Macroscopic diffusive fluctuations for generalized hard rods dynamics

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Abstract: We study the fluctuations in equilibrium for a dynamics of rods with random length. This includes the classical hard rod elastic collisions, when rod lengths are constant and equal to a positive value. We prove that in the diffusive space-time scaling, an initial fluctuation of density of particles of velocity v, after recentering on its Euler evolution, evolve randomly shifted by a Brownian motion of variance $\mathcal{D}(v)$.

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1. Introduction

The mechanical system of one-dimensional hard rods is the simplest non trivial completely integrable dynamics where the macroscopic behavior can be described by generalized hydrodynamics. The density of particles of each given velocity is conserved and in the Euler scaling the macroscopic evolution of such densities have been studied by the pioneristic work of Percus [16] and Boldrighini, Dobrushin and Suhov [4]. Fluctuations around this Euler limit have been studied by Boldrighini and Wick in [6]. Recently these results have been generalized to a completely integrable dynamics of rods of random length (even negative length) where lengths and velocities are exchanged at collision [11]. The elastic collisions are recovered in the particular case that all rods have the same positive length. Similar dynamics were considered in [7], while in [2] velocities are exchanged but not the lengths (i.e. the classical elastic collision).

In this article we investigate the evolution of the densities fluctuations in the diffusive space-time scale for the generalized dynamics considered in [11]. We consider the system in a *stationary homogeneous* initial condition. We will discuss initial inhomogeneous non-stationary state in Section 4.

The result we prove in the present article is that the initial fluctuations of the density of particles of velocity v, after recentering on its Euler evolution, evolve randomly shifted by a Brownian motion of variance $\mathcal{D}(v)$. This diffusion coefficient $\mathcal{D}(v)$ has an explicit expression depending on v and on the particular stationary measure (cf. (3.2)). In the case of rods of constant length $\mathcal{D}(v)$ is the same as computed by Spohn in [18], as well as it appears in the first order diffusive correction to the Euler Hydrodynamic limit [3] [5].

This result corresponds to the following stochastic partial differential equation for the evolution of the macroscopic fluctuation $\Xi_t(y,v,r)$ of the density of particles of velocity v and length r at position y:

$$\partial_t \Xi_t(y,v,r) = \frac{1}{2}\mathcal{D}(v)\partial_y^2 \Xi_t(y,v,r) + \sqrt{\mathcal{D}(v)}\partial_y \Xi_t(y,v,r)\dot{W}_t(v), \tag{1.1}$$

where $(\dot{W}_t(v) : t, v \in \mathbb{R})$ is a centered gaussian field with covariance

$$\mathbb{E}\big(\dot{W}_t(v)\dot{W}_s(w)\big) = \delta(t-s)\frac{\Gamma(v,w)}{\sqrt{\mathcal{D}(v)\mathcal{D}(w)}}$$

with $\Gamma(v,w)$ given in (3.5). Since Ξ_t is a distribution in (y,v,r), (1.1) should be understood in the weak sense.

Notice that in (1.1) the noise term is only white in time and completely correlated in space (i.e. $W_t(v)$ does not depends on y). This is in contrast with the typical diffusive evolution of fluctuations in chaotic systems, where it is expected an additive space-time white noise driving the fluctuations and the equation would be of the type [19]

$$\partial_t \bar{\Xi}_t(y,v,r) = \frac{1}{2}\mathcal{D}(v)\partial_y^2 \bar{\Xi}_t(y,v,r) + \sqrt{\mathcal{D}(v)}\partial_y \dot{W}_{y,t}(v), \tag{1.2}$$

with $\mathbb{E}\big(\dot{W}_{y,t}(v)\dot{W}_{y',s}(w)\big) = \delta(t-s)\delta(y-y')\delta(v-w)$. Notice that the equilibrium solutions of (1.1) and (1.2) have the same space-time covariance, i.e. the space-time covariance does not give informations about the martingale term of the evolution equation. About the uncorrelation in r, this persists in the macroscopic fluctuations as consequence of the decorrelation at the microscopic scale.

In fact we believe, as Herbert Spohn suggested us, that (1.1) is a typical (universal) macroscopic behaviour for the diffusive fluctuations of completely integrable many-body systems [20].

In order to understand why (1.1) arise, we follow the behaviour of two tagged quasi-particles with the same velocity. We call here quasi-particles (or impulsions) the particles with the dynamics defined by the exchange of positions at the moment of collision. The standard technique to study such dynamics is to go to the *reduced description* where quasi-particles are mapped to points and evolve without interaction. Then the evolution of the tagged quasi-particle is obtained by the trivial evolution of the corresponding point, shifted by the collisions with quasi-particles of different velocity. Since the points corresponding to the other quasi-particles are distributed by a Poisson field, these collisions happen at random times and the collisional shifts are independent. Consequently at the Euler scale we have a law of large numbers (see Section 2.2) that produce a deterministic evolution of the tagged quasi-particle with an effective velocity given by

(2.15) (such ergodicity was proven first in [1]). Recentering the position of the tagged quasi-particle on the Euler deterministic behavior, we have then a functional central limit theorem so that the position converges in law to a Brownian motion of variance $\mathcal{D}(v)$. Now consider two tagged quasi-particles with the same velocity and initially located at macroscopic distance: this means that there are typically ε^{-1} particles in between, where ε is the scaling parameter going to 0. In the diffusive scaling each tagged quasi-particle at time t has a number of collisions proportional to $\varepsilon^{-2}t$, but most of them with the same quasi-particles, except for an order ε^{-1} of collisions at the beginning and at the end of the time interval $[0, t]$. Consequently the two tagged quasi-particles completely correlate in the limit $\varepsilon \to 0$, i.e. they converge to the same Brownian motion. This causes the rigid motion of the corresponding density fluctuations. Notice that in chaotic system it is expected that the two quasi-particles at initial macroscopic distance converge to two independent Brownian motion, generating the space-time white noise present in (1.2).

As far as we know, equation (1.1) for the diffusive fluctuations for hard rods is new in the literature. We recently discovered the article by Presutti and Wick [17] that concerns the diffusive behavior of travelling wave initial conditions in hard rods systems, where in the remark after Theorem 1 they comment about possible diffusive behaviour of fluctuations and they write: "spatially separated fluctuations in the density of rods with the same velocity move with the same Brownian component". Strangely [17] is never quoted in the following literature about Navier-Stokes corrections for the hard rods hydrodynamics (cf. [5]). In the introduction of [6] it was announced a second article about the *Navier Stokes corrections* for the evolution of the fluctuations, but the authors confirmed us that this has never been written.

Hard rods dynamics with domain wall initial conditions, a particular case of travelling wave, is investigated in [10]; the paper includes the Navier-Stokes corrections and the computation of the covariance $\Gamma(v, v')$ from Green-Kubo formula.

Diffusive corrections to the hydrodynamic Euler scaling in general completely integrable systems have been investigated in the physics literature, see the recent review [9] with the references therein, as well as [13] and [8]. These articles contain general formulas for diffusion coefficients that are in agreement with our formula for the generalized hard rods. A generalization of our equation (1.1) to other integrable systems will give answer about the macroscopic evolution on the diffusive scale of multipoint correlation functions, mentioned in [9] as an open problem.

As we believe that our approach is more elementary than the one used in the previous literature, we have written this article to be completely independent of results on hard rods prior to the paper [11], which is our starting point. Essentially, the only tools we use are the law of large numbers and the central limit theorem for a Poisson field. In Section 2 we prove the macroscopic evolution of the fluctuations in the Euler scaling (recovering the result of [6]). In Section 3 we prove the evolution of the fluctuations in the diffusive scaling. Finally in

Section 5 we prove two lemmas about limits for Poisson field that we need in the proofs.

2. Equilibrium Fluctuations in the Euler scaling

Let X^ε be the Poisson process on $\mathbb{R}^3$ with intensity $\varepsilon^{-1}\rho\, dx\, d\mu(v,r)$, where μ is a probability on $\mathbb{R}^2$ with finite second moments. We should think about x as the *macroscopic* position of the point x, since the typical distance between points is ε. The *macroscopic length of the rod* (x,v,r) is εr.

We define

$$\sigma := \rho \iint r d\mu(v,r), \qquad \text{length density}$$
$$\pi := \rho \iint rv d\mu(v,r), \qquad \text{momentum density.}$$

Denote the empirical length distribution by

$$N^\varepsilon\varphi := \sum_{(x,v,r)\in X^\varepsilon} \varepsilon r \varphi(x,v,r), \qquad N^\varepsilon(A) := N^\varepsilon \mathbb{1}_A.$$

The expectation of N^ε is

$$\mathbb{E}N^\varepsilon\varphi = \langle\!\langle \varphi \rangle\!\rangle,$$

where the length biased measure $\langle\!\langle \cdot \rangle\!\rangle$ is defined by

$$\langle\!\langle \varphi \rangle\!\rangle := \rho \iiint r\varphi(x,v,r)\, dx\, d\mu(v,r).$$

We have the law of large numbers

$$N^\varepsilon\varphi \xrightarrow[\varepsilon\to 0]{\text{a.s.}} \mathbb{E}N^\varepsilon\varphi = \langle\!\langle \varphi \rangle\!\rangle.$$

If $\varphi(x,v,r) = 1_{[0,1]}(x)$, we have $\sum_{(x,v,r)\in X^\varepsilon} \varepsilon r\varphi(x) \to \sigma$.

The central limit theorem states that

$$\xi^{X,\varepsilon}(\varphi) := \varepsilon^{-1/2}\big(N^\varepsilon - \langle\!\langle \varphi \rangle\!\rangle\big) \xrightarrow[\varepsilon\to 0]{\text{law}} \xi^X(\varphi) \tag{2.1}$$

where ξ^X is the centered gaussian white noise with covariance

$$\mathbb{E}(\xi^X(\varphi)\xi^X(\psi)) = \langle\!\langle \varphi\psi \rangle\!\rangle_2 - \langle\!\langle \varphi \rangle\!\rangle\langle\!\langle \psi \rangle\!\rangle,$$

where

$$\langle\!\langle \varphi \rangle\!\rangle_2 := \rho \iiint r^2\varphi(x,v,r) dx\, d\mu(v,r).$$

Notice that

$$\mathbb{E}(\xi^{X,\varepsilon}\varphi)^2 = \langle\!\langle \varphi^2 \rangle\!\rangle_2 - \langle\!\langle \varphi \rangle\!\rangle^2,$$

so that for any $\varphi \in L^2(\langle\!\langle\cdot\rangle\!\rangle_2)$ we have the bound

$$\mathbb{E}(\xi^{X,\varepsilon}\varphi)^2 \leqslant \langle\!\langle \varphi^2 \rangle\!\rangle_2.$$

Define the *mass* (length) measure by

$$m_a^b(X^\varepsilon) = \varepsilon \sum_{(x,v,r)\in X^\varepsilon} r\big(1_{[x\in[a,b)]} - 1_{[x\in[b,a]]}\big).$$

Consequently,

$$m_a^b(X^\varepsilon) \xrightarrow[\varepsilon\to 0]{\text{a.s.}} \mathbb{E}m_a^b(X^\varepsilon) = (b-a)\sigma. \tag{2.2}$$

To each configuration X^ε and a point $a \in \mathbb{R}$, there are a dilated point and configuration

$$D_a^\varepsilon(b) := b - a + m_a^b(X^\varepsilon)$$
$$Y^\varepsilon = D_0^\varepsilon(X^\varepsilon) := \{(D_0^\varepsilon(x), v, r) : (x,v,r) \in X^\varepsilon\}.$$

Remark 2.1. The distribution of X^ε is space shift invariant, but the distribution of the rod configuration Y^ε is not because Y^ε has no rod containing the origin. Our results can be extended to random rod configurations with space shift invariant distribution, by using Palm transforms [21] and Harris theorem [14]; see for instance [12].

If $r \geqslant 0$ for all $(x,v,r) \in X^\varepsilon$, and the origin does not belong to a rod of Y^ε, then we can define the inverse $D_a^{-1}(Y^\varepsilon) = X^\varepsilon$. The macroscopic dilation of the point b with respect to a is given by

$$\mathbb{E}D_a^\varepsilon(b) = (b-a)(1+\sigma).$$

Denote the length empirical measure induced by Y^ε by

$$K^\varepsilon\varphi := \varepsilon \sum_{(y,v,r)\in Y^\varepsilon} r\varphi(y,v,r).$$

We have the law of large numbers:

$$\begin{aligned}
K^\varepsilon\varphi &= \varepsilon \sum_{(x,v,r)\in X^\varepsilon} r\varphi\big(x + m_a^x(X^\varepsilon), v, r\big) \\
&\xrightarrow[\varepsilon\to 0]{\text{a.s.}} \rho \iiint r\varphi\big(x + (x-a)\sigma, v, r\big)dx\, d\mu(v,r) \\
&= \frac{\rho}{1+\sigma} \iiint r\varphi(x,v,r)dx\, d\mu(v,r) \\
&= \frac{1}{1+\sigma}\langle\!\langle \varphi \rangle\!\rangle.
\end{aligned} \tag{2.3}$$

2.1. *Static CLT for the dilated configuration*

We define the fluctuation field

$$\xi^{Y,\varepsilon}(\varphi) = \varepsilon^{-1/2}\big(K^\varepsilon\varphi - \mathbb{E}K^\varepsilon\varphi\big).$$

We have

$$K^\varepsilon\varphi - \mathbb{E}K^\varepsilon\varphi = (K^\varepsilon\varphi - A^\varepsilon\varphi) + (A^\varepsilon\varphi - \mathbb{E}A^\varepsilon\varphi) - (\mathbb{E}K^\varepsilon\varphi - \mathbb{E}A^\varepsilon\varphi). \tag{2.4}$$

where

$$\begin{aligned} A^\varepsilon\varphi &:= \varepsilon \sum_{(x,v,r)\in X^\varepsilon} r\varphi(x(1+\sigma),v,r),\\ \mathbb{E}A^\varepsilon\varphi &= \frac{\rho}{1+\sigma}\iiint r\varphi(x,v,r)dx\,d\mu(v,r). \end{aligned}$$

The last term in (2.4) gives

$$\begin{aligned} &\varepsilon^{-1/2}(\mathbb{E}K^\varepsilon\varphi - \mathbb{E}A^\varepsilon\varphi)\\ &= \mathbb{E}\Big(\varepsilon^{1/2}\sum_{(x,v,r)\in X^\varepsilon} r\big[\varphi\big(x+m_0^x(X^\varepsilon),v,r\big) - \varphi(x(1+\sigma),v,r)\big]\Big) &(2.5)\\ &= \mathbb{E}\Big(\varepsilon^{1/2}\sum_{(x,v,r)\in X^\varepsilon} r(\partial_y\varphi)(x(1+\sigma),v,r)\big(m_0^x(X^\varepsilon) - x\sigma\big)\Big) + R^\varepsilon\\ &= \iiint r(\partial_y\varphi)(x(1+\sigma),v,r)\varepsilon^{-1/2}\big[\mathbb{E}(m_0^x(X^\varepsilon) - x\sigma)\big]dx\,d\mu(v,r) + R^\varepsilon &(2.6)\\ &= R^\varepsilon, &(2.7) \end{aligned}$$

where R^ε denotes a generic term small with ε. Identity (2.6) follows from Slyvniak-Mecke formula (Theorem 3.2 in [15]). Since by (2.2) the first term in (2.6) is null, identity (2.7) follows.

The second term in (2.4) gives

$$\begin{aligned} &\varepsilon^{1/2}(A^\varepsilon - \mathbb{E}A^\varepsilon)\\ &= \varepsilon^{-1/2}\Big[\varepsilon\sum_{(x,v,r)\in X^\varepsilon} r\breve{\varphi}(x,v,r) - \rho\iiint r\breve{\varphi}(x,v,r)dx\,d\mu(v,r)\Big]\\ &\xrightarrow[\varepsilon\to 0]{\text{law}} \xi^X(\breve{\varphi}), &(2.8) \end{aligned}$$

where $\breve{\varphi}(x,v,r) := \varphi((1+\sigma)x,v,r)$. Notice that

$$\begin{aligned} &\mathbb{E}(\xi^X(\breve{\varphi})\xi^X(\breve{\psi}))\\ &= \rho\iiint r^2\breve{\varphi}(x,v,r)\breve{\psi}(x,v,r)dxd\mu(v,r) - \langle\!\langle\breve{\varphi}\rangle\!\rangle\langle\!\langle\breve{\psi}\rangle\!\rangle\\ &= \frac{\rho}{1+\sigma}\iiint r^2\varphi(y,v,r)\psi(y,v,r)dyd\mu(v,r) - \frac{1}{(1+\sigma)^2}\langle\!\langle\varphi\rangle\!\rangle\langle\!\langle\psi\rangle\!\rangle\\ &= \frac{1}{1+\sigma}\langle\!\langle\varphi\psi\rangle\!\rangle_2 - \frac{1}{(1+\sigma)^2}\langle\!\langle\varphi\rangle\!\rangle\langle\!\langle\psi\rangle\!\rangle. \end{aligned}$$

Finally expand the first term of the RHS of (2.4):

$$\begin{aligned} \varepsilon^{-1/2}(K^\varepsilon\varphi - A^\varepsilon\varphi) &= \varepsilon^{1/2}\sum_{(x,v,r)\in X^\varepsilon} r\big(\varphi\big(x+m_a^x(X^\varepsilon),v,r\big) - \varphi(x(1+\sigma),v,r)\big)\\ &= \frac{1}{1+\sigma}\varepsilon^{1/2}\sum_{(x,v,r)\in X^\varepsilon} r(\partial_x\breve{\varphi})(x,v,r)\big(m_0^x(X^\varepsilon) - x\sigma\big) + R^\varepsilon. \end{aligned}$$

By the functional central limit theorem (2.1) we have

$$\varepsilon^{-1/2}\big(m_0^x(X^\varepsilon) - x\sigma\big) \coloneqq B^\varepsilon(x) \ \xrightarrow[\varepsilon\to 0]{\text{law}}\ B(x) \coloneqq \begin{cases} \xi^X(1_{[0,x]}), & x > 0 \\ -\xi^X(1_{[x,0]}), & x < 0 \end{cases}$$

Since $(B(x) : x \in \mathbb{R})$ is a bilateral Brownian motion, using Lemma 5.1 we have

$$\begin{aligned}
&\varepsilon^{-1/2}(K^\varepsilon\varphi - A^\varepsilon\varphi)\\
&\xrightarrow[\varepsilon\to 0]{\text{law}} \frac{\rho}{1+\sigma}\iiint r(\partial_x\breve\varphi)(x,v,r)B(x)dx\, d\mu(v,r)\\
&= \frac{1}{1+\sigma}\int dx B(x)\partial_x\Big(\rho\iint r\breve\varphi(x,v,r)\, d\mu(v,r)\Big)\\
&= \frac{1}{1+\sigma}\int dx \xi^X\big(1_{[0,x]}1_{x>0} - 1_{[x,0]}1_{x<0}\big)\partial_x\Big(\rho\iint r\breve\varphi(x,v,r)\, d\mu(v,r)\Big)\\
&= \frac{1}{1+\sigma}\xi^X\Big(\int dx\big(1_{[0,x]}1_{x>0} - 1_{[x,0]}1_{x<0}\big)\partial_x\Big(\rho\iint r\breve\varphi(x,v,r)\, d\mu(v,r)\Big)\Big)\\
&= -\frac{1}{1+\sigma}\xi^X\Big(\rho\iint r'\breve\varphi(\cdot,v',r')\, d\mu(v',r')\Big).
\end{aligned}$$

We conclude that

$$\varepsilon^{1/2}(K^\varepsilon\varphi - A^\varepsilon\varphi) \xrightarrow[\varepsilon\to 0]{\text{law}} -\xi^X(P\breve\varphi), \tag{2.9}$$

where

$$P\varphi(x) = \frac{\rho}{1+\sigma}\iint r\varphi(x,v',r')\, d\mu(v',r'). \tag{2.10}$$

Putting together (2.9), (2.5)-(2.7) and (2.8) we have shown that

$$\xi^{Y,\varepsilon}(\varphi) \xrightarrow[\varepsilon\to 0]{\text{law}} \xi^Y(\varphi) = \xi^X\Big(\breve\varphi - P\breve\varphi\Big) = \xi^X\big(C\breve\varphi\big), \tag{2.11}$$

where $C = I - P$. This identifies ξ^Y as the centered gaussian field with covariance

$$\begin{aligned}
\mathbb{E}(\xi^Y(\varphi)\xi^Y(\psi)) &= \mathbb{E}(\xi^X(C\breve\varphi)\xi^X(C\breve\psi))\\
&= \rho\iiint r^2 C\varphi(x(1+\sigma),v,r)C\psi(x(1+\sigma),v,r)dxd\mu(v,r),\\
&= \frac{\rho}{1+\sigma}\iiint r^2 C\varphi(y,v,r)C\psi(y,v,r)dyd\mu(v,r)\\
&= \frac{\rho}{1+\sigma}\langle\!\langle C\varphi C\psi\rangle\!\rangle_2.
\end{aligned}$$

That means for the Fourier transforms

$$\widehat\varphi(k,v,r) = \int e^{i2\pi ky}\varphi(y,v,r)dy$$

$$\mathbb{E}(\xi^Y(\varphi)\xi^Y(\psi)) = \frac{\rho}{1+\sigma}\iiint r^2 C\widehat\varphi(k,v,r)^* C\widehat\psi(k,v,r)dkd\mu(v,r).$$

i.e. a covariance operator

$$\mathcal{C} = \frac{\rho}{1+\sigma}r^2C^2 = \frac{\rho}{1+\sigma}r^2\Big(I + \Big(\frac{\sigma^2}{(1+\sigma)^2} - 2\frac{\sigma}{1+\sigma}\Big)P\Big).$$

Remark 2.2. *This is in agreement with formula (7.61) in Spohn's book [19].*

Example: in the case $d\mu(v,r) = \frac{1}{2}\big(\delta_{v_0}(dv) + \delta_{-v_0}(dv)\big)\delta_a(dr)$ we have $\sigma = \rho a$ and $\pi = 0$. Then,

$$\begin{aligned} P\varphi(x) &= \frac{\rho a}{2(1+\rho a)}\left(\varphi(x,v_0) + \varphi(x,-v_0)\right) \\ C\varphi(x,\pm v_0) &= \frac{2+\rho a}{2(1+\rho a)}\varphi(x,\pm v_0) - \frac{\rho a}{2(1+\rho a)}\varphi(x,\mp v_0). \end{aligned}$$

2.2. Equilibrium fluctuations in the Euler scaling

Recall that the interparticle distance is of order ε, i.e. the coordinates (x,v,r) are already rescaled in the Euler scale. Let X_t^ε denote the free gas configuration at time t:

$$X_t^\varepsilon := \{(x+vt,v,r) : (x,v,t) \in X^\varepsilon\}.$$

Define the flow

$$\begin{aligned} j^\varepsilon(x,v,t) &:= \varepsilon \sum_{(\tilde{x},\tilde{v},\tilde{r})\in X^\varepsilon} \tilde{r}\big(1_{[\tilde{v}<v]}1_{[x<\tilde{x}<x+(v-\tilde{v})t]} - 1_{[\tilde{v}>v]}1_{[x+(v-\tilde{v})t<\tilde{x}<x]}\big) \\ j(x,v,t) &:= \mathbb{E}j^\varepsilon(x,v,t) \qquad (2.12) \\ &= \iiint \rho\, dx\, d\mu(\tilde{v},\tilde{r})\, \tilde{r}\left(1_{[\tilde{v}<v]}1_{[x<\tilde{x}<x+(v-\tilde{v})t]} - 1_{[\tilde{v}>v]}1_{[x+(v-\tilde{v})t<\tilde{x}<x]}\right) \\ &= \rho\int \tilde{r}\int_v^{+\infty}(v-\tilde{v})\, t\, d\mu(\tilde{v},\tilde{r}) + \rho\int\tilde{r}\int_{-\infty}^{v}(v-\tilde{v})\, t\, d\mu(\tilde{v},\tilde{r}) \\ &= tv\sigma - t\pi. \end{aligned}$$

Here $j^\varepsilon(x,v,t)$ is the ideal gas integrated flow along the segment $(x+vs)_{s\in[0,t]}$, and $j(x,v,t)$ is its expectation.

The position of the quasi particle $y_{v,t}^\varepsilon(x)$ associated to (x,v,r) is given by

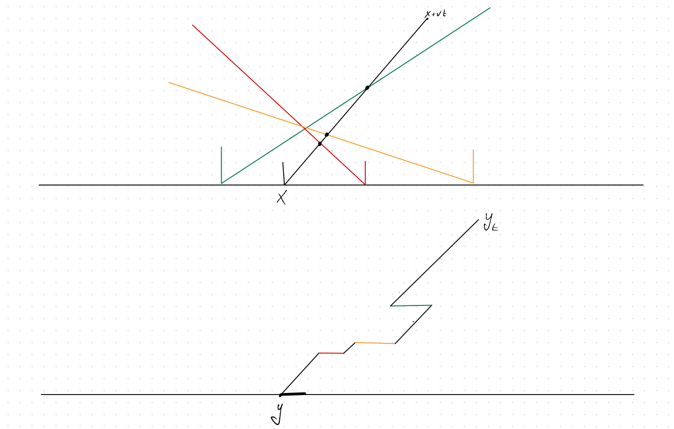
$$\begin{aligned} y_{v,t}^\varepsilon(x) &:= D_0^\varepsilon(x) + vt + j^\varepsilon(x,v,t) \qquad (2.13) \\ y_{v,t}(x) &:= \mathbb{E}y_{v,t}^\varepsilon(x) = x(1+\sigma) + vt + j(x,v,t). \end{aligned}$$

We have the following limit as a consequence of the law of large numbers

$$y_{v,t}^\varepsilon - y \xrightarrow[\varepsilon\to 0]{\text{a.s.}} v^{\text{eff}}(v)t, \qquad (2.14)$$

where the effective velocity is given by

$$v^{\text{eff}}(v) := v(1+\sigma) - \pi. \qquad (2.15)$$



To see (2.14), observe that by (2.13), if $y \in Y^\varepsilon$, there is an $x \in X^\varepsilon$ such that $y = D_0^\varepsilon(x)$ and $y_{v,t}^\varepsilon - y = j^\varepsilon(x, v, t)$, implying that (2.14) is equivalent to

$$\frac{1}{t} j^\varepsilon(x, v, t) \xrightarrow[\varepsilon \to 0]{\text{a.s.}} \frac{1}{t} j(x, v, t) = v\sigma - \pi.$$

We have that the free gas empirical length measure at time t satisfies

$$N_t^\varepsilon \varphi := \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r\varphi(x + vt, v, r) \xrightarrow[\varepsilon \to 0]{\text{a.s.}} \mathbb{E} N_t^\varepsilon \varphi = \langle\!\langle \varphi \rangle\!\rangle.$$

The X-fluctuation field at time t satisfies

$$\xi_t^{X,\varepsilon}(\varphi) := \varepsilon^{-1/2} \big(N_t^\varepsilon \varphi - \langle\!\langle \varphi \rangle\!\rangle \big) \xrightarrow[\varepsilon \to 0]{\text{law}} \xi^X(\varphi_t),$$

where

$$\varphi_t(x, v, r) := \varphi(x + tv, v, r).$$

The hard rod configuration and empirical measure at time t are given by

$$\begin{aligned} Y_t^\varepsilon &:= \big\{ (y_t^\varepsilon(x), v, r) : (x, v, r) \in X^\varepsilon \big\}, \\ K_t^\varepsilon \varphi &:= \varepsilon \sum_{(y,v,r) \in Y_t^\varepsilon} r\varphi(y, v, r) = \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r\varphi(y_t^\varepsilon(x), v, r). \end{aligned}$$

Using (2.3) we have the LLN for K_t^ε:

$$\begin{aligned} K_t^\varepsilon \varphi \xrightarrow[\varepsilon \to 0]{} \rho \iiint r\varphi(y_{v,t}(x), v, r) dx\, d\mu(v, r) \\ = \rho \iint r \Big(\int \varphi\big(x(1 + \sigma) + v^{\text{eff}}(v) t, v, r \big) dx \Big)\, d\mu(v, r) \\ = \frac{\rho}{1 + \sigma} \iiint r\varphi(x, v, r) dx\, d\mu(v, r) = \frac{\rho}{1 + \sigma} \langle\!\langle \varphi \rangle\!\rangle. \end{aligned}$$

We define the Y-fluctuation field at time t by

$$\xi_t^{Y,\varepsilon}(\varphi) := \varepsilon^{-1/2}\big(K_t^\varepsilon \varphi - \mathbb{E}K_t^\varepsilon \varphi\big).$$

We have

$$K_t^\varepsilon \varphi - \mathbb{E}K_t^\varepsilon \varphi = (K_t^\varepsilon \varphi - A_t^\varepsilon \varphi) + (A_t^\varepsilon \varphi - \mathbb{E}A_t^\varepsilon \varphi) - (\mathbb{E}K_t^\varepsilon \varphi - \mathbb{E}A_t^\varepsilon \varphi). \quad (2.16)$$

where

$$A_t^\varepsilon \varphi := \varepsilon \sum_{(x,v,r)\in X^\varepsilon} r\varphi\big(x(1+\sigma) + vt + j^\varepsilon(x,v,t), v, r\big).$$

The last term in (2.16) gives

$$\begin{aligned}
&\varepsilon^{-1/2}(\mathbb{E}K_t^\varepsilon\ \varphi - \mathbb{E}A_t^\varepsilon\ \varphi) \\
&\quad = \mathbb{E}\Big(\varepsilon^{1/2} \sum_{(x,v,r)\in X^\varepsilon} r\big[\varphi\big(x + m_0^x(X^\varepsilon) + vt + j^\varepsilon(x,v,t), v, r\big) \\
&\qquad\qquad - \varphi\big(x(1+\sigma) + vt + j^\varepsilon(x,v,t), v, r\big)\big]\Big) \qquad (2.17) \\
&\quad = \mathbb{E}\Big(\varepsilon^{1/2} \sum_{(x,v,r)\in X^\varepsilon} r(\partial_y\varphi)\big(x(1+\sigma) + vt + j^\varepsilon(x,v,t), v, r\big)\big(m_0^x(X_t^\varepsilon) - x\sigma\big)\Big) + R_t^\varepsilon \\
&\quad = \iiint r(\partial_y\varphi)\big(x(1+\sigma) + vt + j^\varepsilon(x,v,t), v, r\big) \\
&\qquad\qquad \times \varepsilon^{-1/2}\mathbb{E}\big[m_0^x(X^\varepsilon) - x\sigma\big] dx\, d\mu(v,r) + R_t^\varepsilon = R_t^\varepsilon, \qquad (2.18)
\end{aligned}$$

where R_t^ε is of smaller order in ε. The last two identities follow from the Slyvniak-Mecke formula and from (2.2).

By Lemma 5.2 the second term in (2.16) gives

$$\begin{aligned}
&\varepsilon^{1/2}(A_t^\varepsilon - \mathbb{E}A_t^\varepsilon) \\
&\quad = \varepsilon^{-1/2}\Big[\varepsilon \sum_{(x,v,r)\in X^\varepsilon} r\varphi\big(x(1+\sigma) + vt + j^\varepsilon(x,v,t), v, r\big) \\
&\qquad\qquad - \rho \iiint r\varphi\big(x(1+\sigma) + vt + j^\varepsilon(x,v,t), v, r\big) dx\, d\mu(v,r)\Big] \\
&\quad \xrightarrow[\varepsilon\to 0]{\text{law}} \xi_t^X(\breve{\varphi}_t), \qquad (2.19)
\end{aligned}$$

where

$$\breve{\varphi}_t(x,v,r) := \varphi\big(x(1+\sigma) + v^{\text{eff}}(v)t, v, r\big).$$

Finally, the first term in (2.16) gives

$$\begin{aligned}
&\varepsilon^{1/2}(K_t^\varepsilon\varphi - A_t^\varepsilon\varphi) \\
&\quad = \varepsilon^{1/2} \sum_{(x,v,r)\in X^\varepsilon} r\big[\varphi\big(x + m_0^x(X^\varepsilon) + vt + j^\varepsilon(x,v,t), v, r\big)
\end{aligned}$$

$$
\begin{aligned}
&\qquad - \varphi\big(x(1+\sigma)+vt+j^\varepsilon(x,v,t),v,r\big)\big] \\
&= \varepsilon^{1/2} \sum_{(x,v,r)\in X^\varepsilon} r(\partial_x\varphi)\big(x(1+\sigma)+vt+j^\varepsilon(x,v,t),v,r\big)(m_0^x(X^\varepsilon)-\sigma x)+R_t^\varepsilon \\
&= \varepsilon \sum_{(x,v,r)\in X^\varepsilon} r(\partial_x\varphi)\big(x(1+\sigma)+vt+j^\varepsilon(x,v,t),v,r\big)\varepsilon^{-1/2}(m_0^x(X^\varepsilon)-\sigma x)+R_t^\varepsilon,
\end{aligned}
$$

and combining Lemmas 5.1 and 5.2 we obtain that the limit in law of this last process is

$$
\begin{aligned}
&\rho \iiint r(\partial_x\varphi)\big(x(1+\sigma)+v^{\mathrm{eff}}(v)t,v,r\big)\, B(x)\, dx\, d\mu(v,r) \\
&= \frac{1}{1+\sigma}\int dx\, B(x)\, \partial_x\Big(\rho \iint r\breve{\varphi}_t(x,v,r)\, d\mu(v,r)\Big) \\
&= \frac{1}{1+\sigma}\int dx\, \xi^X\big(1_{[0,x]}1_{x>0}-1_{[x,0]}1_{x<0}\big)\, \partial_x\Big(\rho \iint r\breve{\varphi}_t(x,v,r)\, d\mu(v,r)\Big) \\
&= \frac{1}{1+\sigma}\, \xi^X\Big(\int dx\, \big(1_{[0,x]}1_{x>0}-1_{[x,0]}1_{x<0}\big)\, \partial_x\Big(\rho \iint r\breve{\varphi}_t(x,v,r)\, d\mu(v,r)\Big)\Big) \\
&= -\frac{1}{1+\sigma}\, \xi^X\Big(\rho \iint r\breve{\varphi}_t(\cdot,\tilde v,\tilde r)\, d\mu(\tilde v,\tilde r)\Big),
\end{aligned}
$$

where we used $\partial_y\varphi = \frac{1}{1+\sigma}\partial_x\breve{\varphi}_t$, and that R_t^ε is smaller order.

Recalling $P\varphi$ defined in (2.10), we conclude that

$$
\varepsilon^{1/2}(K_t^\varepsilon\varphi - A_t^\varepsilon\varphi) \xrightarrow[\varepsilon\to 0]{\text{law}} -\xi^X(P\breve{\varphi}_t). \tag{2.20}
$$

Putting together (2.20), (2.17)-(2.18) and (2.19) we have shown that

$$
\xi_t^{Y,\varepsilon}(\varphi) \xrightarrow[\varepsilon\to 0]{\text{law}} \xi_t^Y(\varphi) := \xi^X(\breve{\varphi}_t - P\breve{\varphi}_t) = \xi^X(C\breve{\varphi}_t),
$$

where $C := I - P$. Recalling (2.11), we have proven that

$$
\xi_t^Y(\varphi) = \xi_0^Y(\varphi_t),
$$

i.e.

$$
\partial_t\xi_t^Y(\varphi) = \xi_0^Y(v^{\mathrm{eff}}\partial_x\varphi_t) = \xi_t^Y(v^{\mathrm{eff}}\partial_x\varphi).
$$

In other words, in a weak sense ξ_t^Y satisfies the equation

$$
\partial_t\xi_t^Y + v^{\mathrm{eff}}\partial_x\xi_t^Y = 0,
$$

which is the expected equation.

3. Equilibrium Fluctuations in the diffusive scaling

3.1. Quasi-particles in the diffusing scaling

Given a point $(x,v,r)\in X^\varepsilon$, recall that $y_{v,t}^\varepsilon(x)$ is the position at time t of the quasiparticle (y,v,r), defined by (2.13). We will show that

$$
y_{\varepsilon^{-1}t}^\varepsilon(x) - v^{\mathrm{eff}}(v)\varepsilon^{-1}t \xrightarrow[\varepsilon\to 0]{\text{law}} y + \sqrt{\mathcal{D}(v)}W_t(v), \tag{3.1}
$$

where $W_t(v)$ is a Wiener process in t. The limit process $W_t(v)$ does not depend on the initial position x. The processes in $(W_t(v) : v \in \mathbb{R})$ are jointly gaussian, and described by a Lévy Chentsov field [11]. We compute explicitely the covariances.

By (2.13) and (2.12), we have

$$y^{\varepsilon}_{v,\varepsilon^{-1}t}(x) - v^{\text{eff}}(v)\varepsilon^{-1}t = D^{\varepsilon}_0(x) - (1+\sigma)x + j^{\varepsilon}(x,v,\varepsilon^{-1}t) - \big(v\sigma - \pi\big)\varepsilon^{-1}t.$$

Since $D^{\varepsilon}_0(x) \xrightarrow[\varepsilon\to 0]{\text{a.s.}} (1+\sigma)x$, the limit in (3.1) is equivalent to

$$j^{\varepsilon}(x,v,\varepsilon^{-1}t) - \big(v\sigma - \pi\big)\varepsilon^{-1}t \ \xrightarrow[\varepsilon\to 0]{\text{law}} \ \sqrt{\mathcal{D}(v)}W_t(v);$$

Observe that

$$\varepsilon \sum_{(\tilde{x},\tilde{v},\tilde{r})\in X^{\varepsilon}} \tilde{r} 1_{[\tilde{v}<v]} 1_{[x<\tilde{x}<x+(v-\tilde{v})\varepsilon^{-1}t]} = \varepsilon \sum_{(\tilde{x},\tilde{v},\tilde{r})\in X^{\varepsilon^2}} \tilde{r} 1_{[\tilde{v}<v]} 1_{[\varepsilon x<\tilde{x}<\varepsilon x+(v-\tilde{v})t]},$$

where $X^{\varepsilon^2} \coloneqq \{(\varepsilon x, v, r) : (x,v,r) \in X^{\varepsilon}\}$, is obtained from X^{ε} by rescaling all positions by a factor ε, so that X^{ε^2} is a Poisson process of intensity measure $\varepsilon^{-2}\rho\, d\mu(v,r)$. We have that

$$\mathbb{E}\Big(\varepsilon^2 \sum_{(x',v',r')\in X^{\varepsilon^2}} r' 1_{[v'<v]} 1_{[\varepsilon x<x'<\varepsilon x+(v-v')t]}\Big) = t\rho \int r \int_{-\infty}^{v} (v-v') d\mu(v,r).$$

Applying (2.1) to the function

$$\varphi_{\varepsilon x,v,t}(x',v') = 1_{[v'<v]} 1_{[\varepsilon x<x'<\varepsilon x+(v-v')t]} - 1_{[v'>v]} 1_{[\varepsilon x+(v-v')t<x'<\varepsilon x]},$$

we have that

$$j_{X^{\varepsilon}}(x,v,\varepsilon^{-1}t) - (v\sigma-\pi)\varepsilon^{-1}t \ \xrightarrow[\varepsilon\to 0]{\text{law}} \ \xi^{X}(\varphi_{0,v,t}),$$

which has variance

$$\begin{aligned} &\rho \iiint \tilde{r}^2 \big(1_{[\tilde{v}<v]} 1_{[x<\tilde{x}<x+(v-\tilde{v})t]} + 1_{[\tilde{v}>v]} 1_{[x+(v-\tilde{v})t<\tilde{x}<x]}\big) d\tilde{x}\, d\mu(\tilde{v},\tilde{r}) \\ &= \rho \int \tilde{r}^2 \int_{v}^{+\infty} (v-\tilde{v}) t d\mu(\tilde{v},\tilde{r}) - \rho \int \tilde{r}^2 \int_{-\infty}^{v} (v-\tilde{v}) t d\mu(\tilde{v},\tilde{r}) \\ &= t\rho \iint \tilde{r}^2 |v-\tilde{v}| d\mu(\tilde{v},\tilde{r}) \coloneqq t\mathcal{D}(v). \end{aligned} \tag{3.2}$$

About the correlation for different initial position, assuming $x < \overline{x}$:

$$\begin{aligned} &\mathbb{E}\big(\xi^{X,\varepsilon}(\varphi_{\varepsilon x,v,t})\xi^{X,\varepsilon}(\varphi_{\varepsilon \overline{x},v,t})\big) \\ &= \rho \iiint r^2 \big(1_{[v'<v]} 1_{[\varepsilon x<x'<\varepsilon x+(v-v')t]} - 1_{[v'>v]} 1_{[\varepsilon x+(v-v')t<x'<\varepsilon x]}\big) \\ &\quad \times \big(1_{[v'<v]} 1_{[\varepsilon \overline{x}<x'<\varepsilon \overline{x}+(v-v')t]} - 1_{[v'>v]} 1_{[\varepsilon \overline{x}+(v-v')t<x'<\varepsilon \overline{x}]}\big) dx'\, d\mu(v',r) \end{aligned} \tag{3.3}$$

$$
\begin{aligned}
&= \rho \iiint r^2\Big(1_{[v'<v]}1_{[\varepsilon x<x'<\varepsilon x+(v-v')t]}1_{[\varepsilon\overline{x}<x'<\varepsilon\overline{x}+(v-v')t]} \\
&\qquad + 1_{[v'>v]}1_{[\varepsilon x+(v-v')t<x'<\varepsilon x]}1_{[\varepsilon\overline{x}+(v-v')t<x'<\varepsilon\overline{x}]}\Big)dx'\, d\mu(v',r) \\
&= \rho \int r^2 \int_{-\infty}^{v-\varepsilon(\overline{x}-x)/t}\big[(v-v')t-\varepsilon(\overline{x}-x)\big]d\mu(v',r) \\
&\qquad + \rho\int r^2\int_{v+\varepsilon(\overline{x}-x)/t}^{+\infty}(\varepsilon(x-\overline{x})-(v-v')t)d\mu(v',r) \\
&= \rho\big[vt-\varepsilon(\overline{x}-x)\big]\int r^2\int_{-\infty}^{v-\varepsilon(\overline{x}-x)/t} d\mu(v',r) \\
&\qquad - \rho t\int r^2\int_{-\infty}^{v-\varepsilon(\overline{x}-x)/t} v' d\mu(v',r) \\
&\qquad + \rho\big[\varepsilon(x-\overline{x})-vt\big]\int r^2\int_{v+\varepsilon(\overline{x}-x)/t}^{+\infty} d\mu(v',r) \\
&\qquad + \rho t\int r^2\int_{v+\varepsilon(\overline{x}-x)/t}^{+\infty} v' d\mu(v',r) \\
&= \rho\big[vt-\varepsilon(\overline{x}-x)\big]\int r^2\int_{-\infty}^{v-\varepsilon(\overline{x}-x)/t} d\mu(v',r) \\
&\qquad - \rho t\int r^2\int_{-\infty}^{v-\varepsilon(\overline{x}-x)/t} v' d\mu(v',r) \\
&\qquad - \rho\big[vt+\varepsilon(\overline{x}-x)\big]\int r^2\int_{v+\varepsilon(\overline{x}-x)/t}^{+\infty} d\mu(v',r) \\
&\qquad + \rho t\int r^2\int_{v+\varepsilon(\overline{x}-x)/t}^{+\infty} v' d\mu(v',r) \\
&\xrightarrow[\varepsilon\to 0]{} t\rho\iint (r')^2|v-v'|d\mu(v',r') = t\mathcal{D}(v).
\end{aligned}
$$

It follows from (3.3) that two tagged quasi-particles with the same velocity are asymptotically completely correlated.

Considering the correlation at different velocities $v<\overline{v}$:

$$
\begin{aligned}
&\mathbb{E}\big(\xi^{X,\varepsilon}(\varphi_{\varepsilon x,v,t})\xi^{X,\varepsilon}(\varphi_{\varepsilon x,\overline{v},t})\big) \qquad (3.4)\\
&= \rho\iiint r^2\big(1_{[v'<v]}1_{[\varepsilon x<x'<\varepsilon x+(v-v')t]} - 1_{[v'>v]}1_{[\varepsilon x+(v-v')t<x'<\varepsilon x]}\big) \\
&\qquad \times\big(1_{[v'<\overline{v}]}1_{[\varepsilon x<x'<\varepsilon x+(\overline{v}-v')t]} - 1_{[v'>\overline{v}]}1_{[\varepsilon x+(\overline{v}-v')t<x'<\varepsilon x]}\big)\, dx'\, d\mu(v',r) \\
&= \rho\iiint r^2\Big(1_{[v'<v]}1_{[\varepsilon x<x'<\varepsilon x+(v-v')t]}1_{[\varepsilon x<x'<\varepsilon x+(\overline{v}-v')t]} \\
&\qquad\qquad + 1_{[v'>\overline{v}]}1_{[\varepsilon x+(\overline{v}-v')t<x'<\varepsilon x]}1_{[\varepsilon x+(v-v')t<x'<\varepsilon x]} \\
&\qquad\qquad - 1_{[v<v'<\overline{v}]}1_{[\varepsilon x+(v-v')t<x'<\varepsilon x]}1_{[\varepsilon x<x'<\varepsilon x+(\overline{v}-v')t]}\Big)\, dx'\, d\mu(v',r) \\
&= \rho\iiint r^2\Big(1_{[v'<v]}1_{[\varepsilon x<x'<\varepsilon x+(v-v')t]} + 1_{[v'>\overline{v}]}1_{[\varepsilon x+(\overline{v}-v')t<x'<\varepsilon x]}\Big)\, dx'\, d\mu(v',r) \\
&= t\rho\iint r^2\big(1_{[v'<v]}(v-v') + 1_{[v'>\overline{v}]}(v'-\overline{v})\big)d\mu(v',r) \\
&:= t\Gamma(v,\overline{v}).
\end{aligned}
$$

Noting that

$$1_{[v'<v]}(v-v') + 1_{[v'>\overline{v}]}(v'-\overline{v}) = \frac{1}{2}\Big(|v-v'| + |v'-\overline{v}| - (\overline{v}-v)\Big),$$

we have that

$$\Gamma(v,\overline{v}) = \frac{1}{2}\Big(\mathcal{D}(v) + \mathcal{D}(\overline{v}) - (\overline{v}-v)\rho\int r^2 d\mu(v',r)\Big). \tag{3.5}$$

This last expression for the covariance corresponds to the Lévy Chentsov field, as shown in [11].

Remark 3.1. *From (3.4) we can see immediately that $\Gamma(v,\overline{v}) \geqslant 0$. In the particular case where there are only two velocities admitted, for example $d\mu(v,r) = \delta_a(dr)\frac{1}{2}(\delta_{-1}(dv) + \delta_1(dv))$, we have $\mathcal{D}(\pm 1) = \rho a^2$ and $\Gamma(1,-1) = 0$. Typically this decorrelation happens only when two velocities at most are present.*

3.2. Diffusive evolution of density fluctuations

Define the fluctuation field at diffusive scaling as

$$\Xi_t^{Y,\varepsilon}(\varphi) := \varepsilon^{-1/2}\Big[\varepsilon\sum_{(y,v,r)\in Y^\varepsilon} r\varphi\big[y_{v,\varepsilon^{-1}t} - v^{\mathrm{eff}}(v)\varepsilon^{-1}t, v, r\big] - \frac{1}{1+\sigma}\langle\!\langle\varphi\rangle\!\rangle\Big]. \tag{3.6}$$

Notice that this is recentered on the Euler characteristics. Define

$$\varphi_{W_t}(y,v,r) := \varphi\big(y + \sqrt{\mathcal{D}(v)}W_t(v), v, r\big).$$

Theorem 3.2.

$$\Xi_t^{Y,\varepsilon}(\varphi) \xrightarrow[\varepsilon\to 0]{law} \Xi_t^Y(\varphi) = \Xi^Y\big(\varphi_{W_t}\big) = \xi^X\big(C\breve{\varphi}_{W_t}\big)$$

where $(W_t(v) : v\in\mathbb{R})$ is a family of Wiener processes with covariance

$$\mathbb{E}\big(W_t(v), W_t(w)\big) = \frac{t\Gamma(v,w)}{\sqrt{\mathcal{D}(v)\mathcal{D}(w)}}$$

with Γ defined in (3.4). More formally

$$\begin{aligned}\Xi_t^Y(\varphi) &= \iiint r\varphi\big(y + \sqrt{\mathcal{D}(v)}W_t(v), v, r\big)d\xi_0^Y(y,v,r)\\ &= \iiint rC\varphi\big((1+\sigma)(x + \sqrt{\mathcal{D}(v)}W_t(v)), v, r\big)d\xi_0^X(y,v,r)\end{aligned} \tag{3.7}$$

We prove this theorem after some comments. By (3.7), Ξ_t^Y solves the stochastic differential equation

$$d\Xi_t^Y(\varphi) = \frac{1}{2}\Xi_t^Y(\mathcal{D}\partial_y^2\varphi)dt - \iiint\sqrt{\mathcal{D}(v)}(\partial_y\varphi)(y,v,r)dW_t(v)d\Xi_t^Y(y,v,r)$$

or in the time integrated form:

$$\begin{aligned}\Xi_t^Y(\varphi) &= \Xi_0^Y(\varphi) + \int_0^t \frac{1}{2}\Xi_s^Y(\mathcal{D}\partial_y^2\varphi)ds \\ &\quad - \int_0^t \iiint \sqrt{\mathcal{D}(v)}(\partial_y\varphi)(y,v,r)dW_s(v)d\Xi_s^Y(y,v,r) \\ &= \Xi_0^Y(\varphi) + \int_0^t \frac{1}{2}\Xi_s^Y(\mathcal{D}\partial_y^2\varphi)ds - \int_0^t \Xi_s^Y\big(\sqrt{\mathcal{D}}\,\partial_y\varphi\,dW_s\big),\end{aligned}$$

where the last term is a martingale with quadratic variation

$$\int_0^t \Big(\iiint \sqrt{\mathcal{D}(v)}(\partial_y\varphi)(y,v,r)d\Xi_s^Y(y,v,r)\Big)^2 ds = \int_0^t \Xi_s^Y\big(\sqrt{\mathcal{D}}(\partial_y\varphi)\big)^2 ds. \tag{3.8}$$

Notice that

$$\begin{aligned}\mathbb{E}\big(\Xi_t^{Y,\varepsilon}(\varphi)^2\big) &= \frac{\rho}{1+\sigma}\iiint r^2\mathbb{E}\big[(C\varphi)^2(y+\sqrt{\mathcal{D}(v)}W_t(v),v,r)\big]dyd\mu(v,r) \\ &= \frac{\rho}{1+\sigma}\iiint r^2\int (C\varphi)^2\big(y(1+\sqrt{\mathcal{D}(v)}z,v,r\big)\frac{e^{-z^2/2}}{\sqrt{2\pi t}}dzdyd\mu(v,r) \\ &= \frac{\rho}{1+\sigma}\iiint r^2(C\varphi)^2\big(y,v,r\big)dy\,d\mu(v,r),\end{aligned}$$

independent of t, in agreement with the stationarity of the process.

Similarly the expectation of the quadratic variation (3.8) is given by

$$t\frac{\rho}{1+\sigma}\iiint \mathcal{D}(v)r^2(\partial_y\varphi)^2(y,v,r)dy\,d\mu(v,r).$$

Proof of Theorem 3.2. We can express (3.6) as

$$\begin{aligned}&\Xi_t^{Y,\varepsilon}(\varphi) \\ &= \varepsilon^{-1/2}\Big[\varepsilon\sum_{(x,v,r)\in X^\varepsilon} r\varphi\big[x+m_0^x(X^\varepsilon)+j^\varepsilon(x,v,\varepsilon^{-1}t)-v^{\text{eff}}(v)\varepsilon^{-1}t,v,r\big]-\frac{1}{1+\sigma}\langle\!\langle\varphi\rangle\!\rangle\Big] \\ &= \varepsilon^{-1/2}\Big[\varepsilon\sum_{(x,v,r)\in X^\varepsilon} r\varphi\big(x+m_0^x(X^\varepsilon)+z(\varepsilon^{-1}t,x,v;X^\varepsilon),v,r\big)-\frac{1}{1+\sigma}\langle\!\langle\varphi\rangle\!\rangle\Big] \\ &= \varepsilon^{-1/2}\Big[\varepsilon\sum_{(x,v,r)\in X^\varepsilon} r\varphi\big(x(1+\sigma)+z(\varepsilon^{-1}t,x,v;X^\varepsilon),v,r\big)-\frac{1}{1+\sigma}\langle\!\langle\varphi\rangle\!\rangle\Big] \\ &\quad+\varepsilon^{1/2}\sum_{(x,v,r)\in X^\varepsilon} r(\partial_y\varphi)\big(x(1+\sigma)+z(\varepsilon^{-1}t,x,v;X^\varepsilon),v,r\big)\big(m_0^x(X^\varepsilon)-\sigma x\big)+R_t^\varepsilon.\end{aligned} \tag{3.9}$$

Applying Lemma 5.2 to the first term on the rhs of (3.9) we have

$$\varepsilon^{-1/2}\Big[\varepsilon\sum_{(x,v,r)\in X^\varepsilon} r\varphi\big(x(1+\sigma)+z(\varepsilon^{-1}t,0,v;X^\varepsilon),v,r\big)-\frac{1}{1+\sigma}\langle\!\langle\varphi\rangle\!\rangle\Big] \xrightarrow[\varepsilon\to 0]{\text{law}} \xi^X(\check{\varphi}_{\overline{B}_t}),$$

where we have denoted

$$\breve{\varphi}_{W_t}(x, v, r) = \varphi_{W_t}(x(1+\sigma), v, r) = \varphi\Big(x(1+\sigma) + \sqrt{\mathcal{D}(v)}W_t(v), v, r\Big).$$

Combining Lemma 5.2 and the same argument used in (2.9) the second term converges in law:

$$\varepsilon^{1/2} \sum_{(x,v,r)\in X^\varepsilon} r(\partial_y \varphi)\Big(x(1+\sigma) + z(\varepsilon^{-1}t, x, v; X^\varepsilon), v, r\Big)\Big(m_0^x(X^\varepsilon) - \sigma x\Big)$$
$$\xrightarrow[\varepsilon\to 0]{\text{law}} -\xi^X(P\breve{\varphi}_{W_t}).$$

Putting the two terms together we conclude that

$$\Xi_t^{Y,\varepsilon}(\varphi) \xrightarrow[\varepsilon\to 0]{\text{law}} \xi^X(C\breve{\varphi}_{W_t}) = \xi^Y(\varphi_{W_t}). \qquad \square$$

Formally, choosing $\varphi_{k,w}(x, v, r) = e^{i2\pi x k}\varphi(r)\delta(v - w)$, we have that

$$\begin{aligned}\widehat{\varphi}(k, w, t) \coloneqq \Xi_t^{Y,\varepsilon}(\varphi_{k,w}) &= \int d\xi^Y(y, r, v) e^{i2\pi k(y+\sqrt{D(w)}W_t)}\varphi(r)\delta(v - w) \\ &= \xi^Y\Big(e^{i2\pi k(\cdot+\sqrt{D(w)}W_t)}\varphi(\cdot)\delta(\cdot - w)\Big)\end{aligned}$$

satisfies the SDE

$$d\widehat{\varphi}(k, w, t) = -\frac{(2\pi k)^2}{2}\mathcal{D}(w)\widehat{\varphi}(k, w, t) + i2\pi k\sqrt{\mathcal{D}(w)}\widehat{\varphi}(k, w, t)dW_t(w).$$

Notice that $|\widehat{\varphi}(k, w, t)|^2 = |\widehat{\varphi}(k, w, 0)|^2$ for any k, a persistence on the macroscopic scale of the complete integrability of the dynamics also at the level of these fluctuations.

Remark 3.3. Since we consider also systems where lengths r can be negative, in the case that $\sigma = 0$ and $\pi = 0$ the macroscopic evolution of fluctuations in the Euler scaling is the same as the independent point particles. But in the diffusive scaling the fluctuations have non trivial behaviour.

4. A remark about inhomogeneous initial distribution.

Let $f_0(x, v, r)$ be a nice non-negative bounded function on $\mathbb{R}^3$ and X^ε the Poisson process on $\mathbb{R}^3$ with intensity $\varepsilon^{-1}f_0(x, v, r)dx\, dv\, dr$.

In the Euler scaling, the empirical distribution of the free gas converges to the solution of

$$\partial_t f_t(x, v, r) + v\partial_x f_t(x, v, r) = 0,$$

with initial condition given by f_0. For the rods density this corresponds to the equation

$$\begin{aligned}\partial_t g_t(y, v, r) + \partial_y\Big(v^{\text{eff}}(y, v, t)g_t(y, v, r)\Big) &= 0, \\ v^{\text{eff}}(y, v, t) = v + \frac{\iint r(v - w)g_t(y, w, r)dwdr}{1 - \iint rg_t(y, w, r)dwdr}&.\end{aligned}$$

as proven in [11].

For *generic* initial conditions, we can guess that, if the initial density f is absolutely continuous in the x and v coordinates, then it satisfies that the limit as $t \to \infty$ of $f_t(x,v,r)$ is constant in x, that is, $f_t(x,v,r) \longrightarrow \rho\overline{f}(v,r)$, for some $\rho \in \mathbb{R}_+$ and $\overline{f}(v,r)$. This suggests that in a diffusive time scale the system essentially behaves like if it is a stationary state determined by a Poisson point field $\rho\overline{f}(v,r)dx\,dv\,dr$, and the analysis for the macroscopic fluctuations of Section 3 applies.

5. Two limit theorems for Poisson process

Lemma 5.1. *Let $\varphi(x,v,r)$ a smooth function on $\mathbb{R}^3$ with compact support in x. Then*

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{(x,v,r)\in X^\varepsilon} r\varphi(x,v,r)B^\varepsilon(x) \overset{law}{=} \rho \iiint r\varphi(x,v,r)B(x)\,dx\,d\mu(v,r).$$

Proof. Since φ is a smooth function we can approximate both sides by step functions, so that it is enough to prove that

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{(x,v,r)\in X^\varepsilon} 1_{[0\leqslant x<1]} B^\varepsilon(x) \overset{\text{law}}{=} \rho \int_0^1 B(x)\,dx. \tag{5.1}$$

Since in this limit (v,r) are not involved, to simplify notation we will ignore them. We can also substitute $B^\varepsilon(\cdot)$ with a continuous version $\tilde{B}^\varepsilon(\cdot)$ obtained by linear interpolation that converges in law to the same limit. The positive random measure M^ε on $[0,1]$ defined by

$$M^\varepsilon(\psi) = \varepsilon \sum_{(x,v,r)\in X^\varepsilon} 1_{[0\leqslant x<1]}\psi(x), \qquad \psi \in \mathcal{C}(0,1),$$

converges a.s. to the Lebesgue measure dx on $[0,1]$. Then, by the generalization of Slutsky's theorem in Theorem 2.7 (v) in [22], the the couple $(M^\varepsilon, \tilde{B}^\varepsilon)$ converges in law to (dx, B).

Let $F(\mu,\psi)$ a continuous function on $\mathcal{M}_+([0,1])\times\mathcal{C}(0,1)$. Then $F(M^\varepsilon,\tilde{B}^\varepsilon) \longrightarrow F(dx,B)$ in law. Apply this to

$$F(\mu,\psi) = \int_0^1 \psi(x)d\mu(x),$$

and (5.1) follows.

□

Lemma 5.2. *Let $\tilde{X}^\varepsilon$ be a homogeneous Poisson process on $\mathbb{R}$ with density $\varepsilon^{-1}\tilde{\rho}$ and $\{\tilde{B}^\varepsilon(x), x\in\mathbb{R}\}$ be a family of processes independent of $\tilde{X}^\varepsilon$ and such that they* *converge in law to the same random variable $\tilde{B}$. Then for any smooth compact support function $\varphi(x)$ on $\mathbb{R}$,*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1/2}\Big(\varepsilon\sum_{x\in\tilde{X}^\varepsilon}\varphi(x+\tilde{B}^\varepsilon(x))-\tilde{\rho}\int\varphi(x)dx\Big)\overset{law}{=}\xi^{\tilde{X}}(\varphi_{\tilde{B}}),\tag{5.2}$$

where $\varphi_{\tilde{B}}(x)=\varphi(x+\tilde{B})$.

Proof. In order to shorten notation let's assume $\int\varphi(x)dx=0$. Denote

$$\mathcal{Z}_\varepsilon:=\varepsilon^{1/2}\sum_{x\in\tilde{X}^\varepsilon}\varphi(x+\tilde{B}^\varepsilon(x)).$$

Its characteristic function is

$$\begin{aligned}&\mathbb{E}\big[e^{ik\mathcal{Z}_\varepsilon}\big]=\mathbb{E}\big[\mathbb{E}\big(e^{ik\mathcal{Z}_\varepsilon}|B^\varepsilon\big)\big]=\mathbb{E}\Big[\exp\Big(\rho\varepsilon^{-1}\int\big(e^{ik\varepsilon^{1/2}\varphi(x+\tilde{B}^\varepsilon(x))}-1\big)dx\Big)\Big]\\&=\mathbb{E}\Big[\exp\Big(-\rho k^2\int\varphi(x+\tilde{B}^\varepsilon(x))^2dx\Big)\Big]+O(\varepsilon^{1/2})\\&\underset{\varepsilon\to 0}{\longrightarrow}\mathbb{E}\Big[\exp\Big(-\rho k^2\int\varphi(x+\tilde{B})^2dx\Big)\Big]=\mathbb{E}\Big[\exp\Big(-\rho k^2\int\varphi(x)^2dx\Big)\Big].\end{aligned}$$

The joint characteristic function of the couple $\big(\xi^{\varepsilon,X},\mathcal{Z}_\varepsilon\big)$ is

$$\mathbb{E}\Big[e^{ik_1\xi^{\varepsilon,X}(\varphi)+ik_2\mathcal{Z}_\varepsilon}\Big]=\mathbb{E}\Big[\exp\Big(-\rho\int\Big[k_1\varphi(x)+k_2\varphi(x+\tilde{B}^\varepsilon(x))\Big]^2dx\Big)\Big]+O\big(\varepsilon^{1/2}\big)$$

$$\underset{\varepsilon\to 0}{\longrightarrow}\mathbb{E}\Big[\exp\Big(-\rho\int\Big[k_1\varphi(x)+k_2\varphi(x+\tilde{B})\Big]^2dx\Big)\Big]=\mathbb{E}\Big[e^{ik_1\xi^{\tilde{X}}(\varphi)+ik_2\xi^{\tilde{X}}(\varphi_{\tilde{B}})}\Big],$$

i.e. (5.2). ☐

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