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Ordinal methods: Concepts, applications, new developments, and challenges—In memory of Karsten Keller (1961–2022)

Special Collection: Ordinal Methods: Concepts, Applications, New Developments and Challenges

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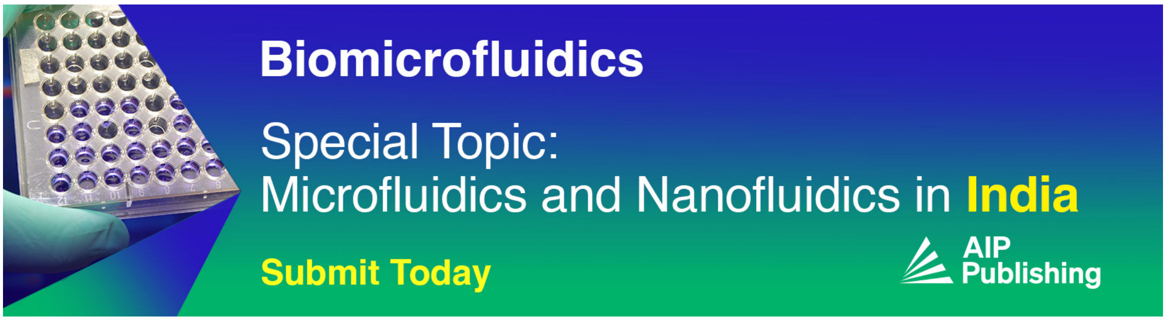

Ordinal methods: Concepts, applications, new developments, and challenges—In memory of Karsten Keller (1961–2022)

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I. INTRODUCTION

In 2013, Karsten Keller, Jürgen Kurths, and one of us (J.M.A.) guest edited an issue of the European Physical Journal Special Topics, entitled *Recent Progress in Symbolic Dynamics and Permutation Complexity*,[1] to celebrate the 10th anniversary of the seminal paper "Permutation entropy: A natural complexity measure for time series" of Christoph Bandt and Bernd Pompe,[2] where the concept of permutation entropy was introduced. That Special Topic comprised 23 contributions that covered both theoretical aspects and applications.

During the 10 years since then, the new "ordinal" methodology has been intensively further developed by theoreticians and practitioners in different directions. However, many of the new concepts and approaches are still not fully understood and there is also a need for a more systematic application of ordinal methods. For this reason, Karsten Keller and the two of us decided to guest edit this Focus Issue of Chaos on *Ordinal Methods: Concepts, Applications, New Developments and Challenges,* where researchers from different disciplines could report on recent advances in the Bandt and Pompe methodology, thus making possible the exchange of new ideas and synergies. Unexpectedly, Karsten passed away on April 19, 2022, a few months after the opening of the Focus Issue.

Karsten was born on April 12, 1961 in Halle (East Germany), where he also attended school until 1979. After 18 months of mandatory army service, he studied Mathematics at the University of Greifswald from 1981 until 1986. He continued with Ph.D. studies in the group of Professor Flachsmeyer, which he completed already in 1988 with a dissertation on orthoposets of extremal points and quantum logics, a subject related to Banach algebras.[3]

Then, he turned to the developing field of fractals and dynamical systems. His Habilitation on Julia equivalences and the abstract Mandelbrot set was defended in 1996 and published in the Springer Lecture Notes. In 2002, he moved from Greifswald to Lübeck, where he became an Assistant Professor. Again, he changed his research topic to the analysis of time series from the viewpoint of dynamical systems and ordinal patterns. At the same time, he began an intense activity that included multidisciplinary research with physicists and life scientists, international cooperations, visits to many universities, and participations in conferences, while his three daughters grew up. Undoubtedly, Karsten belonged to the theoretical camp and, yet, he was one of the first authors to apply ordinal patterns and permutation entropy to the analysis and characterization of real-world data, specifically, EEGs of epileptic patients.[4]

In 2011, Karsten was appointed Professor at the University of Lübeck. Once more, he intensified his activities both in research and teaching. Indeed, in addition to advising Ph.D. students, he worked with mathematically interested high school students, organizing summer schools and supervising junior research projects. He also transmitted his love for mathematics and science to his daughters. Among other academic services, Karsten was a leader of the *Schülerakademie* (student academy) of the University of L übeck. In all his jobs and projects, he was fully engaged with heart and mind.

Both authors of this Editorial collaborated with Karsten for years. Amigó first met him at the University of Göttingen (Germany) in 2008, at a workshop on *Interfaces between Mathematics and its Applications* organized by Professor Manfred Denker, one of the contributors to this Focus Issue.[5] The collaboration between Karsten and Amigó resulted in three research papers,[6–8] three Special Issues, several minisymposia at international conferences, one international workshop at the Max-Planck Institute for the Physics of Complex Systems in Dresden (organized together with Rosso), and research stays at their universities. During these stays in Lübeck, Amigó not only discussed with Karsten the work in progress but also met his Ph.D. students, gave seminars, participated in doctoral exams, and enjoyed nice evenings at his home together with his wife.

Rosso met Karsten at the Mathematical Institute of the University of Lübeck in 2004, where the former was a visiting researcher. It was Karsten who introduced him to Bandt and Pompe's work during that stay. As a result, Karsten and Rosso began a fruitful collaboration on the analysis and characterization on EEG signals that continued during subsequent visits to Lübeck in 2005 and 2006.[9] They also met at several congresses and workshops in Chile, Argentina, and Germany.

Karsten was a friendly and humorous person with two passions: mathematics and running. His character is very well described in the obituary written by his colleagues of the University of Lübeck: "As a mathematician, he had a playful, often unconventional approach to mathematics. Due to his great joy in collaborations, he was a stimulating and popular conversation partner, not only for his colleagues at the Mathematical Institute of the University of Lübeck or at an international level, but also in interdisciplinary exchanges, and last but not least for his students."

With Karsten Keller, we have lost a wonderful colleague and an excellent, internationally recognized scientist. We dedicate the Focus Issue *Ordinal Methods: Concepts, Applications, New Developments and Challenges* to his memory, with contributions from colleagues and friends. In particular, we highlight the personal tributes of Bandt in Ref. 10 and Weiß in Ref. 11.

II. ORDINAL METHODOLOGY: ORDINAL PATTERNS, PERMUTATION ENTROPY, AND BEYOND

The ordinal methodology comprises a number of concepts and tools, some of which are going to be revisited in this section. This methodology can be applied to both dynamical systems and random processes as a symbolization (or discretization) method. For more details on theoretical issues or applications, the interested reader is referred to the book[12] and to the reviews,[1,13,14] respectively.

A. The setting: Chaos theory

The concept of low-dimensional deterministic chaos, derived from the modern theory of nonlinear dynamical systems, has changed our way of understanding and analyzing observational data (time series), leading to a paradigm shift from linear to nonlinear approaches. Linear methods interpret observational signals as being output by an underlying dynamical system that is governed by linear equations, hence small perturbations lead to small effects. Consequently, all irregular behavior must be attributed to random external inputs.[15]

However, chaos theory has shown that random inputs are not the only possible source of irregularities in the outputs of a system. As a matter of fact, nonlinear deterministic autonomous equations can produce very irregular signals, in which case we talk of chaotic systems. Of course, a system that has both nonlinear characteristics and random inputs will most likely produce irregular signals too.[15,16] Chaotic time series are representative of a set of signals exhibiting complex non-periodic traces with continuous, broadband Fourier spectra, as well as displaying exponential sensitivity to small changes in the initial conditions. Clearly, chaotic time series occupy a place intermediate between (a) predictable regular or quasi-periodic signals and (b) totally irregular stochastic signals (noise), which are completely unpredictable. Chaotic time series are irregular in time, barely predictable, and exhibit interesting structures in the phase space.

In sum, chaotic systems display *sensitivity to initial conditions*, which manifests instability everywhere in the phase space and leads to non-periodic motion (chaotic time series). They display long-term unpredictability despite the deterministic character of the temporal trajectory.

In a system undergoing chaotic motion, two neighboring points in the phase space move away exponentially with time. Let $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ be two such points, located within a ball of radius R at time t. Furthermore, assume that these two points cannot be resolved within the ball due to poor instrumental resolution. At some later time t', the distance between the points will typically grow to

$$\left|\mathbf{x}_1(t') - \mathbf{x}_2(t')\right| = \left|\mathbf{x}_1(t) - \mathbf{x}_2(t)\right| \exp\left(\lambda \left|t' - t\right|\right), \qquad (1)$$

where λ is the largest Lyapunov exponent and $\lambda > 0$ for a chaotic dynamics. When this distance at time t' exceeds R, the two points become distinguishable. This implies that instability reveals some information about the phase space population that was not available at earlier times.[16] The above considerations allow one to think of chaos as an *information source*. Moreover, the associated rate of generated information can be formulated in a precise way in terms of Kolmogorov–Sinai's entropy,[17,18] which, in turn, is related to the Lyapunov exponents through the celebrated Pesin's formula.[8,19,20]

The Kolmogorov–Sinai entropy measures the average loss of information rate. Its range of values goes from zero for regular dynamics, it is positive for chaotic systems and infinite for stochastic processes. Consequently, if a dynamical system has at least one positive Lyapunov exponent and a finite positive Kolmogorov–Sinai entropy, one can assert that the system is deterministic–chaotic.

Complex time series are very common in nature and also in man-made systems. The immediate question that arises in connection with the underlying dynamical system of a time series reads: Is the system chaotic (low-dimensional deterministic) or stochastic? Answering this question is important for a proper physical description of irregular dynamics. If one is able to show that the system is dominated by low-dimensional deterministic chaos, then only a few (nonlinear and collective) modes are required to describe the pertinent dynamics.[21] Otherwise, the complex behavior could be modeled by a system dominated by a very large number of excited modes, which are in general better described by stochastic or statistical approaches.

Although several methodologies to evaluate Lyapunov exponents and Kolmogorov–Sinai entropies from time series have been proposed (see, e.g., Refs. 15 and 16), their applicability involves taking into account constraints (stationarity, time series length, parameter selection, etc.) which, in general, make the results inconclusive. Thus, new tools for distinguishing chaos (determinism) from noise (randomness) are needed.

B. Ordinal patterns

The use of quantifiers based on Information Theory, which incorporate in their evaluation the "time causality," are a viable alternative, and is just the methodology proposed by Bandt and Pompe in their cornerstone contribution of 2002,[2] usually known as *ordinal methodology*. This methodology is based on the transformation of a time series into a sequence of symbols called *ordinal patterns of length D*. These patterns (also called permutations or rank vectors) are obtained by means of the "$\leq$" relationship between D successive entries of the series if the *delay time* $\tau = 1$ or τ-spaced data samples for $\tau > 1$. Therefore, the Bandt–Pompe symbolization procedure maps blocks of D data to the set of $D!$ possible ordinal patterns of length D (the "alphabet"), and it is able to capture their temporal structure since ordinal patterns are related to the temporal correlation of the physical phenomena being considered. The transformation of a real-valued time series into a sequence of ordinal patterns (a discrete-valued time series) is called an *ordinal representation*. Therefore, ordinal representations have two parameters: the length of the ordinal patterns D (sometimes called the *embedding dimension*) and the delay time τ. Regarding the selection of the parameters D and τ and the subtleties involved, see, e.g. Ref. 22.

The next step toward time series characterization involves the construction of probability functions from the frequency of ordinal patterns ("empirical probabilities"), enabling the use of information-theoretical quantifiers to account for the dynamics. The main such quantifiers are the normalized permutation Shannon entropy, permutation statistical complexity, and permutation Fisher information. Their suitability to characterize time series dynamics, such as noisy, chaotic, and deterministic behaviors, has already been proven.[23,24] However, to assure the reliability of such measures, the length N of the symbolic time series must be long enough so that the sampling of the alphabet is representative, i.e., $N \gg D!$ for ordinal patterns of length D. Among other problems, this avoids missing patterns.

The transformation proposed by Bandt and Pompe is robust to the presence of observational and dynamical noise, as well as invariant under nonlinear monotonous transformations. Although an ordinal representation loses details of the amplitude of the original time series, it is still suitable for the analysis of experimental data, since it avoids amplitude threshold dependencies that mar other methods based on range partitions, for example. Another advantage in the case of random processes is that the time series need not be stationary for the empirical probabilities to converge to the true probabilities of the patterns (with probability 1) in the limit of arbitrarily long time series; it suffices that the increments of the random process are stationary, which includes non-stationary processes such as the fractional Brownian motion. As an additional asset in practical applications, ordinal patterns (and derived quantities for that matter) can be computed in real time since knowledge of the data range is not required.

C. Tools of the ordinal methodology based on probabilities

In this section, we delve with some relevant information-theoretic quantities that are used in nonlinear time series analysis. So, let $\mathcal{P} = \{p_j; j = 1, \ldots, W\}$ be hereinafter a discrete probability distribution function (PDF), where W is the number of possible states of the system under study. Of course, for the applications we have in mind, $\mathcal{P}$ is the PDF of ordinal patterns of length D, hence, $W = D!$ in such cases.

1. Shannon and generalized entropies

The *Shannon entropy*[25] of the PDF $\mathcal{P}$ is defined as

$$S[\mathcal{P}] = -\sum_{j=1}^{W} p_j \ln p_j. \tag{2}$$

Therefore, the Shannon entropy varies between $S[\mathcal{P}_0] = 0$ for a complete ordered system ($\mathcal{P}_0 = \{p_k = 1$ and $p_j = 0$ for $\forall j \neq k$; $j = 1, \ldots, W$) and $S[\mathcal{P}_e] = \ln W =: S_{\max}$ for a complete disordered system ($\mathcal{P}_e = \{p_j = 1/W; j = 1, \ldots, W\}$). The *normalized Shannon entropy* is given by

$$H[\mathcal{P}] = \frac{S[\mathcal{P}]}{S_{max}} = \frac{S[\mathcal{P}]}{\ln W}. \tag{3}$$

When $\mathcal{P}$ is the PDF of the ordinal patterns of length D of a deterministic or random process, we call $S[\mathcal{P}]$ the (Shannon) *permutation entropy of order D* of that process. This is the concept of permutation entropy (up to a factor) introduced by Bandt and Pompe in Ref. 2 to measure the complexity of time series. Moreover, Bandt and Pompe observed numerically for the logistic map $f_r(x) = rx(1-x)$, with $3.5 \le r \le 4$, that $\frac{1}{D-1} S[\mathcal{P}]$ converges to the Kolmogorov–Sinai (KS) entropy of $f_r(x)$ when D increases. This observation was rigorously proved for any piecewise, strictly monotone interval map f with an invariant measure μ in the paper,[26] i.e.,

$$h^*_\mu(f) \equiv -\lim_{D\to\infty} \frac{1}{D-1} \sum_{j=1}^{D!} p_j \ln p_j = h_\mu(f), \tag{4}$$

where $\{p_j : 1 \le j \le D!\}$ is the PDF of the ordinal patterns of length D in time series generated by the iterations of f (which can be calculated with the measure μ), and $h_\mu(f)$ is the KS entropy of f with respect to the measure μ. The limit $h^*_\mu(f)$ is called the *permutation entropy rate* of the map f (with respect to the invariant measure μ). Let us mention at this point that Unakafov and Keller introduced in Ref. 27 the akin concept of conditional permutation entropy and showed that it converges to the KS entropy more quickly than permutation entropy. This measure was applied to EEGs of epileptic subjects by Keller *et al.*[28]

Equation (4) allows us to estimate the KS entropy of a one-dimensional map via ordinal patterns. It was generalized to countably piecewise monotone maps in Ref. 29. See also Ref. 6 for the generalization of Eq. (4) to higher-dimensional maps via two different approaches.

Shannon's entropy is the fundamental concept in Information Theory and, hence, the most familiar measure of uncertainty and complexity across disciplines. Moreover, $S[\mathcal{P}]$ is unique in that it satisfies the four so-called Shannon–Khinchin axioms: continuity, maximality, expansibility, and strong additivity.[30] Positive probability functionals that satisfy the first three Shannon–Khinchin axioms but not the fourth are called generalized entropies.[31] Among the many generalized entropies, the *Renyi* and *Tsallis entropies* are quite popular in applications; they are defined as

$$R_q[\mathcal{P}] = \frac{1}{1-q} \ln \sum_{j=1}^{W} p_j^q \quad \text{and} \quad T_q[\mathcal{P}] = \frac{1}{1-q} \sum_{j=1}^{W} \left(p_j^q - 1\right), \tag{5}$$

respectively, where $q > 0$ and both $R_1[\mathcal{P}]$ and $T_1[\mathcal{P}]$ are defined by continuity as $S[\mathcal{P}]$. These entropies are useful in nonlinear time series analysis due to the additional leverage provided by the positive parameter q. When $\mathcal{P}$ corresponds to the ordinal representation of a time series, then $R_q[\mathcal{P}]$ and $T_q[\mathcal{P}]$ are called *Renyi* and *Tsallis permutation entropies*.[32,33]

2. The Fisher information measure (FIM)

The FIM[34,35] is a measure of the gradient content of a PDF, thus being quite sensitive even to tiny localized perturbations. The discrete normalized FIM of the PDF $\mathcal{P}$ is given by

$$F[\mathcal{P}] = F_0 \cdot \sum_{j=1}^{W-1} \left(p_{j+1}^{1/2} - p_j^{1/2}\right)^2, \tag{6}$$

where

$$F_0 = \begin{cases} 1 & \text{if } j^* = 1 \text{ or} j^* = W, \text{ and} p_j = 0 \text{ for } \forall\, j \neq j^*, \\ 1/2 & \text{otherwise.} \end{cases}$$

If our system is in a very ordered state, which occurs when almost all the probabilities p_i are zero, we have a normalized Shannon entropy $H \approx 0$, and a normalized FIM $F \approx 1$. On the other hand, when the system under study is in a very disordered state, that is, when all the p_i's oscillate around the same value, we obtain $H \approx 1$ while $F \approx 0$. One can state that the general FIM behavior of the present discrete version is opposite to that of the Shannon entropy, except for periodic motions. The local sensitivity of FIM for discrete PDFs is reflected in the fact that the specific "j-ordering" of the discrete values p_j must be seriously taken into account in evaluating the sum in Eq. (6). The summands can be regarded as a kind of "distance" between two contiguous probabilities. Thus, a different ordering of the summands in (6) would lead to a different value of FIM, hence its local nature. In our works, we follow the lexicographic order described by Lehmer[36] in the generation of the Bandt–Pompe PDF.

3. The statistical complexity measure (SCM)

Several statistical complexity measures (SCMs) have been proposed in the literature. They are the product of an entropic measure times a distance (in probability space) to a fixed reference state Q. The latter quantity is usually called *disequilibrium*; it works as a quantifier of the degree of physical structure of a given time series. The resulting SCM version is able to grasp essential details of the dynamics and capable of discerning among different degrees of periodicity and chaos. This measure, referred to as the "Martín–Plastino–Rosso (MPR) intensive statistical complexity"[23] can be viewed as a functional $C[\mathcal{P}]$ that characterizes the probability distribution $\mathcal{P}$ associated with the time series generated by the considered dynamical system. The *MPR intensive Statistical Complexity Measure* is defined as

$$C[\mathcal{P}] = H[\mathcal{P}] \cdot Q_J[\mathcal{P}, \mathcal{P}_e].$$

Here, $H[\mathcal{P}]$ is the normalized Shannon entropy (3) and the disequilibrium Q_J is defined as

$$Q_J[\mathcal{P}, \mathcal{P}_e] = Q_0 \cdot \mathcal{J}[\mathcal{P}, \mathcal{P}_e],$$

where

$$\mathcal{J}[\mathcal{P}, \mathcal{P}_e] = S\left[\frac{\mathcal{P} + \mathcal{P}_e}{2}\right] - S\left[\frac{\mathcal{P}}{2}\right] - S\left[\frac{\mathcal{P}_e}{2}\right] \tag{7}$$

is the extensive Jensen–Shannon divergence between the PDFs $\mathcal{P}$ and $\mathcal{P}_e = \{p_j = 1/W; j = 1, \ldots, W\}$, and

$$Q_0 = -2\left(\frac{W+1}{W} \ln(W+1) - 2\ln(2W) + \ln W\right)^{-1}$$

is a normalization constant, equal to the inverse of the maximum possible value of $\mathcal{J}[\mathcal{P}, \mathcal{P}_e]$, i.e., the value obtained when one of the components of $\mathcal{P}$, say p_m, is 1 and the remaining p_i's vanish. This intensive quantity reflects the architecture of the system, being different from zero only if there exist privileged, or more likely states among the accessible ones. It quantifies not only randomness but the presence of correlational structures as well. The opposite extremes of perfect order or maximal randomness possess no structure to speak of. Between these two special instances, a wide range of possible degrees of physical structure exist, degrees that should be reflected in the features of the underlying probability distribution.

4. The entropy vs complexity (H × C) plane

We stress that the above SCM is not a trivial function of the entropy in the sense that, for a given entropy H, there exists a range of possible SCM values between a minimum C_{min} and a maximum C_{max},[37] these bounds depending on the dimension of the PDF $\mathcal{P}$. Thus, evaluating the SCM provides important additional information about peculiarities of a probability distribution that are not already carried by the entropy.

This fact is emphasized when $\mathcal{P}$ is a distribution of ordinal patterns. In order to study the time evolution of the permutation statistical complexity measure C, a diagram of C vs the normalized permutation entropy H can be used, where H can be regarded as an arrow of time. Indeed, this kind of diagram, called the *MPR causality plane* $H \times C$, allows visualizing the changes in the dynamics of a system originated by modifications of some characteristic parameters. The range of variation in this 2D-plane is given by $[0, 1] \times [C_{\min}, C_{\max}]$, and the description of the system under analysis is global in both dimensions.[23] See Refs. 38 and 39 for shortcomings of the $H \times C$ plane for high-dimensional systems.

5. The Shannon vs Fisher (H × F) plane

This is a plane formally similar to the previous one, but now one has a description of global (Shannon) vs local characteristics (Fisher) of the dynamical system. The range in this 2D plane is $[0, 1] \times [0, 1]$.[24] The $H \times F$ plane can be used to better visualize the system behavior under different values of the parameters and the associated dynamics.

Needless to say, the $H \times C$ and $H \times F$ planes can be generalized in several ways. For example, the Shannon entropy can be replaced by a generalized entropy, say, the Renyi or Tsallis entropy (5); this adds an additional parameter for classification and discrimination. Furthermore, other combinations of probability functionals, such as Statistical Complexity vs Fisher information, can also be used to define the corresponding plane.

Finally, we would like to mention that several packages exist for the computation of ordinal patterns and tools. In particular, we recommend the excellent tutorial[40] in Python by Pessa and Ribeiro, which is very didactic and includes examples.

D. Forbidden patterns

When a nonlinear dynamics is involved, a deterministic system can generate "random-looking" results that, nevertheless, exhibit persistent trends, cycles (both periodic and non-periodic), and long-term correlations. Our main interest here lies in the emergence of *forbidden/missing patterns*. Why? Because they have the potential ability to distinguish deterministic behavior (chaos) from randomness in finite time series contaminated with observational additive colored noises.[41]

For deterministic one-dimensional maps, Amigó *et al.*[42–44] have conclusively shown that not all possible ordinal patterns (as defined by Bandt–Pompe's methodology) can be effectively materialized into orbits, which in a sense makes these patterns "forbidden." We stress that this is not a conjecture but an established fact. The existence of these forbidden ordinal patterns becomes a persistent feature, a "new" dynamical property. For a fixed pattern length D, the number of forbidden patterns of a time series (unobserved patterns) is independent of the series length N. It must be noted that this independence does not characterize other properties of the series such as proximity and correlation, which die out with time.[12] Furthermore, it follows from the results in Ref. 26 that the number of allowed ordinal patterns grows exponentially with D, hence the number of forbidden patterns grows factorially with D.[12,42]

As for higher-dimensional maps, Amigó and Kennel proved in Ref. 45 that expansive maps have forbidden patterns using lexicographical order to define ordinal patterns. Again, the number of allowed patterns grows exponentially with the pattern length D. Moreover, numerical simulations show that dissipative chaotic maps can have forbidden patterns as well.[46]

Stochastic processes can also have forbidden patterns. However, in the case of uncorrelated (white noise) and certain correlated stochastic processes, it can be numerically shown that *no* forbidden patterns emerge. Such correlated processes include "k-noise" with $k \geq 0$ (noise with power spectrum frequency dependence fitted by f^{-k} values), standard and fractional Brownian motion as well as standard and fractional Gaussian noise.

In the case of time series generated by an *unconstrained stochastic process* (uncorrelated process), every ordinal pattern has the same probability of appearance. If the time series is long enough, all possible ordinal patterns will eventually appear. Therefore, if the number of observations in the time series is sufficiently large, the associated PDF should be the uniform distribution, and the number of observed patterns should depend only on the length N of the time series.

Finally, for correlated stochastic processes, the probability of observing individual patterns depends not only on the correlation structure but also on the time series length N.[47] The non-presence of an ordinal pattern in a finite time series does not qualify the pattern as "forbidden" but only as "missing," since its absence results from the finite sample size. A similar observation also holds for the case of real data series, as they always possess a stochastic component due to the omnipresence of dynamical noise.[48,49]

E. Distinguishing determinism from randomness

In Sec. II D, the existence of "missing ordinal patterns" could be either related to stochastic processes (correlated or uncorrelated) or to deterministic noisy processes, which is the case for observational time series. Amigó[12,44] proposed a test that uses missing ordinal patterns to distinguish determinism (chaos) from pure randomness in finite time series contaminated with observational white noise (uncorrelated noise). This methodology was extended by Carpi *et al.*[47] to the analysis of missing ordinal patterns in stochastic processes with different degrees of correlation. We are speaking of fractional Brownian motion (fBm), fractional Gaussian noise (fGn), and k-noises with $k \geq 0$. Results show that for a fixed pattern length, the decay rate of missing ordinal patterns in stochastic processes depends not only on the series length but also on their correlation structures. More precisely, missing ordinal patterns are more persistent in time series with higher correlation structures. Carpi *et al.*[47] have also shown that the standard deviation of the estimated decay rate of missing ordinal patterns decreases with an increasing length of the patterns. This is due to the fact that longer patterns contain more temporal information and, therefore, they are more effective in capturing the dynamic of time series with correlation structures. An important quantity for us, called $\mathcal{M}(N, D)$, is the number of missing ordinal patterns of length D in a time series with N entries.

As we mentioned before, for correlated stochastic processes the probability of observing an individual pattern of length D depends on the time series length N and on the correlation structure as determined by the type of noise $k > 0$. In fact, as the value of $k > 0$ increases (which implies that correlations grow), greater values of N are needed to reach the "ideal" condition $\mathcal{M}(N, D) = 0$.[47]

If the time series is chaotic but has an additive stochastic component, then one expects that, as the time series length N increases, the number of missing ordinal patterns will decrease and eventually vanish. Whether this happens is independent of the length N, the underlying deterministic components of the time series, or the correlation structure of the added noise. The number of missing ordinal patterns will be $\mathcal{M}(N, D, k)$, where k is the noise characteristic parameter, representing its correlation degree.

In Ref. 41, we dealt with the issue of determinism vs randomness in time series, with the goal of identifying their relative importance in a given time series. We considered time series of the form $\{s_n = x_n + A \cdot \eta_n(k); n = 1, \ldots, N\}$, where x_n is given by the logistic map $x_n = 4x_{n-1}(1 - x_{n-1})$, $\eta_n(k) \in [-1, 1]$ is a k-noise with $k = 0, 1, 2$, and A is the noise amplitude. For the analysis, we used ordinal patterns of length $D = 6$, time lag $\tau = 1$ and $N = 100\,000$. For each time series, the normalized Shannon entropy and the MPR-statistical complexity were evaluated using the PDF of the ordinal patterns, and also its position in the causal $H \times C$ plane as the amplitude A increased.

For contaminating and correlated noise, a new type of planar trajectory-behavior emerges as the noise intensity increases. Starting from a pure deterministic localization, the trajectory of the point P representing a time series in the $H \times C$ plane converges to a pure stochastic localization by following a loop-curve as the noise intensity increases. For a critical value of the noise intensity (A_c), however, this trajectory reverses direction. More precisely, one observes a movement of P that starts at the unperturbed value ($A = 0$) and closely approaches the curve of maximum complexity, from left to right, with increasing entropic values. When $A = A_c$, this displacement reverses direction and takes place now from right to left, below the original curve, converging to the planar location typical of pure-correlated noise. The value of the critical intensity A_c depends on the correlation degree of the noise and the deterministic component.

Three different scenarios have been found here.

(i) The correlated noise acts as a perturbation [mostly $\mathcal{M}(N, D, k) \neq 0$]. The net noise effect is to destroy the forbidden character of some of the patterns. However, due to the low noise intensity and its correlations, the number of affected patterns is relatively low. The dominance of the deterministic component over the noisy one is reflected by low dispersion values for both entropy and statistical complexity.

(ii) The deterministic and the stochastic components have the same hierarchy and $\mathcal{M}(N, D, k) = 0$. However, the persistent character of the forbidden patterns of the deterministic component and their interplay with the correlations present in the noise are reflected in the characteristic point-trajectory of the system, which moves along the curve of maximum complexity, showing that the pertinent patterns do not appear as frequently as the remaining ones. This behavior is indicative of a still active deterministic dynamic. The dispersion values of our two quantifiers increase with the noise intensity. Note that the two scenarios described above correspond to a range of noise intensities given by $A \leq A_c(N, k)$.

(iii) The noisy component is the dominating one and the deterministic component can be considered as a perturbation. This scenario corresponds to the noise intensity range $A \geq A_c(N, k)$ and we have $\mathcal{M}(N, D, k) = 0$ as well, with low dispersion values of entropy and statistical complexity.

III. CONTRIBUTIONS TO THIS FOCUS ISSUE

As said in the Introduction, the year 2022 marked the 20th anniversary of the seminal paper of Bandt and Pompe on permutation entropy.[2] The invitation to contribute to the commemorative Focus Issue *Ordinal Methods: Concepts, Applications, New Developments and Challenges* has resulted in 2 minireviews and 25 research papers. In this section, we summarize the contents of all contributions, grouped by descriptive topics. Papers belonging to overlapping topics have been somewhat arbitrarily assigned to one of them.

A. Analysis of biomedical data

In the paper,[50] Barà *et al.* compare the binning and permutation approach when measuring the coupling between short realizations of random processes via mutual information rate and transfer entropy. The comparison is done with numerical simulations and physiological data. The authors conclude that "while the application to short-term simulated and physiological series provides plausible results, it also evidences troublesome aspects that call for the development of improved entropy estimators and refined embedding strategies."

Guisande *et al.* compare in Ref. 51 the dynamics of human intracranial electroencephalography with two mathematical models: the Hénon map and a q-DG neural firing probability model. Their goal is to investigate the potential of the Hénon map as a model for replicating chaotic brain dynamics in the treatment of Parkinson's and epilepsy patients. The tools used are the Shannon entropy, statistical complexity, and Fisher's information, the probability distributions corresponding to ordinal patterns of lengths 3 and 4. Thus, the dynamic properties of the Hénon map are compared with data from the subthalamic nucleus, the medial frontal cortex, and a q-DG model of neuronal input–output to simulate the local behavior of a population. While the biological data present a much more complex spectrum of dynamical characteristics, the models are able to reproduce some aspects of neural dynamics.

Due to the complexity of the brain dynamics, a data-driven analysis is often the only feasible approach. In the minireview,[52] Lehnertz summarizes the state-of-the-art of uni- and bi-variate techniques of ordinal time series analysis, together with applications in the neurosciences and a list of 159 references. The author also discusses current limitations to stimulate further developments, which would be necessary to advance characterization of evolving functional brain networks during both physiological and pathophysiological conditions. He concludes that "ordinal time series analysis carries the potential to improve characterization of the still poorly understood spatiotemporal dynamics of the human brain."

To detect arrhythmic electrocardiograms, Martínez Coq *et al.* use in Ref. 53 the Shannon permutation entropy vs statistical complexity plane as a feature space to train three machine learning classification algorithms with two databases, one containing normal sinus rhythms and another one containing arrhythmias. The best results were achieved with the Random Forest method after a ten-times tenfold cross-validation scheme was applied to compute the corresponding quality parameters.

B. Applications of ordinal tools

Understanding how the predictability of a streamflow process is affected by human activities is important for making decisions for flood control and water resources management. In the paper,[54] de Carvalho Barreto *et al.* use the complexity-entropy causality plane (CECP) (in standard and weighted forms) with the daily streamflow series of the São Francisco River, Brazil, at several locations upstream of cascade dams and reservoirs. The authors find that the reservoir operations change the temporal variability of streamflow series toward the less predictable regime, corresponding to higher entropy and lower complexity values. This work also suggests that the time-dependent CECP analysis (in sliding windows) could be sensitive to alterations related to the intra-annual variability of reservoir operations.

In Ref. 55, Iaconis *et al.* use ordinal patterns transition networks to identify subjects with dyslexia on simple text reading experiments. To this end, the transitions between ordinal patterns in left-to-right eye movements during text reading were analyzed and characterized. The relative frequency transitions between patterns were used as feature descriptors to train a classifier able to distinguish normal from dyslexic subjects. The classifier is able to distinguish typically developed vs dyslexic subjects with almost 100% accuracy only analyzing the relative frequency of the eye movement transition from one particular permutation pattern to four other patterns including itself.

In the contribution,[56] Martínez *et al.* construct an entropy-time asymmetry plane and evaluate it using both synthetic and real-world time series. This way the authors study the interplay between those important aspects of a system's dynamics. They show that this plane is an adequate tool to better understand situations in which entropy and time asymmetry behave in complementary or independent ways.

Mateos *et al.* apply in Ref. 57 the so-called Rao–Burbea centroids to analyze simulated and real-world times series, as well as real textured 2D images, discretized via ordinal patterns. Rao–Burbea centroids are deformations of the Euclidean metric between discrete probability distributions that include the Jensen–Shannon divergence (7). As a main result, the authors conclude from their work that the best performance in terms of distinguishability is achieved with the Jensen–Shannon divergence.

The paper[58] presents an application of ordinal patterns to linguistics. Indeed, Sánchez *et al.* find in that work that a handful of ordinal patterns suffices to reliably characterize any language. In this application, the underlying time series consists of the frequency ranking of the words (or a monotonic function thereof) in a text, say, a novel. Moreover, fluctuations of the ordinal pattern distributions for a given language can be used to determine the historical period when the text was written as well as its author.

C. Applications to technology

Baba *et al.* use ordinal tools and machine learning in Ref. 59 to detect thermoacoustic instability in a staged single-sector combuster. The ordinal tools consist of the "determinisms" (or degrees of determinism) of the joint symbolic recurrence plots D_J and the ordinal transition pattern-based recurrent plots D_T. The $D_J \times D_T$ plane enables then to detect a precursor of themoacoustic instability with the help of a support vector machine.

The paper by Du *et al.*[60] is a nice application of the ordinal methodology to a complex technical problem, to wit: the characterization of multi-phase flow systems. To this end, the authors resort to a recently proposed tool called an interconnected ordinal pattern complex network. Roughly speaking, this tool associates ordinal networks to multivariate signals from the fluid system to, in turn, construct the interconnected complex network.

D. Concepts and methodology

The minireview[61] by Amigó *et al.* is divided into two parts. The first part is a self-contained survey of the concept of group (or "complexity-based") generalized entropy, which includes perhaps the three best known instances: the Shannon (2), Rényi, and Tsallis (5) entropies. Here, complexity refers to the asymptotic growth of microstates in statistical systems as the number of their constituents increases, e.g., exponential or factorial growths. In the second part, the parallelism between complexity of statistical systems and permutation complexity is exploited to extend the definition of permutation entropy from deterministic processes (exponential growth of the ordinal patterns of length D as $D \to \infty$) to processes with super-exponential growths of the ordinal patterns (e.g., random processes) in a way that the entropy *rate* is positive and finite. Along the way, the basics of the ordinal methodology are revisited.

The paper[10] by Bandt is an insightful blend of new concepts and established applications. Thus, for ordinal patterns of length 3, the author introduces an orthogonal system of four pattern contrasts (i.e., weighted differences of patterns frequencies), the most important of which is the turning rate, already studied by Bienaymè in 1875.[62] Applications include statistical fluctuations of permutation entropy, statistical tests for serial dependence, and the study of EEGs during sleep. In particular, Bandt discusses the need for new models of EEGs due to their serial dependence at all times and scales.

The paper by Denker[5] deals with the fundamental concept of iterated function systems. Iterated function systems are used to model non-autonomous dynamical systems as well as autonomous systems with dynamical noise, so they play an important role in the theory and applications of dynamical systems. In this contribution, Denker studies the Hausdorff dimension of the set of initial values for which indefinite iteration is possible, using the thermodynamic formalism.

E. Innovative approaches

Although discrete-valued and event time series are ubiquitous in practice (think of binary strings or all-or-none signals such as spike trains), most methods of time series analysis to determine the cyclical structure of the data require continuous amplitudes (Wiener-Khinchin theorem) or equidistant time stamps (Walsh transform, Haar wavelets). In Ref. 63, Marwan and Braun propose a novel power spectral analysis for discrete and event series based on the edit distance metric, a tool originally introduced in computer science and also popular in computational neuroscience. This method allows to estimate a power spectrum directly from the event sequence without interpolation. The authors illustrate their method with numerical simulations and apply it to atmospheric rivers in Europe to find typical recurring cycles.

Olivares *et al.* propose in Ref. 64 a model, based on a modulated Markov jitter, to represent ordinal pattern properties of real landing operations in European airports. The parameters of the model are fixed by minimizing the permutation Jensen–Shannon distance between the probability distributions of ordinal patterns generated by the real and synthetic time series. The authors also discuss the application of their model to other aspects of the landing flow dynamics, as well as the applicability of these findings to a real operational environment.

Shahriari *et al.* present in Ref. 65 a novel method for reconstructing the first return maps from time series without the need for embedding. Their method is based on ordinal partitions of the time series and guided by entropy-based measures. Numerical simulations with the Lorenz, Rössler, and Mackey–Glass dynamical systems in chaotic regimes show that this method performs successfully for low-dimensional chaotic systems as well as infinite-dimensional delay differential systems.

F. New ordinal tools

Dagoumguei *et al.* further develop in Ref. 66 the method of permutation largest slope entropy recently proposed by them in order to make it available for real-time analysis of complex systems. They show the correct performance of their method with the logistic map. Also, they implement this technique in a rather simple microcontroller and demonstrate its efficiency for the paradigmatic Duffing oscillator. Finally, the authors compare the results with those from an intense numerical analysis and find a strong agreement.

Stosic *et al.* introduce in Ref. 67 a new ordinal tool to measure the complexity of time series called the generalized weighted permutation entropy. This new tool features a scaling parameter that allows to transform the conventional complexity-entropy causality plane to the complexity-entropy-scale causality box. Numerical simulations with chaotic and random processes as well as real-world data show the enhanced discriminatory power of the new three-dimensional representation.

Zanin generalizes in Ref. 68 the conventional ordinal patterns in that, in his approach, these are evaluated in terms of their distance to ordinal patterns defined in a continuous way. The author tests the performance of continuous ordinal patterns with synthetic and real-world time series. In addition, he shows how continuous ordinal patterns can be used to assess some characteristics of the underlying dynamics, such as time irreversibility.

G. Statistical properties of permutation entropy

The statistical analysis of ordinal patterns is a technically difficult task whose objective is to characterize the distribution of the features they induce. In Ref. 69, Chagas *et al.* study the statistical properties of the perhaps most important functional of the ordinal patterns: the permutation entropy. Specifically, the authors calculate exact and approximate first-, second-, and third-order moments and its asymptotic distribution. Using these results, the authors also present a bilateral test to reject the hypothesis that two signals have the same permutation entropy.

H. Tests for serial dependence

Cánovas *et al.* propose in Ref. 70 a refined test to determine whether a time series is independent and identically distributed. For this, the authors complement the usual representation of a time series by ordinal patterns of length D with additional symbols (0–1 vectors of length D) that contain quantitative information of the corresponding data window. Numerical results using the chi-squared test show that the new approach outperforms the conventional (permutation-only) one even with much fewer symbols.

de Sousa and Hlinka[71] also use the classical chi-squared test and study symbolic processes in the space of ordinal patterns whose maximum dependence range is m. Such ordinal patterns are derived from random walks, white noise, and moving average processes. The authors describe chi-squared asymptotically distributed statistics for such processes and propose a test for m-dependence. Application of these results to EEGs of epileptic patients suggests that the range of serial dependence decreases during epileptic seizures. As a side remark, de Sousa and Hlinka denote ordinal patterns by antisymmetric matrices with off-diagonal components ± 1 (actually, the triangular submatrix above the diagonal) instead of using the conventional rank vectors.

The paper[11] contributed by Weiß is closely related to the previous one. In his paper, Weiß derives the asymptotic distribution of the vector of sample frequencies of ordinal patterns and that of various corresponding test statistics for distribution-free hypothesis tests in real-valued time series, where the null hypothesis is serial independence. The author provides simple closed-form formulas for the implementation of those tests. The performance of these tests is investigated with simulations, and their usefulness is illustrated by an environmental data example.

I. Topological methods

Haruna applies in Ref. 72 methods of topological data analysis to time series in ordinal representations. Topology is one of the more recent additions to symbolic representation of times series, which include other fields of advanced mathematics, such as graph theory (e.g., ordinal networks[73]) and algebra (e.g., transcripts[74,75]). In his paper, Haruna constructs an increasing sequence of simplicial complexes encoding the information about couplings among the components of a given multivariate time series through the intersection of ordinal patterns. A complexity measure is then defined by making use of the persistent homology groups. He validates the complexity measure both theoretically and numerically.

J. 2D ordinal patterns and images

The papers of Bandt and Wittfeld[76] and Muñoz-Guillermo[77] consider the extension of ordinal patterns from dimension 1 to dimension 2 and its application to the analysis and processing of images.

Specifically, Bandt and Wittfeld introduce in Ref. 76 two new parameters (smoothness and branching structure) to characterize 2×2 patterns based on the observation that neighboring pixels come in three types. Therefore, their approach is different from the $H \times C$-like approach used in Ref. 78. The authors show that their parameters describe textures and are well suited to distinguish different structures. Furthermore, the parameters are most stable and informative for isotropic structures.

The approach of Muñoz-Guillermo in Ref. 77 to 2D ordinal patterns and their application to the encryption of images is motivated by multiscale (weighted and non-weighted) two-dimensional permutation entropy. Therefore, she starts her study by analyzing and comparing the properties of those two multiscale permutation entropies, with special emphasis on their behavior when noise is added. In a second, final part, she explores the possibilities of multiscale analysis in encrypted images, including different security levels and encryption methods.

IV. NEW DEVELOPMENTS AND CHALLENGES

With the aim of attracting new researchers to the field of the ordinal methodology and its applications, we wrap up this Editorial with a brief review of some recent developments and challenges. To this end, we showcase a few topics that we deem will be the subject of active research in the years to come. Needless to say, our selection is necessarily far from complete due to the many new ideas currently popping up in the field. Topics and challenges are inspired by the contributions to this Focus Issue.

Time series analysis has been one of the main applications of the ordinal methodology ever since, not least because ordinal patterns and permutation entropy were precisely incepted for that purpose. Indeed, permutation entropy is applied in the seminal paper[2] to speech signals, as well as to noiseless and noisy chaotic time series. Despite its longevity, the analysis of time series, especially in the case of real-world data, still faces some challenges posed by short lengths,[50] noise, nonuniform time stamps,[63] or missing data, non-stationarity, modeling of EEGs[10,51] and other noisy signals,[79] and more. For specific limitations and necessary developments to improve the characterization of the complex networked dynamics of the human brain, see Ref. 52. To cope with such difficulties, researchers have devised a number of refinements, to mention a few regarding the information-theoretical tools: generalized permutation entropies[32,33,61] and divergences,[57,80] multiscale versions of permutation entropy,[81,82] weighted versions,[83] weighted multiscale versions,[84] and generalizations,[67] as well as other kinds of "entropies" (e.g., distance to white noise[85] and permutation largest slope entropy[66]). Also, the $H \times C$ plane has been generalized in several ways[56,59] and complemented with machine learning techniques with good results. Sharpening and generalizing existing tools is certainly a promising avenue to enhance the power of the ordinal methodology for characterization, discrimination, and classification of time series. Yet, the new tools should not be the result of numerical tinkering but the result of theoretical insight—that is the challenge.

Generalizing the symbolization method can also be a way to open new avenues. Once a time series has been represented by, say, algebraic, graph-theoretical, or topological symbols, one can use the leverage of algebra,[74,75] graph theory,[86–88] or topology[89,90] to further extract information about the underlying system. In particular, this approach can also be applied to a time series in an ordinal representation. Such is the case of the unweighted and weighted ordinal networks[91,92] (which are amenable to both graph-theoretical and probabilistic methods,[71]) as well as the persistent homology of sequences of ordinal patterns.[72] The perhaps simplest way of exploiting the algebraic structure of ordinal patterns (i.e., the group of permutations) is the concept of transcript,[74,75] which can be obtained from a single time series (self-transcripts) or from several ones (cross-transcripts); cross-transcripts have been used as information directionality indicators in coupled dynamics.[93] On the contrary, antisymmetric matrices with components 0 and ± 1 do not build groups under addition or multiplication, so the corresponding ordinal representations[71] are not amenable to the usual algebraic operations. Current research is quite active in the study of generalizations and applications of ordinal representations, with new results in extended alphabets of ordinal patterns[68,70] and generalized ordinal patterns,[94] applications of transcripts,[95,96] applications of ordinal networks[55,59,60,65] as well as applications of persistent homology.[72] In view of the interesting results obtained so far, we encourage new proposals in this line of research, particularly in the algebraic and topological processing of time series in ordinal representations.

Ordinal patterns (notoriously those of length 3) have a simple and intuitive interpretation in terms of smoothness or roughness, which allows the study of textures of one-dimensional structures or one-dimensional sections of surfaces.[79,97] The extension of ordinal patterns to two-dimensional and higher-dimensional structures and their application to the analysis and processing of 2D images was pioneered by Ribeiro *et al.*[78] In the case of 2D images, other approaches include Hilbert–Peano curves[98] and 2×2 ordinal patterns.[76] A direct conclusion of these and other related works (e.g., Ref. 77 in this Focus Issue) is that the permutation entropy of the corresponding probability distribution of ordinal patterns is also suitable to characterize images. Examples include pictures, paintings, satellite images,[98] and textures;[99] see also Ref. 57 for the discrimination of real 2D images textures using the Shannon–Jensen divergence of ordinal patterns distributions (7). Applications range from cryptography[77] to environmental surveillance, automated discrimination of textures, artificial vision, and determination of painting authorship (similarly to the identification of writers in Ref. 58). However, progress in the study of 3D images by means of ordinal patterns seems to lag behind; applications to fMRI imaging immediately come to mind.

Within time series analysis, the study of the statistical properties of ordinal patterns and permutation entropy, along with their application to serial dependence, is a classical topic[44,100–102] and still a hot topic, as can also be seen from this Focus Issue.[10,11,69–71] In this rather theoretical area, the work of statisticians and probabilists is especially welcome.[103] Among the challenging topics here, we highlight the following: (i) the study of asymptotic properties of ordinal patterns and application-based quantifiers; (ii) how to establish confidence intervals evaluation in the case of quantifiers based on ordinal patterns under different dynamical situations and different noise contamination intensities and characteristics;[104] (iii) study of serial dependence in discrete-valued time series via ordinal patterns (proposed in Ref. 11). Ordinal patterns of discrete-valued time series were considered in Refs. 105 and 106.

Finally, a challenge can also be finding new application areas. In this Focus Issue, there are nice examples, such as applying ordinal methods to a hydrological issue,[54] real aeroplane landing operations,[64] and linguistics,[58] together with more technical applications to thermoacoustic instabilities in Ref. 59 and characterization of multiphase flows.[60] Ordinal patterns have been also used to study systems with dynamical noise.[107,108] The choice of other topics is yours.

In conclusion, the Ordinal (or Bandt–Pompe) Methodology offers researchers in data analysis and modeling, nonlinear time series analysis, dynamical systems, and complexity theory a wide range of interesting and challenging topics, both theoretical and practical, which makes the field particularly appealing and active. This Focus Issue dedicated to the memory of Karsten Keller is good proof of that.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

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