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A MULTILINEAR PHELPS' LEMMA

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ABSTRACT. We prove a multilinear version of Phelps' Lemma: if the zero sets of multilinear forms of norm one are 'close', then so are the multilinear forms.

INTRODUCTION

Let X denote a Banach space, and S_X its unit sphere. The following two results are both simple and well-known.

Theorem A. *If $f_j, g : X \to \mathbb{C}, j = 1, \dots, k$, are linear forms, then*

$$\bigcap_{j=1}^{k} Ker\, f_j \subset Ker\, g \Rightarrow g = \sum_{j=1}^{k} a_j f_j \text{ for some } a_1, \dots, a_k \in \mathbb{C}.$$

Theorem B. *If $f, g : X \to \mathbb{C}$ are linear forms of norm one, then*

$$Ker\, f \subset Ker\, g \Rightarrow g = af \text{ for some } |a| = 1.$$

In 1960, R. Phelps [8] proved a continuous version of Theorem B. This result has come to be known as Phelps' Lemma, and also as the Parallel Hyperplane Lemma. It has had several important applications, including a fundamental role in the proof of the Bishop-Phelps Theorem [3]. It has also been used to prove that, for any real Banach space X, the kernel of every $x^{**} \in X^{**} \backslash X$ is a norming hyperplane in X^* [6], and to provide an estimate for the distance from x^{**} to X [7]. Recall that the natural generalization of the Bishop-Phelps Theorem to multilinear mappings is false, in general [1]. Nevertheless, Phelps' Lemma can be extended to the multilinear setting. The purpose of the present paper is to provide a proof of this.

Since our argument will refer to the proof of Phelps' Lemma, we include it here. (Note that a recent, more geometric, proof of this result was given by the second author in [5].)

Phelps' Lemma. *If $f, g : X \to \mathbb{C}$ are linear forms of norm one and $\varepsilon > 0$, then*

$$S_X \cap \{f(x) = 0\} \subset S_X \cap \{|g(x)| \le \varepsilon\} \Rightarrow \|g - \alpha f\| \le 2\varepsilon \text{ for some } |\alpha| = 1.$$

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Proof. Consider $g|_{Ker\, f}$. Since the norm of $g|_{Ker\, f}$ on that subspace is less than ε, we can use the Hahn-Banach Theorem to conclude that there is an $h \in X'$ such that $\|h\| \leq \varepsilon$ and $h = g$ on $Ker\, f$. We have that $Ker\, f \subset Ker\,(g - h)$, so by Theorem A, $g - h = af$ for some $a \in \mathbb{C}$. Thus $g - af = h$, and $\|g - af\| = \|h\| \leq \varepsilon$.

Now $|1 - |a|| = |\|g\| - \|af\|| \leq \|g - af\| \leq \varepsilon$, so $1 - \varepsilon \leq |a| \leq 1 + \varepsilon$. If we now take $\alpha = \frac{a}{|a|}$, then $\|g - \alpha f\| = \|g - af + (a - \alpha)f\| \leq \|g - af\| + |a - \alpha| \leq 2\varepsilon$. □

Clearly, as $\varepsilon \to 0$, one recovers Theorem B. Note also that if we are not concerned about the size of the constant a, then the first paragraph of the above proof shows that $\|g - af\| \leq \varepsilon$. We shall refer to this as the *first part* of Phelps' Lemma. Note also that the *second part* of Phelps' Lemma, which provides the 2ε estimate, follows immediately from $\|g - af\| \leq \varepsilon$ regardless of the fact that f and g are linear forms. We shall use this below for multilinear forms.

It is interesting to note that Theorem A above does *not* hold for multilinear forms, although Theorem B does [2].

Our proof of a multilinear Phelps' Lemma will require the use of constants $c_k > 0$ such that if P and Q are k-homogeneous polynomials, then

$$c_k \|P\| \|Q\| \leq \|PQ\|.$$

The existence of such constants has been observed by C. Benítez, Y. Sarantopoulos, and A. Tonge [4] and also by R. Ryan and B. Turett [9].

Although we write all our proofs for complex Banach spaces, note that the same arguments remain true in the real case, by simply replacing $\mathbb{C}$ by $\mathbb{R}$.

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1. A multilinear version of Phelps' Lemma

We will denote by $X_1, \ldots, X_n$ complex Banach spaces with S_{X_i} being the associated unit spheres. We will use the notation $x = (x_1, \ldots, x_{n-1}) \in X_1 \times \cdots \times X_{n-1}$ and $y \in X_n$. A and B will denote n-linear forms on $X_1 \times \cdots \times X_n$ of norm one, and we will write $A(x, y) = A(x_1, \ldots, x_{n-1}, y)$. $A_x : X_n \to \mathbb{C}$ will be the linear function obtained by fixing x, and $A_y : X_1 \times \cdots \times X_{n-1} \to \mathbb{C}$ will be the $n-1$-linear function obtained by fixing y (and similarly for B). For subsets $D \subset S_{X_1} \times \cdots \times S_{X_n}$, denote $D_x = \{y \in S_{X_n} : (x, y) \in D\}$, $D_y = \{x \in S_{X_1} \times \cdots \times S_{X_{n-1}} : (x, y) \in D\}$. We will also write

$$Z(A) = \{(x, y) \in S_{X_1} \times \cdots \times S_{X_n} : A(x, y) = 0\},$$

and for any $\varepsilon > 0$,

$$\varepsilon(B) = \{(x, y) \in S_{X_1} \times \cdots \times S_{X_n} : |B(x, y)| \leq \varepsilon\}.$$

Also,

$$Z(A_y) = \{x \in S_{X_1} \times \cdots \times S_{X_{n-1}} : A_y(x) = 0\}$$

and

$$\varepsilon(B_y) = \{x \in S_{X_1} \times \cdots \times S_{X_{n-1}} : |B_y(x)| \leq \varepsilon\}.$$

We begin with an algebraic result due to [2]. The proof presented here is different, shorter, and has a bearing on our later generalization of Phelps' Lemma.

Proposition 1.1. *If A and B are n-linear forms of norm one, then*

$$Z(A) \subset Z(B) \Rightarrow B = \alpha A \text{ for some } |\alpha| = 1.$$

Proof. We proceed by induction on n. For $n = 1$, this is Theorem B. For $n > 1$, suppose the result true for $n-1$-linear forms. We have, for each $y \in S_{X_n}$,

$$Z(A_y) = Z(A)_y \subset Z(B)_y = Z(B_y).$$

Thus by the induction hypothesis, $B_y = \alpha_y A_y$ for some $\alpha_y \in \mathbb{C}$. Also, for each $x \in S_{X_1} \times \cdots \times S_{X_{n-1}}$,

$$Z(A_x) = Z(A)_x \subset Z(B)_x = Z(B_x),$$

so by Theorem B, $B_x = \alpha_x A_x$ for some $\alpha_x \in \mathbb{C}$. Note that given any (x, y) with $A(x, y) \neq 0$,

$$\alpha_y = \frac{B(x, y)}{A(x, y)} = \alpha_x.$$

Consider y_1 and y_2 such that $A_{y_1} \neq 0 \neq A_{y_2}$. Then $\alpha_{y_1} = \alpha_{y_2}$. Indeed, taking $x \notin Z(A_{y_1}) \cup Z(A_{y_2})$, we have $\alpha_{y_1} = \alpha_x = \alpha_{y_2}$. So for y's such that $A_y \neq 0$, setting $\alpha = \alpha_y$ produces a well-defined constant. If y is such that $A_y = 0$, then by hypothesis $B_y = 0$, so $B_y = \alpha A_y$ in this case as well. Thus $B_y = \alpha A_y$ for all y, so $B = \alpha A$. Since both have norm one, $|\alpha| = 1$. □

Our multilinear Phelps' Lemma will be a continuous version of Proposition 1.1: if $Z(A) \subset \varepsilon(B)$, then $B \approx \alpha A$ for some α with $|\alpha| = 1$. For each $\varepsilon > 0$, $n > 1$, $x \in S_{X_1} \times \cdots \times S_{X_{n-1}}$, and $y \in S_{X_n}$ we shall need the following sets, which generalize the singletons $\{\alpha_y\}$ in the previous proof:

$$\Lambda_x(\varepsilon) = \{\beta \in \mathbb{C} : \|B_x - \beta A_x\| \leq \varepsilon\},$$
$$\Lambda_y(\varepsilon) = \{\alpha \in \mathbb{C} : \|B_y - \alpha A_y\| \leq \varepsilon\}.$$

Lemma 1.2. *If $Z(A) \subset \varepsilon(B)$, then $\Lambda_x(\varepsilon)$ is non-empty for all $x \in S_{X_1} \times \cdots \times S_{X_{n-1}}$.*

Proof. For all x,

$$Z(A_x) = Z(A)_x \subset \varepsilon(B)_x = \varepsilon(B_x).$$

If $B_x = 0$, any β with $|\beta| \leq \varepsilon$ belongs to $\Lambda_x(\varepsilon)$. If $A_x = 0$, then $\|B_x\| \leq \varepsilon$, and $\|B_x - \beta A_x\| \leq \varepsilon$ for all $\beta \in \mathbb{C}$, so $\Lambda_x(\varepsilon) = \mathbb{C}$. Suppose then that both are non-zero, and normalize:

$$Z\left(\frac{A_x}{\|A_x\|}\right) = Z(A_x) \subset \varepsilon(B_x) = \frac{\varepsilon}{\|B_x\|}\left(\frac{B_x}{\|B_x\|}\right).$$

By the first part of Phelps' Lemma, there is a β for which

$$\left\| \frac{B_x}{\|B_x\|} - \beta \frac{A_x}{\|A_x\|} \right\| \leq \frac{\varepsilon}{\|B_x\|}.$$

Thus $\|B_x - \beta \frac{\|B_x\|}{\|A_x\|} A_x\| \leq \varepsilon$. So $\beta \frac{\|B_x\|}{\|A_x\|} \in \Lambda_x(\varepsilon)$. □

Note that we have not yet proved that the sets $\Lambda_y(\varepsilon)$ are non-empty. Of course for bilinear A this is true by the above lemma, but for $n > 2$, this is an essential part of Theorem 1.4 below.

Lemma 1.3. *If $Z(A) \subset \varepsilon(B)$, then*

i) If $\varepsilon \leq \varepsilon'$, then $\Lambda_y(\varepsilon) \subset \Lambda_y(\varepsilon')$.

ii) If $\alpha_1 \in \Lambda_{y_1}(\varepsilon)$ and $\alpha_2 \in \Lambda_{y_2}(\varepsilon)$, then for all $x \in S_{X_1} \times \cdots \times S_{X_{n-1}}$,

$$|\alpha_1 - \alpha_2| \leq \frac{2\varepsilon}{|A(x, y_1)|} + \frac{2\varepsilon}{|A(x, y_2)|}$$

(where if a denominator is zero, we agree that the fraction equals ∞).

iii) If $\alpha \in \Lambda_y(\varepsilon)$, then $D(\alpha, r) \subset \Lambda_y(\varepsilon + r\|A_y\|)$, where $D(\alpha, r)$ is the closed disc centered at α with radius r.

Proof. ii) Suppose $\alpha \in \Lambda_y(\varepsilon)$ and $\beta \in \Lambda_x(\varepsilon)$. Then

$$\begin{aligned}|\alpha - \beta||A(x, y)| &= |\alpha A(x, y) - \beta A(x, y)| \\ &= |\alpha A(x, y) - B(x, y) + B(x, y) - \beta A(x, y)| \\ &\leq |B(x, y) - \alpha A(x, y)| + |B(x, y) - \beta A(x, y)| \\ &\leq \|B_y - \alpha A_y\| + \|B_x - \beta A_x\| \\ &\leq 2\varepsilon,\end{aligned}$$

so $|\alpha - \beta| \leq \frac{2\varepsilon}{|A(x,y)|}$. Now let $\alpha_1 \in \Lambda_{y_1}(\varepsilon)$ and $\alpha_2 \in \Lambda_{y_2}(\varepsilon)$, and take $\beta \in \Lambda_x(\varepsilon)$, which is non-empty by Lemma 1.2. Therefore,

$$|\alpha_1 - \alpha_2| \leq |\alpha_1 - \beta| + |\beta - \alpha_2| \leq \frac{2\varepsilon}{|A(x, y_1)|} + \frac{2\varepsilon}{|A(x, y_2)|}.$$

iii) Say $\alpha \in \Lambda_y(\varepsilon)$ and $|\lambda| \leq r$. We have

$$\begin{aligned}\|B_y - (\alpha + \lambda)A_y\| &\leq \|B_y - \alpha A_y\| + \|\lambda A_y\| \\ &= \|B_y - \alpha A_y\| + |\lambda|\|A_y\| \\ &\leq \varepsilon + r\|A_y\|,\end{aligned}$$

so $\alpha + \lambda \in \Lambda_y(\varepsilon + r\|A_y\|)$. Thus $D(\alpha, r) \subset \Lambda_y(\varepsilon + r\|A_y\|)$. □

The constants c_n appearing in the following proof are the constants referred to in the Introduction (see [4], [9]). Also, we use the fact that k-linear forms are k-homogeneous polynomials. Indeed, any k-linear form on $X_1 \times \cdots \times X_k$ is a k-homogeneous analytic function on the space $X_1 \times \cdots \times X_k$.

Theorem 1.4. *For all n, there is a constant D_n such that if A and B are n-linear forms of norm one, then*

$$Z(A) \subset \varepsilon(B) \Rightarrow \|B - \alpha A\| \leq D_n\varepsilon \quad \text{for some } |\alpha| = 1.$$

Proof. We first observe that by induction on n, there exist constants d_n such that for such n-linear forms

$$Z(A) \subset \varepsilon(B) \Rightarrow \|B - \alpha_0 A\| \leq d_n\varepsilon,$$

for some complex number α_0. Indeed, the case $d_1 = 1$ follows from the first part of Phelps' Lemma. For $n > 1$, suppose the result true for $(n-1)$-linear forms, and proceed as in Lemma 1.2 to obtain that $\Lambda_y(d_{n-1}\varepsilon)$ is non-empty.

Next, take $\delta \in (0, 1)$. Choose $y_\delta \in S_{X_n}$ such that $\|A_{y_\delta}\| \geq 1-\delta$, and take another $y \in S_{X_n}$ for which $A_y \neq 0$. Since k-linear forms are k-homogeneous polynomials, there is a constant $c_{n-1} > 0$ such that

$$c_{n-1}\|A_y\| \ \|A_{y_\delta}\| \leq \|A_y \, A_{y_\delta}\|.$$

Choose $x \in S_{X_1} \times \cdots \times S_{X_{n-1}}$ such that

$$(c_{n-1} - \delta)\|A_y\| \ \|A_{y_\delta}\| < |A(x, y) \, A(x, y_\delta)|\,,$$

and take $\alpha \in \Lambda_y(d_{n-1}\varepsilon)$ and $\alpha_\delta \in \Lambda_{y_\delta}(d_{n-1}\varepsilon)$, which, as we have just seen, are non-empty sets (note that $A(x,y)$ and $A(x,y_\delta)$ are non-zero). Then by ii) of Lemma 1.3,

$$\begin{aligned}|\alpha - \alpha_\delta| &\le \frac{2d_{n-1}\varepsilon}{|A(x,y)|} + \frac{2d_{n-1}\varepsilon}{|A(x,y_\delta)|} \\ &= \frac{2d_{n-1}\varepsilon(|A(x,y_\delta)| + |A(x,y)|)}{|A(x,y)\,A(x,y_\delta)|} \\ &\le \frac{4d_{n-1}\varepsilon}{|A(x,y)\,A(x,y_\delta)|} \\ &\le \frac{4d_{n-1}\varepsilon}{(c_{n-1}-\delta)\|A_y\|\|A_{y_\delta}\|} = r.\end{aligned}$$

Thus $\alpha_\delta \in D(\alpha, r) \subset \Lambda_y(d_{n-1}\varepsilon + r\|A_y\|) = \Lambda_y\left(d_{n-1}\varepsilon + \frac{4d_{n-1}\varepsilon}{(c_{n-1}-\delta)\|A_{y_\delta}\|}\right)$ by iii) of Lemma 1.3. So

$$\begin{aligned}\alpha_\delta &\in \Lambda_y\left(d_{n-1}\varepsilon + \frac{4d_{n-1}\varepsilon}{(c_{n-1}-\delta)(1-\delta)}\right) \\ &= \Lambda_y\left(\left[1 + \frac{4}{(c_{n-1}-\delta)(1-\delta)}\right] d_{n-1}\varepsilon\right).\end{aligned}$$

On the other hand, if y is such that $A_y = 0$,

$$\mathbb{C} = \Lambda_y(\varepsilon) \subset \Lambda_y\left(\left[1 + \frac{4}{(c_{n-1}-\delta)(1-\delta)}\right] d_{n-1}\varepsilon\right).$$

Thus,

$$\alpha_\delta \in \bigcap_{y \in S_{X_n}} \Lambda_y\left(\left[1 + \frac{4}{(c_{n-1}-\delta)(1-\delta)}\right] d_{n-1}\varepsilon\right) = K_\delta.$$

These K_δ's are non-empty compact sets, decreasing as $\delta \to 0$, so

$$\bigcap_{\delta > 0} K_\delta \ne \emptyset.$$

For $\alpha_0 \in \bigcap_{\delta>0} K_\delta$, we have

$$\begin{aligned}\|B - \alpha_0 A\| &= \sup_y \sup_x |B(x,y) - \alpha_0 A(x,y)| \\ &= \sup_y \|B_y - \alpha_0 A_y\| \\ &\le \left(1 + \frac{4}{c_{n-1}}\right) d_{n-1}\varepsilon, \\ &= d_n\varepsilon,\end{aligned}$$

say. Since A and B have norm one, arguing as in the second part of Phelps' Lemma, with $\alpha = \frac{\alpha_0}{|\alpha_0|}$, we get

$$\|B - \alpha A\| \le 2d_n\varepsilon, \qquad \text{with } |\alpha| = 1.$$

Thus, letting $D_n = 2d_n$, the proof is complete. □

Note that the constant obtained in the theorem is

$$D_n = 2d_n = 2\left(1 + \frac{4}{c_1}\right)\left(1 + \frac{4}{c_2}\right)\cdots\left(1 + \frac{4}{c_{n-1}}\right),$$

where the c_k's are the constants found in [4]. Thus in the complex case

$$D_n = 2 \prod_{k=1}^{n-1} \left(1 + 4^{k+1}\right),$$

while in the real setting,

$$D_n = 2 \prod_{k=1}^{n-1} \left(1 + 2 \cdot 16^{k}\right).$$

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