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CHARACTERIZATIONS OF ITALIAN GRAPHS AND SICILIAN GRAPHS *, **

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Abstract. In this work we deal with three variants of domination in graphs, these are Italian domination (or Roman {2}-domination), {2}-domination and 2-domination. We define Sicilian graphs as those graphs for which the Italian domination and the {2}-domination numbers coincide. Sicilian graphs constitute a superclass of Italian graphs (introduced by Klostermeyer et al. in 2019). First, we give a characterization of Italian graphs in terms of the existence of a special Roman {2}-dominating function. Then, we focus on web graphs for which their {2}-domination number was recently found (Cheng et al., 2020), and we study Sicilian web graphs. We explore also Sicilian co-bipartite graphs. As a by-product, we find the 2-domination number for web graphs and co-bipartite graphs. Finally, we show necessary conditions for non-Italian graphs to be Sicilian as well as characterize Sicilian graphs within some relevant graph classes such as quasi-threshold graphs and cographs.

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1. Introduction and preliminaries

All graphs in this paper are undirected and simple. Let G be a graph, and let $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. Whenever it is clear from the context, we simply write V and E and denote $G = (V, E)$. For basic definitions not included here, we refer the reader to [2]. Two vertices of V are *adjacent* if there is an edge of E between them. The *neighborhood* of $v \in V$, $N(v)$, denotes the set of all adjacent vertices to v, and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. A *clique* in G is a subset of V that induces a complete subgraph of G. A vertex $u \in V$ is *universal* in G if $N[u] = V$.

A graph where the vertices can be divided into two disjoint sets such that all edges connect a vertex in one set to a vertex in the other set is called a *bipartite graph*.

For a subset $S \subseteq V$ and an integer-valued function f defined on V, we denote $f(S) = \sum_{v \in S} f(v)$.

A subset D of V is a *dominating set* in G if every vertex in $V \setminus D$ is adjacent to at least one vertex in D, i.e. $\bigcup_{v \in D} N[v] = V$. The minimum size among all dominating sets in G is called the *domination number* of G and denoted by $\gamma(G)$. A subset D of V is a *2-dominating set* in G if every vertex in $V \setminus D$ has at least two adjacent vertices in D [11]. The minimum size among all 2-dominating sets in G is called the *2-domination number* of G and denoted by $\gamma_2(G)$.

Italian domination (also called Roman $\{2\}$-domination) was introduced in 2016 by Chellali et al. [3] as a variant of 2-domination. Given a graph G with vertex set V, a *Roman $\{2\}$-dominating function* (R2DF) $f : V \to \{0, 1, 2\}$ has the property that for every vertex $v \in V$ with $f(v) = 0$, either there exists a vertex $u \in N(v)$ with $f(u) = 2$, or at least two distinct vertices $x, y \in N(v)$ with $f(x) = f(y) = 1$. The *weight* of an R2DF is the value $f(V)$. The minimum weight among all R2DFs of G is called the *Italian domination (or Roman $\{2\}$-domination) number* of G and denoted by $\gamma_I(G)$. The problem of deciding if, given a graph G and an integer number α, G has an R2DF of weight at most α, is NP-complete [3, 4, 9, 15, 16].

Baghirova et al. [1] proved recently that many problems, including Italian domination, are linear-time solvable on graph classes that have bounded clique-width. In particular, the decision problem associated with Italian domination is a 2-stable 1-locally checkable problem with a check function that can be computed in constant time. Nevertheless, this result is mainly of theoretical interest and does not lead to practical algorithms.

In this work we focus on the one hand, on two graph classes which have bounded clique-width —thus, a linear-time algorithm for Italian domination exists from [1]— and on the other hand, on two graphs classes with unbounded clique-width, thus, we do not even know from scratch the existence of an efficient algorithm for them. We find closed formulas for the Italian domination numbers in both cases.

On another direction, in [3] two independent inequality chains relating the corresponding parameters are presented for comparing Italian domination with the classical dominations. These are: $\gamma(G) \leq \gamma_I(G) \leq 2\gamma(G)$ and $\gamma(G) \leq \gamma_I(G) \leq \gamma_2(G)$, for every graph G. In [12], a characterization of trees T for which $\gamma_I(T) = 2\gamma(T)$

is given. A general graph G is called an *Italian graph* if $\gamma_I(G) = 2\gamma(G)$ [13]. With the purpose of exploring Italian graphs, in [13] a graph G is called an *I1a graph* if the range of some minimum weight R2DF of G is the set $\{0, 1\}$. The classes of Italian graphs and I1a graphs are not comparable. Stars are Italian graphs that are not I1a. The path on four vertices P_4 is an I1a graph that is not Italian. Complete graphs are Italian graphs that are I1a as well. All paths are I1a but the only ones that are Italian are P_2, P_3 and P_6 [13]. We notice that all cycles are I1a but the only one that is Italian is C_3. I1a graphs are characterized as those graphs G for which $\gamma_I(G) = \gamma_2(G)$ [13].

As mentioned in the work by Chellali et al. [3], Roman $\{2\}$-dominating functions are also closely related to $\{2\}$-dominating functions [8]. For a function f to be a $\{2\}$-dominating function, even for vertices $v \in V$ with $f(v) = 1$ it is asked for the existence of a vertex u adjacent to v with $f(u) \neq 0$. Formally, for a function $f : V \to \{0, 1, 2\}$ to be a $\{2\}$*-dominating function* ($\{2\}$DF), the property that $f(N[v]) \geq 2$ must hold for every vertex $v \in V$ (and not only for those v with $f(v) = 0$). From their definitions it is clear that $\gamma_I(G) \leq \gamma_{\{2\}}(G)$. The minimum weight of a $\{2\}$DF in G is called the $\{2\}$*-domination number* of G and denoted by $\gamma_{\{2\}}(G)$. The problem of deciding if, given a graph G and an integer α, G has a $\{2\}$DF of weight at most α, is NP-complete [11]. By taking any dominating set of G and defining a function that assigns 2 to its elements and 0 to the remaining vertices of G, we build a $\{2\}$DF of G and thus $\gamma_{\{2\}}(G) \leq 2\gamma(G)$. Thus, the following inequality chain holds for all graphs G: $\gamma(G) \leq \gamma_I(G) \leq \gamma_{\{2\}}(G) \leq 2\gamma(G)$.

In Section 2, we give a characterization of Italian graphs in terms of the existence of an R2DF with range $\{0, 2\}$. Then, we introduce Sicilian graphs as those graphs G such that $\gamma_I(G) = \gamma_{\{2\}}(G)$, which turn out to be a superclass of Italian graphs. With the aim of exploring Sicilian graphs, we deal with web graphs for which the $\{2\}$-domination number is known from a recent work [5]. In particular, for any web graph we find the Italian domination number and prove that they are all I1a graphs, which allows us to find their 2-domination number as well. We find some Italian web graphs as well as non-Italian web graphs that are Sicilian. In Section 3 we deal with co-bipartite graphs and find the Italian, the $\{2\}$-domination and the 2-domination numbers for them. In Section 4 we find necessary conditions for non-Italian graphs to be Sicilian, as well as completely characterize Sicilian graphs within some relevant graph classes such as quasi-threshold graphs and cographs.

2. Italian graphs and Sicilian graphs

Recall that G is an *Italian graph* if $\gamma_I(G) = 2\gamma(G)$ [13]. Clearly, the smallest non-Italian graph is the trivial graph with one vertex (K_1), since $\gamma_I(K_1) = 1$ and $\gamma(K_1) = 1$, and this clearly implies that being an Italian graph is not a property inherited by its induced subgraphs. Although the characterization in Proposition 2.4 below is not a structural one, it describes Italian graphs by the existence of a special R2DF.

We first characterize Italian graphs with Italian domination number equal to 2.

Proposition 2.1. *Let $G = (V, E)$ be a connected graph with $|V| = n \geq 2$. Then $\gamma_I(G) = 2$ if and only if G has a universal vertex, or two non-adjacent vertices of degree $n-2$.*

Proof. First suppose that $\gamma_I(G) = 2$. If $n = 2$, then $G = P_2$ since G is connected and then it has a universal vertex (in fact two). Consider $n \geq 3$ and let f be an R2DF of G with $f(V) = 2$. Clearly, since f assumes values 0, 1 or 2 and $f(V) = 2$, there exists a vertex $v \in V$ with $f(v) = 0$. Since f is an R2DF of G, there either exists $u \in N(v)$ with $f(u) = 2$ (and then $f(v) = 0$ for all $v \in V \setminus \{u\}$), or else there exist two distinct vertices $x, y \in N(v)$ such that $f(x) = f(y) = 1$ (and then $f(v) = 0$ for all $v \in V \setminus \{x, y\}$). In the first case, G has a vertex of degree $n-1$ (i.e. a universal vertex) and in the second, G has two non-adjacent vertices of degree $n-2$ or one universal vertex (in fact two).

For the converse, if G has a vertex u of degree $n-1$ (i.e. u is a universal vertex), then the function f such that $f(u) = 2$ and $f(v) = 0$ for all $v \in V \setminus \{u\}$ is an R2DF of G, implying $\gamma_I(G) = 2$ since $n \geq 2$. If there exist two distinct and non-adjacent vertices $x, y \in V$ of degree $n-2$, then the function f such that $f(x) = f(y) = 1$ and $f(v) = 0$ for all $v \in V \setminus \{x, y\}$ is an R2DF of G with weight $f(V) = 2$, implying $\gamma_I(G) = 2$ since $n \geq 2$. $\square$

From the proof of Proposition 2.1 we derive:

Corollary 2.2. *Every graph with a universal vertex is Italian and no graph without universal vertices but with two non-adjacent vertices of degree $n-2$ is Italian.*

Also from the proof of Proposition 2.1, we notice that every graph with a universal vertex has a minimum weight R2DF with range $\{0, 2\}$ and no graph without universal vertices but with two non-adjacent vertices of degree $n-2$ has such a minimum weight R2DF.

Concerning paths P_n with $n \geq 1$, we know that $\gamma_I(P_n) = \lceil \frac{n+1}{2} \rceil$ and $\gamma(P_n) = \lceil \frac{n}{3} \rceil$ [3]. Klostermeyer et al. noticed that the only ones that are Italian are P_2, P_3 and P_6 [13]. We observe that for these three paths, there exists a minimum weight R2DF with range $\{0, 2\}$, and that the range of every minimum weight R2DF for P_n with $n \neq 2, 3, 6$ is not the set $\{0, 2\}$ (see Figure 1 for some examples).

Concerning cycles C_n with $n \geq 3$, we know that $\gamma_I(C_n) = \lceil \frac{n}{2} \rceil$ and $\gamma(C_n) = \lceil \frac{n}{3} \rceil$ [3]. We notice that the only cycle that is Italian is C_3 and for it, there exists a minimum weight R2DF with range $\{0, 2\}$, and that the range of every minimum weight R2DF for a cycle C_n with $n \geq 4$ is not the set $\{0, 2\}$ (see Figure 2 for some examples).

Let us then introduce:

Definition 2.3. A graph G is called an *I2a graph* if the range of some minimum weight R2DF of G is the set $\{0, 2\}$.

We can prove that the existence of a minimum weight R2DF with range $\{0, 2\}$ is a necessary and sufficient condition for any graph to be Italian.

Proposition 2.4. *A graph is Italian if and only if it is an I2a graph.*

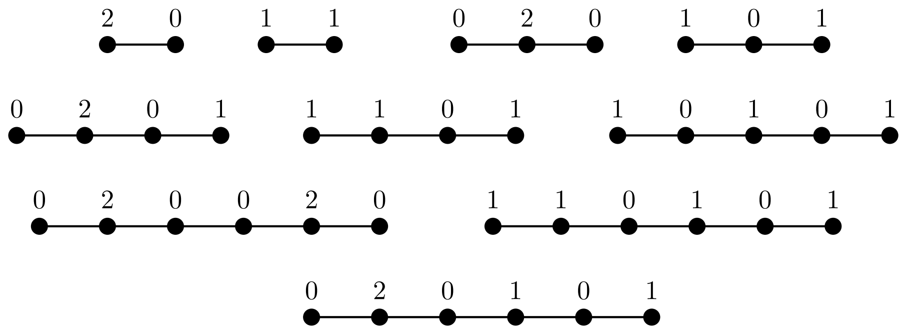

Figure 1. All minimum R2DFs for paths P_n with $2 \leq n \leq 6$

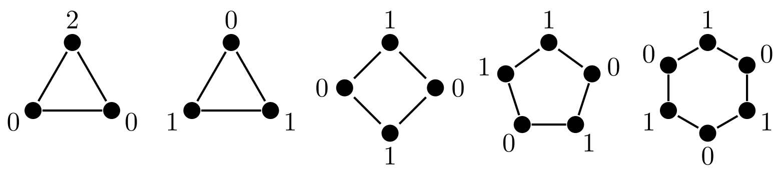

Figure 2. All minimum R2DFs for cycles C_n with $3 \leq n \leq 6$

Proof. Let G be an Italian graph, then by definition $\gamma_I(G) = 2\gamma(G)$. Let D be a minimum size dominating set of G. Let $f : V \to \{0, 2\}$ be the function such that $f(v) = 2$ for all $v \in D$ and $f(u) = 0$ for all $u \in V \setminus D$. Since D is a dominating set of G, f is an R2DF of G. Also, since $\gamma_I(G) = 2\gamma(G)$, f is a minimum weight R2DF of G. Therefore G is an I2a graph.

For the converse, let G be an I2a graph. Let f be a minimum weight R2DF of G with range $\{0, 2\}$. Let $D = \{v \in V : f(v) = 2\}$. Since f is an R2DF of G, D is a dominating set of G. Also, since f is of minimum weight, then D is of minimum size. Therefore, $\gamma_I(G) = 2\gamma(G)$, that is, G is Italian. □

Recall that $\gamma_I(G) \leq \gamma_{\{2\}}(G) \leq 2\gamma(G)$ for every graph G. We introduce the following definition:

Definition 2.5. A graph G is called a *Sicilian graph* if $\gamma_I(G) = \gamma_{\{2\}}(G)$.

Clearly, Sicilian graphs constitute a superclass of Italian graphs, since if a graph G is Italian then the first and second inequalities above Definition 2.5 become equalities, i.e. $\gamma_I(G) = \gamma_{\{2\}}(G) = 2\gamma(G)$.

Example 2.6. For paths P_n with $n \geq 1$, we also know that $\gamma_{\{2\}}(P_n) = 2\lceil \frac{n}{3} \rceil$ [14]. The only Sicilian paths are those three that are Italian (P_2, P_3 and P_6). For cycles C_n with $n \geq 3$, we also know that $\gamma_{\{2\}}(C_n) = \lceil \frac{2n}{3} \rceil$ [14]. We now notice that the only Sicilian cycle is the one that is Italian (C_3).

In other words, there are neither paths nor cycles that are Sicilian but not Italian.

Nevertheless when exploring a closely related graph class, the class of web graphs which are circulant graphs that generalize cycles and complete graphs —in fact, web graphs are precisely defined as power of cycles— we found many Sicilian graphs that are not Italian. Many domination type problems have been widely studied on web graphs in the last years (see for example [5, 7]).

It is known that web graphs have unbounded clique-width, thus obtaining the exact value of the domination parameters for them is even more interesting, provided that it is not possible to apply the results from [1].

Given $n, m \in \mathbb{Z}^+$ with $m \geq 1$ and $n \geq 2m+1$, a *web* graph denoted by W_n^m is a graph where $V(W_n^m) = \{v_0, \ldots, v_{n-1}\}$ and $v_i v_j \in E(W_n^m)$ if and only if $j \equiv i \pm l$ $(mod\, n)$, $l \in \{1, \ldots, m\}$ [17]. It is known that $\gamma(W_n^m) = \lceil \frac{n}{2m+1} \rceil$ [7].

Taking advantage of a recent work [5], the $\{2\}$-domination number of web graphs is known: for a web graph W_n^m, $\gamma_{\{2\}}(W_n^m) = \lceil \frac{2n}{2m+1} \rceil$. However, neither 2-domination nor Italian domination have been studied for web graphs yet.

First we can prove:

Proposition 2.7. *For a web graph W_n^m, $\gamma_I(W_n^m) = \left\lceil \frac{n}{m+1} \right\rceil$.*

Proof. We know from [3] that if G is a connected graph with n vertices and maximum degree $\Delta(G) = \Delta$, then $\gamma_I(G) \geq \frac{2n}{\Delta+2}$. In the case of a web graph W_n^m with $m \geq 1$ and $n \geq 2m+1$, $\Delta(W_n^m) = 2m$ and this lower bound for $\gamma_I(W_n^m)$ becomes $\gamma_I(W_n^m) \geq \frac{2n}{2m+2} = \frac{n}{m+1}$. Since $\gamma_I(W_n^m)$ is an integer number, we have in fact $\gamma_I(W_n^m) \geq \left\lceil \frac{n}{m+1} \right\rceil$.

We will define a function on $V(W_n^m)$ with weight $\left\lceil \frac{n}{m+1} \right\rceil$ and prove that it is an R2DF of W_n^m, and the thesis will follow.

If $v_0, v_1, \ldots, v_{n-1}$ are the vertices of W_n^m and v_i is adjacent to $v_{i\pm l}$, $l \in \{1, \ldots, m\}$ —where sums are taken modulus n— for each $i = 0, \ldots, n-1$, we define the function f on $V(W_n^m)$ such that $f(v_{k(m+1)}) = 1$ for $k = 0, \ldots, \lceil \frac{n}{m+1} \rceil - 1$ and $f(v) = 0$ for the remaining vertices of W_n^m, and prove that it is an R2DF of W_n^m. More precisely, if v_j is a vertex such that $f(v_j) = 0$, let us analyze the two possible cases:

- $k_0(m+1) < j < (k_0+1)(m+1)$ for some $k_0 = 0, \ldots, \lceil \frac{n}{m+1} \rceil - 2$;
- $(\lceil \frac{n}{m+1} \rceil - 1)(m+1) < j \leq n-1$.

In the first case, $v_{k_0(m+1)}$ and $v_{(k_0+1)(m+1)}$ are both adjacent to v_j and $f(v_{k_0(m+1)}) = f(v_{(k_0+1)(m+1)}) = 1$; in the second, $v_{(\lceil \frac{n}{m+1} \rceil - 1)(m+1)}$ and v_0 are both adjacent to v_j, and $f(v_{(\lceil \frac{n}{m+1} \rceil - 1)(m+1)}) = f(v_0) = 1$. It only remains to notice that f defined in this way is an R2DF of W_n^m with weight $\left\lceil \frac{n}{m+1} \right\rceil$. □

Hopefully, we can also derive the 2-domination number of web graphs from [13]:

Corollary 2.8. *For any web graph W_n^m, $\gamma_2(W_n^m) = \lceil \frac{n}{m+1} \rceil$.*

Proof. From the proof of Proposition 2.7 we deduce that W_n^m is an I1a graph for each n and m. Recalling that I1a graphs G are characterized as those for which $\gamma_I(G) = \gamma_2(G)$ [13], it turns out that $\gamma_2(W_n^m) = \gamma_I(W_n^m)$ and the result follows. $\square$

When $m = 1$, W_n^m is isomorphic to a cycle on n vertices for all n. We have already noticed that the only Italian (and thus Sicilian) web graph of the form W_n^1 is C_3. We ask ourselves if there is some Sicilian web graph W_n^m for m greater than 1. For instance, we found that all Sicilian web graphs for the cases $m = 2$ to $m = 4$ are W_5^2, W_7^2, W_{10}^2, W_7^3, W_9^3, W_{10}^3, W_{13}^3, W_{14}^3, W_{17}^3, W_{21}^3, W_9^4, W_{11}^4, W_{12}^4, W_{13}^4, W_{16}^4, W_{17}^4, W_{18}^4. W_{21}^4, W_{22}^4, W_{26}^4, W_{27}^4, W_{31}^4 and W_{36}^4. It can be checked that for $m = 2$, W_7^2 is the only Sicilian web graph that is not Italian. And so are W_9^3, W_{10}^3 and W_{17}^3, for $m = 3$.

3. Italian domination and 2-domination in co-bipartite graphs

In this section we study the Roman {2}-domination and the 2-domination numbers for another graph class that has unbounded clique-width, the class of co-bipartite graphs. Once more, obtaining the exact value of the domination parameters for this other graph class is even more interesting, provided that it is not possible to apply the results from [1].

A graph $G = (V, E)$ is called *co-bipartite* if it is the complement of a bipartite graph. Observe that the vertex set V of a co-bipartite graph G can be partitioned into three sets C_1, C_2 and U, where C_1 and C_2 are non-empty cliques, and U (possibly the empty set) consists of all universal vertices in G.

Theorem 3.1. *Let $G = (V = C_1 \cup C_2 \cup U, E)$ be a connected co-bipartite graph with $|V| = n$, where C_1 and C_2 are non-empty cliques and U is the set of universal vertices in G.*

(1) If $U \neq \emptyset$, or $U = \emptyset$ but there exist two non-adjacent vertices $u \in C_1$ and $w \in C_2$ both of degree $n - 2$, then $\gamma_I(G) = 2$.
(2) If $U = \emptyset$ and there not exist two non-adjacent vertices $u \in C_1$ and $w \in C_2$ both of degree $n - 2$, but there exist distinct vertices $u, w \in C_i$ and $x \in C_j$, with $i, j \in \{1, 2\}$ and $i \neq j$, and such that for all $v \in C_j \setminus \{x\}$ it happens $N[v] \cap \{u, w\} \neq \emptyset$, then $\gamma_I(G) = 3$.
(3) Otherwise, $\gamma_I(G) = 4$.

Proof. (1) Follows from Proposition 2.1.
(2) Let $u, w \in C_i$ be two distinct vertices and $x \in C_j$ $(i \neq j)$ such that for all $v \in C_j \setminus \{x\}$ it happens $N[v] \cap \{u, w\} \neq \emptyset$ (see an example in Figure 3). Notice that Proposition 2.1 implies that $\gamma_I(G) \geq 3$. Let f be the function defined on G by $f(u) = f(w) = f(x) = 1$ and $f(v) = 0$ for all $v \in V \setminus \{u, w, x\}$. It is clear that f is an R2DF of G with $f(V) = 3$. Therefore $\gamma_I(G) = 3$.

(3) Note that if $|C_i| = 1$ for some $i \in \{1, 2\}$, since G is a connected graph and $U = \emptyset$, the only vertex $u \in C_i$ must be adjacent to some $v \in C_j$ with $j \in \{1, 2\}$ and $j \neq i$. But then v is a universal vertex resulting in a contradiction. So we consider from now on that $|C_i| \geq 2$ for each $i \in \{1, 2\}$.

We know from Proposition 2.1 that $\gamma_I(G) \geq 3$. Suppose that $\gamma_I(G) = 3$. Then there exists an R2DF f of G such that $f(V) = 3$. If there were three distinct vertices $u, w, x \in V$ such that $f(u) = f(w) = f(x) = 1$, we have two possibilities:

- If $u, w \in C_i$ and $x \in C_j$, with $i, j \in \{1, 2\}$ and $i \neq j$, then for every vertex v in $C_j \setminus \{x\}$ results $f(v) = 0$ and, since f is an R2DF of G, v would be adjacent to one u or w, arriving to a contradiction since we supposed that this did not happen.
- Now suppose without loss of generality that $u, w, x \in C_1$ and choose $y \in C_2$. Then the function g defined on V such that $g(u) = g(w) = g(y) = 1$ and $g(v) = 0$ for all $v \in V \setminus \{u, w, y\}$ is also an R2DF of G arriving again easily to a contradiction.

If there were two distinct vertices $u, x \in V$ such that $f(u) = 2$ and $f(x) = 1$, then every other vertex would be adjacent to u. But x cannot be adjacent to u, otherwise u would be a universal vertex. Then $u \in C_i$ and $x \in C_j$ with $i, j \in \{1, 2\}$ and $i \neq j$. Observe that since u has degree $n-2$, x does not have degree $n-2$ and therefore there exists $w \in C_i$ distinct from u that is not adjacent to x. But then, for all $v \in C_j \setminus \{x\}, N[v] \cap \{u, w\} \neq \emptyset$, arriving again to a contradiction since we supposed that this did not happen. Therefore $\gamma_I(G) \geq 4$.

Now, let $u, w \in C_1$, $x, y \in C_2$ and let f be the function defined on V such that $f(u) = f(w) = f(x) = f(y) = 1$ and $f(v) = 0$ for all $v \in V \setminus \{u, w, x, y\}$. It is clear that f is a R2DF of G. Therefore $\gamma_I(G) = 4$. □

From Theorem 3.1, we observe the following. For a connected co-bipartite graph G, if $U \neq \emptyset$, then $\gamma_I(G) = 2\gamma(G) = 2$ and then G is Italian (thus also Sicilian). If G satisfies the hypothesis of the first part of Theorem 3.1, item 2, we have $\gamma_{\{2\}}(G) = 3$ (assign 1 to u, w, z with $z \neq u, w$), then G is not Sicilian neither Italian. If G satisfies the hypothesis of the second part of Theorem 3.1, item 2, since $\gamma_I(G) = 3$ is odd, then G is not Italian. If $N[x] \cap \{u, w\} \neq \emptyset$ (see Figure 3 for an example), then $\gamma_{\{2\}}(G) = 3$ (assign 1 to u, w and x), and therefore G is Sicilian. But if $N[x] \cap \{u, w\} = \emptyset$, then $\gamma_{\{2\}}(G) = 4$ and G is not Sicilian. If G satisfies the hypothesis of Theorem 3.1, item 3, $\gamma_I(G) = 2\gamma(G) = 4$ and then G is Italian (thus also Sicilian).

Now, let us turn our attention to 2-domination. We can prove:

Theorem 3.2. *Let $G = (V = C_1 \cup C_2 \cup U, E)$ be a connected co-bipartite graph.*

(1) If $U = \emptyset$ then $\gamma_2(G) = \gamma_I(G)$.

(2) If $|U| = 1$ and there exist $u \in C_1$, $w \in C_2$ with $N[u] = V \setminus \{w\}$ and $N[w] = V \setminus \{u\}$, then $\gamma_2(G) = 2$. Otherwise, $\gamma_2(G) = 3$.

(3) If $|U| \geq 2$ *then* $\gamma_2(G) = 2$.

Proof. (1) If $U = \emptyset$, then from the proof of Theorem 3.1, G is I1a and therefore $\gamma_2(G) = \gamma_I(G)$ (see an example in Figure 3).

(2) If $|U| = 1$ and there exist $u \in C_1$, $w \in C_2$ with $N[u] = V \setminus \{w\}$ and $N[w] = V \setminus \{u\}$, then $\gamma_2(G) = 2$ since $\gamma_2(G) \geq 2$ from its definition and $D = \{u, w\}$ is a 2-dominating set in G.

Now suppose $|U| = 1$ and that there do not exist $u \in C_1$, $w \in C_2$ such that $N[u] = V \setminus \{w\}$ and $N[w] = V \setminus \{u\}$. To prove that $\gamma_2(G) \neq 2$, suppose that there exists a 2-dominating set $D = \{x, y\}$ of G. If one of the elements of D is a universal vertex, then the other one is also a universal vertex, arriving to a contradiction since $|U| = 1$. If x and y belong both to the same clique, then they are both in fact universal vertices, again a contradiction. Finally, if they belong to different cliques, then we must have $N[x] = V \setminus \{y\}$ and $N[y] = V \setminus \{x\}$, arriving to another contradiction. We conclude that $\gamma_2(G) \geq 3$. The set D consisting of one vertex from each clique together with the universal vertex is a 2-dominating set of G, therefore $\gamma_2(G) = 3$.

(3) Suppose $|U| \geq 2$. Let $x, y \in U$ with $x \neq y$. Then $D = \{x, y\}$ is a 2-dominating set of G and therefore $\gamma_2(G) = 2$ since $\gamma_2(G) \geq 2$ from its definition.

□

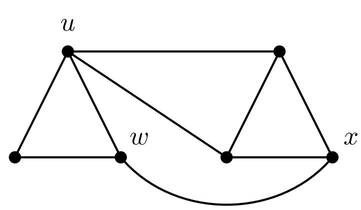

FIGURE 3. A Sicilian non-Italian and I1a co-bipartite graph

4. Characterizations of Sicilian graphs

In this section we analyse two graph classes with bounded clique-width, these are quasi-threshold graphs and cographs. As said in the introduction, we are certain that a linear-time algorithm for the problem of finding a R2DF of minimum weight exists for both of them. Nevertheless, since they are not practically implementable, we find for them the exact values of the domination type problems studied in this work, in addition to characterize Sicilian graphs within these graph classes.

Let us start by showing two examples of Sicilian non-Italian graphs. Notice that the first one (see Figure 4) is the well-known graph S_3 or trampoline and it has a minimum weight R2DF with range $\{0, 1\}$. The second example (see Figure 5) shows a graph G that is also Sicilian but not Italian. Nevertheless, G does not have an R2DF with range $\{0, 1\}$; in fact it can be proved that every minimum weight R2DF of G assigns 2 to the vertex adjacent to both pendant vertices.

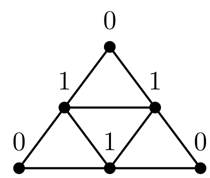

FIGURE 4. S_3 is Sicilian and non-Italian: $\gamma_I(S_3) = \gamma_{\{2\}}(S_3) = 3$ and $2\gamma(S_3) = 4$

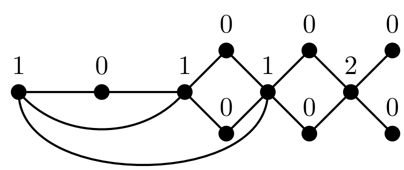

FIGURE 5. A Sicilian non-Italian nor I1a graph G: $\gamma_I(G) = \gamma_{\{2\}}(G) = 5$ and $2\gamma(G) = 6$

We next prove a necessary condition for certain Sicilian non-Italian graphs:

Proposition 4.1. *If G is a Sicilian graph whose minimum weight R2DFs have all range $\{0, 1, 2\}$ then $\gamma_I(G) \geq 5$.*

Proof. From Proposition 2.4, since the range of every minimum weight R2DF of G is $\{0, 1, 2\}$, G is not Italian and $\gamma_I(G) \geq 3$.

Suppose that $\gamma_I(G) = 3$. Since G is Sicilian, $\gamma_{\{2\}}(G) = 3$. Considering that every $\{2\}$DF of G is an R2DF of G, every minimum weight $\{2\}$DF of G must have range $\{0, 1, 2\}$. Let f be a minimum weight $\{2\}$DF of G. Then there exist $u, w \in V$ such that $f(u) = 2$, $f(w) = 1$ and $f(v) = 0$ for all $v \in V \setminus \{u, w\}$. But for f to be a $\{2\}$DF of G, it must happen that every vertex in G is adjacent to u, implying that u is a universal vertex, and this contradicts the fact that $\gamma_{\{2\}}(G) = 3$.

Now suppose $\gamma_I(G) = 4$. Since G is Sicilian, $\gamma_{\{2\}}(G) = 4$. Since every $\{2\}$DF of G is an R2DF of G, then every minimum weight $\{2\}$DF of G has range $\{0, 1, 2\}$. Let f be a minimum weight $\{2\}$DF of G. There exist u, w, $x \in V$ such that $f(u) = 2$, $f(w) = f(x) = 1$ and $f(v) = 0$ for all $v \in V \setminus \{u, w, x\}$.

For f to be a $\{2\}$DF, every vertex apart from u, w and x must be adjacent to u or to both, w and x.

If w and x are adjacent, then the function g defined on V such that $g(u) = g(w) = 2$ and $g(v) = 0$ for all $v \in V \setminus \{u, w\}$ is a minimum weight $\{2\}$DF, contradicting that every minimum weight $\{2\}$DF has range $\{0,1,2\}$. Therefore w and x are non-adjacent.

Since w and x are non-adjacent, they must be adjacent to u, otherwise f would not be a $\{2\}$DF. But again the function g defined on V such that $g(u) = g(w) = 2$ and $g(v) = 0$ for all $v \in V \setminus \{u, w\}$ is an optimal $\{2\}$DF, contradicting that every minimum weight $\{2\}$DF has range $\{0,1,2\}$.

The contradiction came from assuming that $\gamma_I(G) = 4$. Therefore, $\gamma_I(G) \geq 5$.

□

Next, we wonder if we are capable of identifying which are the smallest connected graphs that are not Sicilian. Clearly, the smallest one is the trivial graph with one vertex (K_1), since $\gamma_I(K_1) = 1$ and $\gamma_{\{2\}}(K_1) = 2$, and this clearly implies that being a Sicilian graph is not a property inherited by its induced subgraphs. Concerning connected graphs with two or three vertices — P_2, P_3 and C_3—, we have that they are all Sicilian (see Example 2.6). Concerning the six connected graphs with four vertices, it can be checked that P_4 and C_4 are the only ones that are not Sicilians (see Example 2.6); the remaining are the complete bipartite graph $K_{1,3}$, the paw, the diamond and the complete graph K_4 which are all Sicilian.

Despite the fact that being a Sicilian graph is not a hereditary property to its induced subgraphs, we wonder if having no induced P_4 nor C_4 is a sufficient condition for a graph to be Sicilian. In the remainder, we analyze two graph classes defined by the existence of no induced P_4 and/or C_4.

4.1. Quasi-threshold graphs

Graphs with no induced P_4 nor C_4 are precisely quasi-threshold graphs. More formally, G is a *quasi-threshold graph* if and only if G is (P_4,C_4)-free [18].

Equivalently, a graph is quasi-threshold if it can be constructed recursively as follows [18]: K_1 is a quasi-threshold graph, adding a new vertex adjacent to all vertices of a quasi-threshold graph results in a quasi-threshold graph, and the disjoint union of two quasi-threshold graphs results in a quasi-threshold graph.

Proposition 4.2. *Let G be a quasi-threshold graph. Then $\gamma_I(G) = 2$ when G is connected and, when it is not, $\gamma_I(G) = 2r + s$, where r is the number of connected components of order equal or larger than 2 and s, the number of connected components of order 1.*

Proof. The proof follows by considering in every connected component G_i of G, its recursive construction as mentioned above (given that G_i is a quasi-threshold graph). We notice that the last operation performed in each G_i is the addition of a dominating vertex u_i, when G_i is not K_1. When for some i $G_i = K_1$, then $\gamma_I(G_i) = 1$; otherwise, $\gamma_I(G_i) = 2$ (there exists an optimal R2DF f of G_i with $f(u_i) = 2$ and $f(v) = 0$ for every $v \in V(G_i) \setminus \{u_i\}$). In all, both items follow.

□

The above proposition enables one to characterize quasi-threshold graphs that are Sicilian. In fact, every quasi-threshold graph G with no isolated vertices is

Italian since $\gamma(G) = r$, where r is the number of connected components of G (of order equal or larger than 2).

Theorem 4.3. *Every quasi-threshold graph G with no isolated vertices is an Italian graph and therefore it is also Sicilian.*

Corollary 4.4. *Let G be a quasi-threshold graph with no isolated vertices. Then $\gamma_{\{2\}}(G) = 2r$, where r is the number of connected components of G.*

4.2. Cographs

We now ask ourselves if having no induced P_4 remains a sufficient condition for a given graph to be Sicilian.

Graphs with no induced P_4 (P_4-free) are precisely cographs [6]. Equivalently, G is a cograph if and only if G can be constructed from isolated vertices by disjoint union and join operations (the *join* of two graphs G and H is the graph formed from disjoint copies of G and H by connecting each vertex of G to each vertex of H). Connected cographs have Italian domination number at most 4 [10].

Next we give a characterization of Sicilian cographs. We first prove:

Proposition 4.5. *Let G be a connected cograph.*

1. *If $\gamma_I(G) = 1$ then $G = K_1$, which is not a Sicilian graph;*
2. *If $\gamma_I(G) = 2$ then:*
   - (a) *If G has a universal vertex, then G is an Italian graph.*
   - (b) *If G has no universal vertex, then G is not a Sicilian graph.*
3. *If $\gamma_I(G) = 3$ then G is a Sicilian non-Italian graph.*
4. *If $\gamma_I(G) = 4$ then G is an Italian graph.*

Proof. (1) It is straighforward.

(2) If $\gamma_I(G) = 2$, from Proposition 2.1 G has a universal vertex, or two non-adjacent vertices of degree $n-2$. Clearly, if G has a universal vertex then G is an Italian (and Sicilian) graph. If G has no universal vertex then $\gamma_{\{2\}}(G) \geq 3$ and therefore G is not a Sicilian graph.

(3) Since $\gamma_I(G) = 3$, which is odd, clearly G is not Italian. Let G_1 and G_2 be two induced subgraphs of G —which are also cographs— such that G is the join of G_1 and G_2. We analyze the only two possible cases:

- *Case 1:* there exists an R2DF f of G that assigns a strictly positive weight to exactly two vertices of G. Suppose that $f(v) = 1$, $f(w) = 2$ with $v \in V(G_1)$ and $w \in V(G_2)$, and $f(z) = 0$ for all $z \in V(G) \setminus \{v, w\}$. Then w is a universal vertex in G, which leads to a contradiction since $\gamma_I(G) > 2$.

  Suppose that $f(v) = 1$, $f(w) = 2$ with $v, w \in V(G_1)$. Then every vertex $x \in (V(G_1) \setminus \{v, w\}) \cup V(G_2)$ is adjacent to w. Define a function g on $V(G)$ in the following way: $g(v) = g(w) = 1$, $g(z) = 1$ for some $z \in V(G_2)$ and $g(x) = 0$ otherwise. Since f is an R2DF of G, it is easy to verify that g is a $\{2\}$DF of G. Moreover, since

$\gamma_{\{2\}}(G) \geq \gamma_I(G) = 3$, it holds that $\gamma_{\{2\}}(G) = 3$ and therefore G is a Sicilian graph.

- *Case 2:* there exists an R2DF f of G that assigns a strictly positive weight to exactly three vertices of G. Firstly and without loss of generality, suppose that $v, w, x \in V(G_1)$ where $f(v) = f(w) = f(x) = 1$ and $f(y) = 0$ for all $y \in V(G) \setminus \{v, w, x\}$. Define a function g on V in the following way: $g(v) = g(w) = 1$, $g(z) = 1$ for some $z \in V(G_2)$ and $g(y) = 0$ for all $y \in V(G) \setminus \{v, w, z\}$. Since f is an R2DF of G, g defined in this way is a $\{2\}$DF of G. Moreover, since $\gamma_{\{2\}}(G) \geq \gamma_I(G) = 3$, it holds that $\gamma_{\{2\}}(G) = 3$ and therefore G is a Sicilian graph. Secondly and without loss of generality, suppose that $v, w \in V(G_1)$, $x \in V(G_2)$ where $f(v) = f(w) = f(x) = 1$ and $f(z) = 0$ for all $z \in V(G) \setminus \{v, w, x\}$. In this case, the function f is also a $\{2\}$DF. Moreover, since $\gamma_{\{2\}}(G) \geq \gamma_I(G) = 3$, it holds that $\gamma_{\{2\}}(G) = 3$ and therefore G is a Sicilian graph.

(4) Again, let G_1 and G_2 be two induced subgraphs of G such that G is the join of G_1 and G_2. The set $D = \{v, w\}$ with $v \in V(G_1)$, $w \in V(G_2)$ is a dominating set of G. Then $\gamma(G) = 2$ since G has no universal vertex (otherwise it would be $\gamma_I(G) = 2$), resulting G an Italian (and thus also Sicilian) graph.

□

In all we have:

Theorem 4.6. *Let G be a connected cograph. Then:*

(1) G is an Italian graph if and only if G has a universal vertex or $\gamma_I(G) = 4$.
(2) G is a Sicilian non-Italian graph if and only if $\gamma_I(G) = 3$.
(3) G is not a Sicilian (thus neither Italian) graph if and only if $G = K_1$ or G has two non-adjacent vertices of degree $n - 2$.

5. Conclusions

Given an Italian graph G and a minimum R2DF f of G with range $\{0, 2\}$ (which exists following Proposition 2.4), then the minimum dominating set problem on G can be solved easily since $\{v : f(v) = 2\}$ is an optimal solution for it, and clearly $\gamma(G) = \dfrac{f(V)}{2}$. Besides, since f turns out to be a $\{2\}$-dominating function of G as well and G is Sicilian, the value of $\gamma_{\{2\}}(G)$ can be obtained, which is equal to $f(V)$.

For quasi-threshold graphs, although a linear time algorithm for Italian domination was guaranteed from [1], in this work we went a step further finding closed formulas for their Italian domination numbers (Proposition 4.2). Furthermore, we derived in Corollary 4.4, the $\{2\}$-domination number for any given quasi-threshold without isolated vertices. Concerning cographs, from Proposition 4.5 it can be decided in linear time if a given cograph is Sicilian (and in particular Italian), or otherwise not Sicilian.

Finally, it is known that neither web graphs —studied in Section 2— nor cobipartite graphs —studied in Section 3— have bounded clique-width. Thus, having found in this work the closed formulas for the Italian domination and the 2-domination numbers for them is a significant contribution, provided that it was not even known if an efficient algorithm existed for them.

We wish to reach more characterizations of Sicilian graph classes beyond those studied in this work.

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