FROM WEAK TYPE WEIGHTED INEQUALITY TO POINTWISE ESTIMATE FOR THE DECREASING REARRANGEMENT

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Abstract. We shall prove pointwise estimates for the decreasing rearrangement of Tf, where T covers a wide range of interesting operators in Harmonic Analysis such as operators satisfying a Fefferman-Stein inequality, the Bochner-Riesz operator, rough operators, sparse operators, Fourier multipliers, etc. In particular, our main estimate is of the form

$$(Tf)^*(t) \leq C\left(\frac{1}{t}\int_0^t f^*(s)\,ds + \int_t^\infty \left(1+\log\frac{s}{t}\right)^{-1} \varphi\left(1+\log\frac{s}{t}\right) f^*(s)\,\frac{ds}{s}\right),$$

where φ is determined by the Muckenhoupt A_p-weight norm behaviour of the operator.

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1. Introduction

There are many interesting operators in Harmonic Analysis satisfying that, for every $u \in A_1$,

$$T : L^1(u) \longrightarrow L^{1,\infty}(u) \tag{1.1}$$

is bounded with constant less than or equal to $C||u||_{A_1}^k$, $k \in \mathbb{N}$, where we recall that A_1 is the class of Muckenhoupt weights such that

$$Mu(x) \leq Cu(x), \quad \text{a.e. } x \in \mathbb{R}^n. \tag{1.2}$$

Here, M is the Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy, \qquad f \in L^1_{\text{loc}}(\mathbb{R}^n), \; x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \subseteq \mathbb{R}^n$ containing x, and $||u||_{A_1}$ is the infimum of all constants C in (1.2). In particular, it is known [34] that M satisfies (1.1) with $||M||_{L^1(u) \longrightarrow L^{1,\infty}(u)} \leq c_n ||u||_{A_1}$.

Other examples of such operators are the Hilbert transform, the Riesz transform and, more generally, any Calderón-Zygmund operator (see for instance [31]). Also, sparse operators [25], the Bochner-Riesz operator B_λ at the critical index (that is, $\lambda = \frac{n-1}{2}$) [32, 37], among many others, are known to satisfy condition (1.1).

The main purpose of this paper is to prove pointwise estimates for the decreasing rearrangement of Tf with respect to the Lebesgue measure. In particular, our first main result is the following:

Theorem 1.1. *Let T be a sublinear operator such that, for every $u \in A_1$ and some $k \in \mathbb{N}$, T satisfies (1.1) with constant less than or equal to $C||u||_{A_1}^k$. Then, for every $t > 0$ and every measurable function f,*

$$(Tf)^*(t) \leq C \left(\frac{1}{t} \int_0^t f^*(s)\, ds + \int_t^\infty \left(1 + \log \frac{s}{t}\right)^{k-1} f^*(s) \,\frac{ds}{s} \right).$$

This pointwise estimate is very interesting since obviously it has, as a consequence, boundedness properties of such operators on rearrangement invariant spaces (for more details, we refer to [6]). In particular, it allows us to characterize the weights ω for which they are bounded on weighted Lorentz spaces $\Lambda^{p,q}(\omega)$ [11].

Now, on some occasions, the behaviour of the constant has been improved from, let us say, $||u||_{A_1}^{1+\varepsilon}$, $\varepsilon > 0$, to an expression of the form $\varphi(||u||_{A_1})$, where φ is not a power function. This is, for example, the case when T is a Calderón-Zygmund operator, where the best function φ known up to now is $\varphi(t) = t(1 + \log^+ t)(1 + \log^+ \log^+ t)$ [31].

In order to cover this important class of operators we shall, in fact, prove a more general version of Theorem 1.1 (see Definition 2.1 for the concept of admissible function).

Theorem 1.2. *Let T be a sublinear operator such that, for every $u \in A_1$,*

$$T : L^1(u) \longrightarrow L^{1,\infty}(u),$$

with constant less than or equal to $\varphi(||u||_{A_1})$, where φ is an admissible function. Then, for every $t > 0$ and every measurable function f,

$$(Tf)^*(t) \leq C \left(\frac{1}{t} \int_0^t f^*(s)\, ds + \int_t^\infty \left(1 + \log \frac{s}{t}\right)^{-1} \varphi\left(1 + \log \frac{s}{t}\right) f^*(s) \frac{ds}{s} \right). \tag{1.3}$$

To prove inequality (1.3), our starting point was the following result (see [3]). See Section 2 for the definitions of the classes of weights $B_1^{\mathcal{R}}$ and B_∞^*.

Theorem 1.3. *Let T be an operator satisfying that, for every $u \in A_1$,*

$$T : L^1(u) \to L^{1,\infty}(u)$$

is bounded, with constant less than or equal to $\varphi(||u||_{A_1})$, where φ is an increasing function in $[1, \infty)$. Then, for every $\omega \in B_1^{\mathcal{R}} \cap B_\infty^*$,

$$T : \Lambda^1(\omega) \to \Lambda^{1,\infty}(\omega)$$

is bounded with constant less than or equal to $C_1[\omega]_{B_1^{\mathcal{R}}}\varphi\left(C_2[\omega]_{B_\infty^*}\right)$ for some positive constants C_1, C_2 independent of ω.

Let us now see an easy proposition which helps to motivate what follows:

Proposition 1.4. *Let T be an operator satisfying that, for every $\omega \in B_1^{\mathcal{R}}$,*

$$T : \Lambda^1(\omega) \to \Lambda^{1,\infty}(\omega) \tag{1.4}$$

is bounded with constant less than or equal to $\varphi([\omega]_{B_1^{\mathcal{R}}})$. Then, for every $t > 0$,

$$(Tf)^*(t) \leq \frac{\varphi(1)}{t} \int_0^t f^*(s)\, ds. \tag{1.5}$$

Proof. The hypothesis implies that, for every $\omega \in B_1^{\mathcal{R}}$ and every $t > 0$,

$$(Tf)^*(t)W(t) \leq \varphi([\omega]_{B_1^{\mathcal{R}}}) \int_0^\infty f^*(s)\,\omega(s)\, ds.$$

In particular, since $\omega = \chi_{[0,t]} \in B_1^{\mathcal{R}}$ and $[\omega]_{B_1^{\mathcal{R}}} = 1$, we get that

$$(Tf)^*(t) \leq \frac{\varphi(1)}{t} \int_0^t f^*(s)\, ds.$$

□

Remark 1.5. i) If an operator T satisfies (1.5), we have that (1.4) holds with constant less than or equal to $C[\omega]_{B_1^{\mathcal{R}}}$ and hence we can conclude that under the hypothesis of the previous theorem, $||T|| \leq \tilde{C} \min([\omega]_{B_1^{\mathcal{R}}}, \varphi([\omega]_{B_1^{\mathcal{R}}}))$, for some positive constant $\tilde{C}$ independent of ω.

ii) We also observe that the operator T plays no role and hence the same can be formulated for couples of functions (f, g) in the following sense:

$$||g||_{\Lambda^{1,\infty}(\omega)} \leq \varphi([\omega]_{B_1^{\mathcal{R}}})||f||_{\Lambda^1(\omega)}, \qquad \forall \omega \in B_1^{\mathcal{R}},$$

implies that

$$g^*(t) \leq \varphi(1) f^{**}(t), \qquad \forall t > 0.$$

Taking into account Theorem 1.3, our next goal was to include the hypothesis $\omega \in B_\infty^*$ in its statement:

Theorem 1.6. *Let T be a sublinear operator and let φ be an admissible function. Then, for every $\omega \in B_1^{\mathcal{R}} \cap B_\infty^*$,*

$$T : \Lambda^1(\omega) \to \Lambda^{1,\infty}(\omega)$$

is bounded with norm less than or equal to $C[\omega]_{B_1^{\mathcal{R}}}\, \varphi([\omega]_{B_\infty^*})$ if and only if, for every $t > 0$, and every measurable function f,

$$(Tf)^*(t) \leq C \left(\frac{1}{t} \int_0^t f^*(s)\, ds + \int_t^\infty \left(1 + \log \frac{s}{t}\right)^{-1} \varphi\left(1 + \log \frac{s}{t}\right) f^*(s) \frac{ds}{s} \right).$$

Besides, from the proof of the above theorem, we obtain the following result.

Corollary 1.7. *Let T be a sublinear operator such that, for some admissible function φ and any $1 \leq p < \infty$,*

$$||Tf||_{L^{p,\infty}} \leq C\varphi(p)||f||_{L^{p,1}},$$

with C independent of p, then (1.3) holds.

As a consequence, we have the following result:

Theorem 1.8. *If*

$$T : L^1 \longrightarrow L^{1,\infty}$$

and, there exists $p_0 > 1$ so that

$$T : L^{p_0,1}(u) \longrightarrow L^{p_0,\infty}(u),$$

with constant $C\varphi(||u||_{A_{p_0}})$, for some admissible function φ, then (1.3) holds.

Finally, our technique can be also applied to operators for which condition (1.1) is changed by a weaker one, as the following result shows.

Theorem 1.9. *Let T be a sublinear operator such that, for some $0 < \alpha \leq 1$, some $p_0 \geq 1$ and every $u \in A_1$,*

$$T : L^{p_0,1}(u^\alpha) \longrightarrow L^{p_0,\infty}(u^\alpha),$$

with constant less than or equal to $\varphi(||u||_{A_1})$*, where* φ *is an admissible function. Then, for every* $t>0$ *and every measurable function* f*,*

$$(Tf)^*(t) \le C\left(\frac{1}{t^{\frac{1}{p_0}}}\int_0^t f(s)\,\frac{ds}{s^{1-\frac{1}{p_0}}} + \frac{1}{t^{\frac{1-\alpha}{p_0}}}\int_t^\infty \tilde{\varphi}\left(1+\log\frac{s}{t}\right) f(s)\,\frac{ds}{s^{1-\frac{1-\alpha}{p_0}}}\right),$$

with

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x^{\frac{p_0}{\alpha}}), & 0<\alpha<1, \\ x^{-1}\varphi(x), & \alpha = 1. \end{cases}$$

The paper is organized as follows. In Section 2, we present previous results, the necessary definitions and some technical lemmas which shall be used later on. Section 3 contains the proofs of our results and Section 4 will be devoted to obtain pointwise estimates for the decreasing rearrangement of Tf with respect to the Lebesgue measure for T being operators satisfying a Fefferman-Stein inequality, the Bochner-Riesz operator, a rough operator, a sparse operator or a Fourier multiplier. As usual, we write $A \lesssim B$ if there exists a positive constant $C>0$, independent of A and B, such that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$.

2. Definitions, previous results and lemmas

2.1. Admissible functions.

Definition 2.1. A function $\varphi : [1,\infty] \to [1,\infty]$ is called admissible if satisfies the following conditions:

a) $\varphi(1) = 1$ and it is log-concave, that is

$$\theta \log\varphi(x) + (1-\theta)\log\varphi(y) \le \log\varphi(\theta x + (1-\theta)y), \qquad \forall x,y\ge 1,\ 0\le\theta\le 1,$$

b) and there exist $\gamma, \beta > 0$ such that for every $x \ge 1$,

$$\frac{\gamma}{x} \le \frac{\varphi'(x)}{\varphi(x)} \le \frac{\beta}{x}. \tag{2.1}$$

Observe that (2.1) implies that φ is increasing, as well as that

$$x^\gamma \le \varphi(x) \le x^\beta.$$

Besides, since for every $C>0$,

$$\log\varphi(Cx) = \int_1^x (\log\varphi)'(s)\,ds + \int_x^{Cx}(\log\varphi)'(s)ds \le \log\varphi(x) + \beta\log_+ C,$$

it also holds that

$$\varphi(Cx) \le \max\{1, C^\beta\}\varphi(x). \tag{2.2}$$

From now on, the function φ will be an admissible function.

Examples 2.2. i) Given $\gamma > 0$ and $\beta \geq 0$, the function $\varphi(x) = x^\gamma (1+\log x)^\beta$ is admissible.
ii) Given $n \in \mathbb{N}$, define

$$\log_{(n)} x = \begin{cases} 1+\log x, & \text{if } n=1, \\ 1+\log\big(\log_{(n-1)} x\big), & \text{if } n>1. \end{cases}$$

Using this notation, if $\gamma > 0$ and $\beta_1, \dots, \beta_n \geq 0$, the function

$$\varphi(x) = x^\gamma \prod_{k=1}^{n} \big(\log_{(k)} x\big)^{\beta_k}$$

is also admissible.

The next lemmas are simple computations for admissible functions which shall be fundamental in the proof of our main results.

Lemma 2.3. *Let $0 < q \leq \infty$. If $r \geq 1$ then*

$$\begin{cases} \displaystyle\int_1^r \varphi\big(1+\log s\big)\, \frac{ds}{s^{1-\frac{1}{q}}} \approx \varphi\left(1+\log r\right) r^{\frac{1}{q}} - 1, & q<\infty, \\ \displaystyle\int_1^r \big(1+\log s\big)^{-1} \varphi\big(1+\log s\big)\, \frac{ds}{s} \approx \varphi\left(1+\log r\right) - 1, & q=\infty. \end{cases}$$

Lemma 2.4. *Let $1 < q \leq \infty$. There exists some $\lambda = \lambda(\varphi, q) > 1$ such that for every $t > 0$ and $r \geq \lambda t$,*

$$\begin{cases} \displaystyle\int_r^\infty \varphi\left(1+\log\frac{s}{t}\right) \frac{ds}{s^{2-\frac{1}{q}}} \approx \varphi\left(1+\log\frac{r}{t}\right) \frac{1}{r^{1-\frac{1}{q}}}, & q<\infty, \\ \displaystyle\int_r^\infty \left(1+\log\frac{s}{t}\right)^{-1} \varphi\left(1+\log\frac{s}{t}\right) \frac{ds}{s^2} \approx \left(1+\log\frac{r}{t}\right)^{-1} \varphi\left(1+\log\frac{r}{t}\right) \frac{1}{r}, & q=\infty. \end{cases}$$

Proof. Consider the function

$$g_q(r) = \begin{cases} -\varphi\left(1+\log\frac{r}{t}\right)\ \frac{1}{r^{1-\frac{1}{q}}}, & q<\infty, \\ -\left(1+\log\frac{r}{t}\right)^{-1} \varphi\left(1+\log\frac{r}{t}\right)\ \frac{1}{r}, & q=\infty. \end{cases}$$

Then, straightforward computations show that there exists some $\lambda > 1$ depending only on φ and on q such that for every $r \geq \lambda t$,

$$g_q'(r) \approx \begin{cases} \varphi\left(1+\log\frac{r}{t}\right)\ \frac{1}{r^{2-\frac{1}{q}}}, & q<\infty, \\ \left(1+\log\frac{r}{t}\right)^{-1} \varphi\left(1+\log\frac{r}{t}\right)\ \frac{1}{r^2}, & q=\infty. \end{cases}$$

Thus, since $\lim_{r\to\infty} g_q(r) = 0$, the result follows. ∎

Lemma 2.5. *Given $x \in \mathbb{R}$ and $0 < \mu \leq 1$. Then*

$$\inf_{y \in (0,\mu]} \varphi(y^{-1}) e^{yx} \lesssim \begin{cases} e^{\mu x}, & \text{if } x \leq 0, \\ \varphi\left(\frac{1+x}{\mu}\right), & \text{if } x > 0. \end{cases}$$

Proof. If $x \leq 0$, the infimum is attained at $y = \mu$, and if $x > 0$, we take $y = \mu/(1+x)$. □

Lemma 2.6. *For every $y \geq 1$,*

$$\sup_{x \in [1,\infty)} \varphi(x) e^{-x/y} \lesssim \varphi(y).$$

Proof. By means of (2.2),

$$\varphi(x) e^{-x/y} \leq \max\left\{1, \left(\frac{x}{y}\right)^{\beta} e^{-x/y}\right\} \varphi(y) \leq \max\left\{1, \beta^{\beta} e^{-\beta}\right\} \varphi(y).$$

□

2.2. **Calderón type operators.**

Definition 2.7. Let $1 \leq q_1, q_2 \leq \infty$, and let φ be an admissible function. Then, for every positive and real valued measurable function f, we define

$$P_{q_1} f(t) := \frac{1}{t^{\frac{1}{q_1}}} \int_0^t f(s) \frac{ds}{s^{1-\frac{1}{q_1}}},$$

$$Q_{q_2,\varphi} f(t) := \begin{cases} \dfrac{1}{t^{\frac{1}{q_2}}} \displaystyle\int_t^{\infty} \varphi\left(1 + \log\frac{s}{t}\right) f(s) \frac{ds}{s^{1-\frac{1}{q_2}}}, & q_2 < \infty, \\ \displaystyle\int_t^{\infty} \left(1 + \log\frac{s}{t}\right)^{-1} \varphi\left(1 + \log\frac{s}{t}\right) f(s) \frac{ds}{s}, & q_2 = \infty, \end{cases} \tag{2.3}$$

and

$$S_{q_1,q_2,\varphi} f(t) := P_{q_1} f(t) + Q_{q_2,\varphi} f(t).$$

In particular, if $q_1 = 1$, $q_2 = \infty$, and $\varphi(x) = x$, we recover the Calderón operator [6]

$$Sf(t) := Pf(t) + Qf(t),$$

where P and Q are respectively the Hardy operator and its conjugate

$$Pf(t) = \frac{1}{t} \int_0^t f(s)\, ds, \qquad Qf(t) = \int_t^{\infty} f(s) \frac{ds}{s}.$$

We observe that, in general,

$$S_{q_1,q_2,\varphi} f(t) = \int_0^1 f(st) \frac{ds}{s^{1-\frac{1}{q_1}}} + \begin{cases} \displaystyle\int_1^{\infty} \varphi(1 + \log s) f(st) \frac{ds}{s^{1-\frac{1}{q_2}}}, & q_2 < \infty, \\ \displaystyle\int_1^{\infty} (1 + \log s)^{-1} \varphi(1 + \log s) f(st) \frac{ds}{s}, & q_2 = \infty. \end{cases} \tag{2.4}$$

For every measurable function f, let f^* be its decreasing rearrangement defined by

$$f^*(t) := \inf\{s > 0 : \lambda_f(s) \le t\}, \qquad \lambda_f(t) := |\{|f| > t\}|, \qquad t > 0,$$

and f^{**} the maximal function of f defined by $f^{**}(t) = P(f^*)(t)$, $t > 0$. For further information about these notions and related topics we refer to [6].

Lemma 2.8. *Let $1 \le q_1, q_2 \le \infty$. For every measurable function f,*

$$S_{q_1,q_2,\varphi}(f^*)^{**}(t) = S_{q_1,q_2,\varphi}(f^{**})(t), \qquad t > 0.$$

Proof. By (2.4), clearly, $S_{q_1,q_2,\varphi}(f^*)$ is a decreasing function. Then, it holds that

$$S_{q_1,q_2,\varphi}(f^*)^{**}(t) = P\big(S_{q_1,q_2,\varphi}(f^*)\big)(t), \qquad t > 0,$$

and the result follows immediately by the Fubini's theorem. □

2.3. **Lorentz spaces and some classes of weights.** Let $0 < p < \infty$, and $0 < q \le \infty$. Let ω be a positive locally integrable function defined on $(0, \infty)$, and define $W(t) = \int_0^t \omega(r)\, dr$, $t > 0$. The weighted Lorentz space $\Lambda^{p,q}(\omega)$ is defined by the condition $||f||_{\Lambda^{p,q}(\omega)} < \infty$ where

$$||f||_{\Lambda^{p,q}(\omega)} = \begin{cases} \left(\displaystyle\int_0^\infty f^*(s)^q W(s)^{\frac{q}{p}-1} \omega(s) ds \right)^{\frac{1}{q}}, & q < \infty, \\ \sup_{t>0} f^*(t) W(t)^{\frac{1}{p}}, & q = \infty. \end{cases}$$

For further information about these notions and related topics we refer to [6, 11, 33].

Definition 2.9. The following classes of weights have appeared in the literature concerning the boundedness of Hardy and conjugate Hardy type operators on the class of monotone decreasing functions on $L^{p,q}(\omega)$, denoted by $L^{p,q}_{\text{dec}}(\omega)$, $0 < p < \infty$, $0 < q \le \infty$.

a) $B_p^{\mathcal{R}}$ class: Concerning the Hardy operator P, we have [10, 12] that for $0 < p \le 1$,

$$P : L^p_{\text{dec}}(\omega) \longrightarrow L^{p,\infty}(\omega) \qquad \Longleftrightarrow \qquad \omega \in B_p^{\mathcal{R}},$$

where $\omega \in B_p^{\mathcal{R}}$ is defined by

$$[\omega]_{B_p^{\mathcal{R}}} = \sup_{0<r\le t<\infty} \frac{rW(t)^{\frac{1}{p}}}{tW(r)^{\frac{1}{p}}} < \infty.$$

In this paper, we extend this definition for the whole range $0 < p < \infty$. In fact, B_p^R can be considered to be the 'restricted' class of the well known B_p class [2, 35] and it is easy to see that all weight in B_p is p quasiconcave, that is $B_p \subset B_p^{\mathcal{R}}$.

b) B_q^* class: Concerning the generalized conjugate Hardy-type operator [28, 35] (see (2.3)),

$$Q_{q_2}(t) := Q_{q_2,1}f(t) = \frac{1}{t^{\frac{1}{q_2}}} \int_t^\infty f(s) \frac{ds}{s^{1-\frac{1}{q_2}}}, \qquad t > 0,$$

on $L^p_{\text{dec}}(\omega)$, it holds that for $0<q_2<\infty$,

$$Q_{q_2} : L^p_{\text{dec}}(\omega) \longrightarrow L^p(\omega) \quad \Longleftrightarrow \quad w \in B^*_{\frac{q_2}{p}},$$

where, for $0<q<\infty$, $\omega \in B^*_q$ if

$$[\omega]_{B^*_q} := \sup_{t>0} \frac{1}{W(t)} \int_0^t \left(\frac{t}{s}\right)^{\frac{1}{q}} \omega(s)\, ds < \infty.$$

In particular, if $\omega \in B^*_q$, we have that, for every $0<r<t$,

$$W(r)\left(\frac{t}{r}\right)^{\frac{1}{q}} \leq \int_0^t \left(\frac{t}{s}\right)^{\frac{1}{q}} \omega(s) ds \leq [\omega]_{B^*_q} W(t),$$

and therefore,

$$\frac{W(r)}{W(t)} \leq [\omega]_{B^*_q} \left(\frac{r}{t}\right)^{\frac{1}{q}}. \tag{2.5}$$

c) B^*_∞ class: Concerning the adjoint of the Hardy operator Q, we have [1] that for every $p>0$,

$$Q : L^p_{\text{dec}}(\omega) \longrightarrow L^p(\omega) \iff \omega \in B^*_\infty,$$

where $\omega \in B^*_\infty$ is defined by

$$[\omega]_{B^*_\infty} = \sup_{t>0} \frac{1}{W(t)} \int_0^t \frac{W(s)}{s}\, ds < \infty.$$

Hence, if $\omega \in B^*_\infty$, we have that, for every $0<r<t$,

$$W(r) \log\left(\frac{t}{r}\right) \leq \int_0^t \log\left(\frac{t}{s}\right) \omega(s) ds = \int_0^t \frac{W(s)}{s}\, ds \leq [\omega]_{B^*_\infty} W(t),$$

and therefore, if we define for $\lambda \in (0,\infty)$,

$$\overline{W}(\lambda) := \sup_{t>0} \frac{W(\lambda t)}{W(t)},$$

then

$$\overline{W}(\lambda) \leq [\omega]_{B^*_\infty} \left(\log \frac{1}{\lambda}\right)^{-1}, \qquad 0<\lambda<1. \tag{2.6}$$

From here the following result follows easily:

Lemma 2.10. *If $\omega \in B^*_\infty$ then*

$$\overline{W}(\lambda) \leq e \lambda^{1/(e[\omega]_{B^*_\infty})}, \qquad 0<\lambda<1.$$

Proof. Let $\lambda_0 = e^{-e[\omega]_{B^*_\infty}}$. Since $\overline{W}$ is submultiplicative, by (2.6) and induction on $n \in \mathbb{N}\cup\{0\}$ we get that

$$\overline{W}(\lambda_0^n) \leq \left(\overline{W}(\lambda_0)\right)^n \leq \left(\frac{1}{e}\right)^n = \left(\lambda_0^n\right)^{\frac{1}{e[\omega]_{B^*_\infty}}}.$$

Now take $\lambda \in (0,1)$ and choose $n \in \mathbb{N} \cup \{0\}$ such that $\lambda_0^{n+1} \leq \lambda < \lambda_0^n$. Then, since $\overline{W}$ is increasing,

$$\overline{W}(\lambda) \leq \overline{W}(\lambda_0^n) \leq \left(\lambda_0^n\right)^{\frac{1}{e\,[\omega]_{B^*_\infty}}} \leq e\lambda^{1/(e[\omega]_{B^*_\infty})}.$$

□

A similar result can be obtained for the B^*_q weights.

Lemma 2.11. *Let $1 < q < \infty$. If $\omega \in B^*_q$, then*

$$\overline{W}(\lambda) \leq 4q\,[\omega]_{B^*_q} \lambda^{\frac{1}{q} + \frac{1}{4q[\omega]_{B^*_q}}}.$$

Proof. First of all, C.J. Neugebauer proved in [35] that if $\omega \in B^*_q$ for some $q \in (0,+\infty)$ then there exists some $\varepsilon = \varepsilon(q,\omega) > 0$ such that $\omega \in B^*_{q-\varepsilon}$. In particular, following the estimates used in [35], for $q \geq 1$ and $\omega \in B^*_q$, taking $\varepsilon = \frac{q}{4[\omega]_{B^*_q}}$ we have that

$$[\omega]_{B^*_{q-\varepsilon}} \leq 4q\,[\omega]_{B^*_q}.$$

Hence, from (2.5) we obtain that for every $0 < r < t$,

$$\frac{W(r)}{W(t)} \leq [\omega]_{B^*_{q-\varepsilon}} \left(\frac{r}{t}\right)^{\frac{1}{q}} \left(\frac{r}{t}\right)^{\frac{\varepsilon}{q(q-\varepsilon)}} \leq 4q\,[\omega]_{B^*_q} \left(\frac{r}{t}\right)^{\frac{1}{q}} \left(\frac{r}{t}\right)^{\frac{1}{4q[\omega]_{B^*_q}}}.$$

Therefore, if $\lambda \in (0,1)$ we get that

$$\overline{W}(\lambda) \leq 4q\,[\omega]_{B^*_q} \lambda^{\frac{1}{q} + \frac{1}{4q[\omega]_{B^*_q}}}.$$

□

As an example of the weights presented we have the power weights, which will take an important role in the proofs of the main results.

Lemma 2.12. *Let $\omega(t) = t^{\tau-1}$.*

1) If $0 < p < \infty$ and $0 < \tau \leq p$, then $\omega \in B^{\mathcal{R}}_p \cap B^*_\infty$ with $[\omega]_{B^{\mathcal{R}}_p} = 1$ and $[\omega]_{B^*_\infty} = \tau^{-1}$.

2) If $0 < q < \infty$ and $\tau > \frac{1}{q}$, then $\omega \in B^*_q$ with $[\omega]_{B^*_q} = \dfrac{\tau}{\tau - \frac{1}{q}}$.

3. Proof of our main results

Proof of the necessity of Theorem 1.6. We will first prove the result when $f = \chi_E$ with E a measurable set of finite measure. Then, using our hypothesis with $\omega(t) = t^{\tau-1}$ and Lemma 2.12, we get that, for every $t > 0$ and every $\tau \in (0,1]$,

$$(T\chi_E)^*(t) \lesssim \varphi(\tau^{-1}) \left(\frac{|E|}{t}\right)^{\tau}. \tag{3.1}$$

Taking the infimum in $\tau \in (0,1]$, and using Lemma 2.5 we obtain that

$$(T\chi_E)^*(t) \lesssim \left[\left(\frac{|E|}{t}\right)\chi_{(|E|,\infty)}(t) + \varphi\left(1+\log\frac{|E|}{t}\right)\chi_{(0,|E|)}(t)\right],$$

and by Lemma 2.3,

$$\begin{aligned}(T\chi_E)^*(t) &\lesssim \frac{1}{t}\int_0^t (\chi_E)^*(s)\,ds + \int_t^\infty \left(1+\log\frac{s}{t}\right)^{-1}\varphi\left(1+\log\frac{s}{t}\right)(\chi_E)^*(s)\,\frac{ds}{s}\\ &= S_{1,\infty,\varphi}(\chi_E)^*(t).\end{aligned}$$

The extension to simple functions with compact support follows the same lines as the proof of Theorem III.4.7 of [6]. We include the computations adapted to our case for the sake of completeness. First of all, consider a positive simple function

$$f = \sum_{j=1}^n a_j\chi_{F_j}, \tag{3.2}$$

where $F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n$ have finite measure. Then

$$f^* = \sum_{j=1}^n a_j\chi_{[0,|F_j|)}.$$

Using what we have already proved for characteristic functions we get

$$\begin{aligned}(Tf)^{**}(t) &\le \sum_{j=1}^n a_j\left(T(\chi_{F_j})\right)^{**}(t) \lesssim \sum_{j=1}^n a_j\left(S_{1,\infty,\varphi}(\chi_{[0,|F_j|)})\right)^{**}(t)\\ &= \left(S_{1,\infty,\varphi}\left(\sum_{j=1}^n a_j\chi_{[0,|F_j|)}\right)\right)^{**}(t) = S_{1,\infty,\varphi}(f^*)^{**}(t).\end{aligned} \tag{3.3}$$

Since $S_{1,\infty,\varphi}(f^*)^{**} = S_{1,\infty,\varphi}(f^{**})$ (see Lemma 2.8) we finally obtain that

$$(Tf)^{**}(t) \lesssim S_{1,\infty,\varphi}(f^{**})(t). \tag{3.4}$$

Fix $t > 0$ and consider the set $E = \{x : f(x) > f^*(t)\}$. Using that set define

$$g = (f - f^*(t))^+\chi_E \quad \text{and} \quad h = f^*(t)\chi_E + f\chi_{E^c}. \tag{3.5}$$

Then

$$g^*(r) = (f^*(r) - f^*(t))^+ \quad \text{and} \quad h^*(r) = \min\{f^*(r), f^*(t)\}.$$

Since $\omega = 1$ belong to $B_1^{\mathcal{R}} \cap B_\infty^*$, the corresponding weak inequality leads to

$$(Tg)^*(t/2) \lesssim \frac{1}{t}\int_0^t f^*(s)\,ds - f^*(t).$$

On the other hand, using (3.4) we get

$$(Th)^{**}(t) \lesssim S_{1,\infty,\varphi}(h^{**})(t) = P_1(h^{**})(t) + Q_{\infty,\varphi}(h^{**})(t) = f^*(t) + Q_{\infty,\varphi}(h^{**})(t),$$

where the last equality holds because $h^*(r) = f^*(t)$ for every $r \in [0,t]$. Now, consider the auxiliary function

$$\widetilde{\varphi}(x) = \frac{\varphi(x)}{x}.$$

By Fubini's theorem,

$$\begin{aligned} Q_{\infty,\varphi}(h^{**})(t) &= \int_t^\infty \widetilde{\varphi}\left(1+\log\frac{s}{t}\right) h^{**}(s)\,\frac{ds}{s} \\ &= \int_0^t f^*(t)\left(\int_t^\infty \widetilde{\varphi}\left(1+\log\frac{s}{t}\right)\frac{ds}{s^2}\right) dr \\ &\quad + \int_t^\infty \left(\int_r^\infty \widetilde{\varphi}\left(1+\log\frac{s}{t}\right)\frac{ds}{s^2}\right) f^*(r)\,dr = I_1 + I_2. \end{aligned} \tag{3.6}$$

On the one hand, the first integral is a multiple of $f^*(t)$. Indeed,

$$\begin{aligned} I_1 &= \int_0^t f^*(t)\left(\int_t^\infty \widetilde{\varphi}\left(1+\log\frac{s}{t}\right)\frac{ds}{s^2}\right) dr \\ &= \frac{1}{t}\int_0^t f^*(t)\left(\int_1^\infty \widetilde{\varphi}\left(1+\log u\right)\frac{du}{u^2}\right) dr = C_1\, f^*(t). \end{aligned}$$

On the other hand, to study the second integral we will make use of Lemma 2.4. To do so, we take any $\lambda > 1$ and observe that

$$\begin{aligned} I_2 &= \int_t^\infty \left(\int_r^\infty \widetilde{\varphi}\left(1+\log\frac{s}{t}\right)\frac{ds}{s^2}\right) f^*(r)\,dr \\ &= \int_t^{\lambda t} + \int_{\lambda t}^\infty \left(\int_r^\infty \widetilde{\varphi}\left(1+\log\frac{s}{t}\right)\frac{ds}{s^2}\right) f^*(r)\,dr. \end{aligned}$$

The first part is similar to I_1, and it can be also controlled by a multiple of $f^*(t)$. Indeed, using that f^* is decreasing we get

$$\begin{aligned} \int_t^{\lambda t} \left(\int_r^\infty \widetilde{\varphi}\left(1+\log\frac{s}{t}\right)\frac{ds}{s^2}\right) f^*(r)\,dr &\le \frac{f^*(t)}{t}\int_t^{\lambda t}\left(\int_{r/t}^\infty \widetilde{\varphi}\left(1+\log u\right)\frac{du}{u^2}\right) dr \\ &\le \frac{f^*(t)}{t}\int_t^{\lambda t}\left(\int_1^\infty \widetilde{\varphi}\left(1+\log u\right)\frac{du}{u^2}\right) dr \\ &= C_2\lambda\, f^*(t). \end{aligned}$$

For the second part, by Lemma 2.4 we know that there exists some $\lambda = \lambda(\varphi) > 1$ such that

$$\int_r^\infty \widetilde{\varphi}\left(1+\log\frac{s}{t}\right)\frac{ds}{s^2} \approx \frac{1}{r}\,\widetilde{\varphi}\left(1+\log\frac{r}{t}\right).$$

Therefore,

$$\int_{\lambda t}^\infty \left(\int_r^\infty \widetilde{\varphi}\left(1+\log\frac{s}{t}\right)\frac{ds}{s^2}\right) f^*(r)\,dr \lesssim \int_{\lambda t}^\infty \widetilde{\varphi}\left(1+\log\frac{r}{t}\right) f^*(r)\,\frac{dr}{r}.$$

In conclusion, putting I_1 and I_2 together we obtain that

$$Q_{\infty,\varphi}(h^{**})(t) \lesssim f^*(t) + \int_t^\infty \widetilde{\varphi}\left(1+\log\frac{r}{t}\right) f^*(r)\,\frac{dr}{r} = f^*(t) + Q_{\infty,\varphi}(f^*)(t).$$

Thus,

$$(Tf)^*(t) \leq (Tg)^*(t/2) + (Th)^{**}(t/2) \lesssim S_{1,\infty,\varphi}(f^*)(t).$$

Finally, the general case follows from this particular case dividing the function in its positive and negative parts.

Proof of the sufficiency of Theorem 1.6. Suppose that $(Tf)^*(t) \lesssim S_{1,\infty,\varphi}(f^*)(t)$ for every $t > 0$. The operator $S_{1,\infty,\varphi}$ has the form

$$S_{1,\infty,\varphi}f(t) = \int_0^\infty k(t,s)f(s)ds,$$

where the kernel is

$$k(t,s) = \frac{1}{t}\chi_{[0,t)}(s) + \frac{1}{s}\left(1+\log\frac{s}{t}\right)^{-1}\varphi\left(1+\log\frac{s}{t}\right)\chi_{[t,\infty)}(s).$$

So, using Theorem 3.3 in [12], the norm $\|S_{1,\infty,\varphi}\|_{\Lambda^1(w)\to\Lambda^{1,\infty}(w)}$ can be estimated by

$$A_k := \sup_{t>0}\left(\sup_{r>0}\left(\int_0^r k(t,s)ds\right)W(r)^{-1}\right)W(t).$$

Note that, if $0<r<t$ then we have

$$\int_0^r k(t,s)\,ds = \frac{r}{t}.$$

On the other hand, if $r>t$, by Lemma 2.3 we obtain

$$\int_0^r k(t,s)\,ds \approx \varphi\left(1+\log\frac{r}{t}\right).$$

As a consequence we have that

$$A_k \approx \sup_{t>0}\max\left\{\left(\sup_{0<r<t}\frac{r}{t}\frac{W(t)}{W(r)}\right), \left(\sup_{r>t}\varphi\left(1+\log\frac{r}{t}\right)\frac{W(t)}{W(r)}\right)\right\}.$$

Since

$$\sup_{0<r<t}\frac{r}{t}\frac{W(t)}{W(r)} \leq [\omega]_{B_1^{\mathcal{R}}},$$

and $[\omega]_{B_1^{\mathcal{R}}} \geq 1$ we get that

$$A_k \lesssim \max\left\{[\omega]_{B_1^{\mathcal{R}}}\ ,\ \sup_{t>0}\sup_{r>t}\left\{\varphi\left(1+\log\frac{r}{t}\right)\frac{W(t)}{W(r)}\right\}\right\}.$$

Further, if $\lambda = t/r < 1$, then by Corollary 2.10

$$\frac{W(t)}{W(r)} = \frac{W(\lambda r)}{W(r)} \leq e\lambda^{1/(e[\omega]_{B_\infty^*})}.$$

Therefore,

$$\sup_{t>0}\sup_{r>t}\left\{\varphi\left(1+\log\frac{r}{t}\right)\frac{W(t)}{W(r)}\right\} \leq e\sup_{\lambda<1}\left\{\varphi\left(1+\log\frac{1}{\lambda}\right)\lambda^{1/(e[\omega]_{B^*_\infty})}\right\}$$
$$\leq e^{1+1/e}\sup_{x>1}\left\{\varphi(x)e^{-x/(e[\omega]_{B^*_\infty})}\right\}.$$

Finally, by Lemma 2.6 and the inequality (2.2) we obtain that

$$\sup_{t>0}\sup_{r>t}\left\{\varphi\left(1+\log\frac{r}{t}\right)\frac{W(t)}{W(r)}\right\} \lesssim \varphi(e[\omega]_{B^*_\infty}) \lesssim \varphi([\omega]_{B^*_\infty}),$$

and we arrive to the desired result

$$A_k \lesssim \max\{[\omega]_{B_1^{\mathcal{R}}}, \varphi([\omega]_{B^*_\infty})\}.$$

□

Proof of Theorem 1.2. The proof follows by direct application of Theorems 1.3 and 1.6. □

Proof of Corollary 1.7. We see that the hypothesis is equivalent to equation (3.1), and hence, the proof follows immediately from the proof of the necessity of Theorem 1.6. □

Proof of Theorem 1.8. By the Rubio de Francia's extrapolation [20], the hypothesis implies that, for every $p \geq 1$,

$$||Tf||_{L^{p,\infty}} \lesssim \varphi(p)||f||_{L^{p,1}},$$

and the results follows from Corollary 1.7. □

Now, in order to prove Theorem 1.9, we first need to have the analogue of Theorem 1.3 which can be found in [4].

Theorem 3.1. *Let $1 \leq p_0 < \infty$, $0 < \alpha \leq 1$ and let T be an operator satisfying that, for every $u \in A_1$,*

$$T : L^{p_0,1}(u^\alpha) \to L^{p_0,\infty}(u^\alpha)$$

is bounded with constant less than or equal to $\varphi(||u||_{A_1})$, where φ is an increasing function in $[1,\infty)$. Then, for every $\omega \in B^{\mathcal{R}}_{\frac{1}{p_0}} \cap B^*_{\frac{p_0}{1-\alpha}}$,

$$T : \Lambda^1(\omega) \to \Lambda^{1,\infty}(\omega)$$

is bounded with norm less than or equal to $C_1[\omega]^{\frac{1}{p_0}}_{B^{\mathcal{R}}_{\frac{1}{p_0}}}\,\overline{\varphi}\left(C_2[\omega]_{B^*_{\frac{p_0}{1-\alpha}}}\right)$, for some positive constants C_1, C_2 independent of ω and where

$$\overline{\varphi}(x) = \begin{cases} \varphi(x^{\frac{p_0}{\alpha}}), & 0<\alpha<1, \\ \varphi(x), & \alpha = 1. \end{cases} \tag{3.7}$$

Further, we will need the following generalization of Theorem 1.6.

Theorem 3.2. *Let T be a sublinear operator and let φ be an admissible function. If for every exponent $1 \le q_1 < q_2 \le \infty$ and for every weight $\omega \in B^{\mathcal{R}}_{\frac{1}{q_1}} \cap B^*_{q_2}$,*

$$T : \Lambda^1(\omega) \to \Lambda^{1,\infty}(\omega)$$

is bounded with norm less than or equal to $C[\omega]^{\frac{1}{q_1}}_{B^{\mathcal{R}}_{\frac{1}{q_1}}} \varphi\left([\omega]_{B^*_{q_2}}\right)$ then, for every $t > 0$ and every measurable function f,

$$\begin{aligned}(Tf)^*(t) &\lesssim S_{q_1,q_2,\varphi}(f^*)(t) \\ &:= \frac{1}{t^{\frac{1}{q_1}}} \int_0^t f^*(s) \frac{ds}{s^{1-\frac{1}{q_1}}} + \begin{cases} \dfrac{1}{t^{\frac{1}{q_2}}} \displaystyle\int_t^\infty \varphi\left(1 + \log \frac{s}{t}\right) f^*(s) \frac{ds}{s^{1-\frac{1}{q_2}}}, & q_2 < \infty, \\ \displaystyle\int_t^\infty \left(1 + \log \frac{s}{t}\right)^{-1} \varphi\left(1 + \log \frac{s}{t}\right) f^*(s) \frac{ds}{s}, & q_2 = \infty. \end{cases}\end{aligned}$$

Conversely, suppose that $(Tf)^*(t) \lesssim S_{q_1,q_2,\varphi}(f^*)(t)$, $t > 0$. Then

$$\|T\|_{\Lambda^1(\omega)\to\Lambda^{1,\infty}(\omega)} \lesssim \begin{cases} [\omega]^{\frac{1}{q_1}}_{B^{\mathcal{R}}_{\frac{1}{q_1}}} [\omega]_{B^*_{q_2}} \varphi([\omega]_{B^*_{q_2}}), & q_2 < \infty, \\ [\omega]^{\frac{1}{q_1}}_{B^{\mathcal{R}}_{\frac{1}{q_1}}} \varphi([\omega]_{B^*_\infty}), & q_2 = \infty. \end{cases}$$

Proof. First, assume that for every weight $\omega \in B^{\mathcal{R}}_{\frac{1}{q_1}} \cap B^*_{q_2}$,

$$T : \Lambda^1(\omega) \to \Lambda^{1,\infty}(\omega)$$

is bounded with norm less than or equal to $C[\omega]^{\frac{1}{q_1}}_{B^{\mathcal{R}}_{\frac{1}{q_1}}} \varphi\left([\omega]_{B^*_{q_2}}\right)$. Note that by Lemma 2.12, $\omega(t) = t^{\tau-1}$ belong to $B^{\mathcal{R}}_{\frac{1}{q_1}} \cap B^*_{q_2}$ for every $\tau \in \left(\frac{1}{q_2}, \frac{1}{q_1}\right]$. Hence, using our hypothesis, we obtain that for every measurable set E,

$$(T\chi_E)^*(t) \lesssim \varphi\left(\left[\tau - \frac{1}{q_2}\right]^{-1}\right) \left(\frac{|E|}{t}\right)^\tau = \left[\varphi\left(\tilde{\tau}^{-1}\right) \left(\frac{|E|}{t}\right)^{\tilde{\tau}}\right] \left(\frac{|E|}{t}\right)^{\frac{1}{q_2}},$$

with $\tilde{\tau} = \tau - \frac{1}{q_2}$. Hence, taking the infimum in $\tilde{\tau} \in \left(0, \frac{1}{q_1} - \frac{1}{q_2}\right]$ and using Lemma 2.5 we get that

$$(T\chi_E)^*(t) \lesssim \left(\frac{|E|}{t}\right)^{\frac{1}{q_1}} \chi_{(|E|,\infty)}(t) + \varphi\left(1 + \log \frac{|E|}{t}\right) \left(\frac{|E|}{t}\right)^{\frac{1}{q_2}} \chi_{(0,|E|)}(t).$$

Then, by Lemma 2.3,

$$(T\chi_E)^*(t) \lesssim S_{q_1,q_2,\varphi}(\chi_E)^*(t).$$

The extension to positive simple functions with support in a set of finite measure follows the same lines as the proof of the necessity of Theorem 1.6 with few modifications. First of all, we consider a positive simple function like the one in (3.2). Hence, as in (3.3), using what we have already proved for characteristic functions together with the sublinearity of T and the equivalence $S_{q_1,q_2,\varphi}(f^*)(t)^{**} \approx S_{q_1,q_2,\varphi}(f^{**})(t)$ (see Lemma 2.8), we obtain that

$$(Tf)^{**}(t) \lesssim S_{q_1,q_2,\varphi}(f^{**})(t). \tag{3.8}$$

So fix $t > 0$ and take the functions g and h from f defined in (3.5). Since the weight $\omega(r) = r^{\frac{1}{q_1}-1}$ is in $B^{\mathcal{R}}_{\frac{1}{q_1}} \cap B^*_{q_2}$, the corresponding weak inequality leads to

$$(Tg)^*(t/2) \lesssim \frac{1}{t^{\frac{1}{q_1}}} \int_0^t (f^*(s) - f^*(t)) \frac{ds}{s^{1-\frac{1}{q_1}}} \approx \frac{1}{t^{\frac{1}{q_1}}} \int_0^t f^*(s) \frac{ds}{s^{1-\frac{1}{q_1}}} - f^*(t).$$

On the other hand, using (3.8) for h instead of f we get

$$(Th)^{**}(t) \lesssim S_{q_1,q_2,\varphi}(h^{**})(t) = P_{q_1}(h^{**})(t) + Q_{q_2,\varphi}(h^{**})(t).$$

First, since $h^*(r) = f^*(t)$ for every $r \in [0, t]$,

$$P_{q_1}(h^{**})(t) \approx f^*(t).$$

Besides, arguing as we did to bound (3.6) with the auxiliary function

$$\widetilde{\varphi}(x) = \begin{cases} \varphi(x) e^{\frac{x-1}{q_2}}, & 1 < q_2 < \infty, \\ \dfrac{\varphi(x)}{x}, & q_2 = \infty, \end{cases}$$

then we deduce that

$$Q_{q_2,\varphi}(h^{**})(t) \lesssim f^*(t) + Q_{q_2,\varphi}(f^*)(t),$$

and the result follows.

Conversely, assume that $(Tf)^*(t) \lesssim S_{q_1,q_2,\varphi}(f^*)(t)$ for every $t > 0$. If $q_2 < \infty$ then the operator $S_{q_1,q_2,\varphi}$ has the form

$$S_{q_1,q_2,\varphi}(f^*)(t) = \int_0^\infty k(t,s) f^*(s) ds,$$

where the kernel is

$$k(t,s) = \frac{1}{t^{\frac{1}{q_1}}} \chi_{[0,t)}(s) \frac{1}{s^{1-\frac{1}{q_1}}} + \varphi\left(1 + \log \frac{s}{t}\right) \left(\frac{s}{t}\right)^{\frac{1}{q_2}} \chi_{[t,\infty)}(s) \frac{1}{s}.$$

By Theorem 3.3 in [12], the norm the norm $\|S_{q_1,q_2,\varphi}\|_{\Lambda^1(w) \to \Lambda^{1,\infty}(w)}$ can be estimated by

$$A_k := \sup_{t>0} \left(\sup_{r>0} \left(\int_0^r k(t,s) ds \right) W(r)^{-1} \right) W(t).$$

Now if $0 < r < t$ then we have

$$\int_0^r k(t,s)\, ds = \left(\frac{r}{t}\right)^{\frac{1}{q_1}},$$

while if $r > t$, then by Lemma 2.3 we obtain

$$\int_0^r k(t,s)\,ds \approx \varphi\left(1+\log\frac{r}{t}\right)\left(\frac{r}{t}\right)^{\frac{1}{q_2}}.$$

In consequence, we have that

$$A_k \approx \sup_{t>0} \max\left\{\left(\sup_{0<r<t}\left(\frac{r}{t}\right)^{\frac{1}{q_1}}\frac{W(t)}{W(r)}\right), \left(\sup_{r>t}\varphi\left(1+\log\frac{r}{t}\right)\left(\frac{r}{t}\right)^{\frac{1}{q_2}}\frac{W(t)}{W(r)}\right)\right\}.$$

Since

$$\sup_{0<r<t}\left(\frac{r}{t}\right)^{\frac{1}{q_1}}\frac{W(t)}{W(r)} \leq [w]_{B^{\mathcal{R}}_{\frac{1}{q_1}}}^{\frac{1}{q_1}},$$

we get that

$$A_k \lesssim \max\left\{[w]_{B_1^{\mathcal{R}}}^{\frac{1}{q_1}},\ \sup_{t>0}\sup_{r>t}\left\{\varphi\left(1+\log\frac{r}{t}\right)\left(\frac{r}{t}\right)^{\frac{1}{q_2}}\frac{W(t)}{W(r)}\right\}\right\}.$$

Hence, if $\lambda = t/r < 1$, then by Lemma 2.11

$$\left(\frac{r}{t}\right)^{\frac{1}{q_2}}\frac{W(t)}{W(r)} = \lambda^{-\frac{1}{q_2}}\frac{W(\lambda r)}{W(r)} \leq 4q\,[\omega]_{B^*_{q_2}}\lambda^{\frac{1}{4q_2[\omega]_{B^*_{q_2}}}}.$$

Therefore

$$\begin{aligned}\sup_{t>0}\sup_{r>t}\left\{\varphi\left(1+\log\frac{r}{t}\right)\left(\frac{r}{t}\right)^{\frac{1}{q_2}}\frac{W(t)}{W(r)}\right\} &\lesssim [\omega]_{B^*_{q_2}}\sup_{\lambda<1}\left\{\varphi\left(1+\log\frac{1}{\lambda}\right)\lambda^{\frac{1}{4q_2[\omega]_{B^*_{q_2}}}}\right\}\\ &\lesssim [\omega]_{B^*_{q_2}}\sup_{x>1}\left\{\varphi(x)e^{-x/(4q_2[\omega]_{B^*_{q_2}})}\right\}.\end{aligned}$$

Finally, by Lemma 2.6 and the inequality (2.2) we obtain that

$$\begin{aligned}\sup_{t>0}\sup_{r>t}\left\{\varphi\left(1+\log\frac{r}{t}\right)\left(\frac{r}{t}\right)^{\frac{1}{q_2}}\frac{W(t)}{W(r)}\right\} &\lesssim [\omega]_{B^*_{q_2}}\varphi\big(4q_2[\omega]_{B^*_{q_2}}\big)\\ &\lesssim [\omega]_{B^*_{q_2}}\varphi\big([w]_{B^*_{q_2}}\big).\end{aligned}$$

Combining both estimates we get

$$A_k \lesssim \max\left\{[w]_{B_1^{\mathcal{R}}}^{\frac{1}{q_1}},\ [\omega]_{B^*_{q_2}}\varphi\big([w]_{B^*_{q_2}}\big)\right\},$$

which leads to the desired result. If $q_2 = \infty$, the proof is a combination of the proof for the case $q_2 < \infty$ and the proof of the sufficiency in Theorem 1.6. □

We are finally ready to prove our last main result:

Proof of Theorem 1.9. Observe that $\overline{\varphi}$ as in (3.7) satisfies the same properties as φ. Therefore, the proof follows by direct application of Theorems 3.1 and 3.2. □

4. Examples and applications

We shall present several examples of very interesting operators in Harmonic analysis for which our results give a pointwise estimate of the decreasing rearrangement.

4.1. **Fefferman-Stein inequality.** An operator T is said to satisfy a Fefferman-Stein's inequality ([23]) if, for every positive and locally integrable function u,

$$\int_{\{|Tf(x)|>y\}} u(x)dx \lesssim \int |f(x)|Mu(x)dx. \tag{4.1}$$

Clearly, for every operator satisfying (4.1) we have that

$$T : L^1(u) \longrightarrow L^{1,\infty}(u)$$

is bounded with norm less than or equal to $C||u||_{A_1}$ and hence, as a consequence of Theorem 1.2, we get the following:

Corollary 4.1. *For every $t>0$ and every measurable function f,*

$$(Tf)^*(t) \lesssim \frac{1}{t}\int_0^t f^*(s)\,ds + \int_t^\infty f^*(s)\,\frac{ds}{s}.$$

This is the case (among many others operators) of the area function [13] defined by

$$Sf(x) = \left(\int_{|x-y|\le t} |\nabla_{y,t}(f * P_t)(y)|^2\right)^{1/2},$$

where

$$\nabla_{y,t} = \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \cdots, \frac{\partial}{\partial y_n}, \frac{\partial}{\partial t}\right), \qquad P_t(y) = \frac{c_n t}{(t^2+|y|^2)^{(n+1)/2}}.$$

4.2. **Bochner-Riesz and Rough singular operators.** To begin with, we recall the definition of these operators. Then, we present a known quantitative inequality, which leads to pointwise estimations for the decreasing rearrangement of these operators by using our results.

Bochner-Riesz operators. Let

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx, \qquad \xi \in \mathbb{R}^n,$$

be the Fourier transform of $f \in L^1(\mathbb{R}^n)$. Let $a_+ = \max\{a, 0\}$ denote the positive part of $a \in \mathbb{R}$. Given $\lambda > 0$, the Bochner-Riesz operator B_λ is defined by

$$\widehat{B_\lambda f}(\xi) = \left(1-|\xi|^2\right)_+^\lambda \hat{f}(\xi), \qquad \xi \in \mathbb{R}^n.$$

These operators were first introduced by Bochner [7] and, since then, they have been widely studied. The case $\lambda = 0$ corresponds to the so-called disc multiplier, which is unbounded on $L^p(\mathbb{R}^n)$ if $n \ge 2$ and $p \ne 2$ [22]. When $\lambda > \frac{n-1}{2}$, it is known that $B_\lambda f$ is controlled by the Hardy-Littlewood maximal function Mf. As a consequence, all weighted inequalities for M are also satisfied by B_λ. The value $\lambda = \frac{n-1}{2}$ is called the critical index. In this case, Christ [15] showed that $B_{\frac{n-1}{2}}$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, and Vargas [37] proved that $B_{\frac{n-1}{2}}$ is bounded from $L^1(u)$ to $L^{1,\infty}(u)$ for every $u \in A_1$.

Rough singular integrals. For $n > 1$, set $\Sigma^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ and let Ω be an homogeneous function of degree zero such that

$$\int_{\Sigma^{n-1}} \Omega(x)\, dx = 0. \tag{4.2}$$

The rough singular integral operator is defined by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^n} f(x-y)\, dy, \qquad x \in \mathbb{R}^n,$$

with $y' = \frac{y}{|y|}$. This operator was first introduced by Calderón and Zygmund who proved that [8, 9] T_Ω is bounded on L^p if the even part of Ω belongs to $L \log_+ L$ and its odd part belongs to L^1. Since then, this operator has been widely studied [15, 16, 37]. When $\Omega \in L^\infty$, Duoandikoetxea and Rubio de Francia [21] proved that, for $1 < p < \infty$,

$$T_\Omega : L^p(u) \to L^p(u)$$

is bounded whenever $u \in A_p$, later improved in [19], [38] and [32].

Quantitative results and the pointwise estimation. In [32] the authors obtained the following quantitative result.

Theorem 4.2. *Let T be either the Bochner-Riesz operator $B_{\frac{n-1}{2}}$, where $n > 1$ or the rough singular integral T_Ω, where Ω is in $L^\infty(\Sigma^{n-1})$ and satisfies (4.2). Then,*

$$T : L^2(u) \to L^{2,\infty}(u)$$

is bounded with constant less than or equal to $C||u||^2_{A_2}$.

Hence, using Theorem 1.8, we have the following:

Corollary 4.3. *Consider the hypothesis of Theorem 4.2. Then, for every $t > 0$ and every function f such that Tf is well defined, we have that*

$$(Tf)^*(t) \lesssim \frac{1}{t}\int_0^t f^*(s)\, ds + \int_t^\infty \left(1 + \log\frac{s}{t}\right) f^*(s) \frac{ds}{s}.$$

4.3. **Sparse Operators.** These operators have become very popular due to its role in the so called A_2 conjecture consisting in proving that if T is a Calderón-Zygmund operator then

$$||Tf||_{L^2(v)} \lesssim ||v||_{A_2} ||f||_{L^2(v)}.$$

This result was first obtained by Hytönen [24] and then simplified by Lerner [29, 30], who proved that the norm of a Calderón-Zygmund operator in a Banach function space $\mathbb{X}$ is dominated by the supremum of the norm in $\mathbb{X}$ of all the possible sparse operators and then proved that every sparse operator is bounded on $L^2(v)$ for every weight $v \in A_2$ with sharp constant. Let us give the precise definition. A general dyadic grid $\mathcal{D}$ is a collection of cubes in $\mathbb{R}^n$ satisfying the following properties:

(i) For any cube $Q \in \mathcal{D}$, its side length is 2^k for some $k \in \mathbb{Z}$.
(ii) Every two cubes in $\mathcal{D}$ are either disjoint or one is wholly contained in the other.
(iii) For every $k \in \mathbb{Z}$ and given $x \in \mathbb{R}^n$, there is only one cube in $\mathcal{D}$ of side length 2^k containing it.

Let $0 < \eta < 1$, a collection of cubes $\mathcal{S} \subset \mathcal{D}$ is called η-sparse if one can choose pairwise disjoint measurable sets $E_Q \subset Q$ with $|E_Q| \geq \eta |Q|$, where $Q \in \mathcal{S}$. So, given a η-sparse family of cubes $\mathcal{S}$, the sparse operator is defined by

$$\mathcal{A}_{\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q f(y)\, dy \right) \chi_Q(x), \qquad x \in \mathbb{R}^n.$$

Even though, weighted estimates for these operators are known, one can easily compute the norm in L^p directly, using the standard duality technique: for every, $p \geq 1$ and $g \in L^{p',1}$ with norm equal to 1,

$$\begin{aligned}
\int \mathcal{A}_{\mathcal{S}} f(x) g(x) dx &= \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q f(y)\, dy \right) \int_Q g(x) dx \\
\leq \; & \frac{1}{\eta} \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q f(y)\, dy \right) \frac{1}{|Q|} \int_Q g(x) dx |E_Q| \leq \frac{1}{\eta} \int_{\mathbb{R}^n} Mf(x) Mg(x) dx \\
\leq \; & \frac{1}{\eta} ||Mf||_{L^{p,\infty}} ||Mg||_{L^{p',1}} \leq \frac{C_n}{\eta} \left(\frac{p'}{p'-1} \right) ||f||_{L^{p,1}} ||g||_{L^{p',1}} \approx p ||f||_{L^{p,1}}.
\end{aligned}$$

Therefore, as a consequence of Corollary 1.7, we get the following:

Corollary 4.4. *For every $t > 0$ and every measurable function f,*

$$(\mathcal{A}_{\mathcal{S}} f)^*(t) \lesssim \frac{1}{t} \int_0^t f^*(s)\, ds + \int_t^\infty f^*(s) \frac{ds}{s}.$$

4.4. **Fourier multipliers.** Given $m \in L^\infty(\mathbb{R}^n)$, we say that T_m is a Fourier multiplier if, for every Schwartz function f,

$$\widehat{T_m f}(\xi) = m(\xi) f(\xi), \qquad \xi \in \mathbb{R}^n,$$

and m is called a multiplier. It has been of great interest to identify, when possible, for which maximal operators $\mathcal{M}$ the operator T_m satisfies a Fefferman-Stein's type inequality in L^2 of the form

$$\int_{\mathbb{R}^n} |T_m f(x)|^2 u(x)\, dx \leq \int_{\mathbb{R}^n} |f(x)|^2 \mathcal{M} u(x)\, dx,$$

for measurable functions f and positive locally integrable functions u (see for instance [5, 14, 17, 18, 36]).

In particular, we present the following interesting case.

Proposition 4.5 ([5]). *If $m : \mathbb{R} \to \mathbb{C}$ is a bounded function so that*

$$\sup_{R>0} \int_{R\leq|\xi|\leq 2R} |m'(\xi)|\, d\xi < \infty, \tag{4.3}$$

then, for every measurable function f and every positive locally integrable function u,

$$\int_{\mathbb{R}} |T_m f(x)|^2 u(x)\, dx \leq C \int_{\mathbb{R}} |f(x)|^2 M^7 u(x)\, dx,$$

where $M^7 = M \underbrace{\circ \cdots \circ}_{7} M$ is the 7-fold composition of M with itself.

Hence, if $m \in L^\infty(\mathbb{R})$ satisfies (4.3), then

$$T_m : L^2(u) \longrightarrow L^2(u), \qquad C\, \|u\|_{A_1}^7 .$$

As a consequence of Theorem 1.9, we obtain the following:

Corollary 4.6. *For every $t > 0$ and every measurable function f,*

$$(T_m f)^*(t) \lesssim \frac{1}{t^{\frac{1}{2}}} \int_0^t f^*(s)\, \frac{ds}{t^{\frac{1}{2}}} + \int_t^\infty \left(1 + \log \frac{s}{t}\right)^6 f^*(s)\, \frac{ds}{s}.$$

In this context of Fourier multipliers, let us now consider, for each $\gamma, \beta \in \mathbb{R}$, the class $\mathcal{C}(\gamma, \beta)$ of bounded functions $m : \mathbb{R} \to \mathbb{C}$ for which

$$\operatorname{supp}(m) \subseteq \{\xi : |\xi|^\gamma \geq 1\}, \qquad \sup_{\xi\in\mathbb{R}} |\xi|^\beta |m(\xi)| < \infty,$$

and

$$\sup_{R^\gamma \geq 1} \ \sup_{I \subseteq [R,2R],\, \ell(I) = R^{-\gamma+1}} R^\beta \int_{\pm I} |m'(\xi)|\, d\xi < \infty.$$

Proposition 4.7 ([5]). *Let $\gamma, \beta \in \mathbb{R}$ such that $\gamma > 2\beta$. If $m \in \mathcal{C}(\gamma, \beta)$ then, for every measurable function f and every positive locally integrable function u,*

$$\int_{\mathbb{R}} |T_m f(x)|^2 u(x)\, dx \leq C \int_{\mathbb{R}} |f(x)|^2 M^6 \left(\left(M^5 u^{\frac{\gamma}{2\beta}} \right)^{\frac{2\beta}{\gamma}} \right)(x)\, dx.$$

Therefore, under the hypotheses of the previous result,

$$T_m : L^2(u^{\frac{2\beta}{\gamma}}) \longrightarrow L^2(u^{\frac{2\beta}{\gamma}}), \qquad C \left(\frac{\gamma}{\gamma - 2\beta} \right)^6 \|u\|_{A_1}^{\frac{10\beta}{\gamma}} ,$$

so that, as a consequence of Theorem 1.9, we have the following:

Corollary 4.8. *For every $t > 0$ and every measurable function f,*

$$(T_m f)^*(t) \lesssim \frac{1}{t^{\frac{1}{2}}} \int_0^t f^*(s)\, \frac{ds}{t^{\frac{1}{2}}} + \frac{1}{t^{\frac{\gamma-2\beta}{2\gamma}}} \int_t^\infty \left(1 + \log \frac{s}{t}\right)^{10} f^*(s)\, \frac{ds}{s^{\frac{\gamma+2\beta}{2\gamma}}}.$$

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