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Automorphisms of non-singular nilpotent Lie algebras

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Abstract. For a real, non-singular, 2-step nilpotent Lie algebra $\mathfrak{n}$, the group $\mathrm{Aut}(\mathfrak{n})/\mathrm{Aut}_0(\mathfrak{n})$, where $\mathrm{Aut}_0(\mathfrak{n})$ is the group of automorphisms which act trivially on the center, is the direct product of a compact group with the 1-dimensional group of dilations. Maximality of some automorphisms groups of $\mathfrak{n}$ follows and is related to how close is $\mathfrak{n}$ to being of Heisenberg type. For example, at least when the dimension of the center is two, $\dim \mathrm{Aut}(\mathfrak{n})$ is maximal if and only if $\mathfrak{n}$ is of Heisenberg type. The connection with fat distributions is discussed.
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1. Introduction

A 2-step nilpotent real Lie algebra $\mathfrak{n}$ with center $\mathfrak{z}$ is called *non-singular* [E], or said *to satisfy hypothesis (H)* [M], or be the Lie algebra of a *Métivier group* [MS], if $\mathrm{ad}\, x : \mathfrak{n} \to \mathfrak{z}$ is onto for any $x \notin \mathfrak{z}$. Equivalently, the bracket defines a vector-valued antisymmetric form

$$[\ ,\ ] : \mathfrak{v} \times \mathfrak{v} \to \mathfrak{z},$$

$\mathfrak{v} = \mathfrak{n}/\mathfrak{z}$, such that the 2-forms $\lambda([u,v])$ on $\mathfrak{v}$ are non-degenerate for all $\lambda \in \mathfrak{z}^*$, $\lambda \neq 0$. Here we shall call such algebras *fat* for short, since they are the symbols of fat distributions (as opposite to "flat", or integrable, ones [Mo]), which motivate the questions.

Let $m = \dim(\mathfrak{z})$. While for $m = 1$ there is only one fat algebra up to isomorphisms, for $m \geq 2$ there is an uncountable number of isomorphism classes and for $m \geq 3$ they form a wild set.

In this paper we study the size of groups of automorphisms of $\mathfrak{n}$. $\mathrm{Aut}(\mathfrak{n})$ itself is the semidirect product of the group $G(\mathfrak{n})$ of graded automorphisms of $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ with the abelian group $\mathrm{Hom}(\mathfrak{v}, \mathfrak{z})$, times the group of dilations (t, t^2). Hence, we concentrate on $G = G(\mathfrak{n})$.

We prove that there is an exact sequence

$$1 \to G_0 \to G \to \mathrm{O}(m)$$

where G_0 is the subgroup of G of elements that act trivially on the center. In other words, there are positive-definite inner products on $\mathfrak{z}$ which are invariant under all of $\mathrm{Aut}(\mathfrak{n})$.

If a metric g is also given on $\mathfrak{v}$, as in the case of the nilpotentization of a subriemannian structure, we also consider the subgroups K_0, K, of graded automorphisms that leave g invariant, which define a compatible exact sequence

$$1 \to K_0 \to K \to \mathrm{O}(m).$$

Next, we compute the terms in this sequence and the images G/G_0 and K/K_0, proving that the exactness of

$$1 \to \mathrm{Lie}(K_0) \to \mathrm{Lie}(K) \to \mathfrak{so}(m) \to 1$$

is equivalent to $\mathfrak{n}$ being of Heisenberg type, while the exactness of

$$1 \to \mathrm{Lie}(G_0) \to \mathrm{Lie}(G) \to \mathfrak{so}(m) \to 1$$

is strictly more general. As to $G_0(\mathfrak{n})$, we describe it in detail for the case $m = 2$, leading a proof that, at least in that case, $\dim \mathrm{Aut}(\mathfrak{n})$ is maximal if and only if $\mathfrak{n}$ is of Heisenberg type.

In the last section we explain the connection with the Equivalence Problem for fat subriemannian distributions.

Algebras of Heisenberg type are defined as follows [K]. If $\mathfrak{v}$ is a real unitary module over the Clifford algebra $\mathrm{Cl}(\mathfrak{z})$ associated to a quadratic form on $\mathfrak{z}$, the identity

$$< z, [u,v] >_{\mathfrak{z}} = < z \cdot u, v >_{\mathfrak{v}}$$

with $z \in \mathfrak{z} \subset \mathrm{Cl}(\mathfrak{z})$, $u, v \in \mathfrak{v}$, defines a fat $[\ ,\ ] : \mathfrak{v} \times \mathfrak{v} \to \mathfrak{z}$. Alternatively, they are characterized by possessing a positive-definite metric such that the operator $z\cdot$ defined by the above equation satisfies $z \cdot (z \cdot v) = -|z|^2 v$.

It follows from Adam's theorem on frames on spheres [H] that for any fat algebra there is an Heisenberg type algebra with the same $\dim \mathfrak{z}$ and $\dim \mathfrak{v}$. That these were, in some sense, the most symmetric, was expected from the properties of their sublaplacians [BTV] [CGN] [GV] [K], but we found no explicit statements in this regard.

Related properties of the automorphism groups of nilpotent Lie groups are studied in [P] and [MS].

2. Automorphisms of fat algebras

Let $\mathfrak{n}$ be a 2-step Lie algebra with center $\mathfrak{z}$ and let $\mathfrak{v} = \mathfrak{n}/\mathfrak{z}$, so that

$$\mathfrak{n} \cong \mathfrak{v} \oplus \mathfrak{z} \tag{1}$$

and the Lie algebra structure is encoded into the map

$$[\ ,\ ] : \Lambda^2 \mathfrak{v} \to \mathfrak{z}.$$

Let $n = \dim \mathfrak{v}$ and $m = \dim \mathfrak{z}$. Relative to a basis compatible with (1), the bracket becomes an $\mathbb{R}^m$-valued antisymmetric form on $\mathbb{R}^n$ and an automorphism is a matrix of the form

$$\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}, \qquad a \in \mathrm{GL}(n),\ b \in \mathrm{GL}(m),\ c \in \mathbb{R}^{n \times m}$$

such that

$$b([u,v]) = [au, av].$$

$\mathrm{Aut}(\mathfrak{n})$ always contains the normal subgroup $\mathfrak{D}(\mathfrak{n})$ of dilations and translations

$$\begin{pmatrix} tI_n & 0 \\ c & t^2 I_m \end{pmatrix}, \qquad t \in \mathbb{R}^*,\ c \in \mathbb{R}^{n\times m}.$$

Let

$$G = G(\mathfrak{n}) = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},\ a \in \mathrm{SL}(n),\ b \in \mathrm{GL}(m),\ b([u,v]) = [au, av]\}.$$

Then $\mathrm{Aut}(\mathfrak{n})$ is the semidirect product of $G(\mathfrak{n})$ with $\mathfrak{D}(\mathfrak{n})$. Let

$$G_0 = G_0(\mathfrak{n}) = \{\begin{pmatrix} a & 0 \\ 0 & I_m \end{pmatrix},\ a \in \mathrm{SL}(n),\ [au, av] = [u,v]\},$$

the subgroup of automorphisms that act trivially on the center. These are Lie groups, G_0 is normal in G, and the quotient group

$$G/G_0$$

can be identified with the group of $b \in \mathrm{GL}(\mathfrak{z})$ such that $b([u,v]) = [au, av]$ for some $a \in \mathrm{SL}(\mathfrak{v})$. Obviously,

$$\dim \mathrm{Aut}(\mathfrak{n}) = nm + 1 + \dim(G/G_0) + \dim(G_0). \tag{2}$$

Theorem 2.1. *Let $\mathfrak{n}$ be a fat algebra with center $\mathfrak{z}$. Then there is a positive definite metric on $\mathfrak{z}$ invariant under $G(\mathfrak{n})$.*

Proof. Fix arbitrary positive inner products on $\mathfrak{v}$ and $\mathfrak{z}$. For $z \in \mathfrak{z}$, $u, v \in \mathfrak{v}$

$$(T_z u, v)_{\mathfrak{v}} = (z, [u,v])_{\mathfrak{z}}$$

defines a linear map $z \mapsto T_z$ from $\mathfrak{z}$ to $\mathrm{End}(\mathfrak{v})$. Clearly,

$$\mathfrak{n}\ \textit{fat} \ \Leftrightarrow\ T_z \in \mathrm{GL}(\mathfrak{v})\ \forall z \neq 0.$$

Hence the hypothesis insures that the Pfaffian

$$P(z) = \det(T_z)$$

is non-zero on $\mathfrak{z}\setminus\{0\}$. This is a homogeneous polynomial of degree n, so it satisfies

$$k\|z\|^n \le |P(z)| \le K\|z\|^n \tag{3}$$

where k, K are the minimum and maximum values of $|P|$ on the unit sphere, which are positive.

Let now $g_{a,b} := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathrm{Aut}(\mathfrak{n})$. Then

$$T_{b^{\mathfrak{t}} z} = a^{\mathfrak{t}} T_z a$$

because $(T_{b^t z}u, v)_{\mathfrak{v}} = (b^t z, [u,v])_{\mathfrak{z}} = (z, b([u,v]))_{\mathfrak{z}} = (z, [au, av])_{\mathfrak{z}} = (T_z au, av)_{\mathfrak{v}} = (a^t T_z au, v)_{\mathfrak{v}}$. Consequently

$$P(b^t z) = (\det a)^2 P(z).$$

In particular, if $g \in G$ then $P(b^t z) = P(z)$. This implies

$$k\|b^t z\|^n \leq |P(b^t z)| = |P(z)| \leq K\|z\|^n$$

for all z, therefore $\|b\| \leq \sqrt[n]{K/k}$. The group of $b \in \mathrm{GL}(\mathfrak{z})$ such that $g_{a,b} \in \mathrm{Aut}(\mathfrak{n})$ for some $a \in \mathrm{SL}(\mathfrak{v})$, is therefore bounded in $\mathrm{End}(\mathbb{R}^m)$. Its closure is a compact Lie subgroup of $\mathrm{GL}(\mathfrak{z})$, necessarily contained in $\mathrm{O}(\mathfrak{z})$ for some positive definite metric.

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From now on $\mathfrak{z}$ will be assumed endowed with such invariant metric. If a metric g on $\mathfrak{v}$ is also fixed, as in the case of the nilpotentization of a subriemannian structure, define the groups

$$K = K(\mathfrak{n}, g) = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},\ a \in \mathrm{SO}(\mathfrak{v}),\ b \in \mathrm{O}(\mathfrak{z}),\ [au, av] = b[u,v]\}$$

$$K_0 = K_0(\mathfrak{n}, g) = \{\begin{pmatrix} a & 0 \\ 0 & I \end{pmatrix},\ a \in \mathrm{SO}(\mathfrak{v}),\ [au, av] = [u,v]\}.$$

Let $\mathfrak{g}, \mathfrak{g}_o, \mathfrak{k}, \mathfrak{k}_o$ be the Lie algebras of G, G_0, K, K_0 respectively. Then there is the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \to & \mathfrak{g}_0 & \to & \mathfrak{g} & \to & \mathfrak{so}(m) \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \to & \mathfrak{k}_0 & \to & \mathfrak{k} & \to & \mathfrak{so}(m) \end{array}$$

where the vertical arrows are the inclusions. Below we prove that the bottom sequence extends to

$$0 \to \mathfrak{k}_0 \to \mathfrak{k} \to \mathfrak{so}(m) \to 0$$

if and only if $\mathfrak{n}$ is of Heisenberg type. This is not the case for the top one: the condition that

$$0 \to \mathfrak{g}_0 \to \mathfrak{g} \to \mathfrak{so}(m) \to 0$$

is exact defines a class of fat algebras strictly larger than Heisenberg type algebras. We describe it in the next section for $m = 2$.

Proposition 2.2. *Let $\mathfrak{n} = \mathfrak{v} + \mathfrak{z}$ be an algebra of Heisenberg type. There is a metric on $\mathfrak{z}$ such that $\mathfrak{g}/\mathfrak{g}_o \cong \mathfrak{so}(m)$.*

Proof. There is an inner product in $\mathfrak{v}$ such that the $J_i = T_i$'s satisfy the Canonical Anticommutation Relations

$$J_w J_z + J_z J_w = -2 < z, w > I.$$

For $\|z\| = 1$ let $r_z \in \mathrm{O}(\mathfrak{z})$ be the reflection through the hyperplane orthogonal to z and $J_z \in \mathrm{SL}(\mathfrak{v})$ be as above. Then

$$g_{(J_z,-r_z)} = \begin{pmatrix} J_z & 0 \\ 0 & -r_z \end{pmatrix} \in \mathrm{Aut}(\mathfrak{n}).$$

Indeed,

$$\begin{aligned}(w, [J_z u, J_z v]) &= (J_w J_z u, J_z v) = (-J_z J_w u - 2(z,w)u, J_z v) \\ &= -(J_z J_w u, J_z v) - 2(z,w)(u, J_z v) = (J_w u, J_z J_z v) + 2(z,w)(J_z u, v) \\ &= -(J_w u, v) + 2(z,w)(J_z u, v) = (J_{-w+2(z,w)z} u, v) \\ &= (-w + 2(z,w)z, [u,v]) = (-r_z(w), [u,v]) \\ &= (w, -r_z([u,v])),\end{aligned}$$

so that

$$-r_z([u,v]) = [J_z u, J_z v].$$

The Lie group generated by the $-r_z$ has finite index in $\mathrm{O}(\mathfrak{z})$. ■

Corollary 2.3. *Let $\mathfrak{n}$ be a fat algebra with center of dimension m. Then*

$$\dim(K/K_0) \le \dim(G/G_0) \le m(m-1)/2$$

with equality achieved for any Heisenberg type algebra of the same dimension with center of the same dimension.

Since $\mathrm{Aut}(\mathfrak{n})/\mathrm{Aut}_0(\mathfrak{n}) = (G/G_0) \times (\text{dilations})$, one obtains

Corollary 2.4. *Let $\mathfrak{n}$ be a fat algebra with center of dimension m. Then*

$$\dim(\mathrm{Aut}(\mathfrak{n})/\mathrm{Aut}_0(\mathfrak{n})) \le 1 + m(m-1)/2,$$

with equality achieved for any Heisenberg type algebra of the same dimension and with center of the same dimension.

A converse for Corollary 2.3 is

Theorem 2.5. *If $\mathfrak{n}$ is fat with center of dimension m and*

$$\dim(K/K_0) = m(m-1)/2$$

for some metric on $\mathfrak{v}$, then $\mathfrak{n}$ is of Heisenberg type.

Proof. The hypothesis implies that $\mathfrak{k}/\mathfrak{k}_\mathfrak{o} = \mathfrak{g}/\mathfrak{g}_\mathfrak{o} \cong \mathfrak{so}(m)$, so that K/K_0 acts transitively among the $|z| = 1$. For $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ in this group, $-T_{bz} = aT_z a^{-1}$, hence $T_{bz}^2 = aT_z^2 a^{-1}$. Since T_z is invertible, we can choose the metric such that $T_{z_0}^2 = -I$ for any given z_0. Therefore $T_z^2 = -I$ for all $|z| = 1$, which implies the assertion. ■

Maximal dimension means there are isomorphisms

$$\mathrm{Lie}(K/K_0) = \mathrm{Lie}(G/G_0) \cong \mathfrak{so}(m).$$

Therefore the simply connected covers are isomorphic: $\mathrm{Spin}(m) \cong \widetilde{(G/G_0)}_e$. The induced homomorphism

$$\mathrm{Spin}(m) \to (G/G_0)_e$$

may or may not extend to a homomorphism

$$\mathrm{Pin}(m) \to G/G_0.$$

If it does extend, it may or may not be injective, in which case it is an isomorphism. Therefore, among the algebras for which $\dim(G/G_0)$ is maximal, those for which $\mathrm{Pin}(m) \cong G/G_0$ can be regarded as the most symmetric.

Theorem 2.6. *Suppose $\mathfrak{n}$ is a 2-step graded algebra such that $\mathrm{Aut}(\mathfrak{n})$ contains a copy of $\mathrm{Pin}(m)$ inducing the standard action on $\mathfrak{z}$. Then $\mathfrak{n}$ is of Heisenberg type.*

Proof. The assumption implies that there is a linear map $\mathfrak{z} \to \mathrm{End}(\mathfrak{v})$, denoted by $z \mapsto J_z$ such that $J_z^2 = -|z|^2 I$ for all z and

$$[J_z u, J_z v] = r_z([u, v])$$

for $u, v \in \mathfrak{v}$, $z \in \mathfrak{z}$, $|z| = 1$, where r_z is the reflection in $\mathfrak{z}$ with respect of the line spanned by z. $\mathrm{Pin}(m)$ is the group generated by the J_z's with $\|z\| = 1$, which acts linearly on $\mathfrak{v}$ and is compact. Fix a metric on $\mathfrak{v}$ invariant under it.

We get, as in the proof of Theorem 2.1, that if $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathrm{Aut}(\mathfrak{n})$, then

$$T_{b^{\mathfrak{t}} z} = a^{\mathfrak{t}} T_z a.$$

In particular:

$$T_{r_x(z)} = J_x T_z J_x.$$

If $x = z$, we get $T_z = -J_z T_z J_z$, thus $T_z J_z = -J_z^{-1} T_z = J_z T_z$. If $x \perp z$, we get $T_z = J_x T_z J_x$, thus $T_z J_x = J_x^{-1} T_z = -J_x T_z$. It follows that T_z^2 commutes with J_z and with J_w, $w \perp z$.

Now, let $z \in \mathfrak{z}$ and $w \perp z$. Let $R_w(t)$ the $2t$-rotation from z towards w. Then $R_w(t) = r_z r_{w(t)}$, with $w(t) = \cos(t)z + \sin(t)w$. It follows that

$$\begin{pmatrix} J_z J_{w(t)} & 0 \\ 0 & R_w(t) \end{pmatrix}$$

is an orthogonal automorphism and, therefore, satisfies

$$T_{R_w(t)z} = (J_z J_{w(t)})^{\mathfrak{t}} T_z (J_z J_{w(t)}).$$

Since $(J_z J_{w(t)})^{\mathfrak{t}} = (J_z J_{w(t)})^{-1}$,

$$T^2_{R_w(t)z} = (J_z J_{w(t)})^{\mathfrak{t}} T^2_z (J_z J_{w(t)}) = J_{w(t)} J_z T^2_z J_z J_{w(t)}.$$

Since T^2_z commutes with J_z and J_w,

$$T^2_{R_w(t)z} = T^2_z J_{w(t)} J_z J_z J_{w(t)} = -T^2_z J_{w(t)} J_{w(t)}. \tag{4}$$

But $J^2_{w(t)} = -I$, so that (4) implies that

$$T^2_{R_w(t)z} = T^2_z.$$

For all $z' \in \mathfrak{z}$ we can choose $w \in \mathfrak{z}, t \in \mathbb{R}$ such that $R_w(t)z = z'$, so we get

$$T^2_{z'} = T^2_z, \qquad \text{for all } z' \in \mathfrak{z}, |z'| = 1.$$

The antisymmetry of the bracket implies that T_z is skew-symmetric. Rescaling the scalar product on $\mathfrak{v}$ we obtain that $T^2_z = -I$, so $T^2_{z'} = -I$ for all $z' \in \mathfrak{z}$, $|z'| = 1$. Therefore $\mathfrak{n}$ is of Heisenberg type.

∎

3. The case of center of dimension 2

In this section we compute the groups $G, G_0, G/G_0$ in the case $m = 2$. The various types are parametrized by pairs

$$(\mathbf{c}, \mathbf{r}) \in (\mathbb{U}^\ell / \operatorname{SL}(2, \mathbb{R})) \times \mathbb{Z}^\ell_+$$

where $\mathbb{U}$ is the upper-half plane and $2\ell = 2\sum r_j = \dim \mathfrak{n} - 2$. As a corollary we conclude that $\operatorname{Aut}(\mathfrak{n})$ is maximal if and only if $\mathfrak{n}$ is of Heisenberg type. These are complex Heisenberg algebras of various dimensions regarded as real Lie algebras.

First we recall the normal form for fat algebras with $m = 2$ deduced from [LT]. Given $c = a + bi \in \mathbb{C}$, let

$$Z(c) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

If $r \in \mathbb{Z}_+$, set

$$A(c, r) = \begin{pmatrix} Z(c) & & & \\ I_2 & Z(c) & & \\ & & \ddots & \\ & & I_2 & Z(c) \end{pmatrix}$$

a $2r \times 2r$-matrix. If $\mathbf{c} = (c_1, ..., c_\ell) \in \mathbb{C}^\ell$ and $\mathbf{r} = (r_1, ..., r_\ell) \in \mathbb{N}_+^\ell$, set

$$A(\mathbf{c}, \mathbf{r}) = \begin{pmatrix} A(c_1, r_1) & & & \\ & A(c_2, r_2) & & \\ & & \ddots & \\ & & & A(c_\ell, r_\ell) \end{pmatrix}$$

which is a $2s \times 2s$ matrix, $s = r_1 + ... + r_\ell$.

Let now $\phi, \psi_{(\mathbf{c},\mathbf{r})}$ be the 2-forms on $\mathbb{R}^{4s}$ whose matrices in the standard basis are

$$[\phi] = \begin{pmatrix} 0 & -I_{2s} \\ I_{2s} & 0 \end{pmatrix} \qquad [\psi_{(\mathbf{c},\mathbf{r})}] = \begin{pmatrix} 0 & A(\mathbf{c}, \mathbf{r}) \\ -A^{\mathfrak{t}}(\mathbf{c}, \mathbf{r}) & 0 \end{pmatrix}. \tag{5}$$

Then

$$[u, v]_{(\mathbf{c},\mathbf{r})} = (\phi(u, v), \psi_{(\mathbf{c},\mathbf{r})}(u, v)) =< u, [\phi]v > e_1 + < u, [\psi_{(\mathbf{c},\mathbf{r})}]v > e_2$$

is an $\mathbb{R}^2$-valued antisymmetric 2-form on $\mathbb{R}^{4s}$. Let

$$\mathfrak{n}_{(\mathbf{c},\mathbf{r})} = \mathbb{R}^{4s} \oplus \mathbb{R}^2$$

be the corresponding Lie algebra.

Define $M_{(\mathbf{c},\mathbf{r})} \in \mathrm{End}(\mathfrak{v})$ by

$$\phi(M_{(\mathbf{c},\mathbf{r})}u, v) = \psi_{(\mathbf{c},\mathbf{r})}(u, v),$$

whose matrix is

$$[M_{(\mathbf{c},\mathbf{r})}] = \begin{pmatrix} -A^{\mathfrak{t}}_{(\mathbf{c},\mathbf{r})} & 0 \\ 0 & -A_{(\mathbf{c},\mathbf{r})} \end{pmatrix}.$$

then we have

$$[u, v]_{(\mathbf{c},\mathbf{r})} = \phi(u, v)e_1 + \phi(M_{(\mathbf{c},\mathbf{r})}u, v)e_2, \text{ for } u, v \in \mathbb{R}^{4s}. \tag{6}$$

One can deduce [LT]

Proposition 3.1.

(a) Every fat algebra with center of dimension 2 is isomorphic to some $\mathfrak{n}_{(\mathbf{c},\mathbf{r})}$ *with* $\mathbf{c} \in \mathbb{U}^\ell$.

(b) Two of these are isomorphic if and only if the $\mathbf{r}$*'s coincide up to permutations and the* $\mathbf{c}$*'s differ by some Möbius transformation acting componentwise.*

(c) $\mathfrak{n}_{(\mathbf{c},\mathbf{r})}$ *is of Heisenberg type if and only if* $\mathbf{c} = (c, \dots, c)$ *and* $\mathbf{r} = (1, \dots, 1)$

Let now

$$\mathfrak{n} = \mathfrak{n}_{(\mathbf{c},\mathbf{r})}$$

be fat and let $G = G(\mathfrak{n})$, etc. We denote $\hat{\mathfrak{n}}$ the algebra obtained by replacing the matrices $A(c, r)$ by their semisimple parts and setting all $c_j = \sqrt{-1}$. The resulting $\hat{A}(c, r)$ consists of blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ along the diagonal and $\hat{\mathfrak{n}}$ is isomorphic to the Heisenberg type algebra $\mathfrak{n}_{((i,...,i),(1,...,1))}$. The correspondence

$$\mathfrak{n} \mapsto \hat{\mathfrak{n}}$$

is functorial and seems extendable inductively to fat algebras of any dimension, although here we will maintain the assumption $m = 2$.

Lemma 3.2. *$G_0(\mathfrak{n}) \subset G_0(\hat{\mathfrak{n}})$ and $\dim \mathrm{Aut}(\mathfrak{n}) \le \dim \mathrm{Aut}(\hat{\mathfrak{n}})$.*

Proof. Let ϕ, ψ, $M_{(\mathbf{c},\mathbf{r})} \in \mathrm{End}(\mathfrak{v})$ be as above, so that

$$\phi(M_{(\mathbf{c},\mathbf{r})}u, v) = \psi_{(\mathbf{c},\mathbf{r})}(u, v).$$

By formula (6), $g \in G_0(\mathfrak{n}_{(\mathbf{c},\mathbf{r})})$ if and only if

$$\phi(u, v) = \phi(gu, gv), \qquad \phi(M_{(\mathbf{c},\mathbf{r})}u, v) = \phi(M_{(\mathbf{c},\mathbf{r})}gu, gv) = \phi(g^{-1}M_{(\mathbf{c},\mathbf{r})}gu, v),$$

i.e., if and only if $g \in \mathrm{Sp}(\phi)$ and commutes with $M_{(\mathbf{c},\mathbf{r})}$. In particular it commutes with the semisimple part $\hat{M}_{(\mathbf{c},\mathbf{r})}$. This is conjugate to a matrix having blocks $Z(c) = \begin{pmatrix} \Re(c) & \Im(c) \\ -\Im(c) & \Re(c) \end{pmatrix}$ for various $c \in \mathbb{C}$ along the diagonal, and zeros elsewhere. Every matrix commuting with such a matrix will surely commute with that having all $c = 1$. It follows that g also preserves $\phi(\hat{M}_{(\mathbf{c},\mathbf{r})}u, v)$ and, therefore, it is an automorphism of $\hat{\mathfrak{n}}$ as well. Thus,

$$G_0(\mathfrak{n}) \subset G_0(\hat{\mathfrak{n}}).$$

From Corollary 2.3, $\dim(G(\mathfrak{n})/G_0(\mathfrak{n})) \le \dim(G(\hat{\mathfrak{n}})/G_0(\hat{\mathfrak{n}}))$, and therefore
$\dim G(\mathfrak{n}) = \dim(G(\mathfrak{n})/G_0(\mathfrak{n})) + \dim G_0(\mathfrak{n}) \le \dim(G(\hat{\mathfrak{n}})/G_0(\hat{\mathfrak{n}})) + \dim G_0(\hat{\mathfrak{n}}) = \dim G(\hat{\mathfrak{n}})$.

Formula (2) implies $\dim \mathrm{Aut}(\mathfrak{n}) \le \dim \mathrm{Aut}(\hat{\mathfrak{n}})$, as claimed. ■

Next we will describe $\mathfrak{g}_0(\mathfrak{n}_{(c,r)})$ for $c \in \mathbb{U}$ and $r \in \mathbb{N}_+$, i.e., the case when the matrices A consist of a single block. Since c is $\mathrm{SL}(2,\mathbb{C})-$conjugate to i, it is enough to take $c = i$. Define the 2×2-matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and let $M_r(\mathbb{R}\langle \mathbf{1}, \mathbf{i} \rangle)$ and $M_r(\mathbb{R}\langle \mathbf{x}, \mathbf{y} \rangle)$ denote the real vector spaces of $r \times r$ matrices with coefficients in the span of $\mathbf{1}, \mathbf{i}$ and $\mathbf{x}, \mathbf{y}$ respectively. Then the vector space

$$\mathcal{R}(r) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \; A, D \in M_r(\mathbb{R}\langle \mathbf{1}, \mathbf{i} \rangle), \; B, C \in M_r(\mathbb{R}\langle \mathbf{x}, \mathbf{y} \rangle) \right\},$$

is a actually a matrix algebra.

Note that

$$\mathbf{1}^{\mathfrak{t}} = \mathbf{1}, \quad \mathbf{i}^{\mathfrak{t}} = -\mathbf{i}, \quad \mathbf{x}^{\mathfrak{t}} = \mathbf{x}, \quad \mathbf{y}^{\mathfrak{t}} = \mathbf{y}.$$

Letting $A^{\mathfrak{t}}$ denote the transpose or an $\mathbb{R}$-matrix and A^t, A^* the transpose and conjugate transpose of $\mathbb{R}[\mathbf{i}, \mathbf{x}, \mathbf{y}]$-matrices, one obtains

$$A^{\mathfrak{t}} = A^*$$

for $A \in M_r(\mathbb{R}\langle \mathbf{1}, \mathbf{i} \rangle)$ while

$$A^{\mathfrak{t}} = A^t$$

for $A \in M_r(\mathbb{R}\langle \mathbf{x}, \mathbf{y}\rangle)$.

With the notation

$$J_1 = [\phi] \qquad J_2 = [\psi_{((i,\dots,i),(1,\dots,1))}],$$

$$\mathfrak{g}_0(\hat{\mathfrak{n}}) = \left\{X \in \mathbb{R}^{4r\times 4r} : J_1 X + X^{\mathrm{t}} J_1 = 0, J_2 X + X^{\mathrm{t}} J_2 = 0\right\}.$$

From [S] we know that

$$\mathfrak{g}_0(\hat{\mathfrak{n}}) \cong \mathfrak{sp}(r, \mathbb{C})^{\mathbb{R}}$$

Changing basis,

$$\mathfrak{g}_0(\hat{\mathfrak{n}}) = \{X \in \mathcal{R}(r) : \ J_1 X + X^{\mathrm{t}} J_1 = 0, \ J_2 X + X^{\mathrm{t}} J_2 = 0\}$$

where

$$J_1 = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & \mathbf{i} I_r \\ \mathbf{i} I_r & 0 \end{pmatrix}.$$

This gives an alternative description of this algebra:

$$\mathfrak{g}_0(\hat{\mathfrak{n}}) = \left\{ \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} : \ A \in M_r(\mathbb{R}\langle \mathbf{1}, \mathbf{i}\rangle), \ B, C \in M_r(\mathbb{R}\langle \mathbf{x}, \mathbf{y}\rangle), \ B^t = B, \ C^t = C \right\}$$

We now restrict our attention to matrices $\begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}$ in $\mathfrak{g}_0(\hat{\mathfrak{n}})$ where A, B, C have the respective forms

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_2 \\ 0 & \cdots & 0 & a_1 \end{pmatrix} \qquad \begin{pmatrix} b_1 & \cdots & b_{r-1} & b_r \\ \vdots & ⋰ & ⋰ & 0 \\ b_{r-1} & ⋰ & ⋰ & \vdots \\ b_r & 0 & \cdots & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & \cdots & 0 & c_1 \\ \vdots & ⋰ & ⋰ & c_2 \\ 0 & ⋰ & ⋰ & \vdots \\ c_1 & c_2 & \cdots & c_r \end{pmatrix}$$

with coefficients in $\mathbb{R}^{2\times 2}$. Let $\mathbf{A}_k = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}$ having $a_k = \mathbf{1}$ and zero otherwise and $\mathbf{A}'_k$ the matrix of the same form but with $a_k = \mathbf{i}$ and zeros elsewhere. Similarly, let $\mathbf{B}_k$ (resp. $\mathbf{C}_k$) the matrix $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ (resp., $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$) with b_k (resp. c_k) equal to $\mathbf{x}$ and zeros elsewhere, and $\mathbf{B}'_k$ (resp. $\mathbf{C}'_k$) with b_k (resp. c_k) equal to $\mathbf{y}$ and zeros elsewhere.

Theorem 3.3. *Let $\mathfrak{n} = \mathfrak{n}_{(c,r)}$, $(c, r) \in \mathbb{U} \times \mathbb{N}$, and regard $\mathfrak{g}_0(\mathfrak{n})$ as a subalgebra of $\mathfrak{gl}(\mathfrak{v})$. Then,*

1. *$\mathfrak{g}_0(\mathfrak{n})$ is the $\mathbb{R}$-span of $\mathbf{A}_i, \mathbf{A}'_i, \mathbf{B}_i, \mathbf{B}'_i, \mathbf{C}_i, \mathbf{C}'_i$ for $1 \le i \le r$.*
2. *The semisimple part of $\mathfrak{g}_0(\mathfrak{n})$ is the span of $\mathbf{A}_1, \mathbf{A}'_1, \mathbf{B}_1, \mathbf{B}'_1, \mathbf{C}_1, \mathbf{C}'_1$.*
3. *The solvable radical is the span of $\mathbf{A}_i, \mathbf{A}'_i, \mathbf{B}_i, \mathbf{B}'_i, \mathbf{C}_i, \mathbf{C}'_i$ with $1 < i \le r$.*

In particular, the $\mathbb{R}$-dimension the $\mathfrak{g}_0(\mathfrak{n})$ is equal to $6r$ and the semisimple part of $\mathfrak{g}_0(\mathfrak{n})$ is isomorphic to $\mathfrak{sp}(1, \mathbb{C})$.

Proof. It is enough to consider the case $\mathfrak{n} = \mathfrak{n}_{(i,r)}$. Let $T_2 = [\psi_{(i,r)}]$ and write $T_2 = J_2 + N_2$ where

$$N_2 = \begin{pmatrix} 0 & N \\ -N^t & 0 \end{pmatrix}, \text{ with } N = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 1 & \ddots & 0 & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

From Lemma 3.2, $\mathfrak{g}_0(\mathfrak{n}) = \{X \in \mathfrak{g}_0(\hat{\mathfrak{n}}) : \ T_2X + X^{\mathrm{t}}T_2 = 0\}$. As $\mathfrak{g}_0(\mathfrak{n}) \subset \mathfrak{g}_0(\hat{\mathfrak{n}})$ one obtains

$$\mathfrak{g}_0(\mathfrak{n}) = \{X \in \mathfrak{g}_0(\hat{\mathfrak{n}}) : N_2X + X^{\mathrm{t}}N_2 = 0\}.$$

The conditions on $\begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \in \mathfrak{g}_0(\mathfrak{n})$ are, explicitly,

$$0 = NC - C^tN^t = NC - (NC)^t \tag{7}$$
$$0 = N^tA - AN^t \tag{8}$$
$$0 = N^tB - B^tN = N^tB - (N^tB)^t. \tag{9}$$

For the first equation, note that NC symmetric if and only if $c_{i,j+1} = c_{j,i+1}$ and $c_{1,j} = 0$ for $i, j < n$. Since C is symmetric, $c_{i,j+1} = c_{j,i+1} = c_{i+1,j}$ and $c_{1,j} = 0$ for $i, j < n$. We conclude:

If $i + j = k \leq r$, $c_{i,j} = c_{i,k-i} = c_{i-1,k-i+1} = c_{i-2,k-i+2} \cdots = c_{1,k-1} = 0$

If $i + j = k > r$, $c_{i,j} = c_{i,k-i} = c_{i+1,k-i-1} = c_{i+2,k-i-2} \cdots = c_{r,k-i+i-r} = c_{r,k-r}$

Thus, the strict upper antidiagonals are zero and each lower antidiagonal have all its elements equal.

For the second equation, note that N^t and A commute. This is equivalent to $c_{i,j} = c_{t,s}$ when $j - i = s - t$ and $c_{i,1} = 0$ for $i > 1$. The first condition implies that each diagonal have all its elements equal, while the second implies that the strict lower diagonals are zero.

Equation (9) is analogous to equation (7): the condition N^tB symmetric is equivalent to each antidiagonal have all its elements equal and that the strict lower antidiagonals are 0.

From all this we conclude that the span of $\mathbf{A}_i, \mathbf{A}'_i, \mathbf{B}_i, \mathbf{B}'_i, \mathbf{C}_i, \mathbf{C}'_i$ with $1 \leq i \leq r$ is $\mathfrak{g}_0(\mathfrak{n})$ and (1) follows.

(2) and (3) follow from (1) and the explicit presentation of the matrices $\mathbf{A}_i, \mathbf{A}'_i, \mathbf{B}_i, \mathbf{B}'_i, \mathbf{C}_i, \mathbf{C}'_i$.

■

Corollary 3.4. *(of the proof) Let* $\mathfrak{n}$ *be fat. Then* $\dim(\mathfrak{g}_0(\mathfrak{n}))$ *is maximal if and only if* $\mathfrak{n}$ *is of Heisenberg type.*

Proof. Let $(\mathbf{c}, \mathbf{r}) = ((c_1, \ldots, c_l), (r_1, \ldots, r_l))$ be such that $\mathfrak{n} = \mathfrak{n}_{(\mathbf{c},\mathbf{r})}$. We know that $\mathfrak{g}_0(\mathfrak{n}) \subset \mathfrak{g}_0(\hat{\mathfrak{n}})$. If $c_i \neq c_j$ for some i, j, then there is not intertwining operator between the blocks corresponding to these invariants, so $\mathfrak{g}_0(\mathfrak{n}) \neq \mathfrak{g}_0(\hat{\mathfrak{n}})$.

When $c_1 = c_2 = \cdots = c_l$ we can consider $c_j = i$ for all j. Let $r = \sum r_i$. In this case if $\begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \in \mathfrak{g}_0(\mathfrak{n})$ must be satisfy the equations (7), (8), (9) but with N such that coefficients $n_{j+1,j}$ are 0 or $\mathbf{1}$. Suppose now that $\mathfrak{g}_0(\mathfrak{n})$ is not of Heisenberg type, then some $n_{j+1,j}$ is equal to $\mathbf{1}$. We assume that $n_{21} = \mathbf{1}$ and let $A \in M_r(\mathbb{R}\langle \mathbf{1}, \mathbf{i} \rangle)$ such that $a_{12} = \mathbf{1}$ and 0 otherwise, then

$$X = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}$$

belongs to $\mathfrak{g}_0(\hat{\mathfrak{n}})$ but is not in $\mathfrak{g}_0(\mathfrak{n})$.

■

It can be shown in general that the semisimple part of $\mathfrak{g}_0(\mathfrak{n})$ is isomorphic to $\oplus_i \mathfrak{sp}(m_i, \mathbb{C})$, where m_i is the multiplicity of the pair (c_i, r_i) in $(\mathbf{c}, \mathbf{r})$.

In the case $m = 2$, $\mathfrak{g}/\mathfrak{g}_0$ is either 0 or isomorphic to $\mathfrak{so}(2)$.

Theorem 3.5. $\mathfrak{g}(\mathfrak{n})/\mathfrak{g}_0(\mathfrak{n}) \cong \mathfrak{so}(2)$ *if* $c_1 = \cdots = c_\ell$, *and* 0 *otherwise.*

Proof. $\mathfrak{g}/\mathfrak{g}_0$ is a compact subalgebra of $\mathfrak{gl}(2)$, hence of the form $g\mathfrak{so}(2)g^{-1}$ for some $g \in \mathrm{SL}(2, \mathbb{R})$ and it is nonzero if and only if there exists $X \in \mathfrak{sl}(v)$ such that, in the notation of the proof of Theorem 3.3,

$$\begin{pmatrix} X & 0 \\ 0 & g\mathbf{i}g^{-1} \end{pmatrix}$$

is a derivation of $\mathfrak{n}$. For $g = \mathbf{1}$, if T_1, T_2 correspond to the standard basis of $\mathfrak{z}$, the equations for X become

$$\text{(a)} \quad T_1 X + X^{\mathrm{t}} T_1 = T_2, \qquad \text{(b)} \quad T_2 X + X^{\mathrm{t}} T_2 = -T_1$$

In normal form, and for a single block $A_{(i,r)}$,

$$T_1 = J_1 = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & \mathbf{i} I_r + N \\ \mathbf{i} I_r - N^{\mathrm{t}} & 0 \end{pmatrix}.$$

We decompose

$$T_2 = J_2 + N_2, \qquad \text{with} \qquad J_2 = \begin{pmatrix} 0 & \mathbf{i} I_r \\ \mathbf{i} I_r & 0 \end{pmatrix}, \; N_2 = \begin{pmatrix} 0 & N \\ -N^{\mathrm{t}} & 0 \end{pmatrix}$$

and regard J_1, J_2, T_1, T_2, N_2 as matrices with coefficients in $\mathbb{R}^{2\times 2}$. Note that J_1, J_2 correspond to $\hat{\mathfrak{n}}$, of Heisenberg type. Let

$$Y_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & s & 0 \\ 0 & 2\mathbf{i} & 0 & 0 & 0 & & s & 0 \\ 0 & \mathbf{1} & 4\mathbf{i} & 0 & 0 & 0 & s & 0 \\ 0 & 0 & 2\mathbf{1} & 6\mathbf{i} & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 3\mathbf{1} & 8\mathbf{i} & & s & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \vdots & \vdots & 0 & (n-2)\mathbf{1} & 2(n-1)\mathbf{i} \end{pmatrix}.$$

A straightforward calculation shows that

$$X_0 = \begin{pmatrix} -Y_0^{\mathtt{t}} & 0 \\ 0 & -Y_0^{\mathtt{t}} + \mathbf{i} I_r + N \end{pmatrix}$$

is a solution of (a), (b). We conclude that

$$\begin{pmatrix} X_0 & 0 \\ 0 & \mathbf{i} \end{pmatrix}$$

is a derivation of $\mathfrak{n}_{(i,r)}$, which lies in $\mathfrak{g}(\mathfrak{n}_{(i,r)})$ but not in $\mathfrak{g}_0(\mathfrak{n}_{(i,r)})$.

For any $c \in \mathbb{U}$, $\mathfrak{n}_{(c,r)} \cong \mathfrak{n}_{(i,r)}$, hence they have the same $\mathfrak{g}/\mathfrak{g}_0$ up to isomorphisms. In fact, for any $g \in \mathrm{SL}(2,\mathbb{R})$, the algebra $\mathfrak{n}_{(g\cdot i,r)}$ has a derivation of the form

$$\begin{pmatrix} X & 0 \\ 0 & g\mathbf{i}g^{-1} \end{pmatrix}.$$

For a fixed g, these X are unique modulo $\mathfrak{g}_0$ and come in normal form. Clearly, c determines the 2×2 matrix $g\mathbf{i}g^{-1}$ and the complex number $g\cdot i$.

In the case of an arbitrary fat $\mathfrak{n}_{(\mathbf{c},\mathbf{r})}$, each block (c_k, r_k) determines a corresponding X_k such that

$$\begin{pmatrix} X_k & 0 \\ 0 & g_k\mathbf{i}g_k^{-1} \end{pmatrix}$$

is a derivation of $\mathfrak{n}_{(c_k,r_k)}$. If $n_{(\mathbf{c},\mathbf{r})}$ has a derivation in $\mathfrak{g}$ that is not in $\mathfrak{g}_0$, then its must have one which is combination of such, acting on $\mathfrak{v}$ as $X_1 + X_2 + \cdots$. This forces all the $g_k\mathbf{i}g_k^{-1}$ to be the same and all the c_i to be the same. The reciprocal is clear. ■

In particular, all algebras $\mathfrak{n}_{(\mathbf{c},\mathbf{r})}$ with $c_1 = ... = c_\ell$ and $r_i > 1$ maximize the dimension of $\mathfrak{g}/\mathfrak{g}_0$, but they are not Heisenberg type.

Lauret had pointed out to us that there were non Heisenberg type algebras such that $\mathfrak{g}(\mathfrak{n})/\mathfrak{g}_0(\mathfrak{n}) \neq 0$. Independently, Oscari proved that this holds whenever the c_i's all agree.

4. Fat distributions

Let D be a smooth vector distribution on a smooth manifold M, i.e., a subbundle of the tangent bundle $T(M)$. Its nilpotentization, or symbol, is the bundle on M with fiber

$$N^D(M)_p = \bigoplus_j D_p^{(j)}/D_p^{(j-1)}$$

where $D_p^{(1)} = D_p$ and $D_p^{(j+1)} = D_p^{(j)} + [\Gamma(D), \Gamma(D^j)]_p$. The Lie bracket in $\Gamma(T(M))$ induces a graded nilpotent Lie algebra structure on each fiber of $N^D(M)$. If $D^{(j)} = T(M)$ for some j, D is called completely non-integrable. If $D^{(2)} = T(M)$, the nilpotentization is 2-step, which in the notation of the previous section, is

$$\mathfrak{n}_p = N_D(M)_p = D_p \oplus \frac{D_p + [\Gamma(D), \Gamma(D)]_p}{D_p} = \mathfrak{v}_p + \mathfrak{z}_p,$$

It is also easy to see that D is fat in the sense of Weinstein [Mo] if and only if $\mathfrak{n}_p = \mathfrak{v} + \mathfrak{z}$ is non-singular, i.e., fat in the sense defined in the section 1.

A subriemannian metric g defined on D determines a metric on $\mathfrak{v}$. On $\mathfrak{z}$ we put a metric σ invariant under G. Let $\{\phi_1, ..., \phi_m; \psi_1, ..., \psi_n\}$ be a coframe on M such that

$$D = \cap \ker \phi_i,$$

with $\{\phi_1, ..., \phi_m\}$ and $\{\psi_1, ..., \psi_n\}$ orthonormal with respect to $g + \sigma$. Define $T_z \in \mathrm{End}(D)$ as before, by

$$\sigma(z, [u, v]) = g(T_z u, v).$$

Then D is fat if and only if T_z is invertible for all non-zero $z \in \mathfrak{z}$. The structure equations for the coframe can be written

$$d\phi_k \equiv \sum_i (T_k \psi_i) \wedge \psi_i \qquad mod(\phi_\ell)$$

with the T_k's having the property that any non-zero linear combination of them is invertible. This is deduced from the fact that if $u, v \in \mathfrak{v}$, then $d\phi[u, v] = -\phi([u, v])$, since $u(\phi(v)) = u(0) = 0$. The $d\psi$'s are essentially arbitrary.

Let now M be a the simply connected Lie group with a fat Lie algebra $\mathfrak{n}$, D the left-invariant distribution on M such that $D_e = \mathfrak{v}$. For a left-invariant coframe, the structure equations take the form

$$d\phi_k = \sum_i (J_k \psi_i) \wedge \psi_i, \qquad d\psi_i = 0$$

where $J_1, ..., J_m$ are anticommuting complex structures on D.

The results from the previous sections lead to consider fat distributions satisfying

$$d\phi_k = \sum_i (J_k \psi_i) \wedge \psi_i \qquad mod(\phi_\ell) \tag{4.1}$$

where the J_k are sections of $\mathrm{End}(T(M)^*)$ satisfying the Canonical Commutation Relations

$$J_i J_j + J_j J_i = -2\delta_{ij}.$$

The Equivalence Problem for these systems has been discussed for distributions with growth vector $(2n, 2n+1), (4n, 4n+3)$ and $(8, 15)$. In these cases $\mathfrak{n}$ is parabolic, i.e., isomorphic to the Iwasawa subalgebra of a real semisimple Lie algebra $\mathfrak{g}$ of real rank one. The Tanaka [T] subriemannian prolongation of such algebra is $\mathfrak{g}$, while in the non-parabolic case is just

$$\mathfrak{n} + \mathfrak{k}(\mathfrak{n}) + \mathfrak{a}(\mathfrak{n})$$

where $\mathfrak{a}(\mathfrak{n})$ the 1-dimensional Lie algebra of dilations [Su]. In this case,Tanaka's theorem implies that, in the notation of [Z], the first pseudo G-structure P^0 already carries a canonical frame.

As this paper was being written, E. van Erp pointed out to us his article [Er], where fat distributions are called polycontact and those satisfying (4.1) arise by imposing a compatible conformal structure.

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