On *m*-cluster tilted algebras and trivial extensions ☆

Elsa Fernández [a], Nilda Isabel Pratti [b], Sonia Trepode [b],*

[a] *Facultad de Ingeniería, Universidad Nacional de la Patagonia San Juan Bosco, 9120 Puerto Madryn, Argentina*
[b] *Facultad de Ciencias Exactas y Naturales, Universidad Nacional de Mar del Plata, 7600 Mar del Plata, Argentina*

A R T I C L E   I N F O

A B S T R A C T

In this work we study the connection between iterated tilted algebras and *m*-cluster tilted algebras. We show that an iterated tilted algebra induces an *m*-cluster tilted algebra. This *m*-cluster tilted algebra can be seen as a trivial extension of another iterated tilted algebra which is derived equivalent to the original one. We give a procedure to find this new iterated tilted algebra. These *m*-cluster tilted algebras are quotients of higher relation extensions.

© 2013 Elsevier Inc. All rights reserved.

Introduction

Let H be a finite dimensional hereditary algebra. The cluster category $\mathscr{C}_H$ associated with H was defined in [9] as the orbit category of $D^b(H)/\tau^{-1}[1]$, where τ is the Auslander–Reiten translation and [1] is the shift. The special case of Dynkin type A_n has been also studied in [7]. A cluster tilting object T in the cluster category is an object such that $\mathrm{Ext}^1_{\mathscr{C}_H}(T, T) = 0$ and it is maximal with this property.

The cluster tilted algebras were first introduced by Buan, Marsh and Reiten in [7] as endomorphism rings of cluster tilting objects over the cluster category. Since then, this class of algebras has been extensively studied. Examples of works along these lines can be found, for instance, in [2,6–8,10, 12–14,19].

* Corresponding author.
E-mail addresses: elsafer9@gmail.com (E. Fernández), nilprat@mdp.edu.ar (N.I. Pratti), strepode@mdp.edu.ar (S. Trepode).

In particular, Assem, Brüstle and Schiffler have shown in [2] that cluster tilted algebras are trivial extensions. They defined for an algebra B with gl dim B at most two the relation extension $\mathcal{R}(B)$ of B as the trivial extension $B \ltimes \operatorname{Ext}_B^2(DB, B)$, where $DB = \operatorname{Hom}_B(B, k)$. Then the authors proved that an algebra C is a cluster tilted algebra if and only if it is the relation extension of some tilted algebra B. This connection was extended to iterated tilted algebras of global dimension at most two in [4]. In the mentioned work it was shown that an algebra C is cluster tilted if and only if there exist an iterated tilted algebra B with global dimension at most two and a sequence of homomorphisms

$$B \to C \xrightarrow{\pi} \mathcal{R}(B) \to B$$

whose composition is the identity map and such that the kernel of π is contained in $\operatorname{rad}^2 C$. They also proved that C and $\mathcal{R}(B)$ have the same quivers.

In [1], Amiot proved for any algebra B of global dimension at most two, that there exists a cluster category associated with it such that B is a cluster tilting object in this category. The endomorphism ring of B over that cluster category is the tensor algebra over the B–B-bimodule $\operatorname{Ext}_B^2(DB, B)$. When the algebra B is iterated tilted the mentioned endomorphism algebra is a cluster tilted algebra.

The m-cluster category was introduced by Thomas in [22] as the orbit category $\mathscr{C}_m = D^b(H)/\tau^{-1}[m]$. An object $\widetilde{T}$ is said to be an m-cluster tilting object if $\operatorname{Ext}_{\mathscr{C}_m}^i(\widetilde{T}, \widetilde{T}) = 0$ for $i = 1, \ldots, m$ and the number of isomorphism classes of indecomposable summands of $\widetilde{T}$ is equal to the number of isomorphism classes of simple H-modules. The endomorphism algebra C_m of an m-cluster tilting object over the m-cluster category is called an m-cluster tilted algebra. The procedure introduced by Amiot in [1] can be extended to algebras of global dimension at most $m + 1$ (see [17]), and C_m is the tensor algebra over the B–B-bimodule $\operatorname{Ext}_B^{m+1}(DB, B)$. In case the algebra B is iterated tilted, this endomorphism algebra is an m-cluster tilted algebra. Then m-cluster tilted algebras are tensor algebras. We are going to show that, in particular cases, they can also be seen as trivial extensions.

Our aim in this paper is to study the relation between iterated tilted algebras of global dimension at most $m + 1$ and m-cluster tilted algebras. For an algebra B with global dimension at most $m + 1$ the m-relation extension of B is defined to be the algebra $\mathcal{R}_m(B) = B \ltimes \operatorname{Ext}_B^{m+1}(DB, B)$. Following [21], we recall that an object T in $D^b(H)$ is a tilting complex if and only if $\operatorname{Hom}_{D^b(H)}(T, T[i]) = 0$ for all $i \neq 0$ and T has exactly n non-isomorphic summands, where n is the number of vertices of Q. It follows from [16] that B is an iterated tilted algebra if and only if $B = \operatorname{End}_{D^b(H)}(T)$, where T is a tilting complex.

Let $S_m = \bigcup_{i=0}^{m-1} \operatorname{Ind} H[i] \cup H[m]$ be the standard fundamental domain of $\mathscr{C}_m$, and $\widetilde{X}$ be the class in the orbit category of an object X in $D^b(H)$.

The following is our first result.

Theorem 1. *Let T be a tilting complex in $D^b(H)$ which belongs to S_m. Then $\operatorname{End}_{\mathscr{C}_m}(\widetilde{T})$ is an m-cluster tilted algebra. Moreover if the global dimension of $B = \operatorname{End}_{D^b(H)}(T)$ is at most $m + 1$, then $\operatorname{End}_{\mathscr{C}_m}(\widetilde{T})$ is isomorphic to $\mathcal{R}_m(B)$.*

Given an iterated tilted algebra $B = \operatorname{End}_{D^b(H)}(T)$ of global dimension at most $m + 1$, the corresponding tilting complex T can be spread in an arbitrary number of copies of mod H inside the bounded derived category of H. In order to get a tilting complex in the fundamental domain S_m, we show that the rolling procedure introduced in [4] can be generalized to a new procedure ρ_m. We show that, iterating this procedure, we eventually get for some integer h a tilting complex $\rho_m^h(T)$ in the fundamental domain, such that T and $\rho_m^h(T)$ represent the same object in the m-cluster category. Moreover, the global dimension of $\operatorname{End}_{D^b(H)}(\rho_m^h(T))$ is at most $m + 1$.

Let B be an iterated tilted algebra of global dimension at most $m + 1$, and let T be a tilting complex such that $B = \operatorname{End}_{D^b(H)}(T)$. We denote $C_m(B) = \operatorname{End}_{\mathscr{C}_m}(\widetilde{T})$.

The following is our main theorem.

Theorem 2. *Let $H = kQ$ a hereditary algebra. If T is a tilting complex in $D^b(H)$ such that $B = \operatorname{End}_{D^b(H)}(T)$ has global dimension at most $m + 1$ then:*

a) $\widetilde{T}$ *is an m-cluster tilting object in the m-cluster category $\mathscr{C}_m$ and $C_m(B)$ is an m-cluster tilted algebra.*
b) *There exists a sequence of algebra homomorphisms*

$$B \to C_m(B) \xrightarrow{\pi} \mathcal{R}_m(B) \to B$$

   whose composition is the identity map and such that the kernel of π is contained in $\mathrm{rad}^2 C_m(B)$. *Moreover, $C_m(B)$ and $\mathcal{R}_m(B)$ have the same quiver.*
c) *There exists $h > 0$ such that $B' = \mathrm{End}_{\mathrm{D}^b(H)}(\rho_m^h(T))$ is an iterated tilted algebra of type Q,* $\mathrm{gl\,dim}\, B' \leqslant$
   $m + 1$ *and*

$$C_m(B) \simeq \mathcal{R}_m(B') \simeq C_m(B').$$

On the other hand, these m-cluster tilted algebras are in connection with the higher relation extensions introduced by Assem, Gatica and Schiffler in [3]. More precisely, they are quotients of higher relations extensions. Then, using results in [3] it is possible to describe, in some particular cases, the quivers with relations of the m-cluster tilted algebras induced by iterated tilted algebras.

A natural question to investigate is whether any m-cluster tilted algebra is induced by iterated tilted algebras. That is, if any m-cluster tilted algebra is a direct product of m_i-relation extensions of iterated tilted algebras. In [20], Murphy has classified m-cluster tilted algebras of type A_n. We observe that using his classification is not difficult to prove that algebras of this type are of the desired form.

In Section 1 we fix the notation and recall some well known facts about m-cluster categories. In Section 2 we prove Theorem 1 and we show with an example that the hypothesis that the tilting complex belongs to the fundamental domain is essential. In Section 3 we define the m-rolling procedure. We show that given an arbitrary tilting complex whose endomorphism algebra has global dimension at most $m + 1$ we eventually reach, by iteration of this procedure, a tilting complex in the fundamental domain such that the global dimension of its endomorphism algebra is also bounded by $m + 1$. Finally we prove our main result, Theorem 2.

1. Preliminaries

Throughout this paper let Q be a finite connected quiver without oriented cycles, and k an algebraically closed field. Then $H = kQ$ is a hereditary finite dimensional algebra. We denote by $\mathrm{mod}\,H$ the category of finitely generated right modules over H. As usual, $\mathrm{ind}\,H$ denotes a full subcategory of $\mathrm{mod}\,H$ whose objects are a full set of representatives of the isomorphism classes of indecomposable H-modules.

We consider $\mathrm{D}^b(H) = \mathrm{D}^b(\mathrm{mod}\,H)$ the derived category of bounded complexes of finitely generated H-modules. Recall that in $\mathrm{D}^b(H)$ Serre Duality holds, that is,

$$\mathrm{Hom}_{\mathrm{D}^b(H)}(X, \tau Y) \simeq \mathrm{D\,Hom}_{\mathrm{D}^b(H)}(Y, X[1])$$

for X, Y in $\mathrm{D}^b(H)$, where [1] is the shift functor and τ is the AR-translation in $\mathrm{D}^b(H)$.

It follows from [16] and [21] that if T is a tilting complex and $B = \mathrm{End}_{\mathrm{D}^b(H)}(T)$ then there exists an equivalence of triangulated categories $G : \mathrm{D}^b(H) \to \mathrm{D}^b(B)$ such that $G(T) = B$ and $G(\tau T[1]) = \mathrm{D}B$. Following [4] we denote:

$$P_{X,T} = G(X) \quad \text{and} \quad I_{X,T} = G(\tau X[1])$$

for any direct summand X of T.

The cluster category associated with H was defined and investigated in [9]. Later, for a positive integer m, the m-cluster category $\mathscr{C}_m$ was studied in [22] (see [5,19,23]). By [18] we know that $\mathscr{C}_m$ is a triangulated category. The objects in the m-cluster category are the $F_m = \tau^{-1}[m]$ orbits of objects in $\mathrm{D}^b(H)$ and the morphisms are given by $\mathrm{Hom}_{\mathscr{C}_m}(\widetilde{X}, \widetilde{Y}) = \bigoplus_{i \in Z} \mathrm{Hom}_{\mathrm{D}^b(H)}(X, F_m^i Y)$ where X, Y are objects in $\mathrm{D}^b(H)$ and $\widetilde{X}$, $\widetilde{Y}$ are their respective F_m-orbits.

It is clear that $S_m = \bigcup_{i=0}^{m-1} \operatorname{Ind} H[i] \cup H[m]$ is a fundamental domain for the action of F_m in $\operatorname{Ind} D^b(H)$. The set S_m contains exactly one representative from each F_m-orbit in $\operatorname{Ind} D^b(H)$.

A characterization of an object $\widetilde{T}$ of $\mathscr{C}_m$ as an m-cluster tilting object is given by $\operatorname{Ext}^i_{\mathscr{C}_m}(\widetilde{T}, \widetilde{T}) = 0$ for $i = 1, \ldots, m$ and the number of isomorphism classes of indecomposable summands of T is equal to the number of isomorphism classes of simple H-modules. From now on, we will assume for simplicity that $\widetilde{T}$ is basic in the usual sense.

2. Trivial extensions and *m*-cluster tilted algebras

In all that follows let B be a connected finite dimensional algebra over an algebraically closed field k and M a B–B-bimodule. Recall that the trivial extension of B by M is the algebra $B \ltimes M$ with underlying vector space $B \oplus M$, and multiplication given by $(a, m)(a', m') = (aa', am' + ma')$, for any $a, a' \in B$ and $m, m' \in M$. For further properties of trivial extensions we refer the reader to [15]. Let B be an algebra of global dimension at most $m + 1$ and $DB = \operatorname{Hom}_k(B, k)$. The trivial extension $\mathcal{R}_m(B) = B \ltimes \operatorname{Ext}^{m+1}_B(DB, B)$ is called the m-relation extension of B. This concept is the m-ified analogue of the notion of relation extension introduced in [2]. In this work the authors proved that an algebra C is a cluster tilted algebra if and only if it is the relation extension of some tilted algebra B.

We are going to show that there exists a connection between the m-relation extensions of iterated tilted algebras of global dimension at most $m + 1$ and m-cluster tilted algebras. In fact, we are going to prove that if the tilting complex T belongs to the fundamental domain S_m, then the m-relation extension of the iterated tilted algebra is an m-cluster tilted algebra.

We start with the following technical lemma, which will be needed in the sequel. For $m = 1$ this result is proven in [9, Proposition 1.5(a)].

Lemma 2.1. *Let X and Y be objects of $S_m = \bigcup_{i=0}^{m-1} \operatorname{Ind} H[i] \cup H[m]$. Then $\operatorname{Hom}_{D^b(H)}(F_m^i X, Y) = 0$ for all $i \neq -1, 0$.*

Proof. We have $\operatorname{Hom}_{D^b(H)}(F_m^i X, Y) = \operatorname{Hom}_{D^b(H)}(\tau^{-i} X[im], Y)$. The statement can be proved by considering eight cases, according the integer $i \neq -1, 0$ is positive or negative, and the objects $\tau^{-1} X, Y$ in S_m are in $\bigcup_{i=0}^{m-1} \operatorname{Ind} H[i]$ or in $H[m]$. In each case the proof is straightforward. $\square$

In analogy with [2] we have the following.

Remark 2.2. Let X be an object in $D^b(H)$. We can consider the k-vector space $\operatorname{Hom}_{D^b(H)}(X, F_m X)$ as an $\operatorname{End}_{D^b(H)}(X)$-bimodule defining $u f v = F_m(u) f v$ for u, v in $\operatorname{End}_{D^b(H)} X$ and f in $\operatorname{Hom}_{D^b(H)}(X, F_m X)$.

Next we show that the endomorphism algebra $\operatorname{End}_{\mathscr{C}_m}(\widetilde{X})$ is a trivial extension. This result is an m-ified analogue of Lemma 3.3 of [2].

Proposition 2.3. *Let $\widetilde{X}$ be an object in $\mathscr{C}_m$ induced by an object X in S_m. Then $\operatorname{End}_{\mathscr{C}_m}(\widetilde{X}) = \operatorname{End}_{D^b(H)} X \ltimes \operatorname{Hom}_{D^b(H)}(X, F_m X)$.*

Proof. We know that $\operatorname{End}_{\mathscr{C}_m}(\widetilde{X}) = \bigoplus_{i \in Z} \operatorname{Hom}_{D^b(H)}(X, F_m^i X)$ as k-vector spaces and the product is given by

$$(g_i)_{i \in Z}(f_j)_{j \in Z} = \left(\sum_{i+j=l} F_m^j g_i f_j \right)_{l \in Z}.$$

Using Lemma 2.1, we get that $\operatorname{End}_{\mathscr{C}_m}(\widetilde{X}) = \operatorname{Hom}_{D^b(H)}(X, X) \oplus \operatorname{Hom}_{D^b(H)}(X, F_m X)$. Finally, the claim follows from the fact that $\operatorname{Hom}_{D^b(H)}(X, F_m^2 X) = \operatorname{Hom}_{D^b(H)}(X, \tau^{-2} X[2]) = 0$. $\square$

Lemma 2.4. *Let T be a tilting complex in $D^b(H)$, and let $B = \text{End}_{D^b(H)} T$ an iterated tilted algebra. Then $\text{Hom}_{D^b(H)}(T, F_m T) \simeq \text{Ext}_B^{m+1}(DB, B)$.*

Proof. For $m = 1$ this statement is proven in [2], as part of the proof of their main result. It follows from [16] that $\text{Hom}_{D^b(H)}(T, F_m T) \simeq \text{Hom}_{D^b(B)}(B, F'_m B)$, where $F'_m = \tau^{-1}_{D^b(B)}[m]$ is the functor induced by $F_m = \tau^{-1}[m]$ in $D^b(B)$. We have

$$\text{Hom}_{D^b(B)}\big(B, F'B\big) \simeq \text{Hom}_{D^b(B)}\big(\tau B[1], B[m+1]\big)$$
$$\simeq \text{Hom}_{D^b(B)}\big(DB, B[m+1]\big)$$
$$\simeq \text{Ext}_B^{m+1}(DB, B). \qquad \square$$

Recall that $\mathcal{R}_m(B) = B \ltimes \text{Ext}_B^{m+1}(DB, B)$ is the m-relation extension of B. Now, we are in position to state our first main result.

Theorem 2.5. *Let T be a tilting complex in $D^b(H)$ which belongs to S_m. Then $\text{End}_{\mathscr{C}_m}(\widetilde{T})$ is an m-cluster tilted algebra. Moreover if the global dimension of $B = \text{End}_{D^b(H)}(T)$ is at most $m + 1$, then $\text{End}_{\mathscr{C}_m}(\widetilde{T})$ is isomorphic to $\mathcal{R}_m(B)$.*

Proof. It follows from [11, Proposition 2.4] that $\widetilde{T}$ is an m-cluster tilting object in $\mathscr{C}_m$. Then $\text{End}_{\mathscr{C}_m}(\widetilde{T})$ is an m-cluster tilted algebra. The second fact follows from Proposition 2.3 and Lemma 2.4. $\square$

The following example shows that $C_m(B) = \text{End}_{\mathscr{C}_m}(\widetilde{T})$ is not necessarily isomorphic to the trivial extension $\mathcal{R}_m(B)$ when T is not in the fundamental domain S_m.

Example 2.6. Let Q be a quiver of type $\mathbb{A}_7$ with some orientation and $H = kQ$. Consider the tilting complex T in $D^b(H)$, where the indecomposable summands T_i of T are indicated in the following picture. In the picture the Auslander–Reiten quiver Γ of the derived category $D^b(H)$ is indicated; the arrows are going from left to right and are drawn as lines to simplify the picture. The indecomposable summand T_i of the tilting complex $T = \bigoplus_{i=1}^7 T_i$ has been indicated by the number i inside a circle, that is, the symbol $\circledi$. Furthermore, $F_2^{-1} T_i$, resp. $F_2 T_i$, has been indicated by the symbol $\dot{\circledi}$, resp. $\ddot{\circledi}$.
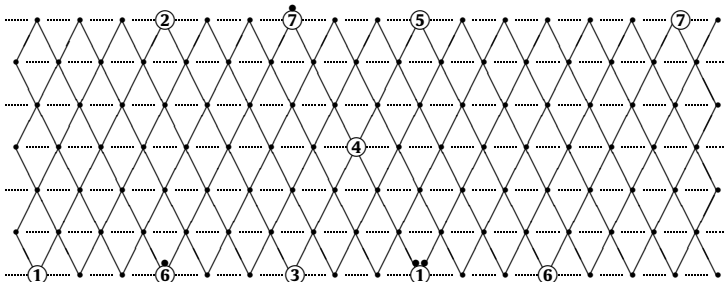
The corresponding iterated tilted algebra $B = \text{End}_{D^b(H)}(T)$ is given by the quiver with relations
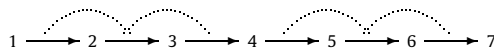
Clearly gl dim $B = 3$. Let $m = 2$ and $\mathscr{C}_2$ be the 2-cluster category of H. Then $\mathcal{C}_2(B) = \text{End}_{\mathscr{C}_2}(\widetilde{T})$ is given by the quiver
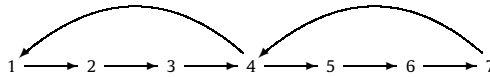
where the composition of any two consecutive arrows inside each cycle is zero.

We have that $C_2(B) = B \oplus \mathrm{Ext}_B^3(B, DB) \oplus \mathrm{Ext}_B^3(B, DB) \otimes \mathrm{Ext}_B^3(B, DB)$.

Since there exists a sectional path in $D^b(H)$ between the vertices $F_2^{-1}T_7$ and F_2T_1 passing through the vertex T_4, we have that the path $7 \to 4 \to 1$ is not zero. It follows that $\mathrm{Ext}_B^3(B, DB) \otimes \mathrm{Ext}_B^3(B, DB) \neq 0$ and $C_2(B)$ is not isomorphic to the trivial extension $\mathcal{R}_2(B)$. We observe that $C_2(B)$ and $\mathcal{R}_2(B)$ have the same quivers.

## 3. The m-rolling of tilting complexes

A very useful tool introduced in [4] to study tilting complexes and iterated tilted algebras of global dimension at most two, is a procedure called the rolling of tilting complexes. In this section we want to show how the mentioned procedure can be carried on in our context. Now, we extend most of the notions and facts from [4, Section 3].

We define for each tilting complex T a new tilting complex $\rho_m(T)$ such that $T \simeq \rho_m(T)$ in the m-cluster category $\mathscr{C}_m$, and such that iterating this procedure we eventually obtain a tilting complex in the fundamental domain of the cluster category.

Let Q be a Dynkin quiver and T a tilting complex of $D^b(kQ)$. Let $<$ denote the partial order in the vertices of the Auslander–Reiten quiver Γ of $D^b(kQ)$ induced by the arrows. That is, if $L \to M$ is an arrow in Γ then we write $L < M$.

Since $T = \bigoplus_{i=1}^n T_i$ has only finitely many summands we can easily find a section $\Sigma = \{\Sigma_1, \ldots, \Sigma_n\}$ such that $T \leqslant \Sigma$, that is, $T_i \leqslant \Sigma_j$ for all i and j. If Σ_j is maximal in Σ with respect to the order $\leqslant$, and $\Sigma_j \notin \{T_1, \ldots, T_n\}$ then $\Sigma' = \Sigma \setminus \{\Sigma_j\} \cup \{\tau \Sigma_j\}$ is also a section satisfying $T \leqslant \Sigma'$. After finitely many steps we get a section $\Sigma(T)$ with $T \leqslant \Sigma(T)$ and all maximal elements in $\Sigma(T)$ belong to add T. Notice that the section $\Sigma(T)$ is uniquely defined by T.

We illustrate how to get $\Sigma(T)$ in Example 2.6.
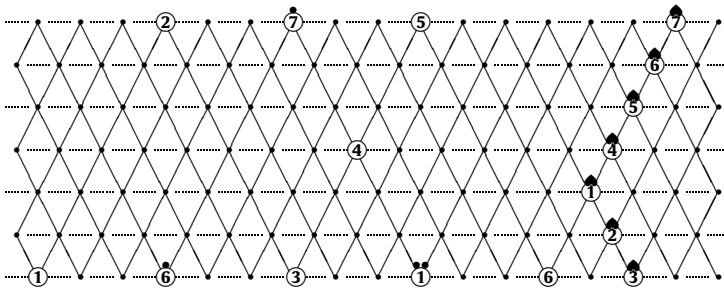
We can start with the section $\Sigma = \{\Sigma_1, \ldots, \Sigma_7\}$. The object Σ_i has been indicated by ⓘ.

Then the maximal elements in Σ are Σ_3 and $\Sigma_7 = T_7$. We replace Σ_3 by $\tau \Sigma_3$ and obtain the new section Σ' having Σ_2 and T_7 as maximal elements. In the next step we replace Σ_2 by $\tau \Sigma_2$ and get the section Σ''. Finally we exchange $\tau \Sigma_3$ by $\tau^2 \Sigma_3 = T_6$ and obtain $\Sigma(T)$, whose only maximal element $\Sigma_7 = T_7$ is in add T.

Definition 3.1 (*m-Rolling of tilting complex, the Dynkin case*). With the previous notations, let X be the sum of those summands of T which belong to $\Sigma(T)$ and T' a complement of X in T. Then define the *m-rolling* of T to be $\rho_m(T) = T' \oplus F_m^{-1} X$.

Now consider the case where Q is not Dynkin. We will use the description of the Auslander–Reiten quiver of $D^b(H)$ given in [4, Section 3]. We transcribe it here for the convenience of the reader. Denote by $\mathcal{P}$ (resp. $\mathcal{I}$) the preprojective (resp. preinjective) component of the Auslander–Reiten quiver of H

and by $\mathcal{R}$ the full subcategory of mod H given by the regular components. For each $r \in \mathbb{Z}$ the regular part $\mathcal{R}$ gives rise to $\mathcal{R}[r]$, given by the complexes $X \in D^b(H)$ concentrated in degree r with $X_r \in \mathcal{R}$. Moreover, for each $r \in \mathbb{Z}$ there is a transjective component $\mathcal{I}[r-1] \vee \mathcal{P}[r]$ of Γ which we shall denote by $\mathcal{R}[r - \frac{1}{2}]$ and each component of Γ is contained in $\mathcal{R}[r]$ for some half-integer r. The notation has the advantage that the different parts are ordered in the sense that $\mathrm{Hom}(\mathcal{R}[a], \mathcal{R}[b]) = 0$ for any two half-integers $a > b$. Also note that $\mathrm{Hom}(\mathcal{R}[a], \mathcal{R}[b]) = 0$ if $a < b - 1$.

We know that $D^b(kQ)$ is composed by the parts $\mathcal{R}[r]$ for $r \in \mathbb{Z}/2$ where $\mathcal{R}[r]$ denotes the regular (resp. transjective) part if r is an integer (resp. not an integer). Now, write $T = \bigoplus_{a \in \mathbb{Z}/2} T_{\mathcal{R}[a]}$, where $T_{\mathcal{R}[a]} \in \mathcal{R}[a]$.

Definition 3.2 (*m-Rolling of tilting complex, the non-Dynkin case*). With the previous notation let s be the largest half-integer such that $T_{\mathcal{R}[s]}$ is non-zero. Then define $X = T_{\mathcal{R}[s]}$ and T' to be the complement of X in T. Define the *m-rolling* of T to be $\rho_m(T) = T' \oplus F_m^{-1}X$.

Remark 3.3. If $T = T' \oplus X$ is a tilting complex in $D^b(H)$ and $\rho_m(T) = T' \oplus F_m^{-1}X$ then we have $\mathrm{Hom}_{D^b(H)}(X, T') = 0$.

Definition 3.4 (*m-Rolling of iterated tilted algebras*). Let B be an iterated tilted algebra. Then define $\rho_m(B)$ to be the endomorphism algebra $\mathrm{End}_{D^b(H)}(\rho_m(T))$, where H is a hereditary algebra with $D^b(B) \simeq D^b(H)$ and T a tilting complex in $D^b(H)$ with $B = \mathrm{End}_{D^b(H)}(T)$.

Notice that $\rho_m(B)$ does not depend on the choice of H or T. See [4, Section 3.2].
The proofs of the following results are similar to those in [4, Section 3.3], so we shall omit them.

Lemma 3.5. *We consider* $T = T' \oplus X$ *a tilting complex in* $D^b(H)$ *such that* $\mathrm{Hom}_{D^b(H)}(X, T') = 0$ *and let* $B = \mathrm{End}_{D^b(H)}(T)$. *Then* $T' \oplus F_m^{-1}X$ *is a tilting complex if and only if* $\mathrm{Hom}_{D^b(H)}(F_m^{-1}X, T'[j]) = 0$ *for all* $j \neq 0$ *if and only if* $\mathrm{Ext}_B^j(I_{X,T}, P_{T',T}) = 0$ *for each* $j \neq m+1$.

Lemma 3.6. *We consider* $T = T' \oplus X$ *a tilting complex in* $D^b(H)$ *such that* $\mathrm{Hom}_{D^b(H)}(X, T') = 0$ *and let* $B = \mathrm{End}_{D^b(H)}(T)$. *If* $\mathrm{gl\,dim}\, B \leqslant m+1$, *then* $\rho_m(T) = T' \oplus F_m^{-1}X$ *is a tilting complex in* $D^b(H)$ *if and only if* $\mathrm{Hom}_{D^b(H)}(\tau X, T'[k]) = 0$ *for* $k = 0, -1, -2, \ldots, -(m+1)$.

Lemma 3.7. *Let* Q *be a Dynkin quiver and* T *a tilting complex in* $D^b(H)$. *Then* $\rho_m(T) < \tau(\Sigma(T))$.

Proposition 3.8. *Let* T *be a tilting complex in* $D^b(H)$ *such that global dimension of* $\mathrm{End}_{D^b(H)}(T)$ *is at most* $m+1$. *Then* $\rho_m(T)$ *is again a tilting complex.*

The next result is analogous to Proposition 3.11 in [4]. We include the proof for the convenience of the reader.

Proposition 3.9. *Let* B *be an iterated tilted algebra. If* $\mathrm{gl\,dim}\, B \leqslant m+1$ *then* $\mathrm{gl\,dim}\, \rho_m(B) \leqslant m+1$.

Proof. Let H be a hereditary algebra and $T = T' \oplus X$ a tilting complex in $D^b(H)$ such that $B = \mathrm{End}_{D^b(H)}(T)$ and $\rho_m(T) = T' \oplus F_m^{-1}X$. Then we have $\mathrm{Hom}_{D^b(H)}(X, T') = 0$ by Remark 3.3 and by Proposition 3.8 the complex $\rho_m(T)$ is a tilting complex in $D^b(H)$. To shorten notations we set $\overline{T} = \rho_m(T)$ and $\overline{B} = \rho_m(B)$. We shall prove that $\mathrm{Ext}_{\overline{B}}^j(D\overline{B}, \overline{B}) = 0$ for all $j \geqslant m+2$. Since $\overline{T}$ is a tilting complex, we can show this by proving that $\mathrm{Hom}_{D^b(H)}(\tau \overline{T}[1], \overline{T}[j])$ is zero for $j \geqslant m+1$.

First note that

$$\mathrm{Hom}_{D^b(H)}\big(\tau T[1], T[i]\big) = 0 \quad \text{for all } i \neq 0, 1, 2, \ldots, m+1, \tag{1}$$

since $\mathrm{Hom}_{D^b(H)}(\tau T[1], T[i]) \simeq \mathrm{Ext}_B^i(DB, B)$.

Therefore $\mathrm{Hom}_{\mathrm{D}^b(H)}(\tau F_m^{-1}X[1], F_m^{-1}X[j]) = \mathrm{Hom}_{\mathrm{D}^b(H)}(\tau X, X[j-1]) = 0$ and $\mathrm{Hom}_{\mathrm{D}^b(H)}(\tau T'[1],$ $T'[j]) = 0$ for $j \geqslant m+2$. Also,

$$\mathrm{Hom}_{\mathrm{D}^b(H)}\big(\tau T'[1], F_m^{-1}X[j]\big) = \mathrm{Hom}_{\mathrm{D}^b(H)}\big(T'[1], X[j-m]\big),$$

which is zero for all $j \neq m+2$ since T is a tilting complex.

Hence, it remains to see that $\mathrm{Hom}_{\mathrm{D}^b(H)}(\tau^2 X[1], T'[j]) = 0$ for $j > 2m+1$. The minimal projective resolution of $I_{X,T}$ in mod B

$$0 \to P_{m+1} \to P_m \to P_{m-1} \xrightarrow{\varphi_{m-1}} \cdots \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} I_{X,T} \to 0$$

gives rise to exact triangles

$$\Delta_1: \quad K_1 \to P_0 \to I_{X,T} \to K_1[1],$$

$$\Delta_i: \quad K_i \to P_{i-1} \to K_{i-1} \to K_i[1]$$

where K_i denotes the kernel of φ_i for $i = 0, \ldots, m-1$ and

$$\Delta_{m+1}: \quad P_{m+1} \to P_m \to K_m \to P_{m+1}[1].$$

Apply first the inverse of the equivalence $G: \mathrm{D}^b(H) \to \mathrm{D}^b(B)$ and then τ, to obtain exact triangles of the form $S_1 \to \tau T_0 \to \tau^2 X[1] \to S_1[1]$, $S_i \to \tau T_{i-1} \to S_{i-1} \to S_i[1]$ for $i = 2, \ldots, m$ and $\tau T_{m+1} \to \tau T_m \to S_m \to \tau T_{m+1}[1]$ with $S_i = \tau G^{-1}(K_i)$ and some $T_0, \ldots, T_{m+1} \in \mathrm{add}\,T$. To these triangles apply the homological functor $\mathrm{Hom}_{\mathrm{D}^b(H)}(-, T'[j])$ to get exact sequences

$$\big(\tau T_0[1], T'[j]\big) \to \big(S_1[1], T'[j]\big) \to \big(\tau^2 X[1], T'[j]\big) \to \big(\tau T_0, T'[j]\big),$$

for $i = 2, \ldots, m$

$$\big(\tau T_{i-1}[i], T'[j]\big) \to \big(S_i[i], T'[j]\big) \to \big(\tau S_{i-1}[i-1], T'[j]\big) \to \big(\tau T_{i-1}[i-1], T'[j]\big)$$

and

$$\big(\tau T_m[m+1], T'[j]\big) \to \big(T_{m+1}[m+1], T'[j]\big) \to \big(\tau S_m[m], T'[j]\big) \to \big(\tau T_m[m], T'[j]\big)$$

where we abbreviated $(Y, Z) = \mathrm{Hom}_{\mathrm{D}^b(H)}(Y, Z)$. By (1), the end terms of the previous sequences are zero for $j > 2m+1$ and hence we get

$$\mathrm{Hom}_{\mathrm{D}^b(H)}\big(\tau^2 X[1], T'[j]\big) \simeq \mathrm{Hom}_{\mathrm{D}^b(H)}\big(S_m[m], T'[j]\big) = 0$$

for $j > 2m+1$, which is what we wanted to prove. $\quad\square$

Using ideas similar to those in Section 3.5 in [4] we can state the following theorem.

Theorem 3.10. *Let $B = \mathrm{End}_{\mathrm{D}^b(H)}(T)$ be an iterated tilted algebra of type Q with $\mathrm{gl\,dim}\,B \leqslant m+1$. Then for sufficiently large h the tilting complex $\rho_m^h(T)$ is in S_m.*

4. The main result

We may now state our main theorem.

Theorem 4.1. *Let $H = kQ$ a hereditary algebra. If T is a tilting complex in $D^b(H)$ such that $B = \text{End}_{D^b(H)}(T)$ has global dimension at most $m + 1$, then:*

a) *$\widetilde{T}$ is an m-cluster tilting object in the m-cluster category $\mathscr{C}_m$ and $C_m(B) = \text{End}_{\mathscr{C}_m}(\widetilde{T})$ is an m-cluster tilted algebra.*

b) *There exists a sequence of algebra homomorphisms*

$$B \to C_m(B) \xrightarrow{\pi} \mathcal{R}_m(B) \to B$$

whose composition is the identity map and such that the kernel of π is contained in $\text{rad}^2 C_m(B)$. Moreover $C_m(B)$ and $\mathcal{R}_m(B)$ have the same quiver.

c) *There exists $h > 0$ such that $B' = \text{End}_{D^b(H)}(\rho_m^h(T))$ is an iterated tilted algebra of type Q, $\text{gl dim } B' \leqslant m + 1$ and*

$$C_m(B) \simeq \mathcal{R}_m(B') \simeq C_m(B').$$

Proof. Let $H = kQ$ be a hereditary algebra. By Theorem 3.10 we have that there exists a number h such that $\rho_m^h(T)$ is in $\mathcal{S}_m$. It follows from Theorem 3.10 that the object $\rho_m^h(T)$ defines an m-cluster tilting object in the m-cluster category $\mathscr{C}_m$. Hence $\text{End}_{\mathscr{C}_m}(\rho_m^h(T))$ is an m-cluster tilted algebra. Since $\rho_m^h(T) \simeq T$ in $\mathscr{C}_m$ we get that $C_m(B) = \text{End}_{\mathscr{C}_m}(T)$ is an m-cluster tilted algebra. This proves a).

The proof of b) goes through exactly as in [4].

For c) note that by iterating the rolling procedure we show in Theorem 3.10 that there exists a number h such that $T' = \rho_m^h(T)$ is a tilting complex in $\mathcal{S}_m$. Then the algebra $B' = \text{End}_{D^b(H)}(\rho_m^h(T))$ is an iterated tilted algebra such that $\text{gl dim } B' \leqslant m + 1$. By Theorem 2.5 we have that $C_m(B') = \text{End}_{\mathscr{C}_m}(\rho_m^h(T)) \simeq \mathcal{R}_m(B')$. Since $\rho_m^h(T) \simeq T$ the result follows. $\square$

We illustrate the former result by an example.

Example 4.2. We come back to Example 2.6 and adopt the same notation. In that example it was shown that $C_2(B)$ is not isomorphic to the trivial extension $\mathcal{R}_2(B)$.

Now we use the m-rolling procedure to obtain an iterated tilted algebra B' as in Theorem 4.1, that is, such that $C_2(B) \simeq \mathcal{R}_2(B') \simeq C_2(B')$.

Applying the m-rolling procedure we get that $T' = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5 \oplus F_2^{-1} T_6 \oplus F_2^{-1} T_7$ is a tilting complex such that the indecomposable summands of it are in the fundamental domain $\mathcal{S}_2$. The endomorphism algebra $B' = \text{End}_{D^b(H)}(T')$ is given by the following quiver with relations
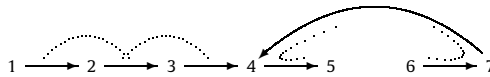
By computing the 2-cluster tilted algebra $C_2(B') = \text{End}_{\mathscr{C}_2}(T')$ we get that

$$C_2(B') = \mathcal{R}_2(B') = B' \ltimes \text{Ext}_{B'}^3(DB', B') \simeq C_2(B).$$
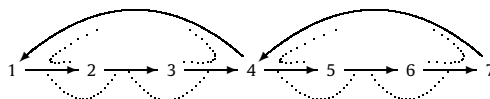
Remark 4.3. In [20], Murphy has proved that m-cluster tilted algebras of type A_n are gentle and that if the quiver of an m-cluster tilted algebra of type A_n contains cycles they must be of length $m+2$ and must have full relations, that is the composition of any two consecutive arrows in the cycle must be a relation. We observe that every connected component of an m-cluster tilted algebra of type A_n is induced by a tilting complex. In fact, we can choose the summands of the tilting complex in a similar way as we did in the example above. The ones which give rise to the cycles in an analogous way, and for the hereditary parts we choose the summands over a complete slice. Then, any connected component of an m-cluster tilted algebra of type A_n is an m_i-relation extension of an iterated tilted algebra of the same type. We conjecture that any m-cluster tilted algebra is a direct product of m_i-relation extensions of iterated tilted algebras.

References

[1] C. Amiot, Cluster categories for algebras of global dimension 2 and quivers with potential, Ann. Inst. Fourier 59 (6) (2009) 2525–2590.
[2] I. Assem, T. Brüstle, R. Schiffler, Cluster tilted algebras as trivial extensions, Bull. Lond. Math. Soc. 40 (1) (2008) 151–162.
[3] I. Assem, A. Gatica, R. Schiffler, The higher relation bimodule, Algebr. Represent. Theory 15 (2012) 1–21.
[4] M. Barot, E. Fernández, M.I. Platzeck, N.I. Pratti, S. Trepode, From iterated tilted algebras to cluster-tilted algebras, Adv. Math. 223 (2010) 1468–1494.
[5] K. Baur, R. Marsh, A geometric description of the m-cluster categories, Trans. Amer. Math. Soc. 360 (11) (2008) 5789–5803.
[6] A.B. Buan, R. Marsh, I. Reiten, Cluster mutation via quiver representations, Comment. Math. Helv. 83 (1) (2008) 143–177.
[7] A.B. Buan, R. Marsh, I. Reiten, Cluster-tilted algebras, Trans. Amer. Math. Soc. 359 (1) (2007) 323–332.
[8] A.B. Buan, R. Marsh, I. Reiten, Cluster-tilted algebras of finite representation type, J. Algebra 306 (2) (2006) 412–431.
[9] A.B. Buan, R. Marsh, I. Reiten, M. Reineke, G. Todorov, Tilting theory and cluster combinatorics, Adv. Math. 204 (2) (2006) 572–618.
[10] A.B. Buan, I. Reiten, From tilted to cluster-tilted algebras of Dynkin type, arXiv:math.RT/0510445, 2005.
[11] A.B. Buan, I. Reiten, H. Thomas, Three kinds of mutation, J. Algebra 339 (1) (2011) 97–113.
[12] A.B. Buan, I. Reiten, A.I. Seven, Tame concealed algebras and cluster quivers of minimal infinite type, J. Pure Appl. Algebra 211 (1) (2007) 71–82.
[13] P. Caldero, F. Chapoton, R. Schiffler, Quivers with relations arising from clusters (A_n case), Trans. Amer. Math. Soc. 358 (3) (2006) 1347–1364.
[14] P. Caldero, F. Chapoton, R. Schiffler, Quivers with relations and cluster tilted algebras, Algebr. Represent. Theory 9 (4) (2006) 359–367.
[15] R. Fossum, P. Griffith, I. Reiten, Trivial extensions of abelian categories and applications to rings: an expository account, in: Ring Theory, Proc. Conf., Park City, Utah, 1971, Academic Press, New York, 1972, pp. 125–151.
[16] D. Happel, Triangulated categories in the representation theory of finite dimensional algebras, Lond. Math. Soc. Lect. Note Ser., vol. 119, Cambridge University Press, 1988.
[17] O. Iyama, S. Oppermann, Stable categories of higher preprojective algebras, Adv. Math. 244 (2013) 23–68.
[18] K. Keller, On triangulated orbit categories, Doc. Math. 10 (2005) 551–581 (electronic).
[19] B. Keller, I. Reiten, Cluster-tilted algebras are Gorenstein and stably Calabi–Yau, Adv. Math. 211 (1) (2007) 123–151.
[20] G.J. Murphy, Derived equivalence classification of m-cluster tilted algebras of type A_n, J. Algebra 323 (4) (2010) 920–965.
[21] J. Rickard, Morita theory for derived categories, J. Lond. Math. Soc. (2) 39 (1989) 436–456.
[22] H. Thomas, Defining an m-cluster category, J. Algebra 318 (1) (2007) 37–46.
[23] B. Zhu, Generalized cluster complexes via quiver representations, J. Algebraic Combin. 27 (1) (2008) 35–54.