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Optimal quantum teleportation protocols for fixed average fidelity

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We demonstrate that, among all quantum teleportation protocols giving rise to the same average fidelity, those with aligned Bloch vectors between the input and output states exhibit the minimum average trace distance. This defines optimal protocols. Furthermore, we show that optimal protocols can be interpreted as the perfect quantum teleportation protocol under the action of correlated one-qubit channels. In particular, we focus on the deterministic case for which the final Bloch vector length is equal for all measurement outcomes. Within these protocols, there exists one type that corresponds to the action of uncorrelated channels: these are depolarizing channels. Thus, we established the optimal quantum teleportation protocol under a very common experimental noise.

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I. INTRODUCTION

Among the most astonishing techniques in quantum information theory are the quantum teleportation protocols (QTPs), which consist of two distant parties, usually called Alice and Bob, aiming to transmit an unknown qubit state $\hat{\rho}^{\rm in}$ from Alice's qubit system $\bar{a}$ to Bob's qubit system b, exploiting the features of quantum states and quantum measurements [1].

QTPs are a paradigmatic example of local operations and classical communication (LOCC) protocols, defined on a system composed of three qubits: the system $\bar{a}$, an additional qubit a, and the target system b [2]. The most general teleportation protocol operates on the total system $\hat{\rho}^{\rm in} \otimes \hat{\rho}^{ab}$, where the joint state $\hat{\rho}^{ab}$ is usually referred to as the resource state. The protocol goes as follows. First, Alice performs a joint measurement on her qubits $\bar{a}$ and a, followed by the classical communication of the corresponding measurement outcome (labelled by m) to Bob, who finally applies local unitary operations on his qubit b according to the communicated result. The noiseless standard quantum teleportation is the only scheme that allows perfect transmission, i.e., $\hat{\rho}^{\rm out}_m = \hat{\rho}^{\rm in}\ \forall\, m$ and for any input state being $\hat{\rho}^{\rm out}_m$ the output states in the target system b [1,3,4]. This protocol consists of a Bell measurement, i.e., a projection onto the Bell basis $\{\hat{\beta}_m\}_1^4$ on qubits $\bar{a}$ and a, and a Bell state as quantum resource, $\hat{\rho}^{ab} = \hat{\beta}$.

In realistic teleportation implementations, states and measurements are typically not perfect. The average fidelity between input and output states is generally employed as a figure of merit of the transmission process [3–6]. In noisy standard QTPs, Alice implements a Bell measurement, where the resource state $\hat{\rho}^{ab}$ is taken to be an arbitrary mixed state. Within these standard protocols, one approach is to maximize the average fidelity over all Bob's unitary operations, the so-called strategies, to determine what kind of mixed resource states give rise to quantum teleportation, i.e., when the average fidelity exceeds the bound $\frac{2}{3}$ for classical teleportation [3,7]. Another approach is, for any given initial resource state, to maximize the singlet fraction, i.e., the fidelity between the resource state and the singlet Bell state, by LOCC, to produce a state $\hat{\rho}^{ab}$ with the highest average fidelity, to be used with the standard QTP [4,8]. These are called optimal standard QTPs.

Furthermore, for general resource states and positive operator-valued measures (POVMs), the optimal protocol was given in Ref. [9] using the same framework for the average fidelity as in Ref. [8]. However, as we show below, we identify several protocols that give rise to the same average fidelity, but that can produce significantly different output states. In Ref. [10] the limited effectiveness of fidelity as a tool for evaluating quantum resources was demonstrated. Here, we employ the trace distance as an additional quantum distinguishability measure to define the set of optimal QTPs in the following sense: they minimize the average trace distance for a fixed value of the average fidelity. One of our main findings is showing that this set is given by the teleportation protocols that align, i.e., those for which the direction of the Bloch vector of the output states is the same as that of the initial state to be teleported.

II. GENERAL TELEPORTATION PROTOCOLS

Let us introduce the main elements for our analysis and fix the notation. The input state of Alice's qubit system $\bar{a}$

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can be written as $\hat{\rho}^{\text{in}} = \frac{1}{2}(\hat{\mathbb{1}} + \mathbf{t}^{\mathsf{T}}\hat{\boldsymbol{\sigma}})$, where $\mathbf{t} = (t_1, t_2, t_3)^{\mathsf{T}}$ is the Bloch vector of $\hat{\rho}^{\text{in}}$ with euclidean norm $t = \|\mathbf{t}\| \leqslant 1$, $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)^{\mathsf{T}}$ is the vector of Pauli operators, $\cdot^{\mathsf{T}}$ denotes transposition, and $\hat{\mathbb{1}}$ is the identity operator. The resource state can be written as

$$\hat{\rho}^{ab} = \frac{1}{4}\left(\hat{\mathbb{1}}_4 + (\mathbf{r}^a)^{\mathsf{T}}\hat{\boldsymbol{\sigma}} \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes (\mathbf{r}^b)^{\mathsf{T}}\hat{\boldsymbol{\sigma}} + \sum_{i,j=1}^{3} \mathbb{r}_{ij}\hat{\sigma}_i \otimes \hat{\sigma}_j\right), \tag{1}$$

where $\mathbf{r}^a$ and $\mathbf{r}^b$ are, respectively, the Bloch vectors of the reduced states $\hat{\rho}^a = \text{Tr}_b(\hat{\rho}^{ab})$ and $\hat{\rho}^b = \text{Tr}_a(\hat{\rho}^{ab})$ and $\mathbb{r}_{ij}$ are the elements of the correlation matrix $\mathbb{r} = \text{Tr}(\hat{\rho}^{ab}\,\hat{\boldsymbol{\sigma}} \otimes \hat{\boldsymbol{\sigma}})$. The parametrization (1) defines the Fano form of a two-qubit state [11].

We shall consider general measurements on Alice's qubit systems $\bar{a}$ and a described by POVMs, that is, a set $\{\hat{E}_m^{\bar{a}a}\}$ of positive-definite operators acting on the Hilbert space $\mathcal{H}^{\bar{a}a}$ such that $\sum_m \hat{E}_m^{\bar{a}a} = \hat{\mathbb{1}} \otimes \hat{\mathbb{1}}$. Each POVM element $\hat{E}_m^{\bar{a}a}$ defines univocally a two-qubit POVM state by means of $\hat{\omega}_m^{\bar{a}a} = \frac{1}{4\bar{P}_m}\hat{E}_m^{\bar{a}a}$ where $\bar{P}_m = \frac{1}{4}\text{Tr}(\hat{E}_m^{\bar{a}a})$. Each POVM state $\hat{\omega}_m^{\bar{a}a}$ is completely characterized by its Fano form, in terms of the Bloch vectors $\boldsymbol{\omega}_m^{\bar{a}}$ and $\boldsymbol{\omega}_m^{a}$ of the reduced states $\hat{\omega}_m^{\bar{a}} = \text{Tr}_a(\hat{\omega}_m^{\bar{a}a})$ and $\hat{\omega}_m^{a} = \text{Tr}_{\bar{a}}(\hat{\omega}_m^{\bar{a}a})$, respectively, and the correlation matrix $\mathbb{w}_m = \text{Tr}(\hat{\omega}_m^{\bar{a}a}\,\hat{\boldsymbol{\sigma}} \otimes \hat{\boldsymbol{\sigma}})$. Note that because the POVM elements add up to the identity, the following POVM conditions have to be fulfilled:

$$\sum_m \bar{P}_m = 1, \tag{2a}$$

$$\sum_m \bar{P}_m\left(\boldsymbol{\omega}_m^{a}\right)^{\mathsf{T}} = \mathbf{0}^{\mathsf{T}}, \tag{2b}$$

$$\sum_m \bar{P}_m\boldsymbol{\omega}_m^{\bar{a}} = \mathbf{0}, \tag{2c}$$

$$\sum_m \bar{P}_m\mathbb{w}_m = \mathbb{0}, \tag{2d}$$

where $\mathbf{0}$ and $\mathbb{0}$ denote the null vector and null matrix, respectively.

As a result of Alice's measurement, the qubit b of Bob's collapses to $\hat{\rho}_m^b = \frac{1}{2}[\hat{\mathbb{1}} + (\mathbf{t}_m^b)^{\mathsf{T}}\hat{\boldsymbol{\sigma}}]$ with probability $P_m = \text{Tr}(\hat{E}_m^{\bar{a}a} \otimes \hat{\mathbb{1}}^b\hat{\rho}^{\text{in}} \otimes \hat{\rho}^{ab}) = \bar{P}_m\, g_m(\mathbf{t})$, where $g_m(\mathbf{t}) = 1 + (\boldsymbol{\omega}_m^{a})^{\mathsf{T}}\mathbf{r}^a + (\mathbb{w}_m\mathbf{r}^a + \boldsymbol{\omega}_m^{\bar{a}})^{\mathsf{T}}\mathbf{t}$. The Bloch vector of $\hat{\rho}_m^b$ is

$$\mathbf{t}_m^b = \frac{\mathbb{o}_m}{g_m(\mathbf{t})}\,\mathbf{t} + \frac{\boldsymbol{\kappa}_m}{g_m(\mathbf{t})}, \tag{3}$$

where $\mathbb{o}_m = \mathbf{r}^b(\boldsymbol{\omega}_m^{\bar{a}})^{\mathsf{T}} + \mathbb{r}^{\mathsf{T}}\mathbb{w}_m^{\mathsf{T}}$ and $\boldsymbol{\kappa}_m = \mathbf{r}^b + \mathbb{r}^{\mathsf{T}}\boldsymbol{\omega}_m^{a}$. Finally, Alice communicates to Bob her measurement result m and Bob applies a unitary operation $\hat{U}_m$ on qubit b. The output quantum state is $\hat{\rho}_m^{\text{out}} = \hat{U}_m\hat{\rho}_m^b\hat{U}_m^{\dagger} = \frac{1}{2}(\hat{\mathbb{1}} + \mathbf{t}_m^{\mathsf{T}}\hat{\boldsymbol{\sigma}})$ with Bloch vector

$$\mathbf{t}_m = \mathbb{R}_m\mathbf{t}_m^b, \tag{4}$$

where $\mathbb{R}_m$ is the unique rotation matrix such that $\hat{U}_m\,\mathbf{n}^{\mathsf{T}}\hat{\boldsymbol{\sigma}}\,\hat{U}_m^{\dagger} = (\mathbb{R}_m\mathbf{n})^{\mathsf{T}}\hat{\boldsymbol{\sigma}}$ with $\mathbf{n}$ the unit real column vector. Thus, for each QTP there is an associated channel Λ that yields $\Lambda(\hat{\rho}^{\text{in}}) = \sum_m P_m\hat{\rho}_m^{\text{out}}$ whose Bloch vector is

$$\mathbf{t}_\Lambda = \sum_m P_m\mathbf{t}_m = \mathbb{C}_\Lambda\mathbf{t} + \mathbf{v}_\Lambda,$$

with $\mathbb{C}_\Lambda = \sum_m \bar{P}_m\mathbb{R}_m\mathbb{o}_m$ and $\mathbf{v}_\Lambda = \sum_m \bar{P}_m\mathbb{R}_m\boldsymbol{\kappa}_m$.

III. GENERALIZED ERROR MEASURES IN QUANTUM TELEPORTATION

The performance of a general QTP can be quantified by taking a measure of distinguishability between the input state and the ensemble of output states in the form $\bar{D}(\hat{\rho}^{\text{in}}) = \sum_m P_m\, D(\hat{\rho}^{\text{in}}, \hat{\rho}_m^{\text{out}})$ where $D(\cdot,\cdot)$ stands for a distance measure between quantum states. Being $P_m = \bar{P}_m\, g_m(\mathbf{t})$, for any choice of D the previous quantity can be expressed as a function of the initial Bloch vector $\mathbf{t}$, so we write $\bar{D}(\hat{\rho}^{\text{in}}) = \bar{D}(\mathbf{t}) \equiv \bar{D}$.

The final figure of merit is the average distance defined as the expectation value of $\bar{D}$ over the uniform distribution of pure input states: $\langle\bar{D}\rangle = \frac{1}{4\pi}\iint_{S(\mathcal{B})}\bar{D}(\mathbf{t})\,d\Omega$, where $d\Omega = \sin\theta\, d\theta\, d\phi$ ($0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi < 2\pi$) is the differential solid angle in the Bloch sphere $S(\mathcal{B})$. The distance deviation $\Delta\bar{D}$ is defined as the standard deviation of the function $\bar{D}$, that is, $\Delta\bar{D} = \sqrt{\langle\bar{D}^2\rangle - \langle\bar{D}\rangle^2}$.

In this work, we shall consider the following distance measures: the trace distance $D_{\text{T}}(\hat{\rho},\hat{\sigma}) = \frac{1}{2}\text{Tr}(\|\hat{\rho} - \hat{\sigma}\|)$ where $\|\hat{A}\| = \sqrt{\hat{A}\hat{A}^{\dagger}}$ stands for the operator norm [12], and the Uhlmann-Jozsa quantum fidelity $F(\hat{\rho},\hat{\sigma}) = [\text{Tr}(\sqrt{\sqrt{\hat{\rho}}\,\hat{\sigma}\,\sqrt{\hat{\rho}}})]^2$ [13]. For qubit states characterized by Bloch vectors $\mathbf{t}$ and $\mathbf{t}_m$, they give $D_{\text{T}}(\hat{\rho}^{\text{in}}, \hat{\rho}_m^{\text{out}}) = \frac{1}{2}\|\mathbf{t} - \mathbf{t}_m\|$, and $F(\hat{\rho}^{\text{in}}, \hat{\rho}_m^{\text{out}}) = \frac{1}{2}(1 + \mathbf{t}^{\mathsf{T}}\mathbf{t}_m + \sqrt{1-t^2}\sqrt{1-t_m^2})$ where $t = \|\mathbf{t}\|$ and $t_m = \|\mathbf{t}_m\|$.

The average fidelity takes the following form for general QTPs:

$$\langle\bar{F}\rangle = \tfrac{1}{2}\big[1 + \tfrac{1}{3}\text{tr}(\mathbb{C}_\Lambda)\big] \tag{5}$$

(where tr denotes the trace of matrices to differentiate from the trace of operators Tr), and the squared fidelity deviation is given by

$$\big(\Delta\bar{F}\big)^2 = \tfrac{1}{4}\Big\{\tfrac{1}{15}\big[\text{tr}\big(\mathbb{C}_\Lambda^2\big) + (\text{tr}(\mathbb{C}_\Lambda))^2 + \text{tr}\big(\mathbb{C}_\Lambda\mathbb{C}_\Lambda^{\mathsf{T}}\big)\big] - \big[\tfrac{1}{3}\text{tr}(\mathbb{C}_\Lambda)\big]^2\Big\} + \tfrac{1}{12}\text{tr}\big(\mathbf{v}_\Lambda\mathbf{v}_\Lambda^{\mathsf{T}}\big).$$

Note that different QTPs can result in the same matrix $\mathbb{C}_\Lambda$ in Eq. (5), producing the same average fidelity. These protocols in general are not equivalent because they can yield physically distinct output states.

IV. OPTIMAL PROTOCOLS FOR FIXED AVERAGE FIDELITY

Let us consider a set of arbitrary teleportation protocols that yield the same average fidelity. The following theorem characterizes the optimal protocols within this set.

Theorem 1. Among all QTPs such that $\langle\bar{F}\rangle = \alpha \in (0, 1]$, the average trace distance $\langle\bar{D}_{\text{T}}\rangle$ takes its minimum value for those protocols that align, i.e., when the corresponding Bloch vectors of the output states $\hat{\rho}_m^{\text{out}}$ are given by $\mathbf{t}_m^{\text{alig}} = s_m\mathbf{t}\ \forall m$

with $s_m \in (0, 1]$ satisfying $\sum_m P_m s_m = 2\alpha - 1$. These protocols are defined as optimal.

Proof. For arbitrary QTPs, we have that

$$\langle \bar{D}_{\mathrm{T}} \rangle = \left\langle \frac{1}{2} \sum_m P_m \|\mathbf{t} - \mathbf{t}_m\| \right\rangle$$
$$\geqslant \left\langle \frac{1}{2} \|\mathbf{t} - \mathbf{t}_\Lambda\| \right\rangle \geqslant 1 - \langle F(\hat{\rho}^{\mathrm{in}}, \Lambda(\hat{\rho}^{\mathrm{in}})) \rangle$$
$$= \frac{1}{2}\left(1 - \frac{1}{3}\mathrm{tr}(\mathbb{C}_\Lambda)\right) = 1 - \langle \bar{F} \rangle, \tag{6}$$

where we used consecutively Jensen's inequality $\sum_n P_n \|\mathbf{a}_n\| \geqslant \|\sum_n P_n \mathbf{a}_n\|$ with $\sum_n P_n = 1$ (because every norm is a convex function), $D_{\mathrm{T}} \geqslant 1 - F$ [12], $F[\hat{\rho}^{\mathrm{in}}, \Lambda(\hat{\rho}^{\mathrm{in}})] = \bar{F}$ because $\hat{\rho}^{\mathrm{in}}$ is a pure state, and Eq. (5). Let us now consider fidelity-equivalent protocols, in the sense that $\langle \bar{F} \rangle = \alpha$ is satisfied for given $\alpha \in (0, 1]$. The average trace distance can take different values, with a fixed lower bound, $\langle \bar{D}_{\mathrm{T}} \rangle \geqslant 1 - \alpha$, as deduced from Eq. (6). It is straightforward to see that this lower bound is attained by protocols such that $\mathbf{t}_m^{\mathrm{alig}} = s_m \mathbf{t}\ \forall m$ with $\sum_m P_m s_m = 2\alpha - 1$. ■

Let us now give a comprehensive characterization of the optimal protocols, defined in Theorem 1. In this context, the following result establishes the necessary and sufficient conditions to have a protocol that aligns.

Theorem 2. An arbitrary QTP aligns if and only if it satisfies, for all m, that: (i) $\mathbb{w}_m \mathbf{r}^a + \boldsymbol{\omega}_m^{\bar{a}} = \mathbf{0}$, (ii) $\mathbf{r}^b = \mathbf{0}$ and $\boldsymbol{\omega}_m^a = \mathbf{0}$, and (iii) $\mathbb{R}_m = s_m \mathbb{w}_m^{-\intercal} \mathbb{r}^{-\intercal}$ with s_m such that $\mathbf{t}_m = s_m \mathbf{t}\ \forall m$, being $\mathbf{t}_m$ the Bloch vector of the output state of the protocol.

Corollary 1. The quantum channel associated to a protocol that aligns is characterized by $\mathbb{C}_{\Lambda^{\mathrm{alig}}} = \frac{1}{3}\mathrm{tr}(\mathbb{C}_{\Lambda^{\mathrm{alig}}})\mathbb{1}$ and $\mathbf{v}_\Lambda^{\mathrm{alig}} = \mathbf{0}$. Thus, this kind of protocol yields null fidelity deviation, $\Delta^{\mathrm{alig}} \bar{F} = 0$.

Proof. The final Bloch vector $\mathbf{t}_m$ of Bob's qubit is given in Eq. (4) with $\mathbf{t}_m^b$ in Eq. (3). Therefore, if $\mathbf{t}_m = s_m \mathbf{t}$, then g_m must be independent of $\mathbf{t}$ and $\boldsymbol{\kappa}_m$ must vanish, for all m. The first condition happens iff the statement (i) of the theorem is true. Applying the POVM conditions (2a) and (2b) to the equations $\boldsymbol{\kappa}_m = \mathbf{r}^b + \mathbb{r}^\intercal \boldsymbol{\omega}_m^a = \mathbf{0}\ \forall m$, we arrive at $\mathbf{r}^b = \mathbf{0}$, so $\mathbb{r}^\intercal \boldsymbol{\omega}_m^a = 0\ \forall m$. Because $\boldsymbol{\kappa}_m = \mathbf{0}\ \forall m$ and $\mathbf{r}^b = \mathbf{0}$, we must have that $\frac{\mathbb{R}_m \mathbb{r}^\intercal \mathbb{w}_m^\intercal}{g_m(\mathbf{t})} = s_m \mathbb{1}$ to align, i.e., $\mathbf{t} = s_m \mathbf{t}_m\ \forall m$. Therefore, the matrices $\mathbb{r}$ and $\mathbb{w}_m$ must be invertible and from the condition $\mathbb{r}^\intercal \boldsymbol{\omega}_m^a = 0\ \forall m$, we obtain that $\boldsymbol{\omega}_m^a = \mathbf{0}\ \forall m$. At this point, we demonstrated statement (ii) of the theorem. Note that we arrived at $g_m(\mathbf{t}) = 1\ \forall m$, which implies $\mathbb{R}_m \mathbb{r}^\intercal \mathbb{w}_m^\intercal = s_m \mathbb{1}$. This proves statement (iii) of the theorem.

Note that statement (iii) of Theorem 2 implies that $\mathbb{C}_{\Lambda^{\mathrm{alig}}} = \sum_m P_m s_m \mathbb{1} = \frac{1}{3}\mathrm{tr}(\mathbb{C}_{\Lambda^{\mathrm{alig}}})\mathbb{1}$ and $\boldsymbol{\kappa}_m = \mathbf{0}\ \forall m$ leads to $\mathbf{v}_\Lambda = \mathbf{0}$. These are the statements of Corollary 1. On the other hand, since the lower bound in Eq. (6) is achieved for protocols that align, we have that for average fidelity α it holds

$$2\alpha - 1 = \frac{1}{3}\mathrm{tr}(\mathbb{C}_{\Lambda^{\mathrm{alig}}}) = \sum_m \bar{P}_m s_m. \tag{7}$$

■

Before establishing the next theorem, we recall that, under suitable local unitary transformations, i.e.,

$$\hat{\rho}_{\mathrm{c}}^{ab} = \hat{U}^a \otimes \hat{U}^b \hat{\rho}^{ab} (\hat{U}^a)^\dagger \otimes (\hat{U}^b)^\dagger,$$

every two-qubit state $\hat{\rho}^{ab}$ can be transformed into a canonical form $\hat{\rho}_{\mathrm{c}}^{ab}$, with correlation matrix $\mathbb{r}_{\mathrm{d}} = (\mathbb{o}^a)\mathbb{r}(\mathbb{o}^b)^\intercal = \mathrm{diag}(r_1, r_2, r_3)$, where $\mathbb{o}^a$ and $\mathbb{o}^b$ are rotation matrices, and the transformed marginal Bloch vectors $\mathbf{r}_{\mathrm{c}}^a = \mathbb{o}^a \mathbf{r}^a$ and $\mathbf{r}_{\mathrm{c}}^b = \mathbb{o}^b \mathbf{r}^b$ [14]. Furthermore, the positivity condition on the density operators $\hat{\rho}^{ab}$ and $\hat{\rho}_{\mathrm{c}}^{ab}$, when $\mathbf{r}^b = \mathbf{0}$, corresponds to the inequalities [15]

$$-2\det(\mathbb{r}_{\mathrm{d}}) - (\|\mathbb{r}_{\mathrm{d}}\|^2 - 1) \geqslant \left\|\mathbf{r}_{\mathrm{c}}^a\right\|^2, \tag{8a}$$

$$f(r_1, r_2, r_3) \geqslant 4\left\|\left(\mathbf{r}_{\mathrm{c}}^a\right)^\intercal \mathbb{r}_{\mathrm{d}}\right\|^2 + \left\|\mathbf{r}_{\mathrm{c}}^a\right\|^2\left[2(1 - \|\mathbb{r}_{\mathrm{d}}\|^2) - \left\|\mathbf{r}_{\mathrm{c}}^a\right\|^2\right], \tag{8b}$$

where $f(r_1, r_2, r_3) = -8\det(\mathbb{r}_{\mathrm{d}}) + (\|\mathbb{r}_{\mathrm{d}}\|^2 - 1)^2 - 4\|\tilde{\mathbb{r}}_{\mathrm{d}}\|^2 = (1 - r_1 - r_2 - r_3)(1 - r_1 + r_2 + r_3)(1 + r_1 - r_2 + r_3)(1 + r_1 + r_2 - r_3)$, $\|\mathbb{r}_{\mathrm{d}}\|^2 = \mathrm{tr}(\mathbb{r}_{\mathrm{d}}^2)$, $\tilde{\mathbb{r}}_{\mathrm{d}} = \det(\mathbb{r}_{\mathrm{d}})\mathbb{r}_{\mathrm{d}}^{-1}$, and $-1 \leqslant \det(\mathbb{r}_{\mathrm{d}}) \leqslant 1$ [here we are assuming that $\det(\mathbb{r}_{\mathrm{d}}) \neq 0$]. Thus, the diagonal elements r_1, r_2, r_3 belong to a convex subset, defined by Eqs. (8), inside the tetrahedron given by inequality (8b) with $\mathbf{r}_{\mathrm{c}}^a = \mathbf{0}$ [14].

We are now in a position to present the following theorem which fully characterizes the QTPs that align.

Theorem 3. All QTPs that align verify the following. (i) The POVM states $\hat{\omega}_m^{\bar{a}a}$ have correlation matrices $\mathbb{w}_m = (\mathbb{o}_m^{\bar{a}})^\intercal \mathbb{w}_{\mathrm{d}m} \mathbb{o}^a$ with $\mathbb{w}_{\mathrm{d}m} = s_m \mathbb{r}_{\mathrm{d}}^{-1}$ and where $\mathbb{r} = (\mathbb{o}^a)^\intercal \mathbb{r}_{\mathrm{d}} \mathbb{o}^b$ is the correlation matrix of the resource state $\hat{\rho}^{ab}$, with $(\mathbb{o}^a)^\intercal$ the rotation matrix that simultaneously diagonalizes the positive definite matrices $\mathbb{r}\mathbb{r}^\intercal$ and $\mathbb{w}_m^\intercal \mathbb{w}_m$, while $\mathbb{o}^b$ and $\mathbb{o}_m^{\bar{a}}$ are the rotation matrices that diagonalize $\mathbb{r}^\intercal \mathbb{r}$ and $\mathbb{w}_m \mathbb{w}_m^\intercal$ respectively. (ii) Bob's rotation matrices are of the form $\mathbb{R}_m = \mathbb{o}_m^{\bar{a}} \mathbb{o}^b\ \forall m$. Finally, (iii) the rotation matrices $\mathbb{o}_m^{\bar{a}}$ must fulfill the POVM condition (d) that in this case reduces to $\sum \bar{P}_m s_m (\mathbb{o}_m^{\bar{a}})^\intercal = \mathbb{0}$.

Corollary 2. All the protocols that align have $\det(\mathbb{r}_{\mathrm{d}}) < 0$.

Proof. From the canonical decomposition of the states $\hat{\rho}^{ab}$ and $\hat{\omega}_m^{\bar{a}a}$ we have that $\mathbb{r} = (\mathbb{o}^a)^\intercal \mathbb{r}_{\mathrm{d}} \mathbb{o}^b$ and $\mathbb{w}_m = (\mathbb{o}_m^{\bar{a}})^\intercal \mathbb{w}_{\mathrm{d}m} \mathbb{o}_m^a$, where the columns of $\mathbb{o}^a$ are eigenvectors of $\mathbb{r}\mathbb{r}^\intercal$ and the columns of $\mathbb{o}_m^a$ are eigenvectors of $\mathbb{w}_m^\intercal \mathbb{w}_m$. Note that $\mathbb{r}\mathbb{r}^\intercal$ and $\mathbb{w}_m^\intercal \mathbb{w}_m$ are positive-definite matrices because $\mathbb{r}$ and $\mathbb{w}_m$ are full rank. They are diagonalized by orthogonal matrices. From the orthogonality of $\mathbb{R}_m$ and condition (iii) in Theorem 2, we get $\mathbb{r}\mathbb{r}^\intercal(\frac{\mathbb{w}_m}{s_m})^\intercal \frac{\mathbb{w}_m}{s_m} = \mathbb{1}$, leading to $[\mathbb{r}\mathbb{r}^\intercal, \mathbb{w}_m^\intercal \mathbb{w}_m] = \mathbb{0}$. Then, $\mathbb{r}\mathbb{r}^\intercal$ and $\mathbb{w}_m^\intercal \mathbb{w}_m$ are diagonalized by a single orthogonal matrix [16] that we can choose to be one of the possible matrices $(\mathbb{o}^a)^\intercal$ in the canonical decomposition of $\mathbb{r}$, i.e., $\mathbb{r}\mathbb{r}^\intercal = (\mathbb{o}^a)^\intercal \mathbb{r}_{\mathrm{d}}^2 \mathbb{o}^a$ and $\mathbb{w}_m^\intercal \mathbb{w}_m = (\mathbb{o}^a)^\intercal \mathbb{w}_{\mathrm{d}m}^2 \mathbb{o}^a$, so $\mathbb{o}_m^a = \mathbb{o}^a\ \forall m$. Thus, we immediately arrive at $\mathbb{w}_{\mathrm{d}m}^2 = s_m^2 \mathbb{r}_{\mathrm{d}}^{-2}$. Finally, we can write $\mathbb{R}_m = \mathbb{o}_m^{\bar{a}} \mathbb{w}_{\mathrm{d}m}^{-1} s_m \mathbb{r}_{\mathrm{d}}^{-1} \mathbb{o}^b$, and then $\mathbb{w}_{\mathrm{d}m} = s_m \mathbb{r}_{\mathrm{d}}^{-1}$ must be true [17]. This proves statement (i). Statements (ii) and (iii) follow straightforwardly. ■

From Theorem 3 it is possible to conclude that the only QTP such that $\mathbf{t}_m = \mathbf{t}\ \forall m$ is, up to local unitaries on the qubit systems, the perfect QTP defined by performing a Bell measurement on qubits $\bar{a}$ and a and a Bell state as resource for qubits a and b. Specifically, the positivity conditions on the density operators $\hat{\rho}^{ab}$ and $\hat{\omega}_m^{\bar{a}a}$ correspond, respectively, to the set of inequalities (8) for the matrix elements of $\mathbb{r}_{\mathrm{d}}$ with $\mathbf{r}_{\mathrm{c}}^b = \mathbf{0}$, and for the matrix elements of $\mathbb{w}_{\mathrm{d}m} = s_m \mathbb{r}_{\mathrm{d}}^{-1}$ with Bloch vectors $\boldsymbol{\omega}_{\mathrm{c}m}^{\bar{a}} = -\mathbb{r}_{\mathrm{d}}^{-1} \mathbf{r}_{\mathrm{c}}^a$ and $\boldsymbol{\omega}_{\mathrm{c}m}^a = \mathbf{0}\ \forall m$ [this follows from conditions (i) and (ii) of Theorem 2; see Eq. (A4) in

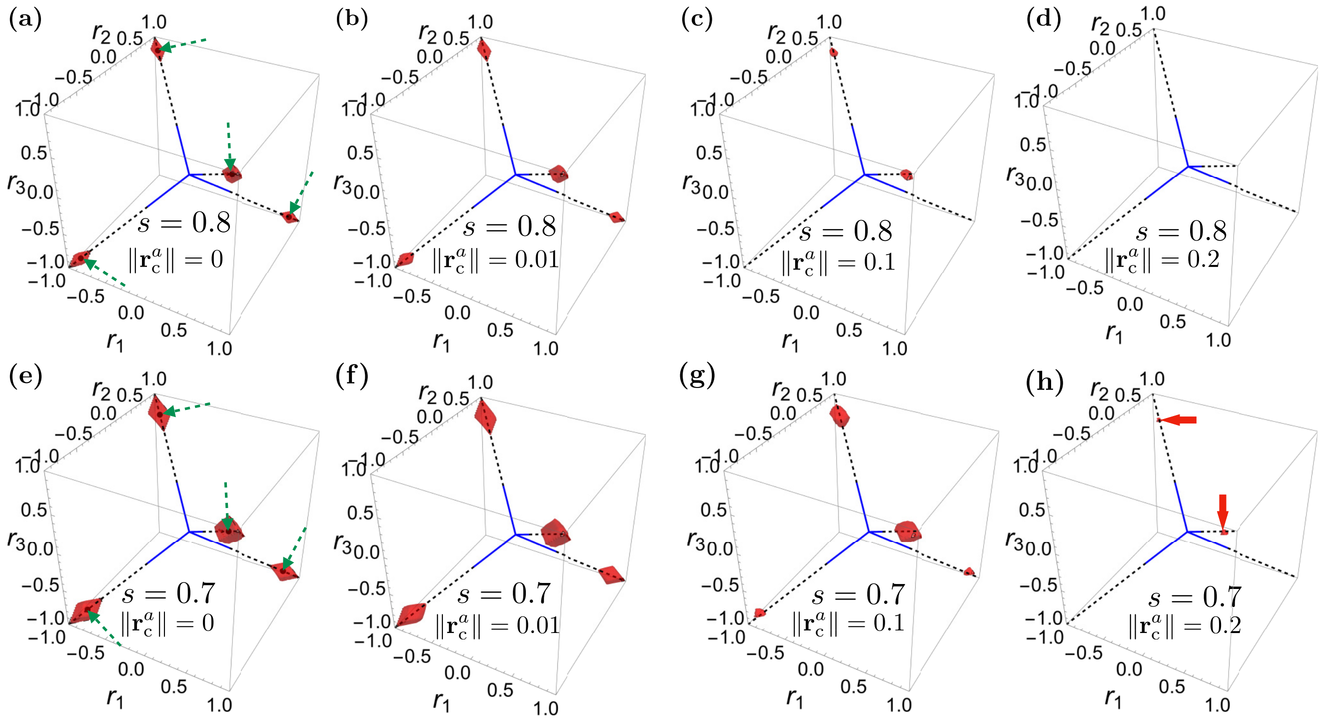

FIG. 1. Diagonal matrix elements (r_1, r_2, r_3) of DQTPs that align, with fixed value of average fidelity $\langle \bar{F} \rangle^{\text{alig}} = \frac{1}{2}(1+s) = \alpha$, for different values of s and Bloch vectors $\mathbf{r}_c^a$ (see text for explanation). The angular spherical coordinates of $\mathbf{r}_c^a$ are $\theta = \phi = 0$ in panels [(a)–(d)], and $\theta = \phi = \frac{\pi}{2}$ in panels [(e)–(h)]. In (d) there is no solution; in (h) the solid arrows (red online) indicate the tiny set of solutions. The lines (blue online) correspond to the four types of Werner states $\hat{W} = \frac{(1-p)}{4}\hat{\mathbb{1}} \otimes \hat{\mathbb{1}} + p\hat{\beta}$ where $\hat{\beta}$ is one of the four Bell states. The solid lines correspond to separable states, i.e., $-\frac{1}{3} \leqslant p \leqslant \frac{1}{3}$, and the dashed lines to entangled states, i.e., $\frac{1}{3} < p \leqslant 1$. In (a) and (e), the dashed arrows (green online) indicate the special cases with $p = s^{\frac{1}{2}}$.

Appendix A]. The only solutions to these sets of inequalities, when $s_m = 1\ \forall m$, correspond to $\mathbb{r}_{\text{d}}^{\text{Bell}} = (\mathbb{r}_{\text{d}}^{\text{Bell}})^{-1} = \mathbb{w}_{dm}^{\text{Bell}} \in \{\mathbb{r}_{\Phi_+}^{\text{Bell}} = -\text{diag}(1,1,1),\ \mathbb{r}_{\Phi_-}^{\text{Bell}} = \text{diag}(-1,1,1), \mathbb{r}_{\Psi_+}^{\text{Bell}} = \text{diag}(1,-1,1), \mathbb{r}_{\Psi_-}^{\text{Bell}} = \text{diag}(1,1,-1)\}$ with $\mathbf{r}_c^a = \mathbf{0}\ \ \forall m$. These solutions are Bell states for the resource $\hat{\rho}_c^{ab} = \hat{\beta}$ and for the POVM operators $\hat{\omega}_{cm}^{\bar{a}a} = \hat{\beta}_m$ with $m = 1, \dots, 4$, in the canonical form. Therefore, from condition (i) of Theorem 3 we have that the correlation matrix of $\hat{\omega}_{cm}^{\bar{a}a}$ is $\mathbb{w}_m = \mathbb{w}_{cm} = (\mathbb{o}_m^{\bar{a}})^{\intercal}\mathbb{w}_{dm}\mathbb{o}^a$ with $\mathbb{o}^a = \mathbb{1}$ and $\mathbb{o}_m^{\bar{a}} = \mathbb{b}_m^{\bar{a}} \in \{\text{diag}(1,1,1), \text{diag}(1,-1,-1), \text{diag}(-1,1,-1), \text{diag}(-1,-1,1)\}$, that are the only diagonal orthogonal matrices in $\mathbb{R}^{3\times 3}$ with $\det(\mathbb{b}_m^{\bar{a}}) = 1$. Note that these matrices satisfy condition (iii) of Theorem 3. All perfect QTPs, therefore, are those with resource state $\hat{\rho}^{ab} = \hat{U}^a \otimes \hat{U}^b\, \beta\, (\hat{U}^a)^\dagger \otimes (\hat{U}^b)^\dagger$, with $\mathbb{r} = (\mathbb{o}^a)^\intercal \mathbb{r}_{\text{d}}^{\text{Bell}} \mathbb{o}^b$ being its correlation matrix and with a POVM composed by $\hat{\omega}_m^{\bar{a}a} = \hat{U}^{\bar{a}} \otimes \hat{U}^a \hat{\omega}_{cm}^{\bar{a}a} (\hat{U}^{\bar{a}})^\dagger \otimes (\hat{U}^a)^\dagger$, with $\hat{\omega}_{cm}^{\bar{a}a} = \hat{\beta}_m$, whose correlation matrices are $\mathbb{w} = (\mathbb{o}^{\bar{a}})^\intercal \mathbb{b}_m^{\bar{a}} \mathbb{r}_{\text{d}}^{\text{Bell}} \mathbb{o}^a$.

It is worth noting that, according to Theorem 3, for teleportation protocols that align, the POVM states can be written as $\hat{\omega}_m^{\bar{a}a} = \hat{U}_m^{\bar{a}} \otimes \hat{U}^a \hat{\omega}_{cm}^{\bar{a}a} (\hat{U}_m^{\bar{a}})^\dagger \otimes (\hat{U}^a)^\dagger$, where $\hat{U}^a$ is one of the local unitary operations that carries $\hat{\rho}^{ab}$ into its canonical form. Therefore, the Bob qubit state $\hat{\rho}_m^b$, after Alice's measurement, does not depend on $\hat{U}^a$. So, we can ignore this local unitary operation.

Now, let us examine the scenario where $\hat{U}_m^{\bar{a}}$ is a unitary matrix such that $\hat{U}_m^{\bar{a}}\, \mathbf{n}^\intercal \hat{\boldsymbol{\sigma}}\, (\hat{U}_m^{\bar{a}})^\dagger = (\mathbb{o}_m^{\bar{a}}\mathbf{n})^\intercal \hat{\boldsymbol{\sigma}}$ with $\mathbb{o}_m^{\bar{a}}$ a diagonal matrix. In the case of diagonal matrices $\mathbb{o}_m^{\bar{a}}$, the only possible way to satisfy condition (iii) of Theorem 3 is when $\mathbb{o}_m^{\bar{a}} = \mathbb{b}_m^{\bar{a}}$ and $s_m = s$ for $m = 1, \dots, 4$.

Therefore, for these particular protocols considered, the resource state has a correlation matrix $\mathbb{r} = \mathbb{r}_{\text{d}}\mathbb{o}^b$ and the POVM states have $\mathbb{w}_m = \mathbb{w}_{cm} = \mathbb{b}_m^{\bar{a}} s\, \mathbb{r}_{\text{d}}^{-1}$ with $m = 1, \dots, 4$, i.e., $\hat{\omega}_m^{\bar{a}a} = \hat{\omega}_{cm}^{\bar{a}a}$. We refer to these kinds of protocols as deterministic quantum teleportation protocols (DQTPs) that align. For these protocols the Bloch vectors of the reduced states $\hat{\rho}^a$ and $\hat{\omega}_m^{\bar{a}}$ are, respectively, $\mathbf{r}^a = \mathbf{r}_c^a$ and $\boldsymbol{\omega}_m^{\bar{a}} = (\mathbb{b}_m^{\bar{a}})^\intercal \boldsymbol{\omega}_{cm}^{\bar{a}}$ with $\boldsymbol{\omega}_{cm}^{\bar{a}} = \boldsymbol{\omega}_c^{\bar{a}} = -s\, \mathbb{r}_{\text{d}}^{-1}\mathbf{r}_c^a$ for $m = 1, \dots, 4$.

The perfect QTP, $s = 1$, is a special case of a DQTP that aligns corresponding to $\mathbf{r}_c^a = \mathbf{0}$ and $\mathbb{r}_{\text{d}} = \mathbb{r}_{\text{d}}^{\text{Bell}}$.

In the case of imperfect alignment of the DQTP, where $s < 1$, the set of allowed values for the diagonal elements of $\mathbb{r}_{\text{d}}$ and $s\,\mathbb{r}_{\text{d}}^{-1}$, as determined by the positivity conditions for the density operators of the resource and POVM states, is quite extensive.

Let us consider, as an example, all the protocols that align for different values of s and Bloch vectors $\mathbf{r}_c^a$ in the scenario $s_m = s$ for all m. Figure 1 illustrates the sets of values (r_1, r_2, r_3), represented by the shaded red volume, for which there exists a POVM that aligns for different values of $\|\mathbf{r}_c^a\|$ and considering two different values of s. These regions are determined by the positivity conditions on $\hat{\rho}^{ab}$ and $\hat{\omega}_m^{\bar{a}a}$ [see inequalities (8)]. It can be observed that, as $\|\mathbf{r}_c^a\|$ increases, the set of solutions becomes smaller. In Figs. 1(a) to 1(d) we consider $s = 0.8$; notice that in Fig. 1(d), corresponding to $\|\mathbf{r}_c^a\| = 0.2$, there is no solution. However, when we reduce the average fidelity value as in Figs. 1(e) to 1(h) where we take $s = 0.7$, a tiny set of solutions appears for $\|\mathbf{r}_c^a\| = 0.2$ [indicated by the solid arrows (red online) in Fig. 1(h)].

V. NOISE IN DQTP THAT ALIGN

For a deterministic QTP meeting the conditions in Theorems 2 and 3, from Eq. (7) we have that $\langle \bar{F}\rangle^{\rm alig} = \frac{1}{2}(1+s) = \alpha$. Therefore, we see that for a fixed value $\alpha < 1$, i.e., fixed $s < 1$, there exist different DQTPs that align giving rise to the same average fidelity (see Fig. 1). These different protocols can be identified as the action of one-qubit channel over the perfect DQTP that aligns: $\hat{\rho}_c^{ab} = (\varepsilon^a \otimes \varepsilon^b)[\hat{\beta}]$ and $\hat{\omega}_{cm}^{\bar{a}a} = (\varepsilon^{\bar{a}} \otimes \varepsilon^a)[\hat{\beta}_m]$ with $m = 1,\ldots,4$.

A generic one-qubit channel ε can be described by the affine transformation $\mathbf{t}^{\rm out} = \mathbb{M}\mathbf{t}^{\rm in} + \mathbf{v}$ of the vectors $\mathbf{t}^{\rm in}$ in the Bloch sphere, where $\mathbb{M}$ and $\mathbf{v}$ are the matrix and the translation vector of the channel, respectively [18].

Using the result in Appendix B it is shown that the correlation matrix of $\hat{\rho}_c^{ab} = (\varepsilon^a \otimes \varepsilon^b)[\hat{\beta}]$ is $\mathbb{T}_{\rm d} = \mathbb{M}_d^a\, \mathbb{M}_d^b\, \mathbb{T}_{\rm d}^{\rm Bell}$, where $\mathbb{M}_{\rm d}^a$ and $\mathbb{M}_{\rm d}^b$ are the diagonal matrices of the affine description of the channels ε^a and ε^b, respectively. Note that, because the values of the diagonal entries of $\mathbb{M}_{\rm d}^a\mathbb{M}_{\rm d}^b$ are inside the tetrahedron of allowed values for channels, the values of the diagonal entries of $\mathbb{T}_{\rm d} = \mathbb{M}_d^a\, \mathbb{M}_d^b\, \mathbb{T}_{\rm d}^{\rm Bell}$ are inside the tetrahedron of allowed values for correlation matrices of two-qubit states [18]. The Bloch vectors of the reduced states of $\hat{\rho}_c^{ab} = (\varepsilon^a \otimes \varepsilon^b)[\hat{\beta}]$ are $\mathbf{v}^a = \mathbf{r}_c^a$ and $\mathbf{v}^b = \mathbf{r}_c^b = \mathbf{0}$, with $\mathbf{v}^a$ and $\mathbf{v}^b$ the affine vectors of the channels ε^a and ε^b, respectively. It follows that the channel ε^b must be unital. The correlation matrix of the POVM states $\hat{\omega}_{cm}^{\bar{a}a} = (\varepsilon^{\bar{a}} \otimes \varepsilon^a)[\hat{\beta}_m]$ are $\mathbb{W}_{cm} = s\,\mathbb{b}_m^{\bar{a}}\, \mathbb{T}_{\rm d}^{-1} = (\mathbb{M}_d^a\, \mathbb{M}_d^b)^{-1}\, s\, \mathbb{b}_m^{\bar{a}}\, \mathbb{T}_{\rm d}^{\rm Bell} = \mathbb{M}_d^{\bar{a}}\, \mathbb{M}_d^a \mathbb{b}_m^{\bar{a}}\, \mathbb{T}_{\rm d}^{\rm Bell}$, with $\mathbb{M}_{\rm d}^{\bar{a}}$ and $\mathbb{M}_{\rm d}^a$ the matrices of the affine description of $\varepsilon^{\bar{a}}$ and ε^a, respectively. Then, we arrive at first condition

$$\mathbb{M}_{\rm d}^{\bar{a}}\left(\mathbb{M}_{\rm d}^a\right)^2 \mathbb{M}_d^b = s\,\mathbb{1}. \tag{9}$$

The second condition, correlating channels on qubits $\bar{a}$, a and b, is

$$\mathbf{v}^{\bar{a}} = \boldsymbol{\omega}_c^{\bar{a}} = -s\left(\mathbb{M}_d^a\, \mathbb{M}_d^b\right)^{-1} \mathbb{T}_{\rm d}^{\rm Bell}\, \mathbf{r}_c^a. \tag{10}$$

Notice that this last condition disappears if the channel ε^a is unital, i.e., with affine vector $\mathbf{v}^a = \mathbf{r}_c^a = \mathbf{0}$. Thus in this case all the three qubit channels ε^a, $\varepsilon^{\bar{a}}$ and ε^b must be unital to have a DQTP that aligns. It is worth noting that all the more common noisy one-qubit quantum channels are of this kind [18].

Conditions (9) and (10) show that, in general, the channels $\varepsilon^{\bar{a}}$, ε^a, and ε^b are correlated. Uncorrelated solutions of Eq. (9) occur only when the channel matrices are independent. If none of the channels is the identity (no noise), uncorrelated solutions are only achieved when all the channels are the same, i.e., $\mathbb{M}_{\rm d}^{\bar{a}} = \mathbb{M}_d^b = \mathbb{M}_d^a = \mathbb{M}_{\rm d}$ and $\mathbb{M}_{\rm d} = s^{\frac{1}{4}}\mathbb{1}$ (which, in turn, defines a depolarizing channel [18]). Because these channels are unital, $\mathbf{r}_c^a = \mathbf{0}$ so $\boldsymbol{\omega}_c^{\bar{a}} = \mathbf{0}$, condition (10) is automatically satisfied. In this case, both the resource and POVM states are Werner states, i.e., $\hat{\rho}^{ab} = \hat{W}$ and $\hat{\omega}_m^{\bar{a}a} = \hat{W}_m$ where $\hat{W}_m = \frac{(1-p)}{4}\hat{\mathbb{1}} \otimes \hat{\mathbb{1}} + p\,\hat{\beta}_m$, with $m = 1,\ldots,4$, and $\hat{W}$ being one the previous states. The noise parameter p satisfies $p = s^{\frac{1}{2}}$, for $\hat{W}$ and $\hat{W}_m\ \forall m$. For each fixed value of s, i.e., fixed average fidelity, these DQTPs that align are spotted in Figs. 1(a) and 1(e) with dashed (green online) arrows. These are also the solutions of DQTPs that align corresponding to uncorrelated channels, but with noise only in one or two of the qubit systems of the protocol. In this case, the only difference is that the depolarizing channels have a matrix $\mathbb{M}_{\rm d} = s^{\frac{1}{2}}\mathbb{1}$.

It is worth noting that the DQTP that align corresponding to uncorrelated noise in all the qubits are formed by entangled Werner states when $\frac{1}{3} \leqslant p = s^{\frac{1}{2}} \leqslant 1$, and by separable when $0 < p = s^{\frac{1}{2}} \leqslant \frac{1}{3}$. In the case of separable states, the average fidelity of the protocols ranges $0 < \langle\bar{F}\rangle^{\rm alig} \leqslant \frac{5}{9} < \frac{2}{3} = \langle\bar{F}\rangle^{\rm cl}$, with $\langle\bar{F}\rangle^{\rm cl}$ the average fidelity corresponding to the classical protocol [19,20]. These show that entanglement is needed to surpass the average fidelity of the classical protocol, both in the resource state and also in the POVM states.

DQTPs that align with Werner states, i.e., $\hat{\rho}^{ab} = \hat{W}$ and $\hat{\omega}_m^{\bar{a}a} = \hat{W}_m$, exist if the parameter that defines all $\hat{W}_m$ states is $p' = \frac{s}{p}$, where p is the parameter that defines $\hat{W}$. In these protocols, the correlation matrix of $\hat{\rho}^{ab} = \hat{W}$ is $\mathbb{T}_{\rm d} = p\mathbb{T}_{\rm d}^{\rm Bell}$, and those of $\hat{\omega}_m^{\bar{a}a} = \hat{W}_m$ are $\mathbb{W}_{cm} = \mathbb{b}_m^{\bar{a}}\frac{s}{p}\mathbb{T}_{\rm d}^{\rm Bell}$. Replacing in the positivity condition (8b), $\mathbb{T}_{\rm d}$ by $\mathbb{W}_{dm} = \frac{s}{p}\mathbb{T}_{\rm d}^{\rm Bell}$ and $\mathbf{r}_c^a$ by $\boldsymbol{\omega}_{cm}^{\bar{a}} = -\mathbb{T}_{\rm d}^{-1}\mathbf{r}_c^a = \mathbf{0}$, we can rewrite this inequality as $p^8\,(p-s)^3\,(p+3s) \geqslant 0$. The solution of this inequality, together with $0 < s \leqslant 1$ and $-\frac{1}{3} \leqslant p \leqslant 1$ [21], corresponds to two cases: case (I) when $s \leqslant p \leqslant 1$ with $0 < s \leqslant 1$, and case (II) when $-\frac{1}{3} \leqslant p \leqslant -3s$ with $0 < s \leqslant \frac{1}{9}$. We stress that only when $p' = p = s^{\frac{1}{2}}$ the DQTP that align is associated with uncorrelated noise in the qubits. This is a particular solution included in the case (I). Also, note that the DQTPs that align with Werner states become standard noisy QTPs when $p = s$ and $s < 1$ so $\hat{W}_m = \hat{\beta}_m\ \forall m$. When $s = 1$ it becomes the perfect QTP. All these DQTPs that align with Werner states need entanglement, both in the resource state and also in the POVM states, to surpass the average fidelity of the classical protocol.

VI. CONCLUSION

We demonstrate that the optimal quantum teleportation protocols over pure random states, with a fixed average fidelity, are those that align the Bloch vectors of the input and output states. In other words, $\mathbf{t}_m = s_m\mathbf{t}$, where s_m is independent of the initial Bloch vector $\mathbf{t}$, for any outcome m of Alice's measurement. This alignment results in output states that are diagonal in the same basis as the initial state. In addition, these protocols effectively act as depolarizing channels $\hat{\rho}_m = \Lambda_m^{\rm dep}(\hat{\rho}^{\rm in})$, for each m. We characterize all the resource states and POVM measures of these optimal protocols, which, in turn, determine the rotation operation in the output state of the protocols.

A remarkable type of aligned QTP is when $s_m = s$ for all m. These deterministic protocols are particularly relevant as they emerge when attempting to implement the perfect QTP under the influence of correlated noise in qubit systems. Among these protocols, we demonstrate the existence of one with uncorrelated noise, corresponding to the same depolarizing channel in the qubits. The amount of noise in this protocol determines the average fidelity of the teleportation process, a situation commonly encountered in experimental implementations [22]. Therefore, in such experimental scenarios, we establish that the optimal QTP involves preparing a Bell state as the resource state and employing a Bell measurement as a POVM.

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APPENDIX A: POSITIVITY CONDITIONS ON THE DENSITY OPERATORS $\hat{\rho}^{ab}$ AND $\hat{\omega}_m^{\bar{a}a}$ THAT SATISFY THEOREM 2

Here we explicitly write down the inequalities that define the positivity conditions on the density operators $\hat{\rho}^{ab}$ and $\hat{\omega}_m^{\bar{a}a}$ that satisfy Theorem 2, following Ref. [15].

The positivity conditions on density operators $\hat{\rho}^{ab}$ of the form in Eq. (1) were given in Ref. [15]. When the marginal Bloch vector $\mathbf{r}^b$ is null these inequalities are

$$3 - \|\mathbb{r}_{\rm d}\|^2 \geqslant \left\|\mathbf{r}_{\rm c}^a\right\|^2, \tag{A1a}$$

$$-2\det(\mathbb{r}_{\rm d}) - (\|\mathbb{r}_{\rm d}\|^2 - 1) \geqslant \left\|\mathbf{r}_{\rm c}^a\right\|^2, \tag{A1b}$$

$$-8\det(\mathbb{r}_{\rm d}) + (\|\mathbb{r}_{\rm d}\|^2 - 1)^2 - 4\|\tilde{\mathbb{r}}_{\rm d}\|^2 \geqslant 4\left\|\mathbb{r}_{\rm d}\mathbf{r}_{\rm c}^a\right\|^2 + \left\|\mathbf{r}_{\rm c}^a\right\|^2\left[2(1 - \|\mathbb{r}_{\rm d}\|^2) - \left\|\mathbf{r}_{\rm c}^a\right\|^2\right], \tag{A1c}$$

where $\tilde{\mathbb{r}}_{\rm d} = \det(\mathbb{r}_{\rm d})\mathbb{r}_{\rm d}^{-1}$. The correlation matrix $\mathbb{r}_{\rm d} = (\mathbb{o}^a)\mathbb{r}(\mathbb{o}^b)^{\intercal} = \mathrm{diag}(r_1, r_2, r_3)$ and the marginal Bloch vector $\mathbf{r}_{\rm c}^a = \mathbb{o}^a\mathbf{r}^a$ correspond to the state in the canonical form $\hat{\rho}_{\rm c}^{ab}$. It is straightforward to show that when the matrix $\mathbb{r}_{\rm d}$ is invertible, i.e., $\det(\mathbb{r}_{\rm d}) \neq 0$, the first equation is redundant.

Equivalently, the relevant positivity conditions on the density operators $\hat{\omega}_m^{\bar{a}a}$ that satisfy Theorem 2 are

$$-2\det(\mathbb{w}_{{\rm d}m}) - (\|\mathbb{w}_{{\rm d}m}\|^2 - 1) \geqslant \left\|\boldsymbol{\omega}_{m,{\rm c}}^{\bar{a}}\right\|^2, \tag{A2a}$$

$$-8\det(\mathbb{w}_{{\rm d}m}) + (\|\mathbb{w}_{{\rm d}m}\|^2 - 1)^2 - 4\|(\tilde{\mathbb{w}}_m)_{\rm d}\|^2 \geqslant 4\left\|\mathbb{w}_{{\rm d}m}\boldsymbol{\omega}_{m,{\rm c}}^{\bar{a}}\right\|^2 + \left\|\boldsymbol{\omega}_{m,{\rm c}}^{\bar{a}}\right\|^2\left[2(1 - \|\mathbb{w}_{{\rm d}m}\|^2) - \left\|\boldsymbol{\omega}_{m,{\rm c}}^{\bar{a}}\right\|^2\right]. \tag{A2b}$$

Now we know that

$$\mathbb{w}_{{\rm d}m} = s_m\mathbb{r}_{\rm d}^{-1},$$

$$(\tilde{\mathbb{w}}_m)_{\rm d} = \det(\mathbb{w}_{{\rm d}m})\,\mathbb{w}_{{\rm d}m}^{-1} = \frac{s_m^3}{\det(\mathbb{r}_{\rm d})}\frac{1}{s_m}\mathbb{r}_{\rm d} = \frac{s_m^2}{\det(\mathbb{r}_{\rm d})}\mathbb{r}_{\rm d},$$

and

$$\boldsymbol{\omega}_{m,{\rm c}}^{\bar{a}} = -s_m\mathbb{r}_{\rm d}^{-1}\mathbf{r}_{\rm c}^a.$$

Therefore,

$$\|\mathbb{w}_{{\rm d}m}\|^2 = s_m^2\left\|\mathbb{r}_{\rm d}^{-1}\right\|^2 = \frac{s_m^2}{[\det(\mathbb{r}_{\rm d})]^2}\|\tilde{\mathbb{r}}_{\rm d}\|^2,$$

$$\|(\tilde{\mathbb{w}}_m)_{\rm d}\|^2 = \frac{s_m^4}{[\det(\mathbb{r}_{\rm d})]^2}\|\mathbb{r}_{\rm d}\|^2,$$

$$\left\|\boldsymbol{\omega}_{m,{\rm c}}^{\bar{a}}\right\|^2 = s_m^2\left\|\mathbb{r}_{\rm d}^{-1}\mathbf{r}_{\rm c}^a\right\|^2 = \frac{s_m^2}{[\det(\mathbb{r}_{\rm d})]^2}\left\|\tilde{\mathbb{r}}_{\rm d}\mathbf{r}_{\rm c}^a\right\|^2,$$

and

$$\left\|\mathbb{w}_{{\rm d}m}\,\boldsymbol{\omega}_{m,{\rm c}}^{\bar{a}}\right\|^2 = s_m^4\left\|\mathbb{r}_{\rm d}^{-2}\mathbf{r}_{\rm c}^a\right\|^2 = \frac{s_m^4}{[\det(\mathbb{r}_{\rm d})]^4}\left\|\tilde{\mathbb{r}}_{\rm d}^2\mathbf{r}_{\rm c}^a\right\|^2.$$

Replacing these expressions into Eq. (A2) we arrive at the set of inequalities

$$-2s_m^3\det(\mathbb{r}_\mathrm{d})-\{s_m^2\|\tilde{\mathbb{r}}_\mathrm{d}\|^2-[\det(\mathbb{r}_\mathrm{d})]^2\}\geqslant s_m^2\|\tilde{\mathbb{r}}_\mathrm{d}\mathbf{r}_\mathrm{c}^a\|^2, \tag{A3a}$$

$$-8s_m^3[\det(\mathbb{r}_\mathrm{d})]^3+\{s_m^2\|\tilde{\mathbb{r}}_\mathrm{d}\|^2-[\det(\mathbb{r}_\mathrm{d})]^2\}^2-4s_m[\det(\mathbb{r}_\mathrm{d})]^2\|\mathbb{r}_\mathrm{d}\|^2$$
$$\geqslant 4s_m^4\|\tilde{\mathbb{r}}_\mathrm{d}^2\mathbf{r}_\mathrm{c}^a\|^2-s_m^2\|\tilde{\mathbb{r}}_\mathrm{d}\mathbf{r}_\mathrm{c}^a\|^2(2\{s_m^2\|\tilde{\mathbb{r}}_\mathrm{d}\|^2-[\det(\mathbb{r}_\mathrm{d})]^2\}+s_m^2\|\tilde{\mathbb{r}}_\mathrm{d}\mathbf{r}_\mathrm{c}^a\|^2). \tag{A3b}$$

Therefore, for given values of the Bloch vector $\mathbf{r}_\mathrm{c}^a$ and the parameter s_m, the set of allowed values of the matrix elements r_i with $i=1,2,3$ are defined by the inequalities (A1b), (A1c), and (A3), i.e.,

$$-2\det(\mathbb{r}_\mathrm{d})-(\|\mathbb{r}_\mathrm{d}\|^2-1)\geqslant\|\mathbf{r}_\mathrm{c}^a\|^2, \tag{A4a}$$

$$-8\det(\mathbb{r}_\mathrm{d})+(\|\mathbb{r}_\mathrm{d}\|^2-1)^2-4\|\tilde{\mathbb{r}}_\mathrm{d}\|^2\geqslant 4\|\mathbb{r}_\mathrm{d}\mathbf{r}_\mathrm{c}^a\|^2+\|\mathbf{r}_\mathrm{c}^a\|^2[2(1-\|\mathbb{r}_\mathrm{d}\|^2)-\|\mathbf{r}_\mathrm{c}^a\|^2], \tag{A4b}$$

$$-2s_m^3\det(\mathbb{r}_\mathrm{d})-\{s_m^2\|\tilde{\mathbb{r}}_\mathrm{d}\|^2-[\det(\mathbb{r}_\mathrm{d})]^2\}\geqslant s_m^2\|\tilde{\mathbb{r}}_\mathrm{d}\mathbf{r}_\mathrm{c}^a\|^2, \tag{A4c}$$

$$-8s_m^3[\det(\mathbb{r}_\mathrm{d})]^3+\{s_m^2\|\tilde{\mathbb{r}}_\mathrm{d}\|^2-[\det(\mathbb{r}_\mathrm{d})]^2\}^2-4s_m^4[\det(\mathbb{r}_\mathrm{d})]^2\|\mathbb{r}_\mathrm{d}\|^2\geqslant$$
$$\geqslant 4s_m^4\|\tilde{\mathbb{r}}_\mathrm{d}^2\mathbf{r}_\mathrm{c}^a\|^2-s_m^2\|\tilde{\mathbb{r}}_\mathrm{d}\mathbf{r}_\mathrm{c}^a\|^2\Big(2\{s_m^2\|\tilde{\mathbb{r}}_\mathrm{d}\|^2-[\det(\mathbb{r}_\mathrm{d})]^2\}+s_m^2\|\tilde{\mathbb{r}}_\mathrm{d}\mathbf{r}_\mathrm{c}^a\|^2\Big). \tag{A4d}$$

Note that the left-hand side of inequality (A4b) is

$$\begin{aligned}f(r_1,r_2,r_3)&=(1-r_1-r_2-r_3)(1-r_1+r_2+r_3)(1+r_1-r_2+r_3)(1+r_1+r_2-r_3)\\&=-8\det(\mathbb{r}_\mathrm{d})+(\|\mathbb{r}_\mathrm{d}\|^2-1)^2-4\|\tilde{\mathbb{r}}_\mathrm{d}\|^2.\end{aligned} \tag{A5}$$

APPENDIX B: CALCULATION OF THE FANO FORM OF $(\varepsilon^a\otimes\varepsilon^b)[\hat{\rho}^{ab}]$

Here we show the action of local arbitrary one-qubit channels on a two-qubit state given in the Fano form (1). An analogous calculation with only one-qubit channel was performed in Ref. [23].

Lemma 1. Let ε^a and ε^b be one-qubit channels described by the affine parameters Λ^a, $\mathbf{v}^a$, and Λ^b, $\mathbf{v}^b$, respectively, and let $\hat{\rho}^{ab}$ be an arbitrary two-qubit state given in Fano form in Eq. (1), then

$$\begin{aligned}(\varepsilon^a\otimes\varepsilon^b)[\hat{\rho}^{ab}]=\frac{1}{4}\Bigg(&\hat{\mathbb{1}}\otimes\hat{\mathbb{1}}+[(\mathbf{r}^a)^\intercal\mathbb{\Lambda}^a+(\mathbf{v}^a)^\intercal]\hat{\boldsymbol{\sigma}}\otimes\hat{\mathbb{1}}+\hat{\mathbb{1}}\otimes[(\mathbf{r}^b)^\intercal\mathbb{\Lambda}^b+(\mathbf{v}^b)^\intercal]\hat{\boldsymbol{\sigma}}\\&+\sum_{i=1}^3\sum_{j=1}^3\big[\mathbf{v}^a(\mathbf{v}^b)^\intercal+(\mathbb{\Lambda}^a)^\intercal\mathbf{r}^a(\mathbf{v}^b)^\intercal+\mathbf{v}_i^a(\mathbf{r}^b)^\intercal\mathbb{\Lambda}^b+(\mathbb{\Lambda}^a)^\intercal\,\mathbb{r}\,\mathbb{\Lambda}^b\big]_{ij}\hat{\sigma}_i\otimes\hat{\sigma}_j\Bigg).\end{aligned} \tag{B1}$$

Using the linear property of the quantum channels, we get

$$\begin{aligned}(\varepsilon^a\otimes\varepsilon^b)[\hat{\rho}^{ab}]&=\frac{1}{4}\Bigg(\varepsilon^a[\hat{\mathbb{1}}]\otimes\varepsilon^b[\hat{\mathbb{1}}]+\varepsilon^a[(\mathbf{r}^a)^\intercal\hat{\boldsymbol{\sigma}}]\otimes\varepsilon^b[\hat{\mathbb{1}}]+\varepsilon^a[\hat{\mathbb{1}}]\otimes\varepsilon^b[(\mathbf{r}^b)^\intercal\hat{\boldsymbol{\sigma}}]+\sum_{i,j=1}^3\mathbb{r}_{ij}\,\varepsilon^b[\hat{\sigma}_i]\otimes\varepsilon^b[\hat{\sigma}_j]\Bigg)\\&=\frac{1}{4}\Bigg(\hat{\mathbb{1}}\otimes\hat{\mathbb{1}}+[(\mathbf{r}^a)^\intercal\mathbb{\Lambda}^a+(\mathbf{v}^a)^\intercal]\hat{\boldsymbol{\sigma}}\otimes\hat{\mathbb{1}}+\hat{\mathbb{1}}\otimes[(\mathbf{r}^b)^\intercal\mathbb{\Lambda}^b+(\mathbf{v}^b)^\intercal]\hat{\boldsymbol{\sigma}}(\mathbf{v}^a)^\intercal\hat{\boldsymbol{\sigma}}\otimes(\mathbf{v}^b)^\intercal\hat{\boldsymbol{\sigma}}\\&\quad+(\mathbf{r}^a)^\intercal\mathbb{\Lambda}^a\hat{\boldsymbol{\sigma}}\otimes(\mathbf{v}^b)^\intercal\hat{\boldsymbol{\sigma}}+(\mathbf{v}^a)^\intercal\hat{\boldsymbol{\sigma}}\otimes(\mathbf{r}^b)^\intercal\mathbb{\Lambda}^b\hat{\boldsymbol{\sigma}}+\sum_{k,l=1}^3[(\mathbb{\Lambda}^a)^\intercal\,\mathbb{r}\,\mathbb{\Lambda}^b]_{kl}\hat{\sigma}_k\otimes\hat{\sigma}_l\Bigg)\\&=\frac{1}{4}\Bigg(\hat{\mathbb{1}}\otimes\hat{\mathbb{1}}+[(\mathbf{r}^a)^\intercal\mathbb{\Lambda}^a+(\mathbf{v}^a)^\intercal]\hat{\boldsymbol{\sigma}}\otimes\hat{\mathbb{1}}+\hat{\mathbb{1}}\otimes[(\mathbf{r}^b)^\intercal\mathbb{\Lambda}^b+(\mathbf{v}^b)^\intercal]\hat{\boldsymbol{\sigma}}\\&\quad+\sum_{i=1}^3\sum_{j=1}^3[\mathbf{v}^a(\mathbf{v}^b)^\intercal+(\mathbb{\Lambda}^a)^\intercal\mathbf{r}^a(\mathbf{v}^b)^\intercal+\mathbf{v}_i^a(\mathbf{r}^b)^\intercal\mathbb{\Lambda}^b+(\mathbb{\Lambda}^a)^\intercal\,\mathbb{r}\,\mathbb{\Lambda}^b]_{ij}\hat{\sigma}_i\otimes\hat{\sigma}_j\Bigg),\end{aligned} \tag{B2}$$

where we used that $\varepsilon[\hat{\mathbb{1}}]=\hat{\mathbb{1}}+(\mathbf{v})^\intercal\hat{\boldsymbol{\sigma}}$ and $\varepsilon[\hat{\sigma}_i]=\sum_{j=1}^3\mathbb{\Lambda}_{ij}\hat{\sigma}_j$ (so $\varepsilon[\hat{\boldsymbol{\sigma}}]=\mathbb{\Lambda}\hat{\boldsymbol{\sigma}}$) that can be easily proven using the affine representation of ε.

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