

# The large $D$ limit of dimensionally continued gravity

Gaston Giribet

Physics Department, University of Buenos Aires, and IFIBA-CONICET

*Ciudad Universitaria, Pabellón 1, 1428, Buenos Aires, Argentina.*

Instituto de Física, Pontificia Universidad Católica de Valparaíso

*Casilla 4950, Valparaíso, Chile.*

## Abstract

In a recent paper [1] Emparan, Suzuki, and Tanabe studied general relativity in the limit in which the number of spacetime dimensions  $D$  tends to infinity. They showed that, in such limit, the theory simplifies notably. It reduces to a theory whose fundamental objects, black holes and black branes, behave as non-interacting particles. Here, we consider a different way of extending gravity to  $D$  dimensions. We present a special limit of dimensionally continued gravity in which black holes retain their gravitational interaction at large  $D$  and still have entropy proportional to the mass. The similarities and differences with the limit considered in [1] are discussed.

# 1 Introduction

There exist several ways of extending general relativity (GR) to higher dimensions. The simplest one is retaining the form of Einstein-Hilbert Lagrangian density and then extend the action to  $D \geq 4$  dimensions. However, this proposal encounters a naturalness problem since in  $D > 4$  dimensions Einstein tensor is not as special as it is in  $D = 4$ . In  $D > 4$ , the requirement of the equations of motion to be symmetric rank-two covariantly conserved equations of second order does not select Einstein tensor uniquely. In addition, there exists the possibility to supplement Einstein-Hilbert action with dimensionally extended characters of the form<sup>1</sup>

$$\chi_n = \int \varepsilon_{a_1 a_2 \dots a_{2n} \dots a_D} R^{a_1 a_2} \wedge R^{a_3 a_4} \wedge \dots \wedge R^{a_{2n-1} a_{2n}} \wedge e^{a_{2n+1}} \wedge e^{a_{2n+2}} \wedge \dots \wedge e^{a_D}, \quad (1)$$

which, despite of being of order  $R^n$ , yield second-order field equations. Then, it is natural to inquire about why not to include the whole hierarchy of characters  $\chi_n$  up to order  $(D - 1)/2$  in the gravity action. Similarly to how Einstein-Hilbert action  $\chi_1$  can be thought of as the dimensional extension of Euler characteristic in  $D = 2$  dimensions, in  $D > 4$  it is natural to define the gravity action by including the dimensional extension of the other Chern-Weil topological invariants. In  $D = 4$ , for instance, the Gauss-Bonnet theorem implies that  $R^2$  terms of this sort do not modify Einstein equations, as early noticed by Lanczos [2]; however, in  $D > 4$  it is natural to include such terms. The same happens with  $\chi_n$  in higher dimensions. The theory of gravity in  $D$  dimensions whose action consists of all the dimensionally extended topological densities (1) up to  $n = (D - 1)/2$  is known under the rubric of Lovelock, after D. Lovelock have found in [3] the generalization of the Einstein tensor to  $D$  dimensions. Apart from being the most general metric theory of gravity yielding second order equations of motion, Lovelock theory describes next-to-leading order contributions to string theory [4] and M-theory [5, 6] effective actions. This theory exhibits very interesting features, like its property of being free of ghosts about certain maximally symmetric backgrounds. In the last few years, Lovelock theory was considered in the context of holography, providing a prototypical example to investigate how the introduction of higher-curvature terms affects certain aspects of AdS/CFT correspondence [7].

The above digression about which is the natural extension of GR to  $D$  dimensions acquires particular importance in relation to recent studies on the behavior of gravity in the large  $D$  limit

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<sup>1</sup>Here we will work in the first order formalism; see Section 2 for conventions.

[1]. This limit had already been considered in the literature, for instance in Refs. [8, 9, 10, 11], and it was recently revisited in [1] by Emparan et al., who observed that GR simplifies notably when  $D$  goes to infinity. In particular, they observed that in this limit the theory reduces to a theory of non-interacting particles. The fundamental objects of the theory, black holes and black  $p$ -branes, exhibit vanishing cross-section and behave like dust matter.

The idea of considering the large  $D$  limit of gravity theory can be motivated by the large  $N$  limit of gauge theories. The latter has shown to be a fruitful tool to investigate the structure of both Yang-Mills and Chern-Simons theories. Exceptis excipiendis, gravity theory can also be considered as a gauge theory for the local Lorentz group  $SO(D-1, 1)$ . In turn, it is natural to explore whether one can extract relevant information from studying its  $1/D$  expansion. Of course, besides the mathematical analogy with the large  $N$  limit of gauge theories, the fact that  $D$  represents the dimensionality of the spacetime itself introduces additional conceptual difficulties. Nevertheless, as explained in [1], this limit may still be considered and interesting physical information can be extracted from studying it.

Here, we will consider a different way of extending gravity to  $D$  dimensions and study the limit of large  $D$ . More precisely, we will consider the gravity theory defined by the action that includes all terms (1) up to a given order  $R^k$ , with  $k \leq (D-1)/2$ . For this type of theories, the mentioned analogy between the large  $N$  limit of gauge theories and the large  $D$  limit of gravity is even more direct since in the particular case  $2k+1 = D$  the actions we will consider coincide with Chern-Simons actions (CS) for the gauge group  $SO(D-1, 2)$ , and then they correspond to actual gauge theories. This can be regarded as an additional motivation to study these models. For  $3 < 2k+1 < D$ , instead, one is in an intermediate situation, between GR and CS. This will allow us to play between two extremes, between  $k=1$  and  $k=(D-1)/2$ . The fact of having now two parameters,  $D$  and  $k$ , allows us to take the large  $D$  limit in different manners. For instance, we can take  $D$  going to infinity by keeping  $k$  fixed, but we also can take both  $D$  and  $k$  large in such a way that the quotient  $D/k$  remains fixed. In the latter case we will find that, contrary to the limit considered in [1], the black holes happen to retain their gravitational potential in a finite region outside the horizon. At first, this might sound surprising since the  $R^k$  terms of Lovelock theory are expected to introduce ultraviolet effects merely. In the words of [1], the fact that Riemann curvature tends to strongly localize close to the horizon indicates that the dust picture should still apply [in Lovelock theory] at least in some situations. We will

see that, although this is the case in certain situations, it is not true in general and Lovelock black holes may actually retain the interactions at large  $D$ .

## 2 Dimensionally continued gravity

As said, we will be concerned with Lovelock theory of gravity. The idea of considering Lovelock theory in relation to the large  $D$  limit of gravity was already proposed in [1]. The action of the theory can be written as follows

$$S = \kappa^{-1} \sum_{n=0}^{D/2} \alpha_n \chi_n \quad (2)$$

where the terms  $\chi_n$  are given by

$$\chi_n = \int \varepsilon_{a_1 a_2 \dots a_{2n} \dots a_D} R^{a_1 a_2} \wedge R^{a_3 a_4} \wedge \dots \wedge R^{a_{2n-1} a_{2n}} \wedge e^{a_{2n+1}} \wedge e^{a_{2n+2}} \wedge \dots \wedge e^{a_D} \quad (3)$$

where  $R^{ab} = R_{\mu\nu}^{ab} dx^\mu \wedge dx^\nu$  is the curvature two-form,  $R^{ab} = d\omega^{ab} + \omega_c^a \wedge \omega^{cb}$ , with  $\omega^{ab} = \omega_\mu^{ab} dx^\mu$  being the spin connection one-form, and  $e^a = e_\mu^a dx^\mu$  is the vierbein one-form. Latin indices refer to indices in the tangent bundle while Greek indices refer to indices in the spacetime. In (2)  $\kappa$  and  $\alpha_n$  are dimensionful constant that introduce new fundamental scales in the theory. We will discuss these scales below.

The equations of motion are obtained by varying (2) with respect to the vierbein and the spin connection. Varying with respect to  $e^a$  yields

$$\sum_{n=0}^{D/2} \alpha_n (D - 2n) \varepsilon_{a a_2 a_3 \dots a_D} R^{a_2 a_3} \wedge \dots \wedge R^{a_{2n} a_{2n+1}} \wedge e^{a_{2n+2}} \wedge \dots \wedge e^{a_D} = 0, \quad (4)$$

while varying with respect to  $\omega^{ab}$  yields

$$\sum_{n=0}^{D/2} \alpha_n n (D - 2n) \varepsilon_{a b a_3 a_4 \dots a_D} R^{a_3 a_4} \wedge \dots \wedge R^{a_{2n-1} a_{2n}} \wedge T^{a_{2n+1}} \wedge e^{a_{2n+2}} \wedge \dots \wedge e^{a_D} = 0, \quad (5)$$

where  $T^a = de^a + \omega_b^a \wedge e^b$  is the torsion two-form. Equations (5) vanish if torsion is taken to be zero. Notice this is sufficient but not necessary condition if  $D \geq 4$ . Here we will consider  $T^a = 0$ . Then, the equations that remain to be solved are (4).

In addition to considering (4) we will define our theory by specifying a criterion to choose special sets of coupling constants  $\alpha_n$ . We will follow the criterion of Ref. [12]. That is, we will demand the theory to admit a unique maximally symmetric vacuum. This prevents the theory

from suffering from ghost instabilities [13] and other type of pathologies [14]. This requirement of a unique vacuum leads to the following choice of couplings constants [12]

$$\alpha_{n \leq k} = \frac{L^{2(n-k)}}{(D-2n)} \frac{\Gamma(k+1)}{\Gamma(n+1)\Gamma(k-n+1)}, \quad (6)$$

while  $\alpha_{n > k} = 0$ . Ipso facto, this introduces an additional parameter of the theory,  $k$ , which represents the highest order  $R^k$  in the action. This invites to define the critical dimension  $D_c \equiv 2k + 1$ , which represents the minimum number of dimensions such that a term  $\chi_k$  in the action would contribute non-trivially to the equations of motion. In other words,  $\chi_k$  is the Chern-Weil topological invariant in  $D_c - 1$  dimensions. In the particular case  $D = D_c$  (i.e.  $D = 2k + 1$ ) the theory defined by (2)-(6) coincides with the Chern-Simons theory of gravity [15]. In the case  $D = D_c + 1$  the action admits to be written as a Pfaffian, and then it is often referred to as the Born-Infeld action [16]. Hereafter, we will be viewing the gravity theory as a biparametric model, and consequently we will express all the formulae below as functions of  $D$  and  $D_c$ .

At first glance it might seem remarkable that demanding the theory to admit a unique maximally symmetric vacuum yields a relation between the coupling constants  $\alpha_n$  that makes all of them to be determined by a unique fundamental scale  $L$ . However, due to the plethora of vacua in higher-curvature theory, such a requirement turns out to be actually very restrictive and this is why, apart from Planck scale  $\kappa$ ,  $L$  appears as the only relevant scale.

About Planck scale, we find convenient to define Newton constant as follows

$$\kappa = 2\Gamma(D-1)\Omega_{D-2}G_{D,D_c} \quad (7)$$

where  $G_{D,D_c}$  has dimensions of  $(\text{length})^{D-D_c+1}$ , such that the coefficient of the Einstein-Hilbert term,  $\alpha_1/\kappa$ , has dimensions of  $(\text{length})^{2-D}$  as required. In (7),

$$\Omega_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma(\frac{D-1}{2})} \quad (8)$$

is the volume of the unit  $(D-2)$ -sphere.

We also recognize the cosmological constant

$$\Lambda = -\frac{(D-1)(D-2)}{2L^2}, \quad (9)$$

which is given by the coefficient  $\alpha_0/\kappa$  in the action above.

### 3 Dimensionally continued black holes

#### Classical black holes

Another interesting features of the set of theories defined by the choice (6) is the fact that they can be solved analytically in a variety of examples. In particular, their spherically symmetric solutions can be found explicitly for generic values of  $D$  and  $D_c$ . These metrics take the form [12]

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega_{D-2}^2 \quad (10)$$

with

$$f(r) = 1 + \frac{r^2}{L^2} - \left(\frac{r_0}{r}\right)^{2(D-D_c)/(D_c-1)}. \quad (11)$$

In the particular case  $D_c = 3$  ( $k = 1$ ) this solution reduces to Schwarzschild-Tangherlini solution of GR, as expected. In the cases  $D = D_c$ , on the other hand, this solution coincides with the Bañados-Teitelboim-Zanelli solution for Chern-Simons gravity [17].

The mass of solutions (10)-(11) can be computed by resorting to the Hamiltonian formalism [12]. The result is expressed in terms of the horizon radius  $r_H$  as follows

$$M = \frac{r_H^{D-D_c}}{2G_{D,D_c}} \left(1 + \frac{r_H^2}{L^2}\right)^{(D_c-1)/2} \quad (12)$$

up to an additive constant that can be set to zero for simplicity.

At this stage we are ready to study the geometry of these black holes in the large  $D$  limit. In this limit the volume of the  $(D - 2)$ -sphere exhibits the Stirling scaling  $\Omega_{D-2} \sim D^{-D/2}$ , so that it tends to zero. This means that the base manifold of the black hole shrinks in the large  $D$  limit. This was rephrased in [1] as the black holes having vanishing cross-section when  $D$  goes to infinity.

Outside the horizon, the gravitational potential damps off faster as  $D$  increases. This implies that the gravitational interactions between Schwarzschild-Tangherlini black holes extinguishes in the large  $D$  limit. In the general case (11), the way the gravitational potential scales with  $D$  also depends on how  $D_c$  scales. If  $D_c$  remains finite in the large  $D$  limit, the behavior of solutions (10)-(11) would be qualitatively similar to that of [1]. However, if, instead, both  $D$  and  $D_c$  are taken to infinity in a way that the quotient  $D/D_c$  remains fixed, then the black

holes happen to retain their gravitational interaction outside the horizon. In this limit  $\Omega_{D-2}$  still vanishes, but metric function (11) has a large  $D$  behavior

$$f(r) \simeq 1 + \frac{r^2}{L^2} - \left(\frac{r_0}{r}\right)^{2(D/D_c-1)}, \quad (13)$$

and the gravitational potential remains finite.

Let us discuss the location of the horizon radius  $r_H$  of solution (11) in the large  $D$  limit. For large  $D$  with finite  $D_c$  and  $L$  the horizon radius is

$$r_H = r_0 \left(1 + \frac{1 - D_c}{D} \log(r_0/L)\right) + \mathcal{O}(1/D^2), \quad (14)$$

and, as in [1], one finds that  $r_H$  tends to  $r_0$  when  $D$  goes to infinity with  $D_c$  fixed. Then, the interpretation in this case is similar to that in GR: the strong suppression of the gravitational field due to the term  $\sim -(r_0/r)^{2D/D_c}$  in the Newtonian potential makes the interactions of the black holes outside the horizon to vanish. Nevertheless, also as in the case of large  $D$  GR, the notion of *large* AdS black holes still makes sense, as we will discuss below when we come to discuss thermodynamics.

Different features are exhibited by solution (11) in the limit in which both  $D$  and  $D_c$  tend to infinity by keeping the ration  $D/D_c$  finite. In this case one finds

$$r_0^2 \simeq r_H^2 \left(1 + \frac{r_H^2}{L^2}\right)^{\frac{D_c}{D-D_c}}, \quad (15)$$

and then  $r_H$  does not tend to  $r_0$ , but the ratio  $r_0^2/r_H^2 > 1$  remains finite. This is consistent with the black hole retaining the gravitational effects in its vicinity.

## Quantum black holes

Now, let us turn to discuss black holes in the quantum regime. The Hawking temperature associated to black holes (10)-(11) can easily be calculated to be

$$T = \frac{\hbar}{2\pi(D_c - 1)} \left( \frac{(D - 1)r_H}{L^2} + \frac{(D - D_c)}{r_H} \right), \quad (16)$$

which reproduces the GR result for  $D_c = 3$ . We observe that the theory for generic  $D$  and  $D_c$  seems to exhibit Hawking-Page transition, provided  $L$  is finite. If  $D$  goes to infinity and  $D_c$

remains fixed, temperature (16) diverges. Still, there is a point at which the specific heat changes its sign and the transition occurs. This happens at the scale  $r = L\sqrt{(D - D_c)/(D - 1)} \simeq L$ . On the other hand, in contrast to what happens in GR, the presence of higher-curvature terms permits to take the large  $D$  limit in a way that  $T$  remains finite. This is achieved by taking  $D_c$  to infinity as well by keeping  $D/D_c$  fixed. For instance, if we define  $D_c = D(1 - \alpha)$ , then the scale at which the transition takes place is governed by  $\alpha$ , obtaining  $r \simeq L\sqrt{\alpha}$ .

Let us study the case of asymptotically flat solutions. This is obtained by taking the large  $L$  limit. In the theories defined by (2)-(6) this corresponds to having only the highest curvature term  $R^k$  turned on. In this limit, we find

$$T = \frac{\hbar(D - D_c)}{2\pi(D_c - 1)r_H}. \quad (17)$$

The entropy, on the other hand, is

$$S = \frac{\pi r_H^{D-D_c+1}(D_c - 1)}{\hbar G_{D,D_c}(D - D_c + 1)}. \quad (18)$$

Because of the presence of higher-curvature terms in the action, these black holes happen not to obey the Bekenstein-Hawking area law. Instead, entropy is a different monotonic function of the horizon area  $A$ , namely  $S \propto A^{\frac{D-D_c+1}{D-2}}$ . From (18) and (12) we also observe that even in the particular limit in which the black holes retain their gravitational potential, the entropy and the mass go  $S \propto M$  when  $D$  is large. This implies that such a behavior is not necessarily associated to the non-interacting picture, at least not in a simple way.

## Black $p$ -branes

The study of the thermodynamics of black holes (10)-(11) enables to study the thermodynamical stability of other black objects of the theory. For instance, consider black  $p$ -branes. That is, consider solutions of the form  $\Sigma_{D-p} \times T^p$ , with  $T^p$  being a  $p$ -torus and  $\Sigma_{D-p}$  being a black hole of the type discussed above. This type of solutions was considered in Refs. [18, 19], where it was shown that metric

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega_{D-2-p}^2 + \sum_{i=1}^p dz_i^2$$

with

$$f(r) = 1 - \left(\frac{r_0}{r}\right)^{2(D-p-D_c)/(D_c-1)}$$



are solutions of the theory (2)-(6) in the limit  $L \rightarrow \infty$ .

One can analyze the thermodynamical instability of black  $p$ -branes by comparing the entropy of such a configuration with that of a black hole. This requires a careful analysis of the parameters involved in each configuration when comparing them in the microcanonical ensemble. The thermodynamical stability analysis yields the following result for the quotient of entropies [18]

$$\frac{S^{\text{Black } p\text{-brane}}}{S^{\text{Black hole}}} = \frac{(D - D_c + 1)}{(D - D_c - p)} (2G_{D,D_c})^{\lambda_1} M^{\lambda_2} (A_{D,D_c,p})^{\lambda_3}, \quad (19)$$

with

$$A_{D,D_c,p} = \frac{\Gamma(D - D_c - p + 1)\Gamma(D - 1)\Gamma((D - p - 1)/2)}{\Gamma(D - D_c + 1)\Gamma(D - 1 - p)\Gamma((D - 1)/2)} \frac{\pi^{p/2}}{\text{Vol}(T^p)} \quad (20)$$

and with critical exponents

$$\lambda_1 = \frac{1}{D - D_c - p} - \frac{1}{D - D_c} \quad (21)$$

$$\lambda_2 = \frac{D - D_c + 1 - p}{D - D_c - p} - \frac{D - D_c + 1}{D - D_c} \quad (22)$$

$$\lambda_3 = \frac{1}{D - D_c - p} \quad (23)$$

From (19) we observe that the thermodynamical analysis of the black hole / black brane transition in this theory is qualitatively similar to that of GR: There always exists a critical mass above which the black  $p$ -brane is the preferable configuration. The natural question arises as to how this picture is modified in the large  $D$  limit. For instance, in the large  $D$  limit with  $D_c$  fixed, all the exponents  $\lambda_{1,2,3}$  tend to zero. This behavior is actually expected because here we are considering  $p$  fixed. A similar behavior is exhibited also in the limit in which the quotient  $D/D_c$  remains fixed. An interesting limit is given by taking both parameters to infinity by keeping the difference  $D - D_c$  finite. In this limit, exponents  $\lambda_{1,2,3}$  remain finite while  $A_{D,D_c,p}$  scales as  $\sim D^{p/2}/\text{Vol}(T^p)$ . It would be interesting to study the instability of  $p$ -brane solutions of this theory in a similar way to what has been done in Refs. [20, 21] at large  $D$ . The analysis of mechanical stability, on the other hand, can hardly be accomplished for these theories. This is mainly because of two reasons: First, the higher-curvature terms in the action introduce higher powers of the derivatives that make the complexity of the equations to grow dramatically even for large  $D$ . Secondly, the special theories that are being selected by demanding (6) have the property of having a unique maximally symmetric vacuum, and this produces that the

equations of motion factorize in a way that the first orders in perturbation theory identically vanish, making necessary to go beyond the linear approximation. It would also be interesting to study the large  $D$  limit of Lovelock theory in relation to other issues as holographic applications, causality bounds [22, 23], and other subjects in which this theory presents remarkable curiosities as well. This is matter of further study.

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