as functions of $\nu \in [d/2, d + 1/2]$.

found [11], it is not suited for a small temperature expansion. We found the first two corrections, the first analytical of order T^d in (3.9), and the second numerical of order T^{2d} via (4.4) and (4.9). The order T^d agrees with [8] for general d and with [5, 6] for $d = 4$ (summarised in (3.10)). The order T^{2d} reproduces the well known result [15] for $d = 2$. This last check is summarised in figure 1. We then predicted the T^8 contributions in the $d = 4$ case in figure 3. Finally we showed in figure 4 explicitly that the large ν approximation does not properly describe the small ν limit, a result which is not surprising but had to be checked. The method presented here is thus appropriate for this range of ν, not described by the geodesic approximation.

Our approach can deal with general spacetime dimensions d. It is not tied to the Heun equation as is the case of the analytical solution [11]. In fact in $d > 4$ apparently more singular points are present and so it is not clear if the equation to be solved is still of the Heun type. In this paper we found for arbitrary d an explicit expression (3.9) for the propagator at order T^d for arbitrary ν. At the order T^{2d} we conjecture an analytic formula (5.12), valid for large ν.

The perturbative expansion in T/k, which we do in d-dimensional Euclidean spacetime, seems to be compatible with the analytical continuation to Minkowski spacetime. This is checked for example if we compare our Euclidean result with the Minkowski all-order result in $d = 2$.

Few words of caution are perhaps important at this point. As studied in [1], the Heun equation has one potential problem and one real problem. The potential one is connected to the uniform convergence of the asymptotic expansion in the whole coordinate interval, also close to the singularities. The authors of [1] use successfully the method of Langer and Olver [18–20]. In our case this would roughly mean that the coefficients (2.40)–(2.41) need to be finite, which we easily prove they are. More precisely, in the limit $k = \omega$ we start from the same equation of [1]. The difference with them is that we automatically find an expansion in positive integer powers of T^d, while in [1] this turns out to be true only after some cancellations. For our more general case of $k \neq \omega$ we do not know of any method equivalent to Langer and Olver. Finding such a method is clearly beyond the scope of this paper. We however check our results by comparing them to the known ones in the literature and find perfect agreement whenever available. The second, real problem, is the appearance in the solution of terms proportional to positive powers of $\exp(-1/T)$. These terms, as in [1], are not available in our method. We would like here to stress that these terms are certainly one of the reasons why the full solution [11] mentioned above is not suited for a low T expansion.

Acknowledgments

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A Some useful identities

We first present some useful formulae regarding Bessel functions for indices $\nu \notin \mathbb{Z}$. By using the expansions given in [21], we have for $q \ll 1$,

$$\begin{aligned} I_\nu(q) &= \frac{1}{\Gamma(1+\nu)} \left(\frac{q}{2}\right)^{\nu} {}_0F_1\left(1+\nu; \frac{q^2}{4}\right), \\ K_\nu(q) &= \frac{\Gamma(\nu)}{2} \left(\frac{q}{2}\right)^{-\nu} {}_0F_1\left(1-\nu; \frac{q^2}{4}\right) + \frac{\Gamma(-\nu)}{2} \left(\frac{q}{2}\right)^{\nu} {}_0F_1\left(1+\nu; \frac{q^2}{4}\right), \end{aligned} \tag{A.1}$$

and for $q \gg 1$ the asymptotic expansions is

$$\begin{aligned} I_\nu(q) &\sim \frac{e^q}{\sqrt{2\,\pi\,q}}\, {}_2F_0\left(\frac{1}{2}+\nu, \frac{1}{2}-\nu; \frac{1}{2\,q}\right) + i\, e^{i\,\pi\,\nu} \frac{e^{-q}}{\sqrt{2\,\pi\,q}}\, {}_2F_0\left(\frac{1}{2}+\nu, \frac{1}{2}-\nu; -\frac{1}{2\,q}\right), \\ K_\nu(q) &\sim \sqrt{\frac{\pi}{2\,q}}\, e^{-q}\, {}_2F_0\left(\frac{1}{2}+\nu, \frac{1}{2}-\nu; -\frac{1}{2\,q}\right). \end{aligned} \tag{A.2}$$

The elements of the matrix $\mathbf{b}(q)$ in (2.28) are defined by,

$$\begin{aligned} 2\,\nu\, b_{11}(q) &\equiv 2\,\nu\, I_\nu(q)\, K_\nu(q) \\ &= {}_1F_2\left(\begin{matrix} \frac{1}{2} \\ 1+\nu, 1-\nu \end{matrix}; q^2\right) + \frac{\Gamma(-\nu)}{\Gamma(\nu)} \left(\frac{q}{2}\right)^{2\,\nu} {}_1F_2\left(\begin{matrix} \frac{1}{2}+\nu \\ 1+2\,\nu, 1+\nu \end{matrix}; q^2\right), \\ b_{22}(q) &\equiv -b_{11}(q)\,, \\ \frac{4}{\alpha}\, b_{12}(q) &\equiv 4\, K_\nu(q)^2 = \Gamma(-\nu)^2 \left(\frac{q}{2}\right)^{2\,\nu} {}_1F_2\left(\begin{matrix} \frac{1}{2}+\nu \\ 1+2\,\nu, 1+\nu \end{matrix}; q^2\right) \\ &\qquad + \Gamma(\nu)\,\Gamma(-\nu)\, {}_1F_2\left(\begin{matrix} \frac{1}{2} \\ 1+\nu, 1-\nu \end{matrix}; q^2\right) + (\nu \to -\nu)\,, \\ -\alpha\, b_{21}(q) &\equiv I_\nu(q)^2 = \frac{1}{\Gamma(1+\nu)^2} \left(\frac{q}{2}\right)^{2\,\nu} {}_1F_2\left(\begin{matrix} \frac{1}{2}+\nu \\ 1+2\,\nu, 1+\nu \end{matrix}; q^2\right), \end{aligned} \tag{A.3}$$

where we have used (A.1), the notable product identity

$$ {}_0F_1\,(a;z)\ {}_0F_1\,(b;z) = {}_2F_3 \left(\begin{matrix} \frac{1}{2}(a+b), \frac{1}{2}(a+b-1) \\ a, b, a+b-1 \end{matrix} ; 4\,z \right) , \tag{A.4} $$

and the fact that

$$ {}_{p+1}F_{q+1} \left(\begin{matrix} a_1, \dots a_p, c \\ b_1, \dots b_q, c \end{matrix} ; z \right) = {}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} ; z \right) . \tag{A.5} $$

Using

$$ \int^q dz\, z^{s-1}\ {}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} ; z^2 \right) = \frac{q^s}{s}\ {}_{p+1}F_{q+1} \left(\begin{matrix} a_1, \dots, a_p, \frac{s}{2} \\ b_1, \dots, b_q, 1+\frac{s}{2} \end{matrix} ; q^2 \right) , \tag{A.6} $$

the integrals (3.2) are (for positive s)

$$ \begin{aligned} I_s[b_{11}; q, 0] &= \frac{1}{2\,\nu} \frac{q^s}{s} \left(f_s(q) + g_s^{(+)}(q) \right) , \\ I_s[b_{22}; q, 0] &= -\frac{1}{2\,\nu} \frac{q^s}{s} \left(f_s(q) + g_s^{(+)}(q) \right) , \\ I_s[b_{12}; q, \infty] &= \frac{G_2^{(0)}(k)}{2\,\nu} \left(\frac{q^s}{s} \left(2\, f_s(q) + g_s^{(+)}(q) + g_s^{(-)}(q) \right) + 2\nu\alpha_s \right) , \\ I_s[b_{21}; q, 0] &= -\frac{1}{2\,\nu G_2^{(0)}(k)} \frac{q^s}{s} g_s^{(+)}(q) , \end{aligned} \tag{A.7} $$

where we defined the functions $f_s(q)$ and $g_s^{(\pm)}(q)$ as

$$ \begin{aligned} f_s(q) &\equiv {}_2F_3 \left(\begin{matrix} \frac{s}{2}, \frac{1}{2} \\ 1+\nu, 1+\frac{s}{2}, 1-\nu \end{matrix} ; q^2 \right) , \\ g_s^{(\pm)}(q) &\equiv \gamma_s^{(\pm)}\ q^{\pm 2\,\nu}\ {}_2F_3 \left(\begin{matrix} \frac{s}{2} \pm \nu, \frac{1}{2} \pm \nu \\ 1 \pm 2\,\nu, 1 \pm \nu + \frac{s}{2}, 1 \pm \nu \end{matrix} ; q^2 \right) , \quad \gamma_s^{(\pm)} \equiv \frac{2^{\mp 2\,\nu}\, s}{s \pm 2\,\nu} \frac{\Gamma(\mp\nu)}{\Gamma(\pm\nu)} , \end{aligned} \tag{A.8} $$

and α_s in (2.29). Eq. (A.7) can be used to compute the elements of the matrix (3.1)

$$ \int_{q_0}^q dz\, z\, u^{(m)}(z)\ \mathbf{b}(z) = A_m\, I_{d\,m}[\mathbf{b}; q, q_0] + B_m\, I_{d\,m+2}[\mathbf{b}; q, q_0] , \quad m = 1, 2, \dots . \tag{A.9} $$

B From x to k space

In reference [7], Rodríguez-Gómez and Russo computed the low temperature expansion of the propagator up to second order in T^4, in four dimensions and large operator dimension limit. Their result, given in coordinate space $(\tau, \vec{x})$, can be written as

$$ G_2(x, \eta) = G_2^{(0)}(x) + G_2^{(1)}(x, \eta) + G_2^{(2)}(x, \eta) + \dots , \tag{B.1} $$

where $x \equiv \sqrt{\tau^2 + \vec{x}^2}$, $\eta \equiv \frac{\tau}{x}$, and

$$\begin{aligned} G_2^{(0)}(x) &= x^{-2\,\Delta}\,, \\ G_2^{(1)}(x,\eta) &= \frac{2\,\Delta}{15}\left(\frac{\pi\,T}{2}\right)^4 C_2^{(1)}(\eta)\,x^{4-2\,\Delta}\,, \\ G_2^{(2)}(x,\eta) &= \frac{2\,\Delta^2}{15^2}\left(\frac{\pi\,T}{2}\right)^8 \left(C_0^{(1)}(\eta) + C_2^{(1)}(\eta) + C_4^{(1)}(\eta)\right) x^{8-2\,\Delta}\,. \end{aligned} \tag{B.2}$$

Here $C_m^{(a)}(\eta)$ are Gegenbauer polynomials, the relevant ones are

$$C_0^{(1)}(\eta) = 1\,, \quad C_2^{(1)}(\eta) = -1 + 4\,\eta^2\,, \quad C_4^{(1)}(\eta) = 1 - 12\,\eta^2 + 16\,\eta^4\,. \tag{B.3}$$

To compare (B.1) with our results, we need (B.2) in momentum space $(\omega, \vec{q})$. To this end, we first introduce the Fourier transform

$$f_{2a}(k) \equiv \int d^d x\, e^{-i\,k\cdot x}\, x^{-2\,a} = c_{2a}\, k^{2\,a-d}\,, \tag{B.4}$$

where $k \equiv \sqrt{\omega^2 + \vec{q}^2}$. By using the representation

$$x^{-2\,a} = \frac{1}{\Gamma(a)} \int_0^\infty dt\, t^{a-1}\, e^{-x^2\,t}\,, \qquad \Re(a), \Re(x^2) > 0\,, \tag{B.5}$$

we get

$$\begin{aligned} c_{2a} &= \int d^d x\, e^{-i\,x_\mu k^\mu/k}\, x^{-2\,a} = \frac{1}{\Gamma(a)} \int_0^\infty dt\, t^{a-1} \int d^d x\, e^{-x^2\,t - i\,x_\mu k^\mu/k} \\ &= \frac{\pi^{\frac{d}{2}}}{\Gamma(a)} \int_0^\infty dt\, t^{a-\frac{d}{2}-1}\, e^{-\frac{1}{4\,t}} = (4\pi)^{\frac{d}{2}}\, \frac{\Gamma\left(\frac{d}{2} - a\right)}{2^{2\,a}\,\Gamma(a)}\,. \end{aligned} \tag{B.6}$$

A useful relation that follows is (for $d = 4$)

$$\begin{aligned} \frac{2^{-2\,m}\,c_{2a}}{c_{2a+2m}} &= \frac{\Gamma(2-a)}{\Gamma(2-a-m)}\,\frac{\Gamma(a+m)}{\Gamma(a)} = (-)^m\,(a-1)_m\,(a)_m \\ &= (-)^m\,(a-1)\,(a-1+m) \prod_{l=1}^{m-1} (a-1+l)^2\,. \end{aligned} \tag{B.7}$$

So, from (B.2) we get:

Zero order

$$G_2^{(0)}(k) = f_{2\,\Delta}(k) = c_{2\Delta}\, k^{2\,\Delta-d} = \pi^{\frac{d}{2}}\, \frac{\Gamma(-\nu)}{\Gamma(\Delta)} \left(\frac{k}{2}\right)^{2\,\nu}\,. \tag{B.8}$$

Note the different normalization of (B.2) with respect to the one used in the paper, equation (2.25).

First order

$$
\begin{aligned}
G_2^{(1)}(k) &= \frac{1}{k^4}\,\frac{2\,\Delta\,k^4}{15\,2^4}\,\left(-f_{2\,\nu}(k) - 4\,\partial_\omega^2 f_{2\,\nu+2}(k)\right) \\
&= \frac{1}{k^4}\,\frac{2\,\Delta\,k^{2\nu}}{15\,2^4}\,\left(-c_{2\,\nu} - 8\,(\nu-1)\,c_{2\,\nu+2}\,\left(1 + 2\,(\nu-2)\,\left(\frac{\omega}{k}\right)^2\right)\right) \\
&= \frac{G_2^{(0)}(k)}{k^4}\,\frac{2\,\Delta}{15\,2^4}\,\frac{c_{2\,\nu}}{c_{2\,\nu+4}}\,\left(-1 - 8\,(\nu-1)\,\frac{c_{2\,\nu+2}}{c_{2\,\nu}}\,\left(1 + 2\,(\nu-2)\,\left(\frac{\omega}{k}\right)^2\right)\right) \\
&= \frac{G_2^{(0)}(k)}{k^4}\,\frac{2}{15}\,\nu\,(\nu^2-1)\,(\nu^2-4)\,C_2^{(1)}\left(\frac{\omega}{k}\right)\,,
\end{aligned}
\tag{B.9}
$$

where we have used (B.7) and the relation

$$
\int d^4x\, e^{-i\,x_\mu k^\mu/x}\, x^{-2\,a}\,\eta^m = i^m\, c_{2a+m}\, k^{4-2a}\,\partial_\omega^m k^{2a-4+m}\,. \tag{B.10}
$$

Second order

$$
\begin{aligned}
G_2^{(2)}(k) &= \frac{1}{(kz_h)^8}\,\frac{2\,\Delta^2\,k^8}{15^2\,2^8}\,\int d^4x\, e^{-i\,k\cdot x}\, x^{-2\,\Delta+8}\,\left(C_0^{(1)}(\eta) + C_2^{(1)}(\eta) + C_4^{(1)}(\eta)\right) \\
&= \frac{G_2^{(0)}(k)}{(kz_h)^8}\,\frac{2\,\Delta^2}{15^2\,2^8}\,\frac{c_{2\nu-4}}{c_{2\nu+4}}\,\sum_{l=0}^{2} I_{2\nu}^{(2l)}(\omega/k)\,,
\end{aligned}
\tag{B.11}
$$

where

$$
I_{2\nu}^{(2l)}(\omega/k) \equiv \frac{k^{8-2\nu}}{c_{2\nu-4}}\,\int d^4x\, e^{-i\,k\cdot x}\, x^{-2\,\nu+4}\, C_{2\,l}^{(1)}(\eta)\,. \tag{B.12}
$$

We get

$$
\begin{aligned}
I_{2\nu}^{(0)}(x) &= 1\,, \\
I_{2\,\nu}^{(2)}(x) &= \frac{\nu-4}{\nu-2}\,C_2^{(1)}(x)\,, \\
I_{2\,\nu}^{(4)}(x) &= \frac{(\nu-4)\,(\nu-5)}{(\nu-1)\,(\nu-2)}\,C_4^{(1)}(x)\,.
\end{aligned}
\tag{B.13}
$$

From (B.11) we obtain

$$
\begin{aligned}
G_2^{(2)}(k) = \frac{G_2^{(0)}(k)}{(kz_h)^8}\,\frac{2}{15^2}\,(\nu+2)^2\,(\nu-3)\,(\nu+1)\,\nu^2\,(\nu-1)^2\,(\nu-2)^2\,\Bigg(1 + \frac{\nu-4}{\nu-2}\,C_2^{(1)}(\omega/k) \\
+ \frac{(\nu-4)\,(\nu-5)}{(\nu-1)\,(\nu-2)}\,C_4^{(1)}(\omega/k)\Bigg)\,.
\end{aligned}
\tag{B.14}
$$

For large ν we have

$$
\begin{aligned}
\left.\frac{(kz_h)^8}{G_2^{(0)}(k)}\,G_2^{(2)}(k)\right|_{\nu\to\infty} &= \frac{2}{15^2}\,\nu^{10}\,\left(1 + C_2^{(1)}(\omega/k) + C_4^{(1)}(\omega/k)\right) \\
&= \frac{2}{15^2}\,\nu^{10}\,\left(4\,\left(\frac{\omega}{k}\right)^2 - 1\right)^2\,.
\end{aligned}
\tag{B.15}
$$

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