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An interpolation result for A_1 weights with applications to fractional Poincaré inequalities

Irene Drelichman

Abstract. We characterize the real interpolation space between weighted L^1 and $W^{1,1}$ spaces on arbitrary domains different from $\mathbb{R}^n$, when the weights are positive powers of the distance to the boundary multiplied by an A_1 weight. As an application of this result we obtain weighted fractional Poincaré inequalities with sharp dependence on the fractional parameter s (for s close to 1) and show that they are equivalent to a weighted Poincaré inequality for the gradient.

A_1-painoja koskeva interpolointitulos ja sovelluksia murtoasteisiin Poincarén epäyhtälöihin

Tiivistelmä. Tässä työssä kuvaillaan avaruuden $\mathbb{R}^n$ mielivaltaisen aidon osa-alueen painollisten L^1- ja $W^{1,1}$-avaruuksien väliset reaaliset interpolointiavaruudet, kun tarkasteltavat painot ovat A_1-painolla kerrottuja reunaetäisyyden positiivisia potensseja. Tuloksen sovelluksena saadaan painollisia murtoasteisia Poincarén epäyhtälöitä, joilla on tarkka riippuvuus murtoasteisesta sileydestä s (lähellä arvoa 1), ja osoitetaan, että nämä ovat yhtäpitäviä gradienttia koskevan painollisen Poincarén epäyhtälön kanssa.

1. Introduction

Given a domain $\Omega \subset \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$, we denote by $d(x) = d(x, \partial\Omega)$ the distance from x to the boundary. Let $\alpha, \beta \geq 0$, and ω be a weight in Muckenhoupt's class A_1, that is, such that $M\omega(x) \leq C\omega(x)$ a.e., where M is the Hardy–Littlewood maximal function. We consider the weighted Sobolev space

$$W^{1,1}_\omega(\Omega, d^\alpha, d^\beta) = \{f \in L^1_\omega(\Omega, d^\alpha) \colon \|\nabla f\|_{L^1_\omega(\Omega, d^\beta)} < \infty\}$$

where $\|f\|_{L^1_\omega(\Omega,d^\beta)} = \|f\omega d^\beta\|_{L^1(\Omega)}$.

The first goal of this paper is to show that, for any such domain, and any $\alpha \geq 0$, one has

$$(L^1_\omega(\Omega, d^\alpha), W^{1,1}_\omega(\Omega, d^\alpha, d^{\alpha+1}))_{s,1} = \widetilde{W}^{s,1}_\omega(\Omega, d^\alpha, d^{\alpha+s}) \tag{1.1}$$

with equivalence of norms, where

$$\widetilde{W}^{s,1}_\omega(\Omega, d^\alpha, d^\beta) = \left\{f \in L^1_\omega(\Omega, d^\alpha) \colon |f|_{\widetilde{W}^{s,1}_\omega(\Omega,d^\beta)} < \infty\right\}$$

and

$$|f|_{\widetilde{W}^{s,1}_\omega(\Omega,d^\beta)} = \int_\Omega \int_{|x-y|<\tau d(x)} \frac{|f(x)-f(y)|}{|x-y|^{n+s}}\, dy\, d(x)^\beta \omega(x)\, dx.$$

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This result generalizes the one in [1, Theorem 1.1], which corresponds to the case $\omega \equiv 1$ (notice that that result is written for bounded domains, but the same arguments apply as long as $\Omega \neq \mathbb{R}^n$). The proof of the embedding $\widetilde{W}^{s,1}_\omega(\Omega, d^\alpha, d^{\alpha+s}) \subseteq (L^1_\omega(\Omega, d^\alpha), W^{1,1}_\omega(\Omega, d^\alpha, d^{\alpha+1}))_{s,1}$ follows closely the one in that paper, modifying it to include the A_1 weight. But, because of the presence of the weight, the opposite embedding requires a completely different proof. We borrow some ideas from [11], but we adapt them to our seminorm and to the presence of different powers of the distance to the boundary. We remark that, among other differences, in [11] both the function and its (generalized) gradient belong to the same weighted space, which is not our case.

The characterization in (1.1) is strongly related to the obtention of fractional Poincaré inequalities with sharp dependence on the fractional parameter s, for s close to 1.

Recall that, for a cube Q, $1 \leq p < \frac{1}{s}$, and $\frac{1}{2} \leq s < 1$, it was proved in [3, Theorem 1] that

$$\|f - f_Q\|^p_{L^p(Q)} \lesssim \frac{(1-s)}{(n-sp)^{p-1}} \int_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\, dx\, dy, \tag{1.2}$$

where f_Q stands for the integral average of f over Q. Here, the implicit constant depends on the side-length of Q but, in what follows, we will not be interested in such dependence. Other proofs and extensions of this inequality can be found in [14, 22, 18, 23].

For irregular domains, a more suitable fractional norm was introduced in [15], and it was shown that for any bounded John domain $\Omega \subset \mathbb{R}^n$ (see definition below) and any fixed constant $0 < \tau < 1$,

$$\|f - f_\Omega\|^p_{L^p(\Omega)} \lesssim \int_\Omega \int_{|x-y|<\tau d(x)} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\, dy\, dx. \tag{1.3}$$

Generalizations of this result can be found in [7, 9, 12, 17], but it should be noted that the scaling factor $(1-s)$ in the right-hand side of (1.2) cannot be obtained with any of those proofs. This turns out to be a drawback, since this factor plays a key role in the limiting behavior of the seminorm when $s \to 1^-$, and it relates fractional and classical Poincaré inequalities. Indeed, it was proved in [2] (see also [19]) that, for a bounded extension domain Ω, $1 \leq p < \infty$, and $f \in W^{1,p}(\Omega)$,

$$\lim_{s\to 1^-} (1-s) \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\, dx\, dy = K_{n,p} \|\nabla f\|_{L^p(\Omega)}$$

where $K_{n,p}$ is an explicit constant, so that one can recover from (1.2) the classical Poincaré inequality for the gradient in Q.

For arbitrary bounded domains, the analogous result holds using the restricted fractional seminorm. Namely, it was proved in [8] that, for $f \in W^{1,p}(\Omega)$, $1 < p < \infty$,

$$\lim_{s\to 1^-} (1-s) \int_\Omega \int_{|x-y|<\tau d(x)} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\, dy\, dx = K_{n,p} \|\nabla f\|_{L^p(\Omega)},$$

and this result was extended to $p = 1$ in [20]. This suggests that (1.3) should also hold with the $(1-s)$ factor. The second goal of this paper is to show that this is indeed the case when $p = 1$, in the more general weighted setting. More precisely, we prove that for bounded John domains one has

$$\inf_{c\in\mathbb{R}} \|f-c\|_{L^1_\omega(\Omega,d^\alpha)} \lesssim \frac{(1-s)}{s(n+s)} \int_\Omega \int_{|x-y|<\tau d(x)} \frac{|f(x)-f(y)|}{|x-y|^{n+s}}\, dy\, d(x)^{\alpha+s}\omega(x)\, dx$$

whenever $\omega \in A_1$. This is done by showing that this inequality is equivalent to a weighted Poincaré inequality for the gradient, which is known. The proof of this equivalence uses some ideas from Oscar Domínguez Bonilla, which relate bounds for the K-functional corresponding to (1.1) to the obtention of sharp inequalities, so the author would like to thank him for generously sharing them. It is worth noting that K-functionals have also been recently used in a different way to derive self-improving type inequalities of several classical inequalities in [5].

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2. Notation and preliminary results

As usual, we will write $A \lesssim B$ to mean $A \le CX$ whenever C is a positive constant independent of relevant quantities. Throughout this paper we shall only keep track of the dependence of the constants with respect to the interpolation parameter s, that we will use later in our arguments.

Let $L(\Omega)$ denote the collection of measurable functions $f\colon \Omega \to \mathbb{R}$. In what follows, we will consider the following weighted Lebesgue and Sobolev spaces

$$L^1_\omega(\Omega,d^\alpha) = \{f \in L(\Omega)\colon \|f\|_{L^1_\omega(\Omega,d^\alpha)} = \|f\omega d^\alpha\|_{L^1(\Omega)} < \infty\},$$
$$W^{1,1}_\omega(\Omega,d^\alpha,d^{\alpha+1}) = \{f \in L^1_\omega(\Omega,d^\alpha)\colon \|\nabla f\|_{L^1_\omega(\Omega,d^{\alpha+1})} < \infty\}$$

and their fractional counterparts

$$W^{s,1}_\omega(\Omega,d^\alpha,d^{\alpha+s}) = \{f \in L^1_\omega(\Omega,d^\alpha)\colon |f|_{W^{s,1}_\omega(\Omega,d^{\alpha+s})} < \infty\},$$
$$\widetilde{W}^{s,1}_\omega(\Omega,d^\alpha,d^{\alpha+s}) = \{f \in L^1_\omega(\Omega,d^\alpha)\colon |f|_{\widetilde{W}^{s,1}_\omega(\Omega,d^{\alpha+s})} < \infty\}$$

where

$$|f|_{W^{s,1}_\omega(\Omega,d^{\alpha+s})} = \int_\Omega \int_\Omega \frac{|f(x)-f(y)|}{|x-y|^{n+s}}\, dy\, d(x)^{\alpha+s}\omega(x)\, dx,$$

and

$$|f|_{\widetilde{W}^{s,1}_\omega(\Omega,d^{\alpha+s})} = \int_\Omega \int_{|x-y|<\tau d(x)} \frac{|f(x)-f(y)|}{|x-y|^{n+s}}\, dy\, d(x)^{\alpha+s}\omega(x)\, dx.$$

By definition, for $0<s<1$, the real interpolation space between $L^1_\omega(\Omega,d^\alpha)$ and $W^{1,1}_\omega(\Omega,d^\alpha,d^{\alpha+1})$ is given by

$$(L^1_\omega(\Omega,d^\alpha), W^{1,1}_\omega(\Omega,d^\alpha,d^{\alpha+1}))_{s,1} = \{f \in L^1_\omega(\Omega,d^\alpha)\colon \|f\|_{(L^1_\omega(\Omega,d^\alpha),W^{1,1}_\omega(\Omega,d^\alpha,d^{\alpha+1}))_{s,1}} < \infty\}$$

with

$$\|f\|_{(L^1_\omega(\Omega,d^\alpha),W^{1,1}_\omega(\Omega,d^\alpha,d^{\alpha+1}))_{s,1}} = \int_0^\infty \lambda^{-s} K(f,\lambda)\, \frac{d\lambda}{\lambda} \tag{2.1}$$

and

$$\begin{aligned} K(f,\lambda) = \inf\{&\|g\|_{L^1_\omega(\Omega,d^\alpha)} + \lambda\|h\|_{W^{1,1}_\omega(\Omega,d^\alpha,d^{\alpha+1})}\colon f = g+h,\\ &g \in L^1_\omega(\Omega,d^\alpha),\ h \in W^{1,1}_\omega(\Omega,d^\alpha,d^{\alpha+1})\}. \end{aligned} \tag{2.2}$$

As announced, we will obtain the characterization

$$(L^1_\omega(\Omega,d^\alpha), W^{1,1}_\omega(\Omega,d^\alpha,d^{\alpha+1}))_{s,1} = \widetilde{W}^{s,1}_\omega(\Omega,d^\alpha,d^{\alpha+s})$$

with equivalence of norms. The norm of the latter space is also equivalent to that of

$$W^{s,1}_\omega(\Omega, d^\alpha, \delta^{\alpha+s}) = \{f \in L^1_\omega(\Omega, d^\alpha) \colon |f|_{W^{s,1}_\omega(\Omega,\delta^{\alpha+s})} < \infty\}$$

where $\delta(x,y) = \min\{d(x), d(y)\}$ and

$$|f|_{W^{s,1}_\omega(\Omega,\delta^{\alpha+s})} = \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{|x-y|^{n+s}} \, \delta(x,y)^{\alpha+s} \, dy \, \omega(x) \, dx.$$

The proof of this result is contained in the following lemma. Observe that it implies, in particular, that the norms of the spaces $\widetilde{W}^{s,1}_\omega(\Omega, d^\alpha, d^{\alpha+s})$ for different values of $0 < \tau < 1$ are all equivalent.

Lemma 2.1. *Let Ω be a domain, $\Omega \neq \mathbb{R}^n$, $0 < s < 1$, and $\alpha \geq 0$. Then,*

$$\widetilde{W}^{s,1}_\omega(\Omega, d^\alpha, d^{\alpha+s}) = W^{s,1}_\omega(\Omega, d^\alpha, \delta^{\alpha+s})$$

with equivalent norms.

Proof. Fix $0 < \tau < 1$. Observe that, whenever $|x - y| < \tau d(x)$, one has $d(x) \sim d(y)$ and, therefore,

$$\begin{aligned}
&\int_\Omega \int_{|x-y|<\tau d(x)} \frac{|f(x) - f(y)|}{|x-y|^{n+s}} \, dy \, d(x)^{\alpha+s} \omega(x) \, dx \\
&\lesssim \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{|x-y|^{n+s}} \delta(x,y)^{\alpha+s} \, dy \, \omega(x) \, dx.
\end{aligned}$$

For the other inequality, we have

$$\begin{aligned}
&\int_\Omega \int_{|x-y|\geq\tau d(x)} \frac{|f(x) - f(y)|}{|x-y|^{n+s}} \delta(x,y)^{\alpha+s} \, dy \, \omega(x) \, dx \\
&\lesssim \int_\Omega \int_{|x-y|\geq\tau d(x)} \frac{|f(x)| + |f(y)|}{|x-y|^{n+s}} \delta(x,y)^{\alpha+s} \, dy \, \omega(x) \, dx.
\end{aligned}$$

Now,

$$\begin{aligned}
&\int_\Omega \int_{|x-y|\geq\tau d(x)} \frac{|f(x)|}{|x-y|^{n+s}} \delta(x,y)^{\alpha+s} \, dy \, \omega(x) \, dx \\
&\lesssim \int_\Omega \left(\int_{|x-y|\geq\tau d(x)} \frac{1}{|x-y|^{n+s}} \, dy \right) |f(x)| d(x)^{\alpha+s} \omega(x) \, dx \\
&\lesssim \|f\|_{L^1_\omega(\Omega, d^\alpha)}.
\end{aligned}$$

And, since $|x-y| \geq \tau d(x) \Rightarrow d(y) \leq |x-y| + d(x) \leq (1 + \frac{1}{\tau})|x-y|$, by Fubini and [13, Lemma (b)]

$$\begin{aligned}
&\int_\Omega \int_{|x-y|\geq\tau d(x)} \frac{|f(y)|}{|x-y|^{n+s}} \delta(x,y)^{\alpha+s} \, dy \, \omega(x) \, dx \\
&\lesssim \int_\Omega \left(\int_{|x-y|\geq\frac{\tau}{1+\tau} d(y)} \frac{\omega(x)}{|x-y|^{n+s}} \, dx \right) |f(y)| d(y)^{\alpha+s} \, dy \\
&\lesssim \int_\Omega M\omega(y) \, |f(y)| \, d(y)^\alpha \, dy \lesssim \|f\|_{L^1_\omega(\Omega, d^\alpha)}.
\end{aligned}$$

□

Although our interpolation result holds for arbitrary domains, we will then apply it to domains where a certain weighted Poincaré inequality holds (see Theorem 4.1). A full characterization of domains supporting such inequalities is still missing, but they are known to hold in bounded John domains. Moreover, under the additional assumption of a *separation property*, this is exactly the larger class where they hold (see [16, Theorem 2.1]), so we recall their definition below.

Definition 2.1. A bounded domain $\Omega \subset \mathbb{R}^n$ is a John domain if for a fixed $x_0 \in \Omega$ and any $y \in \Omega$ there exists a rectifiable curve given by $\gamma(\cdot, y)\colon [0,1] \to \Omega$ such that $\gamma(0,y) = y$ and $\gamma(1,y) = x_0$, and there exist constants δ and K, depending only on the domain Ω and on x_0, such that $d(\gamma(s,y)) \geq \delta s$ and $|\frac{\partial \gamma}{\partial s}(s,y)| \leq K$.

3. Proof of our main theorem

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $\Omega \neq \mathbb{R}^n$, $0 < s < 1$, $\alpha \geq 0$, and $\omega \in A_1$. Then*

$$(L^1_\omega(\Omega, d^\alpha), W^{1,1}_\omega(\Omega, d^\alpha, d^{\alpha+1}))_{s,1} = \widetilde{W}^{s,1}_\omega(\Omega, d^\alpha, d^{\alpha+s}) = W^{s,1}_\omega(\Omega, d^\alpha, \delta^{\alpha+s})$$

with equivalence of norms.

Proof. The proof follows by Lemma 2.1 and the following two lemmas. □

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $\Omega \neq \mathbb{R}^n$, $0 < s < 1$, $\alpha \geq 0$, and $\omega \in A_1$. Then*

$$(L^1_\omega(\Omega), W^{1,1}_\omega(\Omega, d^\alpha, d^{\alpha+1}))_{s,1} \subseteq \widetilde{W}^{s,1}_\omega(\Omega, d^{\alpha+s}).$$

Proof. By Lemma 2.1 we may take $\tau = \frac{1}{16}$ and, rewriting the seminorm in a similar fashion as in [11, Theorem 5.2], we have

$$\begin{aligned}
&\int_\Omega \int_{|x-y|<\frac{d(x)}{16}} \frac{|f(y)-f(x)|}{|x-y|^{n+s}}\, dy\, d(x)^{\alpha+s}\omega(x)\, dx \\
&= \int_\Omega \sum_{i=4}^\infty \int_{B(x,\frac{d(x)}{2^i})\setminus B(x,\frac{d(x)}{2^{i+1}})} \frac{|f(y)-f(x)|}{|x-y|^{n+s}}\, dy\, d(x)^{\alpha+s}\omega(x)\, dx \\
&\lesssim \int_\Omega \sum_{i=4}^\infty \Big(\frac{d(x)}{2^i}\Big)^{-(n+s)} \int_{B(x,\frac{d(x)}{2^i})\setminus B(x,\frac{d(x)}{2^{i+1}})} |f(y)-f(x)|\, dy\, d(x)^{\alpha+s}\omega(x)\, dx \\
&\lesssim \int_\Omega \sum_{i=4}^\infty \Big(\frac{d(x)}{2^i}\Big)^{-s} \frac{1}{|B(x,\frac{d(x)}{2^i})|} \int_{B(x,\frac{d(x)}{2^i})} |f(y)-f(x)|\, dy\, d(x)^{\alpha+s}\omega(x)\, dx \\
&\lesssim \int_\Omega \sum_{i=4}^\infty 2^{is} \fint_{B(x,\frac{d(x)}{2^i})} |f(y)-f(x)|\, dy\, d(x)^{\alpha}\omega(x)\, dx
\end{aligned} \tag{3.1}$$

Observe that

$$\begin{aligned}
&\int_{2^{-i}}^{2^{-i+1}} \fint_{B(x,\lambda d(x))} |f(x)-f(y)|\, dy \frac{d\lambda}{\lambda^{1+s}} \\
&\gtrsim \int_{2^{-i}}^{2^{-i+1}} \frac{1}{|B(x,2^{-i+1}d(x))|} \int_{B(x,2^{-i}d(x))} |f(x)-f(y)|\, dy \frac{d\lambda}{\lambda^{1+s}} \\
&\gtrsim \int_{2^{-i}}^{2^{-i+1}} \frac{1}{2^{(-i+1)(1+s)}} \frac{1}{|B(x,2^{-i}d(x))|} \int_{B(x,2^{-i}d(x))} |f(x)-f(y)|\, dy\, d\lambda \\
&\gtrsim \frac{2^{-i}}{2^{(-i+1)(1+s)}} \fint_{B(x,2^{-i}d(x))} |f(x)-f(y)|\, dy \\
&\gtrsim 2^{is} \fint_{B(x,2^{-i}d(x))} |f(x)-f(y)|\, dy
\end{aligned}$$

So that, plugging this into (3.1), we obtain

$$\begin{aligned}
&\int_\Omega \int_{|x-y|<\frac{d(x)}{16}} \frac{|f(y)-f(x)|}{|x-y|^{n+s}}\,dy\,d(x)^{\alpha+s}\omega(x)\,dx \\
&\lesssim \int_\Omega \sum_{i=4}^\infty 2^{is} \fint_{B(x,\frac{d(x)}{2^i})} |f(y)-f(x)|\,dy\,d(x)^\alpha\omega(x)\,dx \\
&\lesssim \int_\Omega \sum_{i=4}^\infty \int_{2^{-i}}^{2^{-i+1}} \fint_{B(x,\lambda d(x))} |f(x)-f(y)|\,dy\,\frac{d\lambda}{\lambda^{1+s}}\,d(x)^\alpha\omega(x)\,dx \\
&\lesssim \int_\Omega \int_0^{\frac18} \fint_{B(x,\lambda d(x))} |f(x)-f(y)|\,dy\,\frac{d\lambda}{\lambda^{1+s}}\,d(x)^\alpha\omega(x)\,dx \\
&= \int_0^{\frac18} E(f,\lambda)\frac{d\lambda}{\lambda^{1+s}} \qquad (3.2)
\end{aligned}$$

with

$$E(f,\lambda) := \int_\Omega \fint_{B(x,\lambda d(x))} |f(x)-f(y)|\,dy\,d(x)^\alpha\omega(x)\,dx.$$

Now, for each $\lambda \in (0,\frac18)$, pick a decomposition $f = g_\lambda + h_\lambda$, with $g_\lambda \in L^1_\omega(\Omega, d^\alpha)$, $h_\lambda \in W^{1,1}_\omega(\Omega, d^\alpha, d^{\alpha+1})$, and $\|d^\alpha g_\lambda\|_{L^1_\omega(\Omega)} + \lambda\|d^{\alpha+1}\nabla h_\lambda\|_{L^1_\omega(\Omega)} \le 2K(f,\lambda)$, with $K(f,\lambda)$ as in (2.2). We remark that we may assume that $f \not\equiv 0$ to guarantee that $K(f,\lambda) > 0$. Since $E(f,\lambda) \le E(g_\lambda,\lambda) + E(h_\lambda,\lambda)$, we may bound these terms separately.

Observe that, for $x \in \Omega$, $y \in B(x,\lambda d(x))$ and $\lambda \in (0,\frac18)$,

$$\begin{aligned}
d(x) &\le d(y) + |x-y| < d(y) + \lambda d(x) \Rightarrow d(x) < \frac87 d(y),\\
d(y) &\le d(x) + |x-y| < d(x) + \lambda d(x) < \frac98 d(x).
\end{aligned}$$

Therefore, we have that $x \in B(y, \frac87\lambda d(y))$ and $d(x) \sim d(y)$. Hence, by Fubini,

$$\begin{aligned}
E(g_\lambda,\lambda) &\lesssim \int_\Omega \frac{1}{(\lambda d(x))^n} \int_{B(x,\lambda d(x))} (|g_\lambda(x)| + |g_\lambda(y)|)\,dy\,d(x)^\alpha\omega(x)\,dx \\
&\lesssim \int_\Omega |g_\lambda(x)|\,d(x)^\alpha\omega(x)\,dx + \int_\Omega \frac{1}{(\lambda d(x))^n}\int_{B(x,\lambda d(x))} |g_\lambda(y)|\,dy\,d(x)^\alpha\omega(x)\,dx \\
&\lesssim \int_\Omega |g_\lambda(x)|\,d(x)^\alpha\omega(x)\,dx + \int_\Omega |g_\lambda(y)| \frac{d(y)^\alpha}{(\lambda d(y))^n}\int_{B(y,\frac87\lambda d(y))} \omega(x)\,dx\,dy \\
&\lesssim \int_\Omega |g_\lambda(x)|\,d(x)^\alpha\omega(x)\,dx + \int_\Omega |g_\lambda(y)|\,d(y)^\alpha M\omega(y)\,dy \\
&\lesssim \|g_\lambda\|_{L^1_\omega(\Omega,d^\alpha)}, \qquad (3.3)
\end{aligned}$$

where in the last inequality we have used that $M\omega(y) \lesssim \omega(y)$ almost everywhere, because $\omega \in A_1$.

To bound $E(h_\lambda,\lambda)$, let $B = B(x,\lambda d(x))$ and $h_{\lambda,B} = \frac{1}{|B|}\int_B h_\lambda(z)\,dz$. Then, for any $y \in B$, by [10, Lemma 7.16] we may write

$$\begin{aligned}
|h_\lambda(x) - h_\lambda(y)| &\le |h_\lambda(x) - h_{\lambda,B}| + |h_{\lambda,B} - h_\lambda(y)| \\
&\lesssim \int_B \frac{|\nabla h_\lambda(z)|}{|x-z|^{n-1}}\,dz + \int_B \frac{|\nabla h_\lambda(z)|}{|y-z|^{n-1}}\,dz.
\end{aligned}$$

So, we obtain

$$E(h_\lambda,\lambda) = \int_\Omega \frac{1}{(\lambda d(x))^n}\int_{B(x,\lambda d(x))} |h_\lambda(x)-h_\lambda(y)|\,dy\,d(x)^\alpha\omega(x)\,dx \tag{3.4}$$

$$\lesssim \int_\Omega \frac{1}{(\lambda d(x))^n}\int_{B(x,\lambda d(x))}\left(\int_{B(x,\lambda d(x))}\frac{|\nabla h_\lambda(z)|}{|x-z|^{n-1}}\,dz + \int_{B(x,\lambda d(x))}\frac{|\nabla h_\lambda(z)|}{|y-z|^{n-1}}\,dz\right)dy\,d(x)^\alpha\omega(x)\,dx$$

$$= I + II \tag{3.5}$$

If $z\in B(x,\lambda d(x))$ and $\lambda\in(0,\frac{1}{8})$, observe that by the computations right before (3.3) (replacing y by z), we can deduce that $x\in B(z,\frac{8}{7}\lambda d(z))$ and that $d(x)\sim d(z)$. Hence, by Fubini and [13, Lemma (a)],

$$\begin{aligned} I &\lesssim \int_\Omega\int_{B(x,\lambda d(x))}\frac{|\nabla h_\lambda(z)|}{|x-z|^{n-1}}\,dz\,d(x)^\alpha\omega(x)\,dx\\ &\lesssim \int_\Omega\int_{B(z,\frac{8}{7}\lambda d(z))}\frac{\omega(x)}{|x-z|^{n-1}}\,dx\,|\nabla h_\lambda(z)|\,d(z)^\alpha dz\\ &\lesssim \int_\Omega \lambda d(z)M\omega(z)|\nabla h_\lambda(z)|\,d(z)^\alpha dz\\ &\lesssim \lambda\|\nabla h_\lambda\|_{L^1_\omega(\Omega,d^{\alpha+1})}. \end{aligned}$$

Similarly, to bound II, recall that for $y\in B(x,\lambda d(x))$ and $\lambda\in(0,\frac{1}{8})$, we have that $x\in B(y,\frac{8}{7}\lambda d(y))$ and that $\frac{7}{8}d(x)<d(y)<\frac{9}{8}d(x)$, so that, if $z\in B(x,\lambda d(x))$,

$$|z-y|<|z-x|+|x-y|<\lambda d(x)+\frac{8}{7}\lambda d(y)<\frac{16}{7}\lambda d(y).$$

Therefore, by Fubini,

$$\begin{aligned} II &= \int_\Omega \frac{1}{(\lambda d(x))^n}\int_{B(x,\lambda d(x))}\int_{B(x,\lambda d(x))}\frac{|\nabla h_\lambda(z)|}{|y-z|^{n-1}}\,dz\,dy\,d(x)^\alpha\omega(x)\,dx\\ &\lesssim \int_\Omega\frac{1}{(\lambda d(y))^n}\int_{B(y,\frac{16}{7}\lambda d(y))}\int_{B(y,\frac{8}{7}\lambda d(y))}\omega(x)\,dx\,\frac{|\nabla h_\lambda(z)|}{|y-z|^{n-1}}\,dz\,d(y)^\alpha dy\\ &\lesssim \int_\Omega\int_{B(y,\frac{16}{7}\lambda d(y))} M\omega(y)\,\frac{|\nabla h_\lambda(z)|}{|y-z|^{n-1}}\,dz\,d(y)^\alpha dy\\ &\lesssim \int_\Omega\int_{B(y,\frac{16}{7}\lambda d(y))} \omega(y)\,\frac{|\nabla h_\lambda(z)|}{|y-z|^{n-1}}\,dz\,d(y)^\alpha dy. \end{aligned}$$

Now, observe that for $z\in B(y,\frac{16}{7}\lambda d(y))$ and $\lambda\in(0,\frac{1}{8})$ we have

$$\begin{aligned} &d(y)<d(z)+|z-y|<d(z)+\frac{16}{7}\lambda d(y)<d(z)+\frac{2}{7}d(y)\Rightarrow d(y)<\frac{7}{5}d(z),\\ &d(z)<d(y)+|z-y|<d(y)+\frac{16}{7}\lambda d(y)<\frac{9}{7}d(y), \end{aligned}$$

Hence, $d(y) \sim d(z)$ and $|z-y| < \frac{16}{7}\lambda d(y) < \frac{16}{5}\lambda d(z)$, so by Fubini and [13, Lemma (a)],

$$II \lesssim \int_\Omega \int_{B(z,\frac{16}{5}\lambda d(z))} \frac{\omega(y)}{|y-z|^{n-1}}\, dy\, |\nabla h_\lambda(z)|\, d(z)^\alpha\, dz$$
$$\lesssim \int_\Omega \lambda d(z) M\omega(z)\, |\nabla h_\lambda(z)|\, d(z)^\alpha\, dz \lesssim \lambda \|\nabla h_\lambda\|_{L^1_\omega(\Omega, d^{\alpha+1})}.$$

Finally, we arrive at

$$\int_\Omega \int_{|x-y|<\frac{d(x)}{16}} \frac{|f(y)-f(x)|^p}{|x-y|^{n+sp}}\, dy\, d(x)^{\alpha+s}\, \omega(x)\, dx$$
$$\lesssim \int_0^1 \Big(\|g_\lambda\|_{L^1_\omega(\Omega,d^\alpha)} + \lambda \|\nabla h_\lambda\|_{L^1_\omega(\Omega,d^{\alpha+1})} \Big) \frac{d\lambda}{\lambda^{1+s}} \lesssim \int_0^1 \lambda^{-s} K(f,\lambda)\, \frac{d\lambda}{\lambda}.$$

This completes the proof. □

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $\Omega \neq \mathbb{R}^n$, $0<s<1$, $\alpha \geq 0$, and $\omega \in A_1$. Then*

$$\widetilde{W}^{s,1}_\omega(\Omega, d^{\alpha+s}) \subseteq (L^1_\omega(\Omega), W^{1,1}_\omega(\Omega, d^\alpha, d^{\alpha+1}))_{s,1}.$$

Proof. Observe first that, by the trivial bound $K(f,\lambda) \leq \|f\|_{L^1_\omega(\Omega,d^\alpha)}$, we always have

$$\int_1^\infty \lambda^{-s} K(f,\lambda)\, \frac{d\lambda}{\lambda} \leq \|f\|_{L^1_\omega(\Omega,d^\alpha)} \int_1^\infty \lambda^{-s}\, \frac{d\lambda}{\lambda} \lesssim \|f\|_{L^1_\omega(\Omega,d^\alpha)}.$$

Also, for a given decomposition $f = g + h$ as in (2.2),

$$\int_0^1 \lambda^{1-s} \|h\|_{L^1_\omega(\Omega,d^\alpha)}\, \frac{d\lambda}{\lambda} \lesssim \int_0^1 \lambda^{1-s} \|f\|_{L^1_\omega(\Omega,d^\alpha)} \frac{d\lambda}{\lambda} + \int_0^1 \lambda^{1-s} \|g\|_{L^1_\omega(\Omega,d^\alpha)} \frac{d\lambda}{\lambda}$$
$$\lesssim \|f\|_{L^1_\omega(\Omega,d^\alpha)} + \int_0^1 \lambda^{-s} \|g\|_{L^1_\omega(\Omega,d^\alpha)} \frac{d\lambda}{\lambda}.$$

Therefore,

$$\int_0^\infty \lambda^{-s} K(\lambda, f)\, \frac{d\lambda}{\lambda} \lesssim \|f\|_{L^1_\omega(\Omega,d^\alpha)} + \int_0^1 \lambda^{-s} (\|g\|_{L^1_\omega(\Omega,d^\alpha)} + \lambda \|\nabla h\|_{L^1_\omega(\Omega,d^{\alpha+1})}) \frac{d\lambda}{\lambda}, \tag{3.6}$$

and to prove the claimed embedding it suffices to bound the integral on the right-hand side for specific choices of g and h that we will define below.

As in [1, Section 4], given a cube $Q \subset \mathbb{R}^n$, the distance from Q to the boundary of Ω is denoted by $d(Q, \partial\Omega)$, while $\operatorname{diam}(Q)$ and ℓ_Q are the diameter and length of the edges of Q, respectively. We pick a Whitney decomposition $\mathcal{W} = \{Q\}$ of Ω and, for every fixed $0 < \lambda \leq 1$, we build a new dyadic decomposition $\mathcal{W}^\lambda = \{Q^\lambda\}$ by dividing each $Q \in \mathcal{W}$ in such a way that $\frac{1}{2}\lambda \ell_Q \leq \ell_{Q^\lambda} \leq \lambda \ell_Q$. Notice that, in particular, this means that $\frac{1}{2}\lambda \operatorname{diam}(Q) \leq \operatorname{diam}(Q^\lambda) \leq \lambda \operatorname{diam}(Q)$. The center of Q^λ_j in this new partition is denoted by x^λ_j, and we write ℓ^λ_j instead of $\ell_{Q^\lambda_j}$.

For each $\mathcal{W}^\lambda$ we can define the covering of expanded cubes $\mathcal{W}^{\lambda^*} = \{(Q^\lambda_j)^*\}$ where Q^* is the cube with the same center as Q but expanded by a factor $9/8$. Observe that it satisfies $\sum_j \chi_{(Q^\lambda_j)^*}(x) \leq C$ for every $x \in \Omega$, and that, for $x \in (Q^\lambda_j)^*$,

$$\frac{3}{4} \frac{\operatorname{diam}(Q^\lambda_j)}{\lambda} \leq d(x) \leq \frac{41}{4} \frac{\operatorname{diam}(Q^\lambda_j)}{\lambda}. \tag{3.7}$$

Associated to this covering we consider a smooth partition of unity $\{\psi_j^\lambda\}$ such that $\operatorname{supp}(\psi_j^\lambda) \subset (Q_j^\lambda)^*$, $0 \le \psi_j^\lambda \le 1$, $\sum_j \psi_j^\lambda = 1$ in Ω, and $\|\nabla \psi_j^\lambda\|_\infty \le \frac{C}{\ell_j^\lambda}$.

For a given (fixed) C^∞ function $\varphi \ge 0$ such that $\operatorname{supp}(\varphi) \subset B(0, \frac{1}{4})$ and $\int \varphi = 1$, and for each $t > 0$, we define $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$. Then, for a given $f \in \widetilde{W}_\omega^{s,1}(\Omega, d^\alpha, d^{\alpha+s})$ we define

$$h^\lambda(y) = \sum_j f_j^\lambda \psi_j^\lambda(y), \tag{3.8}$$

with

$$f_j^\lambda = \int_{\mathbb{R}^n} f * \varphi_{\ell_j^\lambda}(z)\varphi_{\ell_j^\lambda}(z - x_j^\lambda)\, dz,$$

which is a smooth approximation of f. Moreover, by [1, page 9], one has that, for $y \in (Q_j^\lambda)^*$,

$$\begin{aligned}
&|f(y) - f_j^\lambda| \\
&\le \int_0^1 \int_{|x-y|<Ct\ell_j^\lambda} \int_{|x-w|<\frac{d(x)}{2}} |f(x) - f(w)| \chi_{|x-w|<\frac{1}{4}t\ell_j^\lambda}\, dw \,\frac{\chi_{(Q_j^\lambda)^*}(x)}{t^{2n+1}}\, dx\, dt\, (\ell_j^\lambda)^{-2n}.
\end{aligned}$$

Since the family $\mathcal{W}^{\lambda^*}$ has finite overlapping, we have that

$$\|f - h^\lambda\|_{L^1_\omega(\Omega, d^\alpha)} \le C \sum_j \|f - f_j^\lambda\|_{L^1_\omega((Q_j^\lambda)^*, d^\alpha)}.$$

Using that, for $x \in (Q_j^\lambda)^*$, $\ell_j^\lambda \sim \lambda d(x)$ and that $|x-y| < Ct\ell_j^\lambda \Rightarrow d(y) \lesssim d(x)$, we have

$$\begin{aligned}
&\int_0^1 \lambda^{-s}\|f - h^\lambda\|_{L^1_\omega(\Omega, d^\alpha)} \frac{d\lambda}{\lambda} \\
&\lesssim \int_0^1 \sum_j \int_{(Q_j^\lambda)^*} \int_0^1 \int_{|x-y|<Ct\ell_j^\lambda} \int_{|x-w|<\frac{d(x)}{2}} |f(x) - f(w)|\, \chi_{|x-w|<\frac{1}{4}t\ell_j^\lambda}\, dw \\
&\quad \cdot \frac{(\ell_j^\lambda)^{-2n}\lambda^{-s}}{t^{2n+1}}\, dt\, dx\, d(y)^\alpha \omega(y)\, dy\, \frac{d\lambda}{\lambda} \\
&\lesssim \int_0^1 \sum_j \int_{(Q_j^\lambda)^*} \int_0^1 \int_{|x-w|<\frac{d(x)}{2}} |f(x) - f(w)|\, \chi_{|x-w|<\frac{1}{4}t\ell_j^\lambda}\, dw \\
&\quad \cdot \left(\int_{|x-y|<Ct\ell_j^\lambda} \omega(y)\, dy \right) \frac{(\ell_j^\lambda)^{-2n}\lambda^{-s}}{t^{2n+1}}\, dt\, d(x)^\alpha dx\, \frac{d\lambda}{\lambda} \\
&\lesssim \int_0^1 \sum_j \int_{(Q_j^\lambda)^*} \int_0^1 \int_{|x-w|<\frac{d(x)}{2}} |f(x) - f(w)|\, \chi_{|x-w|<\frac{1}{4}t\ell_j^\lambda}\, dw (t\ell_j^\lambda)^n \\
&\quad \cdot M\omega(x) \frac{(\ell_j^\lambda)^{-2n}\lambda^{-s}}{t^{2n+1}}\, dt\, d(x)^\alpha dx\, \frac{d\lambda}{\lambda} \\
&\lesssim \int_0^1 \sum_j \int_{(Q_j^\lambda)^*} \int_0^1 \int_{|x-w|<\frac{d(x)}{2}} |f(x) - f(w)|\, \chi_{|x-w|<\frac{1}{4}t\ell_j^\lambda}\, dw\, \omega(x) \\
&\quad \cdot \frac{(\ell_j^\lambda)^{-n}\lambda^{-s}}{t^{n+1}}\, dt\, d(x)^\alpha dx\, \frac{d\lambda}{\lambda}
\end{aligned}$$

$$\lesssim \int_0^1 \sum_j \int_{(Q_j^\lambda)^*} \int_0^1 \int_{|x-w|<\frac{d(x)}{2}} |f(x)-f(w)|\, \chi_{|x-w|<\frac{c}{4}t\lambda d(x)}\, dw\, \omega(x)$$
$$\cdot \frac{d(x)^{\alpha-n}\lambda^{-n-s}}{t^{n+1}}\, dt\, dx\, \frac{d\lambda}{\lambda}$$
$$\lesssim \int_0^1 \int_\Omega \int_0^1 \int_{|x-w|<\frac{d(x)}{2}} |f(x)-f(w)|\, \chi_{|x-w|<\frac{c}{4}t\lambda d(x)}\, dw\, \omega(x) \frac{d(x)^{\alpha-n}\lambda^{-n-s}}{t^{n+1}}\, dt\, dx\, \frac{d\lambda}{\lambda}$$
$$\lesssim \int_\Omega \int_0^1 \int_{|x-w|<\frac{d(x)}{2}} |f(x)-f(w)|\, dw \int_{\frac{4|x-w|}{ctd(x)}}^\infty \lambda^{-n-s-1}\, d\lambda\, \frac{d(x)^{\alpha-n}}{t^{n+1}}\, dt\, \omega(x)\, dx$$
$$\lesssim \frac{1}{n+s} \int_\Omega \int_0^1 \int_{|x-w|<\frac{d(x)}{2}} \frac{|f(x)-f(w)|}{|x-w|^{n+s}}\, dw\, t^{s-1} d(x)^{\alpha+s}\, dt\, \omega(x)\, dx$$
$$\lesssim \frac{1}{s(n+s)} \int_\Omega \int_{|x-w|<\frac{d(x)}{2}} \frac{|f(x)-f(w)|}{|x-w|^{n+s}}\, dw\, d(x)^{\alpha+s}\omega(x)\, dx$$

On the other hand, recalling that $\operatorname{supp}(\psi_j^\lambda) \subset (Q_j^\lambda)^*$, that $\|\nabla\psi_j^\lambda\|_\infty \le \frac{C}{\ell_j^\lambda}$, and that $\nabla(\sum_j \psi_j^\lambda) = 0$,

$$|\nabla h^\lambda(y)| = \Big|\sum_j f_j^\lambda \nabla\psi_j^\lambda(y)\Big| \lesssim \sum_j |f_j^\lambda - f(y)| \frac{1}{\ell_j^\lambda} \chi_{(Q_j^\lambda)^*}(y).$$

Therefore,

$$\begin{aligned}\int_0^1 \lambda^{1-s} \|\nabla h^\lambda\|_{L^1_\omega(\Omega, d^{\alpha+1})} \frac{d\lambda}{\lambda} &\lesssim \int_0^1 \sum_j \lambda^{1-s} \|(f-f_j^\lambda)(\ell_j^\lambda)^{-1}\|_{L^1_\omega((Q_j^\lambda)^*, d^{\alpha+1})} \frac{d\lambda}{\lambda}\\ &\lesssim \int_0^1 \sum_j \lambda^{1-s} \|(f-f_j^\lambda)\lambda^{-1}\|_{L^1_\omega((Q_j^\lambda)^*, d^{\alpha})} \frac{d\lambda}{\lambda}\\ &\lesssim \int_0^1 \sum_j \lambda^{-s} \|(f-f_j^\lambda)\|_{L^1_\omega((Q_j^\lambda)^*, d^{\alpha})} \frac{d\lambda}{\lambda}\end{aligned}$$

so this term can be bounded as before.

Summing up,

$$\begin{aligned}&\int_0^1 \lambda^{-s}\Big(\|f-h^\lambda\|_{L^1_\omega(\Omega,d^\alpha)} + \lambda\|\nabla h^\lambda\|_{L^1_\omega(\Omega,d^{\alpha+1})}\Big)\frac{d\lambda}{\lambda}\\ &\lesssim \frac{1}{s(n+s)} \int_\Omega \int_{|x-y|<\frac{d(x)}{2}} \frac{|f(y)-f(x)|}{|x-y|^{n+s}}\, dy\, d(x)^{\alpha+s}\omega(x)\, dx.\end{aligned} \tag{3.9}$$

This concludes the proof. □

Remark 3.1. It is immediate that inequality (3.9) also holds for every $\frac{1}{2} < \tau < 1$. If one wishes to obtain it for $0 < \tau < \frac{1}{2}$, it suffices to choose $supp(\varphi) \subset B(0,\varepsilon)$ for sufficiently small ε in the above proof, as the reader can check by following the computations in [1, page 8].

4. Applications to fractional Poincaré inequalities

In the forthcoming results we will make use of two well-known properties of weighted norms contained in the following lemma. We include a proof for the sake of completeness.

Lemma 4.1. *Let Ω be a bounded domain, ν a locally integrable nonnegative function, and $f_\nu = \frac{1}{\nu(\Omega)}\int_\Omega f(x)\nu(x)\,dx$. Then,*

(1) $\inf_{c\in\mathbb{R}} \|f-c\|_{L^1_\nu(\Omega)} \sim \|f-f_\nu\|_{L^1_\nu(\Omega)}$,
(2) $\|f-f_\nu\|_{L^1_\nu(\Omega)} \le 2\|f\|_{L^1_\nu(\Omega)}$.

Proof. (1) It is immediate that $\inf_{c\in\mathbb{R}} \|f-c\|_{L^1_\nu(\Omega)} \le \|f-f_\nu\|_{L^1_\nu(\Omega)}$. For the other inequality, it suffices to observe that, for any $c\in\mathbb{R}$,

$$\begin{aligned}\|f-f_\nu\|_{L^1_\nu(\Omega)} &\le \|f-c\|_{L^1_\nu(\Omega)} + \|c-f_\nu\|_{L^1_\nu(\Omega)}\\ &\le \|f-c\|_{L^1_\nu(\Omega)} + \nu(\Omega)\left|c-\frac{1}{\nu(\Omega)}\int_\Omega f(x)\nu(x)\,dx\right|\\ &\le 2\|f-c\|_{L^1_\nu(\Omega)}.\end{aligned}$$

(2) Write

$$\begin{aligned}\|f-f_\nu\|_{L^1_\nu(\Omega)} &\le \frac{1}{\nu(\Omega)}\int_\Omega\int_\Omega |f(x)-f(y)|\,\nu(x)\nu(y)\,dx\,dy\\ &\le \int_\Omega |f(x)|\nu(x)\,dx + \int_\Omega |f(y)|\nu(y)\,dy = 2\|f\|_{L^1_\nu(\Omega)}.\end{aligned}$$ □

Theorem 4.1. *Let Ω be a bounded domain, $\alpha\ge 0$, $\omega\in A_1$ and $\|\nabla f\|_{L^1_\omega(\Omega,d^{\alpha+1})} < \infty$. Then, the following are equivalent:*

(1) $\inf\limits_{c\in\mathbb{R}} \|f-c\|_{L^1_\omega(\Omega,d^\alpha)} \lesssim \|\nabla f\|_{L^1_\omega(\Omega,d^{\alpha+1})}$.
(2) $\inf\limits_{c\in\mathbb{R}} \|f-c\|_{L^1_\omega(\Omega,d^\alpha)} \lesssim \dfrac{(1-s)}{s(n+s)}\displaystyle\int_\Omega\int_{|x-y|<\tau d(x)} \frac{|f(x)-f(y)|}{|x-y|^{n+s}}\, d(x)^{\alpha+s}\omega(x)\,dy\,dx$ *for every $0<\tau<1$ and every $0<s<1$.*
(3) $\inf\limits_{c\in\mathbb{R}} \|f-c\|_{L^1_\omega(\Omega,d^\alpha)} \lesssim \dfrac{(1-s)}{s(n+s)}\displaystyle\int_\Omega\int_{|x-y|<\tau d(x)} \frac{|f(x)-f(y)|}{|x-y|^{n+s}}\, d(x)^{\alpha+s}\omega(x)\,dy\,dx$ *for every $0<\tau<1$ and some $0<s<1$.*

Proof. (1) ⇒ (2) This is a straightforward generalization of an unpublished result by Oscar Domínguez Bonilla for the case $\alpha=0, \omega\equiv 1$. Define h^λ as in (3.8) and $\nu = d^\alpha\omega$. By hypothesis and the previous lemma,

$$\begin{aligned}\|f-f_\nu\|_{L^1_\nu(\Omega)} &\lesssim \|f-h^\lambda-(f_\nu-(h^\lambda)_\nu)\|_{L^1_\nu(\Omega)} + \|h^\lambda-(h^\lambda)_\nu\|_{L^1_\nu(\Omega)}\\ &\lesssim \|f-h^\lambda\|_{L^1_\nu(\Omega)} + \|d\nabla h^\lambda\|_{L^1_\nu(\Omega)}.\end{aligned}$$

Then, for $\lambda\le 1$,

$$\begin{aligned}\lambda\|f-f_\nu\|_{L^1_\nu(\Omega)} &\lesssim \lambda\|f-h^\lambda\|_{L^1_\nu(\Omega)} + \lambda\|d\nabla h^\lambda\|_{L^1_\nu(\Omega)}\\ &\lesssim \|f-h^\lambda\|_{L^1_\omega(\Omega,d^\alpha)} + \lambda\|\nabla h^\lambda\|_{L^1_\omega(\Omega,d^{\alpha+1})}.\end{aligned}$$

Therefore,

$$\int_0^1 \lambda^{-s+1}\|f-f_\nu\|_{L^1_\nu(\Omega)}\frac{d\lambda}{\lambda} \lesssim \int_0^1 \lambda^{-s}\Big(\|f-h^\lambda\|_{L^1_\omega(\Omega,d^\alpha)} + \lambda\|\nabla h^\lambda\|_{L^1_\omega(\Omega,d^{\alpha+1})}\Big)\frac{d\lambda}{\lambda}$$

for every $0 < s < 1$. Then, by (3.9) and Remark 3.1,

$$\frac{1}{(1-s)}\|f - f_\nu\|_{L^1_\nu(\Omega)} \lesssim \int_0^1 \lambda^{-s}\Big(\|f - h^\lambda\|_{L^1_\omega(\Omega,d^\alpha)} + \lambda\|\nabla h^\lambda\|_{L^1_\omega(\Omega,d^{\alpha+1})}\Big)\frac{d\lambda}{\lambda}$$
$$\lesssim \frac{1}{s(n+s)}|f|_{\widetilde{W}^{s,1}_\omega(\Omega,d^{\alpha+s})}$$

for all values of $0 < \tau < 1$.

So that, again by the previous lemma,

$$\inf_{c\in\mathbb{R}}\|f - c\|_{L^1_\omega(\Omega,d^\alpha)} \lesssim \frac{(1-s)}{s(n+s)}|f|_{\widetilde{W}^{s,1}_\omega(\Omega,d^{\alpha+s})}.$$

(2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (1) By (3.2), for $\tau = \frac{1}{16}$ and repeating for f the computations previoulsy made for h_λ in (3.5), we get

$$(1-s)\int_\Omega\int_{|x-y|<\tau d(x)}\frac{|f(y)-f(x)|}{|x-y|^{n+s}}\,dy\,d(x)^{\alpha+s}\omega(x)\,dx \lesssim \|\nabla f\|_{L^1_\omega(\Omega,d^{\alpha+1})},$$

and the result follows. □

By the previous theorem one immediately has:

Corollary 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain, $\alpha \ge 0$, $\omega \in A_1$, and $\|\nabla f\|_{L^1_\omega(\Omega,d^{\alpha+1})} < \infty$. Then,*

$$\inf_{c\in\mathbb{R}}\|f - c\|_{L^1_\omega(\Omega,d^\alpha)} \lesssim \frac{(1-s)}{s(n+s)}\int_\Omega\int_{|x-y|<\tau d(x)}\frac{|f(x)-f(y)|}{|x-y|^{n+s}}\,d(x)^{\alpha+s}\omega(x)\,dy\,dx$$

for every $0 < \tau < 1$ and every $0 < s < 1$.

Proof. It suffices to check that

$$\inf_{c\in\mathbb{R}}\|f - c\|_{L^1_\omega(\Omega,d^\alpha)} \lesssim \|\nabla f\|_{L^1_\omega(\Omega,d^{\alpha+1})}.$$

This can be seen with a slight modification of the proof in [6, Theorem 3.4] (which is the case $\alpha = 0$), we briefly indicate the necessary steps.

Following that proof, by duality it suffices to bound $\int_\Omega (f - f_\varphi)(y)g(y)d(y)^\alpha\,dy$ for any g such that $\|\omega^{-1}g\|_{L^\infty(\Omega)} \le 1$.

As in [6, equation (3.2)] and noting that $|x-y| \le Cd(x) \Rightarrow d(y) \lesssim d(x)$, we have

$$\int_\Omega |(f(y) - f_\varphi)g(y)|d(y)^\alpha\,dy \lesssim \int_\Omega\int_{|x-y|\le Cd(x)}\frac{|g(y)|\chi_\Omega(y)}{|x-y|^{n-1}}\,dy|\nabla f(x)|d(x)^\alpha\,dx$$
$$\lesssim \int_\Omega M(\chi_\Omega g)(x)d^{\alpha+1}(x)|\nabla f(x)|\,dx \tag{4.1}$$
$$\lesssim \|\omega^{-1}M(\chi_\Omega g)\|_{L^\infty(\Omega)}\|\omega\, d^{\alpha+1}\nabla f\|_{L^1(\Omega)}$$
$$\lesssim \|\omega^{-1}g\|_{L^\infty(\Omega)}\|\nabla f\|_{L^1_\omega(\Omega,d^{\alpha+1})} \tag{4.2}$$
$$\lesssim \|\nabla f\|_{L^1_\omega(\Omega,d^{\alpha+1})}$$

where in (4.1) we have used [13, Lemma (a)], and in (4.2) we have used [21, Theorem 4] (actually, the remark at the end of [21, Section 7] regarding its extension to the n-dimensional case). □

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Irene Drelichman
CMaLP, Departamento de Matemática
Facultad de Ciencias Exactas
Universidad Nacional de La Plata
La Plata, Buenos Aires, Argentina
irene@drelichman.com