TWO-WEIGHT BOUNDEDNESS FOR LOCAL FRACTIONAL MAXIMAL AND APPLICATIONS

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Abstract. Given Ω a proper open subset of a metric space with the weak homogeneity property and a measure μ doubling on certain local balls, we give sufficient conditions about local weights for the two-weight boundedness of the local fractional maximal operator acting on weighted Lebesgue spaces. As applications we obtain analogous results for singular and fractional type operators and their commutators. As a further application we present an a priori estimate for solutions of $\Delta^m u = f$ in Ω, acting in weighted Sobolev spaces involving the distance to the boundary and different local weights.

1. Introduction

Let Ω be a proper open and non empty subset of X, where X is a metric space satisfying the weak homogeneity property, that is, if d is a metric on X, there are no more than N points in a ball $B = B(x,r)$ whose distance from each other is greater than $r/2$, for all ball B for some fixed number N. It is clear that this property implies separability of X. Also, we will suppose that all the balls contained in Ω are connected sets and their closure are compact sets.

For $0 < \beta < 1$, we consider the family of balls well-inside of Ω defined by

$$\mathcal{F}_\beta = \{B = B(x_B, r_B) : \ x_B \in \Omega, \ r_B < \beta \, \mathrm{d}(x_B, \Omega^c)\} \ ,$$

where x_B and r_B are the center and the radius of the ball and $\mathrm{d}(x_B, \Omega^c)$ is the distance of x_B to the complementary set of Ω.

As usual, we will denote by λB the ball with same center and radius $\lambda-$times of B. Also, given a Borel measure μ defined on Ω such that $0 < \mu(B) < \infty$ for any ball $B \in \mathcal{F} = \cup_{0<\alpha<1} \mathcal{F}_\alpha$, we will say that μ is doubling on $\mathcal{F}_\beta$ if there is some constant C_β such that

$$\mu(B) \le C_\beta \, \mu(\frac{1}{2}B) \, , \tag{1.1}$$

for any ball $B \in \mathcal{F}_\beta$.

Remark 1.2. Let us remark that the doubling condition on $\mathcal{F}_\beta$ implies the finiteness of the measure $\mu(B)$ for $B \in \mathcal{F}_\beta$, even though this is not so for all the balls contained in Ω.

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Remark 1.3. In [6] it was proved that such a measure is doubling on $\mathcal{F}_\alpha$ too, for $0<\alpha<1$, perhaps with a different constant that may increase to infinity when α increases to 1. For instance, if $\Omega=(0,\infty)$, $d\mu=dx/x$, it is easy to see that μ is doubling on $\mathcal{F}_\alpha$ with constant $C(\alpha)\to\infty$ when $\alpha\to 1^-$. In fact, to see this we consider $x_0\in(0,\infty)$, $\alpha\in(0,1)$ and $0<t<\alpha$. Then, we have

$$\int_{x_0-tx_0}^{x_0}\frac{dx}{x}=\ln\frac{1}{1-t}=\frac{\ln\frac{1}{1-t}}{\ln\frac{1}{1-\frac{t}{2}}}\int_{x_0-\frac{t}{2}x_0}^{x_0}\frac{dx}{x}\,.$$

Now, denoting $C(t)=\frac{\ln\frac{1}{1-t}}{\ln\frac{1}{1-\frac{t}{2}}}$, it can be easily proved that $C(t)$ is a continuous function over $[0,\alpha]$ such that $C(\alpha)\to\infty$ when $\alpha\to 1^-$ which, in turn, proves our statement for the measure given by $d\mu=\frac{dx}{x}$.

In this context, for each $0\le\gamma<1$ we consider the local fractional maximal function on Ω defined as follows

$$M_\beta^\gamma f(x)=\sup_{B:\,x\in B\in\mathcal{F}_\beta}\frac{1}{\mu(B)^{1-\gamma}}\int_B|f(y)|\,d\mu(y)\,, \tag{1.4}$$

for every $f\in L^1_{\mathrm{loc}}(\Omega)$ and every $x\in\Omega$. Moreover, for each $r>1$ we denote $M_\beta^{\gamma,r}f=\left(M_\beta^{\gamma r}(|f|^r)\right)^{1/r}$, whenever $\gamma r<1$. Throughout this work, unless otherwise indicated, all the spaces involved will be defined with the measure μ.

When $\gamma=0$, we will use the notation M_β instead M_β^0. This case is considered by Harboure, Viviani and the second author in [6]. They characterized the class of weights w such that M_β is bounded in $L^p(\Omega,w)$. This class is a local version of the A_p-Muckenhoupt classes and it is denoted by A_p^β. We say that a positive a.e. function w in $L^1_{\mathrm{loc}}(\Omega)$ belongs to A_p^β if there exists a constant C_β such that

$$\left(\frac{1}{\mu(B)}\int_B w\,d\mu\right)\left(\frac{1}{\mu(B)}\int_B w^{-\frac{1}{p-1}}\,d\mu\right)^{p-1}\le C_\beta\,, \tag{1.5}$$

for every ball B in $\mathcal{F}_\beta$.

In our work, we will say that a weight u satisfies a doubling condition on $\mathcal{F}$ if the measure $u(B)=\int_B u\,d\mu$ is doubling in the sense of (1.1) for any ball $B\in\mathcal{F}$. In this case, we will denote $u\in\mathcal{D}(\mathcal{F})$.

Remark 1.6. Let $\Omega=\left\{(x,y)\in\mathbb{R}^2: x>0\right\}$ with the metric $\mathrm{d}((x_1,y_1),(x_2,y_2))=\max\{|x_1-x_2|,|y_1-y_2|\}$. Given $\beta\in(0,1)$ we define on Ω the function

$$w(x,y)=\begin{cases}\mathrm{e}^{-1/x}\,, & \text{if } |y|\le mx\,,\\ 1\,, & \text{if } |y|>mx\,,\end{cases}$$

where $m=\frac{\beta}{4(1+\beta)}$. By considering the balls $B((2^{-i},0),\frac{m}{1+m})$, $i\in\mathbb{N}$, it is not difficult to see that $w\in\mathcal{D}(\mathcal{F}_\beta\backslash\mathcal{F}_{\tilde\beta})$ with $\tilde\beta=\frac{m}{1+m}$. Moreover, $w\notin\mathcal{D}(\mathcal{F}_{\tilde{\tilde\beta}})$ for each $\tilde{\tilde\beta}<\tilde\beta$.

On the other hand, we will consider the class $A_\infty^\beta=\bigcup_p A_p^\beta$. Observe that if a weight u belongs to A_∞^β, then $u\in\mathcal{D}(\mathcal{F}_\beta)$. Moreover, there are positive constants

C and δ such that

$$\frac{u(E)}{u(B)} \leq C \left(\frac{\mu(E)}{\mu(B)} \right)^{\delta} , \tag{1.7}$$

holds for every measurable subset $E \subset B$ and for any $B \in \mathcal{F}_\beta$.

The two-weight boundedness problem for M_β was studied by the first two authors and Viviani in [15]. In the context of $\mathbb{R}^n$ with the Lebesgue measure, they gave a characterization of the pairs of weights (u, v) for which M_β is bounded from $L^p(\Omega, v)$ to $L^q(\Omega, u)$. More precisely, they proved the following theorem

Theorem 1.8. *Given $1 < p \leq q < \infty$ and $0 < \beta < 1$, let (u, v) be a pair of weights such that $u \in \mathcal{D}(\mathcal{F}_\beta)$ and $\sigma = v^{1-p'}$ belongs to A_∞^β. Then*

$$M_\beta : L^p(\Omega, v) \to L^q(\Omega, u) \tag{1.9}$$

if and only if

$$|B|^{-1} \left(\int_B u\, dx \right)^{1/q} \left(\int_B \sigma\, dx \right)^{1/p'} \leq C , \tag{1.10}$$

for every ball $B \in \mathcal{F}_\beta$.

The condition (1.10) was denoted there as $\mathrm{A}_{p,q}^\beta$. Now, related to the aim of this article, for $1 < p \leq q < \infty$, $0 < \beta < 1$ and $0 \leq \gamma < 1$, we will say that a pair of weights u and v lies in the class $\mathrm{A}_{p,q}^{\gamma,\beta}$ if and only if

$$\mu(B)^{\gamma - 1}\, u(B)^{1/q}\, \sigma(B)^{1/p'} \leq C , \tag{1.11}$$

for every ball $B \in \mathcal{F}_\beta$, where $\sigma = v^{1-p'}$. We will write $(u, v) \in \mathrm{A}_{p,q}^{\gamma,\beta}$.

Note that this type of condition usually appears in literature in connection with two-weight boundedness problems (see, for instance, [2] , [4] and [10]). In the particular case $\gamma = 0$ and μ the Lebesgue measure, we obtain (1.10). Moreover, if $\gamma = 0$, $u = v$ and $p = q$, we recover the local Muckenhoupt class of weights A_p^β as in (1.5).

It is important to observe that, as in the one-weight case, (1.10) describes the same class for all the values of β (see Theorem 3.4 in [6] and Lemma 4.1 in [15]). Analogously, it is not difficult to check that $\mathrm{A}_{p,q}^{\gamma,\beta}$ coincides with $\mathrm{A}_{p,q}^{\gamma,\alpha}$ for $0 < \alpha < \beta < 1$, taking into account that μ satisfies the doubling condition on $\mathcal{F}_\beta$, and $u, \sigma \in \mathcal{D}(\mathcal{F}_\beta)$. So, as in the one-weight case, we can refer to those as local weights and denote this class as $\mathrm{A}_{p,q}^{\gamma,loc}$. In the particular case $\gamma = 0$ we write $A_{p,q}^{loc}$.

Remark 1.12. For a domain Ω and $0 < \beta < 1$ we denote $\mathrm{d}(x) = \mathrm{d}(x, \Omega^c)$. Since $\mathrm{d}(x) \simeq \mathrm{d}(x_B)$ for $x \in B$ and x_B the center of B with $B \in \mathcal{F}_\beta$, we have that $\mathrm{d}^\alpha \in A_p^\beta$ for all α. In turn, for the class $A_{p,q}^{\gamma,\beta}$, it is not difficult to see that $(\mathrm{d}^\alpha, \mathrm{d}^\delta) \in A_{p,q}^{\gamma,\beta}$ for a bounded domain Ω whenever $\gamma + \frac{1}{q} - \frac{1}{p} \leq 0$ and $\frac{\alpha}{q} - \frac{\delta}{p} \leq 0$. For a general domain Ω we get the result but when $\gamma + \frac{1}{q} - \frac{1}{p} = \frac{\alpha}{q} - \frac{\delta}{p} = 0$.

Before stating our first result we need to recall some facts about Orlicz spaces. A function $\phi : [0, \infty) \to [0, \infty)$ is called a doubling *Young function* if it is continuous, increasing, convex and such that $\phi(0) = 0$, $\phi(t) \to \infty$ as $t \to \infty$ and $\phi(2t) \leq C\phi(t)$, for all t and some positive constant C. It follows that $\phi(st) \geq s\phi(t)$ for all $s \geq 1$ and $t > 0$. Now, for a given metric space X, we will consider the Orlicz space $L^\phi =$

$L^\phi(X)$ as a subclass of measurable functions on X satisfying $\int_X \phi(|f|)\, d\mu < \infty$. A seminorm in these spaces is given by the Luxemburg norm as follows

$$\|f\|_\phi = \inf\left\{\lambda > 0 \,:\, \int_X \phi\Big(\frac{|f|}{\lambda}\Big)\, d\mu \le 1\right\}.$$

On the other hand, for each ball B belonging on X, we consider the ϕ-average of f over B as above, that is

$$\|f\|_{\phi,B} = \inf\left\{\lambda > 0 \,:\, \frac{1}{\mu(B)}\int_B \phi\Big(\frac{|f|}{\lambda}\Big)\, d\mu \le 1\right\}.$$

We recall that, given a Young function ϕ, the function $\tilde{\phi}(t) = \sup_{s>0}(st - \phi(s))$, known as complementary function of ϕ, satisfies the property

$$t \le \phi^{-1}(t)\, \tilde{\phi}^{-1}(t) \le 2t\,,$$

for all $t > 0$. Thus, we have the generalized Hölder inequality

$$\frac{1}{\mu(B)}\int_B |f\, g|\, d\mu \le C\ \ \|f\|_{\phi,B}\ \|g\|_{\tilde{\phi},B}\ .$$

For a weight u we will say that f belongs to the weighted Orlicz space $L^\phi(u) = L^\phi(X, u)$ if and only if $fu \in L^\phi$ and we denote by $\|f\|_{L^\phi(u)} = \|f\|_{\phi,u} = \|fu\|_\phi$ the norm on this spaces. For more details about Orlicz spaces see, for instance, [16].

Definition 1.13. *Let $1 < p < \infty$. We say that a Young function ϕ belongs to the class B_p if there is a positive constant c such that*

$$\int_c^\infty \frac{\phi(t)}{t^p}\, \frac{dt}{t} < \infty\,.$$

In the case where X is a space of homogeneous type the definition above characterizes the class of functions such that the associated maximal operator

$$M_\phi f(x) = \sup_{B:\, x\in B} \|f\|_{\phi,B} \tag{1.14}$$

is bounded on $L^p(X)$ (see Theorem 1.4 in [13]).

With all this in mind, our first main result can be stated as follows.

Theorem 1.15. *Let us consider $1 < p \le q < \infty$, $0 < \beta < 1$, $0 \le \gamma < 1$ and ψ be a doubling Young function such that $\tilde{\psi}$ belongs to B_p. Then, for every pair of weights (u, v) such that $u \in \mathcal{D}(\mathcal{F}_\beta)$ and $\sigma = v^{1-p'} \in \mathcal{D}(\mathcal{F}_\beta \backslash \mathcal{F}_{\tilde{\beta}})$, with $\tilde{\beta} < \beta/160$, we have*

$$M^\gamma_\beta : L^p(\Omega, v) \to L^q(\Omega, u)\,; \tag{1.16}$$

whenever

$$\mu(B)^{\gamma-1/p+1/q}\left(\frac{1}{\mu(B)}\int_B u\, d\mu\right)^{1/q} \|v^{-1/p}\|_{\psi,B} \le C\,, \tag{1.17}$$

for every ball $B \in \mathcal{F}_\beta$.

Remark 1.18. We consider Ω and d as in the Remark 1.6. Let

$$\sigma(x,y)=\begin{cases}\mathrm{e}^{-1/x}\,, & \text{if } |y|\le mx\,,\\ 1\,, & \text{if } |y|>mx\,,\end{cases}$$

for $0<\beta<1$ with $m=\frac{\beta}{160(1+\beta)}$. Again, it is easy to check that $\sigma\in\mathcal{D}(\mathcal{F}_\beta\backslash\mathcal{F}_{\tilde\beta})$ for $\beta'=\frac{m}{1+m}$. Then, taking $\gamma=0$, $1<p=q<\infty$ and $r>1$ such that $r'<p$ we can to show that $u=1$ in Ω and $v=\sigma^{-\frac{1}{p-1}}$ satisfies (1.17) with $\psi(t)=t^r$.

As a corollary of Theorem 1.15 above we recover Theorem 1.8.

Corollary 1.19. *Given $1<p\le q<\infty$ and $0<\beta<1$, let (u,v) be a pair of weights such that $u\in\mathcal{D}(\mathcal{F}_\beta)$ and $\sigma=v^{1-p'}$ belongs to A_∞^β. Then* (1.9) *if and only if* (1.10) *holds.*

Proof. Clearly, (1.9) imply (1.10). On the other hand, if $\sigma\in A_\infty^\beta$ then it satisfies a reverse Hölder inequality, that is, there exists $r>1$ such that

$$\left(\frac{1}{\mu(B)}\int_B\sigma^r\right)^{1/r}\le\frac{c}{\mu(B)}\int_B\sigma\,,$$

for some positive constant c and every ball $B\in\mathcal{F}_\beta$. Then, for $\gamma=0$ we get that (1.10) implies (1.17) by taking $\psi(t)=t^{p'r}$ and by using the Theorem 1.15 we have (1.9). □

Remark 1.20. It is important to note that this result generalizes Theorem 1.1 in [6], to the two-weight case.

The techniques involved in the proof of Theorem 1.15 strongly rely on establishing a connection between the boundedness problem for operators in the local setting and in spaces of homogeneous type, (see Section 2). So, the following result in the context of spaces of homogeneous type, which is an improvement of Theorem 1.6 in [2], will turn out to be a very important tool.

The definition we will give here are intended to contain cases where the space $\mathcal{X}$ is a subset of a metric space. In this direction, for $0\le\gamma<1$, we denote by $M_{\mathcal{X}}^\gamma$ the fractional maximal operator defined for each $f\in L^1_{\mathrm{loc}}(\mathcal{X})$ by

$$M_{\mathcal{X}}^\gamma f(x)\;=\;\sup_{B:\,x\in B\in F(\mathcal{X})}\frac{1}{\mu(B_{\mathcal{X}})^{1-\gamma}}\int_{B_{\mathcal{X}}}|f(y)|\,d\mu(y)\,,\tag{1.21}$$

where $B_{\mathcal{X}}=B\cap\mathcal{X}$ and $F(\mathcal{X})=\{B(x_B,r_B):x_B\in\mathcal{X};r_B>0\}$. In the case $\gamma>0$, for each $1<r<1/\gamma$ we denote $M_{\mathcal{X}}^{\gamma,r}f=\left(M_{\mathcal{X}}^{\gamma r}(|f|^r)\right)^{1/r}$, in a similar way as in (1.4).

In the following theorems, if the space $\mathcal{X}$ is in particular a space of homogeneous type in itself we will denote it $\mathbb{X}$.

Theorem 1.22. *Let $(\mathbb{X},d,\mu)$ be a space of homogeneous type and let $1<p\le q<\infty$, $0\le\gamma<1$. Let ψ be a doubling Young function such that $\tilde\psi$ belongs to B_p. Suppose (u,v) is a pair of weights such that*

$$\mu(B)^{\gamma-1/p}\Big(\int_B u\,d\mu\Big)^{1/q}\|v^{-1/p}\|_{\psi,B}\le C\,,\tag{1.23}$$

for every ball $B \subset \mathbb{X}$. Then

$$M^{\gamma}_{\mathbb{X}} : L^p(\mathbb{X}, v) \to L^q(\mathbb{X}, u) .$$

Remark 1.24. The particular case $\mathbb{X} = \mathbb{R}^n$ with the Lebesgue measure was proved by Pérez [11] (see Theorem 2.11 in that work). He studied a slighty more general maximal operator M_{φ} where φ is a positive essentially nondecreasing function such that $\varphi(t)/t \to 0$ whenever $t \to \infty$. Roughly speaking the function φ measures the degree of fractionability of the operator. The hypothesis on the pair of weights there can be recovered by taking $\varphi(t) = t^{\gamma}$.

Theorem 1.22 and an adapted version of Welland's inequality (Lemma 2.3) allow us to get the following result about fractional integral operators $T^{\gamma}_{\mathbb{X}}$ in the context of spaces of homogeneous type (see Section 2 for its definition).

Theorem 1.25. *Let $(\mathbb{X}, d, \mu)$ be a space of homogeneous type, $1 < p < q < \infty$ and $0 < \gamma < 1$. Let ψ be as in Theorem 1.22 and let (u, v) be a pair of weights satisfying that there exists $r > 1$ such that*

$$\mu(B)^{\gamma - 1/p + 1/q} \left(\frac{1}{\mu(B)} \int_B u^r \, d\mu \right)^{1/rq} \|v^{-1/p}\|_{\psi, B} \le C \, , \tag{1.26}$$

for every ball $B \subset \mathbb{X}$. Then we have

$$T^{\gamma}_{\mathbb{X}} : L^p(\mathbb{X}, v) \to L^q(\mathbb{X}, u) .$$

It is important to note that this result also provides an extension in some way of Theorem 2.1 in [11], to the case of spaces of homogeneous type and $p < q$.

Now, we introduce the notion of local operators ([7]). For $0 < \beta < 1$ and $0 \le \gamma < 1$ we say that T^{γ}_{β} is a (β, γ)-local operator if it satisfies

(1) There exists a kernel $K : \Omega \times \Omega \to \mathbb{C}$ such that for any $f \in L^{\infty}_c(\Omega)$

$$T^{\gamma}_{\beta} f(x) = \int_{\Omega} K(x, y) \, f(y) \, d\mu(y) \, , \qquad \text{a.e. } x \notin \operatorname{supp} f \, .$$

Moreover, $T^{\gamma}_{\beta} f(x) = 0$ for every x such that $B(x, \beta \mathrm{d}(x, \Omega^c)) \cap \operatorname{supp} f = \emptyset$.

(2) The kernel satisfies:
   (a) for every $x, y \in \Omega$ such that $\mathrm{d}(x, y) < \mathrm{d}(x, \Omega^c)$

$$|K(x, y)| \le \frac{C}{\mu(B(x, \mathrm{d}(x, y)))^{1-\gamma}} \, ;$$

   (b) for some $\varepsilon > 0$ and whenever $\mathrm{d}(x, x_0) \le 2\mathrm{d}(x, y)$

$$|K(x, y) - K(x_0, y)| + |K(y, x) - K(y, x_0)| \le \frac{C}{\mu(B(x, \mathrm{d}(x, y)))^{1-\gamma}} \left(\frac{\mathrm{d}(x, x_0)}{\mathrm{d}(x, y)} \right)^{\varepsilon} .$$

(3) Only in the case $\gamma = 0$: $T^0_{\beta} : L^2(\Omega) \to L^2(\Omega)$.

In the case $\gamma = 0$ in order to simplify the notation, we will use T_{β} instead of T^0_{β}. For obvious reasons, we will refer to this type of operators as "β-local singular operators", while in the case $0 < \gamma < 1$ we will speak of "β-local fractional operators of order γ".

As an example of the last operators, for an open subset Ω of $\mathbb{R}^n$, we consider

$$I_\beta^\gamma f(x) = \int_\Omega \frac{\chi_{B(x,\beta \mathrm{d}(x,\Omega^c))}(y)}{|x-y|^{n(1-\gamma)}} f(y)\, d\mu(y)\,. \tag{1.27}$$

In this work we consider $L_c^\infty(\Omega)$ as the set of functions essentially bounded with respect to μ and support contained in a finite number of balls B such that $\mathrm{d}(B,\Omega^c)>0$. By using the local version of Whitney's Lemma (see Lemma 3.4) it is easy to see that $L_c^\infty(\Omega)$ is a dense subspace of $L^p(\Omega,v)$, where v is a weight.

The class of β-local singular operators was recently studied by Harboure, Viviani and the second author ([7]) in the context of metric spaces. More precisely, they proved that T_β is bounded on $L^p(\Omega,w^p)$, for $p>1$ and weights w in $A_{p,p}^{\mathrm{loc}}$.

This kind of operator has been well studied by several authors in other contexts (see for instance [4] in the Euclidean case and [1] in the context of space of homogeneous type).

In this work we ask ourselves about the two-weight problem for this kind of operators and their commutators. In order to give answers we will adapt ideas given in [7] and combine them with new techniques, specially in the case of the commutators. An important tool of our work is the β-local maximal sharp operator defined by

$$\mathcal{M}_\beta^\sharp f(x) = \sup_{B:\, x\in B\in\mathcal{F}_\beta} \frac{1}{\mu(B)}\int_B |f-m_B f|\, d\mu \;+\; \sup_{B:\, x\in B\in\mathfrak{C}_\beta} \frac{1}{\mu(B)}\int_B |f|\, d\mu\,, \tag{1.28}$$

where $\mathfrak{C}_\beta = \{B = B(x_B, \beta\, \mathrm{d}(x_B,\Omega^c)) : x_B\in\Omega\}$. In a similar way, we consider a local version of the well known BMO space.

Definition 1.29. *We say that a function f belongs to the space* $\mathrm{BMO}_\beta(\Omega)$*, with $0<\beta<1$, if and only if $\mathcal{M}_\beta^\sharp f\in L^\infty(\Omega)$. We denote $\|f\|_{\mathrm{BMO}_\beta(\Omega)} = \|\mathcal{M}_\beta^\sharp f\|_{L^\infty(\Omega)}$.*

Now, for $0<\beta<1$, $0\le\gamma<1$, $m\in\mathbb{N}$ and $b\in\mathrm{BMO}_\beta(\Omega)$ we define the commutator of order m for a (β,γ)-local operator as follows

$$T_{\beta,b}^{\gamma,m} f(x) = \int_\Omega (b(x)-b(y))^m\; K(x,y)\, f(y)\, d\mu(y)\,,$$

where the kernel K is as before both in the singular case and in the fractional case.

Theorem 1.30. *Let $0<\beta<1$, $0\le\gamma<1$, $m\in\mathbb{N}\cup\{0\}$ and $1<p\le q<\infty$. Let u and v be weights such that $u\in A_\infty^\beta$, $\sigma_r = v^{1-(p/r)'}\in\mathcal{D}(\mathcal{F}_\beta\backslash\mathcal{F}_{\tilde\beta})$, with $\tilde\beta<\beta/4000$ and*

$$\mu(B)^{\gamma-1/p+1/q}\left(\frac{1}{\mu(B)}\int_B u\, d\mu\right)^{1/q} \|v^{-r/p}\|_{\psi,B}^{1/r} \le C\,, \tag{1.31}$$

for all $B\in\mathcal{F}_\beta$, for some $r>1$ such that $\gamma r<1$, where ψ is as in Theorem 1.15. Then, for $b\in\mathrm{BMO}_\beta(\Omega)$ we have

$$T_{\beta,b}^{\gamma,m} : L^p(\Omega,v)\to L^q(\Omega,u)\,.$$

Remark 1.32. Theorem 1.30 hold for the operator $T_{\beta,b}^\gamma$ taking $m=0$. This result is interesting in itself, and it is not necessary to give a special proof since it follows in the same way as for $m\in\mathbb{N}$ as the results involved in the proof are also valid when $m=0$.

The following proposition, interesting in itself, is an important tool for the proof of the previous result for small values of β.

Proposition 1.33. *Let $(\mathbb{X}, d, \mu)$ be a space of homogeneous type and $\Omega \subset \mathbb{X}$ as usual. Let $m \in \mathbb{N} \cup \{0\}$, $0 \leq \gamma < 1$ and $1 < r < 1/\gamma$. Then there exist constants β_0 and $0 < \eta < 1$ such that for each $\beta \in (0, \beta_0/3)$, if T^{γ}_{β} is a (β, γ)-local operator, then the commutator $T^{\gamma,m}_{\beta,b}$ satisfies*

$$\int_{\Omega} |T^{\gamma,m}_{\beta,b} f|^q \, u \, d\mu \leq C \, \|b\|^{mq}_{BMO^{\beta}} \int_{\Omega} |M^{\gamma,r}_{\eta} f|^q \, u \, d\mu \, ,$$

for every $f \in L^{\infty}_c(\Omega)$, where $u \in A^{\beta}_{\infty}$ and $b \in \mathrm{BMO}_{\beta}(\Omega)$.

The structure of this paper is as follows. Section 2 contains some useful technical lemmas and the proof of Theorem 1.22 and Theorem 1.25. Section 3 proves several tools about the covering lemma that we use later. Section 4 shows that local BMO functions belong to certain classic BMO spaces in the setting of spaces of homogeneous type. In addition, Theorem 1.15 is proved. Finally, the proofs of Theorem 1.30 and Proposition 1.33 are in Section 5 with some applications of the main result. Throughout this paper C will denote a positive constant, not the same at each occurrence.

2. Results on spaces of homogeneous type

Let $\mathbb{X}$ be a set endowed with a quasi-distance d with constant A such that, for each $x \in \mathbb{X}$ and $r > 0$, the balls $B(x, r) = \{y \in \mathbb{X} : \; d(y, x) < r\}$ are open sets and a positive measure μ satisfying a doubling property. A set like this is called a *space of homogeneous type* and denoted by $(\mathbb{X}, d, \mu)$.

In this section we consider the fractional maximal operator $M^{\gamma}_{\mathbb{X}}$ as in (1.21). We state the following lemma without proof (it can be found in [2]). First, we introduce some notation. For a ball $B = B(x, r)$ we denote $\tilde{B} = B(x, 5A^2 r)$, where A is the quasi-distance constant. If E is a μ-measurable set and f is a nonnegative integrable function we write

$$m_E f = \mu(E)^{\gamma - 1} \int_E f \, d\mu \, ,$$

for a fixed $\gamma \in [0, 1)$. For each $k \in \mathbb{Z}$, let $\Omega_k = \left\{y \in \mathbb{X} : b^{k+1} \geq M^{\gamma}_{\mathbb{X}} f(y) > b^k\right\}$, where $b > 1$ is fixed and depends on the quasi-distance and doubling property constants. In the case $\mu(\mathbb{X}) < \infty$, we take k_0 such that $b^{k_0+1} \geq m_{\mathbb{X}} f > b^{k_0}$. Note that, in this case, $\Omega_k = \emptyset$, for $k < k_0$.

Lemma 2.1. [2] *For any non negative function f with bounded support, and any $k \in \mathbb{Z}$ such that $\Omega_k \neq \emptyset$, there exist a sequence $\{B^k_i\}_{i \in \mathbb{N}}$ of balls satisfying:*

(2.1.1) $\quad \Omega_k \subset \bigcup_{i=1}^{\infty} \tilde{B}^k_i \, ,$

(2.1.2) $\quad B^k_i \cap B^k_j = \emptyset$ *whenever $i \neq j$.*

(2.1.3) *If $\mu(\mathbb{X}) = \infty$, then for every B^k_i, there exists $x^k_i \in B^k_i$ such that if r^k_i is the radius of B^k_i, $r \geq 5A^2 \, r^k_i$ and $x^k_i \in B = B(y, r)$, then $b^{k+1} \geq M^{\gamma}_{\mathbb{X}} f(x^k_i) \geq m_{B^k_i} f > b^k \geq m_B f$.*

(2.1.4) *If $\mu(\mathbb{X}) < \infty$, then (2.1.3) still holds for $k > k_0$, but if $k = k_0$ we only have one ball $B_1^{k_0}$ such that $\Omega_k \subset B_1^{k_0} = \mathbb{X}$ and $b^{k_0+1} \geq M_{\mathbb{X}}^{\gamma} f(x_i^{k_0}) \geq m_{B_i^{k_0}} f > b^{k_0}$, for some $x_i^{k_0} \in B_i^{k_0}$.*

(2.1.5) *If $x \notin \cup_{j=k}^{\infty} \cup_{i=1}^{\infty} \tilde{B}_i^j$ and $M_{\mathbb{X}}^{\gamma} f(x) < \infty$, then $M_{\mathbb{X}}^{\gamma} f(x) \leq b^k$.*

(2.1.6) *Let $I_j^k = \{(l,n) \in \mathbb{Z} \times \mathbb{N} \, : \, l \geq k+2, \, \tilde{B}_n^l \cap \tilde{B}_j^k \neq \emptyset\}$ and let $A_j^k = \bigcup_{(l,n) \in I_j^k} \tilde{B}_n^l$. Then $2\mu(A_j^k) \leq \mu(B_j^k)$.*

(2.1.7) *Let $E_j^k = \tilde{B}_j^k \setminus A_j^k$. Then $2\mu(E_j^k) \geq \mu(\tilde{B}_j^k)$ and $\mu(\mathbb{X} \setminus \cup_{k,j} E_j^k) = 0$. If $x \in E_j^k$ and $M_{\mathbb{X}}^{\gamma} f(x) < \infty$, then $M_{\mathbb{X}}^{\gamma} f(x) \leq b^{k+2}$.*

(2.1.8) *Let $F_j^k = B_j^k \setminus A_j^k$. Then $\mu(F_j^k) \geq C \, \mu(\tilde{B}_j^k)$ and*

$$\sum_{k=-\infty}^{\infty} \sum_{j=l}^{\infty} \chi_{F_j^k}(x) \leq 3 \,,$$

where χ_E denotes the characteristic function of the set E.

Now, we are in a position to proceed with the proof of Theorem 1.22.

Proof of the Theorem 1.22. Let $f \geq 0$, $f \in L^p(\mathbb{X}, v)$ with bounded support. By Lemma 2.1 there exist a sequence of balls $\{B_i^k\}$ and a positive constant C such that

$$\begin{aligned}
&\|M_{\mathbb{X}}^{\gamma} f\|_{L_u^q}^q \\
&\quad = \int_{\mathbb{X}} (M_{\mathbb{X}}^{\gamma} f)^q \, u \, d\mu \\
&\quad \leq C \sum_{i,k} (m_{B_i^k} f)^q \int_{\tilde{B}_i^k} u \, d\mu \\
&\quad = C \sum_{i,k} \mu(B_i^k)^{\gamma q} \int_{\tilde{B}_i^k} u \, d\mu \left(\frac{1}{\mu(B_i^k)} \int_{B_i^k} f \, d\mu \right)^q \\
&\quad \leq C \sum_{i,k} \left(\mu(\tilde{B}_i^k)^{\gamma - 1/p} \Big(\int_{\tilde{B}_i^k} u \, d\mu \Big)^{1/q} \|v^{-1/p}\|_{\psi, \tilde{B}_i^k} \right)^q \mu(\tilde{B}_i^k)^{q/p} \, \|f \, v^{1/p}\|_{\tilde{\psi}, B_i^k}^q \\
&\quad \leq C \sum_{i,k} \mu(F_i^k)^{q/p} \, \|f \, v^{1/p}\|_{\tilde{\psi}, B_i^k}^q \,,
\end{aligned}$$

where in the last step we use (1.23) and the condition *(2.1.8)*. Thus, by the definition of $M_{\tilde{\psi}}$ and the fact that $\{F_i^k\}$ have bounded overlapping we can conclude that, since $p \leq q$

$$\begin{aligned}
\|M_{\mathbb{X}}^{\gamma} f\|_{L_u^q}^q &\leq C \left(\sum_{i,k} \int_{\mathbb{X}} (M_{\tilde{\psi}}(f \, v^{1/p}))^p \, \chi_{F_i^k} \, d\mu \right)^{q/p} \\
&\leq C \left(\int_{\mathbb{X}} (M_{\tilde{\psi}}(f \, v^{1/p}))^p \, d\mu \right)^{q/p} \\
&\leq C \left(\int_{\mathbb{X}} f^p v \, d\mu \right)^{q/p} ,
\end{aligned}$$

where the boundedness of $M_{\tilde{\psi}}$ on $L^p(\mathbb{X})$ follows from the fact that $\tilde{\psi} \in B_p$ (see Theorem 1.4 in [13] and Remark 1.17 here). □

In addition, we will consider integral fractional operators. More precisely, we will say that $T^{\gamma}_{\mathbb{X}}$, with $0 < \gamma < 1$, is a fractional integral operator of order γ if it satisfies:

(1) There exist a kernel $K : \mathbb{X} \times \mathbb{X} \mapsto \mathbb{C}$ such that for any $f \in L^{\infty}_c(\mathbb{X})$

$$T^{\gamma}_{\mathbb{X}} f(x) = \int_{\mathbb{X}} K(x,y)\, f(y)\, d\mu(y)\,, \qquad \text{a.e. } x \notin \operatorname{supp} f\,.$$

(2) The kernel satisfies:
  (a) for every $x, y \in \mathbb{X}$

$$|K(x,y)| \leq \frac{C}{\mu(B(x, \mathrm{d}(x,y)))^{1-\gamma}}\,;$$

  (b) for some $\varepsilon > 0$

$$|K(x,y) - K(x_0,y)| + |K(y,x) - K(y,x_0)| \leq \frac{C}{\mu(B(x, \mathrm{d}(x,y)))^{1-\gamma}} \left(\frac{\mathrm{d}(x,x_0)}{\mathrm{d}(x,y)} \right)^{\varepsilon}.$$

Remark 2.2. It is well known the $L^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)$ boundedness for this class of operators, where $1/p = 1/q + \gamma$. See for instance [1], [2] and references here.

The following lemma gives a version of Welland's inequality (see [17]) in the context of spaces of homogeneous type. This kind of result has already appeared in the literature, with slightly stronger hypotheses (see Proposition 5.1 in [3]). Then we can obtain the proof of Theorem 1.25 proceeding in the same way as for Theorem 6.5 in [5].

Lemma 2.3. *Let $(\mathbb{X}, d, \mu)$ be a space of homogeneous type. Let $0 < \gamma < 1$ and $0 < \varepsilon < \min\{\gamma, 1-\gamma\}$. Then, for every bounded function f with compact support the following inequality holds*

$$|T^{\gamma}_{\mathbb{X}} f(x)| \leq C \left(M^{\gamma-\varepsilon}_{\mathbb{X}} f(x)\, M^{\gamma+\varepsilon}_{\mathbb{X}} f(x) \right)^{1/2},$$

for every $x \in \mathbb{X}$, where $C > 0$ only depends on n, γ and ε.

Proof. In order to prove the lemma, taking $s > 0$ to be chosen later we split the operator as follows

$$\begin{aligned} |T^{\gamma}_{\mathbb{X}} f(x)| &\leq \int_{\mathbb{X}} \frac{|f(y)|}{\mu(B(x,d(x,y)))^{1-\gamma}}\, d\mu(y) \\ &\leq \int\limits_{\{y \in \mathbb{X} :\, \mu(B(x,d(x,y))) < s\}} \frac{|f(y)|}{\mu(B(x,d(x,y)))^{1-\gamma}}\, d\mu(y) \\ &\quad + \int\limits_{\{y \in \mathbb{X} :\, \mu(B(x,d(x,y))) \geq s\}} \frac{|f(y)|}{\mu(B(x,d(x,y)))^{1-\gamma}}\, d\mu(y) \\ &= I + II. \end{aligned}$$

Now, we consider the non-necessarily symmetric quasi-distance δ associated to $(\mathbb{X}, \mathrm{d}, \mu)$ and equivalent to d, introduced by Macías and Segovia in [8], given by

$$\delta(x,y) = \begin{cases} \mu(B(x,\mathrm{d}(x,y)))\,, & x \neq y\,; \\ 0\,, & x = y\,. \end{cases}$$

It satisfies that

$$\delta \leq \mathrm{d} \leq 3\delta\,. \tag{2.4}$$

Now, denoting by B_δ the balls with respect to the metric δ, we have that $B(x,r) \subset B_\delta(x,r) \subset B(x,3r)$, for every $x \in \mathbb{X}$ and $r > 0$.

For I, we assume first that $\mu(\{x\}) = 0$. Therefore, defining $R_k = \{y \in \mathbb{X} : 2^{-k-1}s \leq \delta(x,y) < 2^{-k}s\}$ and using the inclusion $B_\delta(x, 2^{-k}s) \subset B(x, 2^{-k+2}s)$, $\mu(B(x,\delta(x,y))) \leq \delta(x,y)$ and the fact that μ is doubling, we can get

$$\begin{aligned} I &\leq \sum_{k=0}^{\infty} \int_{R_k} \frac{|f(y)|}{\delta(x,y)^{1-\gamma}}\, d\mu(y) \\ &\leq C\ \sum_{k=0}^{\infty} (2^{-k}s)^{\varepsilon} \frac{1}{\mu(B(x,2^{-k-1}s))^{1-\gamma+\varepsilon}} \int_{B(x,2^{-k+2}s)} |f(y)|\, d\mu(y) \\ &\leq C\ \sum_{k=0}^{\infty} (2^{-k}s)^{\varepsilon}\, M_{\mathbb{X}}^{\gamma-\varepsilon} f(x) \\ &\leq C\ s^{\varepsilon}\, M_{\mathbb{X}}^{\gamma-\varepsilon} f(x)\,. \end{aligned}$$

Now, if $\mu(\{x\}) \neq 0$ we have that $2^{-k_1-1}s < \mu(\{x\}) \leq 2^{-k_1}s$, for some k_1 and proceeding as above

$$\begin{aligned} I &\leq \int_{\{x\}} \frac{|f(y)|}{\mu(\{x\})^{1-\gamma}}\, d\mu(y) + \sum_{k=k_1+1}^{\infty} \int_{R_k} \frac{|f(y)|}{\delta(x,y)^{1-\gamma}}\, d\mu(y) \\ &\leq C\ (2^{-k_1}s)^{\varepsilon}\, M_{\mathbb{X}}^{\gamma-\varepsilon} f(x) + C \sum_{k=k_1+1}^{\infty} (2^{-k}s)^{\varepsilon}\, M_{\mathbb{X}}^{\gamma-\varepsilon} f(x) \\ &\leq C\, s^{\varepsilon}\, M_{\mathbb{X}}^{\gamma-\varepsilon} f(x)\,. \end{aligned}$$

To estimate term II, let $N_k = \{y \in \mathbb{X} :\ 2^k s \leq \delta(x,y) < 2^{k+1}s\}$. Assuming that $\mu(\mathbb{X}) = \infty$ and by the doubling property of μ, we have

$$\begin{aligned} II &= \int_{\{y\in\mathbb{X}:\, \delta(x,y)\geq s\}} \frac{|f(y)|}{\delta(x,y)^{1-\gamma}}\, d\mu(y) \\ &= \sum_{k=0}^{\infty} \int_{N_k} \delta(x,y)^{-\varepsilon} \frac{|f(y)|}{\delta(x,y)^{1-\gamma-\varepsilon}}\, d\mu(y) \\ &\leq \sum_{k=0}^{\infty} (2^k s)^{-\varepsilon} \frac{1}{\mu(B(x,2^k s))^{1-\gamma-\varepsilon}} \int_{B_\delta(x,2^{k+1}s)} |f(y)|\, d\mu(y) \\ &\leq C\ \sum_{k=0}^{\infty} (2^k s)^{-\varepsilon} \frac{1}{\mu(B(x,2^{k+3}s))^{1-\gamma-\varepsilon}} \int_{B(x,2^{k+3}s)} |f(y)|\, d\mu(y) \\ &\leq C\, s^{-\varepsilon}\, M_{\mathbb{X}}^{\gamma+\varepsilon} f(x)\,. \end{aligned}$$

On the other hand, if $\mu(\mathbb{X})<\infty$, we take k_1 such that $2^{k_1}s<\mu(\mathbb{X})\leq 2^{k_1+1}s$. Then, for each x we have that $B(x,2^{k_1+1}s)=\mathbb{X}$ and $C\,2^{k+1}s\leq \mu(B_\delta(x,2^{k+1}s))$ (see [9]). Therefore, in a similar way as before we obtain

$$\begin{aligned} II &\leq C\,\mu(\mathbb{X})^{-\varepsilon}\, M_{\mathbb{X}}^{\gamma+\varepsilon} f(x) + \sum_{k=0}^{k_1-1} (2^k s)^{-\varepsilon}\, M_{\mathbb{X}}^{\gamma+\varepsilon} f(x) \\ &\leq C\,2^{-\varepsilon k_1} s^{-\varepsilon}\, M_{\mathbb{X}}^{\gamma+\varepsilon} f(x) + s^{-\varepsilon}\, M_{\mathbb{X}}^{\gamma+\varepsilon} f(x) \\ &\leq C\, s^{-\varepsilon}\, M_{\mathbb{X}}^{\gamma+\varepsilon} f(x)\,. \end{aligned}$$

In both cases we have the same estimate. Finally

$$T_{\mathbb{X}}^{\gamma} f(x) \leq C\, s^{\varepsilon}\, M_{\mathbb{X}}^{\gamma-\varepsilon} f(x) + s^{-\varepsilon}\, M_{\mathbb{X}}^{\gamma+\varepsilon} f(x)\,,$$

and taking $s^{\varepsilon}=(M_{\mathbb{X}}^{\gamma+\varepsilon}f(x))^{1/2}(M_{\mathbb{X}}^{\gamma-\varepsilon}f(x))^{-1/2}$ we get the desired result. □

We end this section with the following lemma, which is a version of the Fefferman-Stein inequality on spaces of homogeneous type. It was proved by Pradolini and the second author in [12].

Lemma 2.5. [12] *Let $(\mathbb{X},d,\mu)$ be a space of homogeneous type such that $\mu(\mathbb{X})<\infty$. Let f be a positive function with bounded support and $w\in A_\infty$. Then, for every p, $1<p<\infty$, there exists a positive constant C such that if $\|M_{\mathbb{X}}f\|_{L^p(\mathbb{X},w)}<\infty$, then*

$$\|M_{\mathbb{X}} f\|_{L^p(\mathbb{X},w)} \leq C \|M_{\mathbb{X}}^{\sharp} f\|_{L^p(\mathbb{X},w)}\,, \tag{2.6}$$

where

$$M_{\mathbb{X}}^{\sharp} f(x) \;=\; \sup_{B:\, x\in B\in F(\mathbb{X})} \frac{1}{\mu(B_{\mathbb{X}})} \int_{B_{\mathbb{X}}} |f(y)-m_{B_{\mathbb{X}}} f|\, d\mu(y) + \frac{1}{\mu(\mathbb{X})}\int_{\mathbb{X}} |f(y)|\, d\mu(y)\,,$$

where $B_{\mathbb{X}}$ and $F(\mathbb{X})$ are as in (1.21).

In this context we say that $f\in \mathrm{BMO}(\mathbb{X})$ if $M_{\mathbb{X}}^{\sharp}f\in L^\infty(\mathbb{X})$, and we denote $\|f\|_{\mathrm{BMO}(\mathbb{X})}=\|M_{\mathbb{X}}^{\sharp}f\|_{L^\infty(\mathbb{X})}$.

3. Covering lemmas

In this section we present some known results that are useful in the context of local operators. We state them below without proof.

Before we need the notion of "cloud" of a given set E in Ω. That is, given $0<\beta<1$ and $E\in\Omega$, the set

$$\mathcal{N}_\beta(E) = \bigcup_{\substack{R\cap E\neq\emptyset \\ R\in\mathcal{F}_\beta}} R\,, \tag{3.1}$$

is the "cloud" of E. In particular, for balls, this idea was introduced in [6] along with the following generalized notion of dilated ball. We denote

$$\widetilde{B}=5B\,,\quad \text{if}\quad 5B\in\mathcal{F}_\beta \qquad\text{or}\qquad \widetilde{B}=\mathcal{N}_\beta(B)\,,\quad\text{if}\quad 5B\notin\mathcal{F}_\beta\,. \tag{3.2}$$

In that article, the authors proved several lemmas. For the sake of completeness, we state the following ones and invite the reader to look for their proofs there.

Lemma 3.3 ([6], local Vitali)**.** *Let X be a separable metric space and Ω an open proper subset of X. Let $0<\beta<1$ and Γ a family of balls belonging to $\mathcal{F}_\beta$ with uniformly bounded radii. Then, there exists a disjoint and at most countable subfamily Λ such that the collection of open sets $\{\tilde{B}\}_{B\in\Lambda}$, with $\tilde{B}$ defined as (3.2), still covers $\bigcup_{B\in\Gamma} B$.*

Lemma 3.4 ([6], Whitney for $\mathcal{F}_\beta$)**.** *Let Ω be an open proper subset of X. Given $0<\beta<1$, for each a, $0<a<\beta/80$, there exists a covering $\mathcal{W}_a$ of Ω by balls belonging to $\mathcal{F}_\beta$ with the following properties*

i) If $P=B(x_P,r_P)\in\mathcal{W}_a$, then $10\,P\in\mathcal{F}_\beta$ and

$$\frac{1}{2}a\;d(x_P,\Omega^c)\le r_P\le a\;d(x_P,\Omega^c)\,.$$

ii) If P and P' belong to $\mathcal{W}_a$ and $P\cap P'\neq\emptyset$ then $P'\subset 5P$ and $P\subset 5P'$.

iii) There is a number M, only depending on β and a, such that for any ball $B_0=B(x_0,r_0)\in\mathcal{F}_\beta$ with $5B_0\notin\mathcal{F}_\beta$, the cardinality of the set

$$\mathcal{W}_a(B_0)=\{P\in\mathcal{W}_a\;:\;P\cap\mathcal{N}_\beta(B_0)\neq\emptyset\}\;,$$

is at most M. We will denote the union of this sets as

$$\mathcal{W}_{a,B_0}=\bigcup_{P\in\mathcal{W}_a(B_0)}P\,.$$

Even though we said before that the proof of the latter lemma can be found in [6], it will be useful to recall here how to build the desired balls in it. For this, we consider the bands $\Omega_k=\{x\in\Omega\;:\;2^{k-1}\le \mathrm{d}(x,\Omega^c)<2^k\}$. Since X is separable, we choose, for each k, a maximal net of points in Ω_k, namely $\{x_i^k\}_{i\in J_k}$ with $J_k\subset\mathbb{N}$, whose distances from each other are at least $a2^{k-1}$. Thus,

$$\mathcal{W}_a=\bigcup_{i\in J_k;k\in\mathbb{N}}B(x_i^k,a2^{k-1})\,.$$

In general, we will denote these balls by $P_i=B(x_i,r_i)$. A careful examination of the proof of the Lemma 3.1 in [6] reveals that the measure does not need to be doubling for all ball sizes. So we can state it in the following way this result.

Lemma 3.5. [6] *Let X be a metric space with the weak homogeneity property and Ω as before. In the same hypothesis on the Lemma 3.4, if μ is doubling on $\mathcal{F}_\beta\backslash\mathcal{F}_{\tilde{\beta}}$, with $\tilde{\beta}<a/2$, then, for any ball B_0 such that $5B_0\notin\mathcal{F}_\beta$ we get*

(1) If P_j and P_i belongs to $\mathcal{W}_a(B_0)$, we have

$$\mu(P_j)\le C\;\mu(P_i)\,,$$

for some positive constants C.

(2) Moreover

$$\mu(\mathcal{N}_\beta(B_0))\le C\;\mu(B_0)\,.$$

where $C>0$ only depend on β, a and the constant of the doubling property of μ.

Remark 3.6. By an analogous reasoning and the same hypotheses of the previous lemma we can prove that $\mu(\mathcal{N}_\beta(\mathcal{N}_\beta(B_0))) \leq C\ \mu(B_0)$, for every ball B_0 such that $5B_0 \notin \mathcal{F}_\beta$.

Remark 3.7. For the constant a fixed in Lemma 3.4 we consider $k \in \mathbb{N}$ such that $\frac{1}{2}\,\frac{\beta}{80k} < a < \frac{\beta}{80k}$ and we denote $\hat{P}_i = 80k\, P_i$ for each $P_i \in \mathcal{W}_a$. Then

$$\frac{1}{2}\frac{1}{2}\frac{\beta}{80k}\mathrm{d}(x_i,\Omega^c) < \frac{1}{2}a\mathrm{d}(x_i,\Omega^c) \leq r_i \leq a\mathrm{d}(x_i,\Omega^c) < \frac{\beta}{80k}\mathrm{d}(x_i,\Omega^c)\,,$$

which implies that $\hat{P}_i \in \mathcal{F}_\beta$ but $5\hat{P}_i \notin \mathcal{F}_\beta$. Consequently, using Lemma 3.5, for every μ doubling on $\mathcal{F}_\beta$

$$\mu(\mathcal{N}_\beta(P_i)) \leq \mu(\mathcal{N}_\beta(\hat{P}_i)) \leq C\ \mu(\hat{P}_i) \leq C\ \mu(P_i)\,,$$

where C does not depend on i.

The following lemma says that a certain element x can not belong to too many clouds of balls of the covering given by Lemma 3.4.

Lemma 3.8. *Let $0<\beta<1$, for a collection of Whitney type balls $\{P_i\}$, there is a natural number M such that $\sum_i \chi_{\mathcal{N}_\beta(P_i)}(x) \leq M$, for all $x \in \Omega$.*

Proof. For $k \in \mathbb{Z}$ we consider the bands $\Omega_k = \left\{y \in \Omega\,:\, 2^{k-1} \leq \mathrm{d}(y,\Omega^c) < 2^k\right\}$. Suppose that $x \in \Omega_k$ and define $\mathcal{N} = \bigcup_{i\colon x\in\mathcal{N}_\beta(P_i)} \mathcal{N}_\beta(P_i)$. If we show that $\mathcal{N}$ is a bounded set in X, since the center of P_i lies in a maximal net of points in each Ω_j whose distances from each other are at least $a2^{j-1}$ the conclusion follows by applying the weak homogeneity property.

In order to prove our statement we will prove two facts. The first is that there exists a constant C such that $\mathrm{d}(y,z) \leq C$, for $y,z \in \mathcal{N}$. The second is the inclusion

$$\mathcal{N} \subset \bigcup_{j=j_0}^{l} \Omega_j\,. \tag{3.9}$$

for certain constants j_0 y l independent of i.

Let us start considering $y,z \in \mathcal{N}$, and balls R, B, R' and B' belonging $\mathcal{F}_\beta$ with the following properties: $y \in R$, $z \in R'$, $R\cap P_i \neq \emptyset$, $B \cap P_i \neq \emptyset$, $R'\cap P_j \neq \emptyset$, $B'\cap P_j \neq \emptyset$ and $x \in B\cap B'$ So, defining $A = \{x_i, x_j, x_R, x_{R'}, x_B, x_{B'}\}$ the set of centers, by (3.9) we can deduce that

$$\mathrm{d}(y,z) \leq 2\beta \max_{z\in A} \mathrm{d}(z,\Omega^c) \leq 2\beta\, 2^{k_{\max}} = C\,,$$

where $k_{\max}$ is the largest of the subindexes corresponding to the bands containing the centers in A. In order to prove the inclusion above we use some claims whose proofs can be found in [6] whit same names.

Claim 1. Given B and B' in $\mathcal{F}_\beta$ such that $B\cap B' \neq \emptyset$, if $x_B \in \Omega_k$ then $x_{B'} \in \Omega_i$ with $k-m \leq i \leq k+m$ for some m only depending on β.

Claim 2. If $B \in \mathcal{F}_\beta$ and $x_B \in \Omega_k$ then $B \subset \cup_{j=k-n}^{k+1}\Omega_j$ for some fixed n depending only on β.

Claim 3. If $P_i \cap \Omega_j \neq \emptyset$, then $x_{P_i} \in \Omega_{j-1}\cup\Omega_j\cup\Omega_{j+1}$.

Now, since x belongs to $\Omega_k \cap B$, by Claim 1 we have that $x_B \in \cup_{j=k-m}^{k+m} \Omega_j$. Then, from Claim 2, $B \subset \cup_{j=k-m-n}^{k+m+1} \Omega_j$. Now, since $B \cap P_i \neq \emptyset$, we apply Claim 3 and conclude that x_{P_i} lies in some Ω_j, $k-m-n-1 \leq j \leq k+m+2$. Finally, for an arbitrary ball R such that $R \cap P_i \neq \emptyset$ we can deduce in analogous way that $R \in \cup_{j=k-2m-2n-2}^{k+2m+3} \Omega_j$ and the proof of (3.9) is complete taking $j_0 = k-2m-2n-2$ and $l = k+2m+3$. □

Remark 3.10. A similar reasoning allows us to prove that $\{\mathcal{N}_\beta(\mathcal{N}_\beta(P_i))\}$ have bounded overlap as well.

4. Estimates for local operators

In this section we prove some results concerning local operators. To do this, as in [7], we consider metric spaces having a particular geometric property. More precisely, we say that X with a metric d have the $\mathcal{P}$ property if there exists $\sigma > 0$ such that $\forall x_0 \in X$, $R > 0$, $y \in B(x_0, R)$ and $0 < r \leq 2R$, there exists $z \in X$ satisfying

$$B(z, \sigma r) \subset B(x_0, R) \cap B(y, r)\,.$$

In general, a metric space (X, d) does not necessarily have this property. However, we can take an equivalent metric δ (introduced by Macias and Segovia (see Section 2)) in such a way that (X, δ) has the $\mathcal{P}$ property.

In [7] the authors proved that given Ω an open proper subset of X, then a measure μ is doubling on $\mathcal{F}_\beta$ with respect to d if and only if it is doubling on $\mathcal{F}_\beta$ with respect to the metric δ. Consequently, it is not difficult to see that $A_{p,q}^{\beta,\gamma}(\mathrm{d})$ is equivalent to $A_{p,q}^{\beta,\gamma}(\delta)$ for each $0 < \beta < 1$ and $0 \leq \gamma < 1$, where $A_{p,q}^{\beta,\gamma}(\mathrm{d})$ denote the class of weights $A_{p,q}^{\beta,\gamma}$ respect to d-balls.

We will denote by $\mathcal{F}_\beta(\delta)$ the local family of balls where the distance of x_B to the complementary set of Ω is measured by the metric δ. Thus, if $\mathcal{M}_\beta^\gamma$ is the local fractional maximal function associated to $\mathcal{F}_\beta(\delta)$, this new function is equivalent to M_β^γ whenever $0 < \beta < 1/3$, that is, there exist positive constants c_1 and c_2 such that

$$c_1\, M_\beta^\gamma f(x) \leq \mathcal{M}_\beta^\gamma f(x) \leq c_2 M_\beta^\gamma f(x) \tag{4.1}$$

for every $x \in \Omega$.

The important role of the $\mathcal{P}$ property can be appreciated in the next lemma.

Lemma 4.2. *Let (X, d) be a metric space satisfying the $\mathcal{P}$ property. Let us consider Ω an open proper subset of X and μ a doubling measure on $\mathcal{F}$. Let $0 < \beta < 1/14$ and let $\{P_i\}$ be a covering of Whitney's type as in Lemma 3.4, then there exist $0 < \tau < 1/7$ and a new covering $\{Q_i\}$ with the following properties:*

1. *$Q_i \in \mathcal{F}_\tau$ and $\displaystyle\bigcup_{x \in P_i} B(x, \beta d(x, \Omega^c)) \subset Q_i$, for every i.*
2. *The space $(Q_i, d|_{Q_i}, \mu|_{Q_i})$ is of homogeneous type, with a uniform doubling constant on i.*

To get the new covering it is sufficient to take $Q_i = \theta P_i$, with $\theta = \frac{2\beta}{a} + \beta + 1$, where $\beta < 1/14$ and a small enough. We can check that $Q_i \in \mathcal{F}_\tau$, for some $0 < \tau < 1/7$. The details can be found inside the proof of Theorem 2.2 in [7].

Lemma 4.3. *Let $\beta \geq 1/42$. We consider the covering $\{P_i'\}$ provided by Lemma 3.4 and its associated $\{Q_i'\}$ as in Lemma 4.2 for $\beta' = \beta/50$. Then, every ball $B = B(x_B, r_B)$ with $x_B \in Q_i'$, $r_B \leq 2R_i$ belongs to $\mathcal{F}_\beta$, where R_i is the radius of Q_i'. In particular, $Q_i' \in \mathcal{F}_\beta$.*

Proof. Let $B = B(x_B, r_B)$ where $x_B \in Q_i'$ and $r_B \leq 2R_i$. Taking into account that $R_i = \theta\, r_i = \theta\, a\, \mathrm{d}(x_i, \Omega^c)$, since $a < \beta'/80$ and $\beta \geq 1/42$ we estimate

$$\begin{aligned}\theta a &= 2\beta' + a(\beta'+1) = \frac{1}{50}\Big(2\beta + a(\beta+50)\Big)\\ &< \frac{1}{50}\Big(2\beta + \frac{51}{80\cdot 50}\Big) < \frac{1}{50}\Big(2\beta + 3\beta\Big) = \frac{1}{10}\beta\,.\end{aligned}$$

Thus, if $r \leq 2R_i$ we proceed as follows

$$R_i < \frac{1}{10}\beta\big(\mathrm{d}(x_B,\Omega^c) + \mathrm{d}(x_B, x_i)\big) < \frac{1}{10}\beta \mathrm{d}(x_B,\Omega^c) + \frac{1}{10}R_i\,;$$

this implies that

$$r_B \leq 2R_i < \beta \mathrm{d}(x_B, \Omega^c)\,,$$

and the lemma is proved. □

Lemma 4.4. *Let $\beta < 1/4$. Then* $\mathrm{BMO}_\beta(\Omega) = \mathrm{BMO}_{4\beta}(\Omega)$*.*

Proof. Clearly $\mathrm{BMO}_{4\beta}(\Omega) \subset \mathrm{BMO}_\beta(\Omega)$, since μ is doubling on $\mathcal{F}$ and this family is increasing on β. In order to prove the converse inclusion, let $\hat\beta = 4\beta$, $f \in \mathrm{BMO}_\beta(\Omega)$ and $B = B(x_B, r_B) \in \mathcal{F}_{\hat\beta}$. If $B \in \mathcal{F}_\beta$ or $B \in \mathfrak{C}_\beta$ (see Definition 1.29 and (1.28)) there is nothing to prove.

Now, suppose that $B \in \mathcal{F}_{\hat\beta} \backslash \mathcal{F}_\beta$. We consider $\mathcal{W}_a$ a covering for Ω given by Lemma 3.4 with $a < \hat\beta/80$. Since $5r_B > \hat\beta \mathrm{d}(x_B, \Omega^c)$, from item *iii)* in that lemma we get that there exists a finite number independent of the ball B, say N, of balls $P_i \in \mathcal{W}_a$ such that $P_i \cap B \neq \emptyset$ and $B \subset \cup_{i=1}^N P_i$. In particular, let $P_{i_0} = B(x_{i_0}, r_{i_0})$ be the one containing x_B. Since $P_{i_0} \subset B(x_B, 2r_{i_0})$ and the fact that $B \notin \mathcal{F}_\beta$, we can deduce $2r_{i_0} < r_B$ and $\mu(P_{i_0}) \leq \mu(B(x_B, r_B))$. Thus

$$\begin{aligned}\frac{1}{\mu(B)}\int_B |f - m_B f|\, d\mu &\leq \frac{2}{\mu(B)}\int_B |f - m_{P_{i_0}} f|\, d\mu\\ &\leq 2\sum_{j=1}^N \frac{1}{\mu(P_{i_0})}\int_{P_j} |f - m_{P_{i_0}} f|\, d\mu\\ &\leq 2\sum_{j=1}^N \frac{1}{\mu(P_{i_0})}\int_{P_j} |f - m_{P_{i_0}} f|\, d\mu \qquad (4.5)\\ &\leq C\, \|f\|_{\mathrm{BMO}_\beta(\Omega)}\,, \qquad (4.6)\end{aligned}$$

where the Lemma 3.5 is used in (4.5) and the last inequality can be obtained by considering a finite chain of balls in $\mathcal{W}_a$, say $P = P_1, P_2, \cdots, P_n = P'$, with $n \leq N$. By using the doubling condition of μ and *ii)* of Lemma 3.4 we get

$$\begin{aligned}&\frac{1}{\mu(P)}\int_P |f - m_{P'} f|\, d\mu\\ &\qquad \leq C\,\frac{1}{\mu(P)}\int_P |f - m_P f|\, d\mu + |m_{P_1} f - m_{P_2} f| + \cdots + |m_{P_{n-1}} f - m_{P_n} f|\end{aligned}$$

$$\leq C\ \|f\|_{\mathrm{BMO}_\beta(\Omega)} + C \sum_{j=2}^{n-1} \frac{1}{\mu(5P_j)} \int_{5P_j} |f - m_{5P_j} f|\, d\mu$$
$$\leq C\ \|f\|_{\mathrm{BMO}_\beta(\Omega)}\ .$$

Now, combining this estimate with (4.6) we complete the proof of lemma. So, the proof is done. □

Lemma 4.7. *Let (X, d) be a metric space satisfying the $\mathcal{P}$ property. Let Ω be an open proper subset of X and μ a doubling measure on $\mathcal{F}$. Let β and τ be such that $0 < \beta < \tau < 1/3$ and let $Q \in \mathcal{F}_\tau$. If $f \in \mathrm{BMO}_\beta(\Omega)$ then $f \in \mathrm{BMO}(Q)$. Moreover, the following inequality holds*

$$\|f\|_{\mathrm{BMO}(Q)} \leq C\ \|f\|_{\mathrm{BMO}_\beta(\Omega)}\ ,$$

where the constant do not dependent on Q.

Proof. Let $B = B(x_B, r_B)$ with $x_B \in Q$ and $r_B > 0$. Let R be the radius of Q. If $r_B < 2R$, since $Q \in \mathcal{F}_\tau$ we get

$$R < \tau\, \mathrm{d}(x_Q, \Omega^c) \leq \tau\, \mathrm{d}(x_B, x_Q) + \tau\, \mathrm{d}(x_B, \Omega^c) \leq \tau\, R + \tau\, \mathrm{d}(x_B, \Omega^c)\, .$$

Thus,

$$R < \frac{\tau}{1-\tau}\, \mathrm{d}(x_B, \Omega^c)\, ,$$

and $B \in \mathcal{F}_{2\tau/(1-\tau)}$. Now, we remember that $B_Q = B \cap Q$ and since the $\mathcal{P}$ property implies $\mu(B) \leq C\, \mu(B_Q)$, by taking the smallest natural number j_0 such that $2\tau/(1-\tau) \leq 4^{j_0}\beta$ we can estimate as follows:

$$\frac{1}{\mu(B_Q)} \int_{B_Q} |f - m_{B_Q} f|\, d\mu \leq \frac{C}{\mu(B)} \int_B |f - m_B f|\, d\mu$$
$$\leq C\ \|f\|_{\mathrm{BMO}_{4^{j_0+1}\beta}(\Omega)} \leq C\ \|f\|_{\mathrm{BMO}_\beta(\Omega)}\ ,$$

where we used the Lemma 4.4. On the other hand, in the case $r_B \geq 2R$, we have that $B_Q = Q$ and the conclusion is obvious. □

Before starting with the proof of one of the most important results of this work, it is necessary to note that the B_p condition is sufficient to prove that (1.17) implies (1.11), that is $(u, v) \in A^{\beta,\gamma}_{p,q}$. It is easy to see that ψ doubling and $\tilde{\psi}$ belonging to B_p imply that $\tilde{\psi}(t^{1/p})$ is a concave function (or equivalent to one concave function). Then its inverse $\left(\tilde{\psi}^{-1}(t)\right)^p$ is convex and by an argument of duality we can see that $L^\psi(B, \frac{d\mu}{\mu(B)}) \subset L^{p'}(B, \frac{d\mu}{\mu(B)})$ with $\|g\|_{p',B} \leq C\ \|g\|_{\psi,B}$. Then condition (1.11) follows from (1.17) by taking $g = v^{-1/p}$ in the last inequality.

Proof of Theorem 1.15. Let β_0 be such that $1/100 < \beta_0 < 1/42$. For the case $\beta \leq \beta_0$, we consider a Whitney's type covering $\{P_i\}$ given by Lemma 3.4 applied in (X, δ) for a such that $\beta/160 < a < \beta/80$. Now, let $\{Q_i\}$ be the covering provided by Lemma 4.2. We have that $\Omega = \cup_i P_i = \cup_i Q_i$. Then, by (4.1), we can estimate as follows

$$\int_\Omega (M^\gamma_\beta f(x))^q\, u(x)\, d\mu(x) \leq \int_\Omega (\mathcal{M}^\gamma_\beta f(x))^q\, u(x)\, d\mu(x)$$

$$\leq \sum_i \int_{P_i} (\mathcal{M}^{\gamma}_{\beta} f(x))^q\, u(x)\, d\mu(x)$$
$$\leq \sum_i \int_{Q_i} (\mathcal{M}^{\gamma}_{Q_i} f(x))^q\, u_i(x)\, d\mu(x)\,,$$

where $u_i = u|_{Q_i}$ and the fact that every ball $B \in \mathcal{F}_\beta$ containing $x \in P_i$ is contained in Q_i by Lemma 4.2 again. Now, since each Q_i we can take $M^{\gamma}_{\mathbb{X}} = \mathcal{M}^{\gamma}_{Q_i}$, by applying Theorem 1.22 on each Q_i and using the bounded overlapping property (see [7]) and the fact that $p \leq q$ we get

$$\int_\Omega (\mathcal{M}^{\gamma}_{\beta} f(x))^q\, u(x)\, d\mu(x) \leq C \sum_i \left(\int_{Q_i} |f(x)|^p\, v_i(x)\, d\mu(x) \right)^{q/p}$$
$$\leq C \left(\int_\Omega |f(x)|^p\, v(x)\, d\mu(x) \right)^{q/p} .$$

Thus, we obtain the desired estimate in this case. Now, if $\beta > \beta_0$ it is sufficient to prove (1.16) for $M^{\gamma}_{(\beta_0,\beta]}$ where

$$M^{\gamma}_{(\beta_0,\beta]} f(x) = \sup_{B:\, x \in B \in \mathcal{F}_\beta \setminus \mathcal{F}_{\beta_0}} \mu(B)^{\gamma-1} \int_B |f|\, d\mu\,. \tag{4.8}$$

In fact, we note that $M^{\gamma}_{\beta} f(x) \leq M^{\gamma}_{\beta_0} f(x) + M^{\gamma}_{(\beta_0,\beta]} f(x)$. For this aim, we consider again a covering $\{P_i\}$ of Ω as in Lemma 3.4 for a such that $\beta/160 < a < \beta/80$ again. Then, given P_i, for each $x \in P_i$, we choose a ball $B_x \in \mathcal{F}_\beta \setminus \mathcal{F}_{\beta_0}$ such that

$$M^{\gamma}_{(\beta_0,\beta]} f(x) \leq 2\, \mu(B_x)^{\gamma-1} \int_{B_x} |f|\, d\mu\,.$$

Since $B_x \cap P_i \neq \emptyset$ we have that $B_x \subset \mathcal{N}_\beta(P_i)$. Now, assuming for the time being that there exists some constant $C > 0$ such that

$$\mu(B_x) \geq C\ \mu(P_i) \tag{4.9}$$

holds, for every $x \in P_i$ and every i, we can obtain

$$\int_\Omega \left(M^{\gamma}_{(\beta_0,\beta]} f(x) \right)^q u(x)\, d\mu(x)$$
$$\leq C\ \sum_i \int_{P_i} \left(\mu(B_x)^{\gamma-1} \int_{B_x} |f|\, d\mu \right)^q u(x)\, d\mu(x)$$
$$\leq C\ \sum_i \mu(P_i)^{(\gamma-1)q}\, u(P_i) \left(\int_{\mathcal{N}_\beta(P_i)} |f|\, d\mu \right)^q$$
$$\leq C\ \sum_i \mu(P_i)^{(\gamma-1)q}\, u(P_i)\, \sigma\big(\mathcal{N}_\beta(P_i)\big)^{q/p'} \left(\int_{\mathcal{N}_\beta(P_i)} |f|^p\, v\, d\mu \right)^{q/p}$$
$$\leq C\ \sum_i \left(\mu(P_i)^{\gamma-1} u(P_i)^{1/q} \sigma(P_i)^{1/p'} \right)^q \left(\int_{\mathcal{N}_\beta(P_i)} |f|^p\, v\, d\mu \right)^{q/p}$$
$$\leq C\ \left(\sum_i \int_\Omega \chi_{\mathcal{N}_\beta(P_i)} |f|^p\, v\, d\mu \right)^{q/p}$$

$$\leq C\left(\int_{\Omega}|f|^p\, v\, d\mu\right)^{q/p},$$

using Hölder's inequality, Remark 3.7 and the fact that (u, v) satisfies (1.11) (see the comment before the start of this proof). The last inequality follows from Lemma 3.8.

Now, we will prove (4.9). Let us start with $B_x = B(x_B, r_B)$ and $P_i = B(x_i, r_i)$. If the ball B_x is such that $5B_x \notin \mathcal{F}_\beta$ then, since $P_i \subset \mathcal{N}_\beta(B_x)$, the Lemma 3.5 implies that $\mu(P_i) \leq \mu(\mathcal{N}_\beta(B_x)) \leq K\mu(B_x)$.

On the other hand, if $5B_x \in \mathcal{F}_\beta$, taking into account that $10P_i \in \mathcal{F}_\beta$ we have

$$\begin{aligned}
\mathrm{d}(x_i, \Omega^c) &\leq \mathrm{d}(x_B, \Omega^c) + \mathrm{d}(x_B, x) + \mathrm{d}(x, x_i) \\
&\leq \mathrm{d}(x_B, \Omega^c) + \frac{\beta}{5}\mathrm{d}(x_B, \Omega^c) + \frac{\beta}{10}\mathrm{d}(x_i, \Omega^c),
\end{aligned}$$

which implies that

$$\mathrm{d}(x_i, \Omega^c) \leq \frac{10 + 2\beta}{10 - \beta}\mathrm{d}(x_B, \Omega^c) < 2\,\mathrm{d}(x_B, \Omega^c).$$

Furthermore, since $\beta_0 > 1/100$ and $B_x \notin \mathcal{F}_{\beta_0}$ we get

$$r_i < \frac{\beta}{80}\mathrm{d}(x_i, \Omega^c) < \frac{\beta}{40}\mathrm{d}(x_B, \Omega^c) < 2\beta_0\,\mathrm{d}(x_B, \Omega^c) < 2\, r_B.$$

Now, it is clear that $P_i \subset 5B_x$ and using Lemma 3.5 again the proof of (4.9) is complete, and the proof of theorem is done. □

We end this section with the following lemma, which reflects the relationship between the fractional maximal operator defined on a space of homogeneous type and the local fractional maximal operator.

Lemma 4.10. *Let Ω be an open subset in a metric space (X, d) satisfying the $\mathcal{P}$ property and let $0 \leq \gamma < 1$. Suppose that the space $(Q, d|_Q, \mu|_Q)$ is of homogeneous type, where $Q \in \mathcal{F}_\xi$, for some $0 < \xi < 1/3$. Then, there exist η, with $0 < \eta < 1$, and $C > 0$ such that for each $x \in Q$ we have*

$$M_Q^\gamma f(x) \leq C M_\eta^\gamma f(x),$$

where M_Q^γ and M_η^γ are as in (1.21) and (1.4) respectively.

Proof. Let $0 < \xi < 1/3$ and $Q = B(x_0, r_0) \in \mathcal{F}_\xi$. Given $x \in Q$, there exist $x_B \in Q$ and $r_B > 0$ such that $B_Q = B(x_B, r_B) \cap Q$ satisfies that

$$M_Q^\gamma f(x) \leq 2\,\frac{1}{\mu(B_Q)^{1-\gamma}}\int_{B_Q} f\, d\mu.$$

Now, by the $\mathcal{P}$ property we can get that $\mu(B_Q) \geq C\,\mu(B(x_B, r))$. If $r_B > 2r_0$ the result is obvious since $B_Q = Q$. On the other hand, if $r_B \leq 2r_0$, in an analogous way as in Lemma 4.7, taking $\eta = 2\xi/(1-\xi) < 1$ the lemma is done. □

5. Applications

In this section we prove Theorem 1.30 and give some other applications. In this direction, we will need Proposition 1.33, which is interesting in itself since it is a local version of an analogous result proved in the context of spaces of homogeneous type (see [14]).

Proof of Proposition 1.33. The first part of the proof is analogous to the one given in [7, Proposition 3.3]. However, for a sake of completeness we include a sketch of it. First, we need to check that T^γ_β is a $(3\beta,\gamma)$-local operator with respect to the metric δ.

The representation of T^γ_β with $0 \leq \gamma < 1$ by a kernel and the boundedness properties are independent of the metric. By using the equivalence of the metric given in (2.4) and the hypothesis on the kernel, we have

$$\operatorname{supp} K \subset \{(x,y) : \mathrm{d}(x,y) < \beta \mathrm{d}(x,\Omega)\} \subset \{(x,y) : \delta(x,y) < 3\beta\, \delta(x,\Omega)\}\,.$$

Also we can deduce that

$$|K(x,y)| \leq \frac{C}{\mu(B_\delta(x,\mathrm{d}(x,y))^{1-\gamma}} \leq \frac{C}{\mu(B_\delta(x,1/3\,\delta(x,y))^{1-\gamma}}\,.$$

Thus, the size condition is obtained. Analogously we can check the smoothness conditions respect to δ.

Now, taking $3\beta < 1/14$ and considering a covering $\{P_i\}$ of Ω as in Lemma 3.4 we consider the associated balls $\{Q_i\}$ which are spaces of homogeneous type, given by Lemma 4.2. Then, for each $x \in P_i$, since $T^\gamma_{\beta,b}$ is a (β,γ)-local operator, we can see $T^{\gamma,m}_{\beta,b}$ as an operator defined as in Section 2 on Q_i. Moreover, for each $x \in P_i$ we have that $T^{\gamma,m}_{\beta,b} f(x) = T^{\gamma,m}_{\beta,b}(f\chi_{Q_i})(x)$.

So, for $\gamma = 0$, the restriction of $T^m_{\beta,b}$ to functions supported on Q_i is the commutator of a Calderón-Zygmund operator in this space. Then, by applying Theorem 1.4 in [14] we obtain

$$\int_{P_i} |T^m_{\beta,b} f(x)|^q\, u(x)\, d\mu(x) \leq C\, \|b\|^{mq}_{BMO(Q_i)} \int_{Q_i} \left((M^{m+1}_{Q_i} f)(x)\right)^q u(x)\, d\mu(x)\,.$$

For $0 < \gamma < 1$, keeping in mind the size condition on the kernel, we get the following pointwise estimate for the restriction of $T^{\gamma,m}_{\beta,b}$ to functions supported on Q_i,

$$|T^{\gamma,m}_{\beta,b} f(x)| \leq C \int_{Q_i} |b(y) - b(x)|^m K(x,y)\, f(y)\, d\mu(y)\,.$$

Now, by using the Theorem 1.1 in [1] we have that

$$\int_{Q_i} |T^{\gamma,m}_{\beta,b} f(x)|^q\, u(x)\, d\mu(x) \leq C\, \|b\|^{mq}_{BMO(Q_i)} \int_{Q_i} |M^\gamma_{Q_i}(M^m_{Q_i} f)(x)|^q\, u(x)\, d\mu(x)\,,$$

for every $f \in L^\infty$ with compact support. Now, for $0 \leq \gamma < 1$ we note that there exists a constant $C > 0$ such that

$$M^\gamma_{Q_i}(M^m_{Q_i} f)(x) \leq C\, M^\gamma_{\phi_m} f(x)\,,$$

for every $x \in Q_i$, where $\phi_m(t) = t\,(\log(e+t))^m$ (see Lemma 4.1 in [1]). So, for each $1 < r < 1/\gamma$, choosing t_0 large enough such that $\phi_m(t) \leq Ct^r$ for every $t \geq t_0$, we have that $\|f\|_{\phi_m,Q_i} \leq C\,\|f\|_{t^r,Q_i}$. Then, by Lemma 4.10,

$$M^\gamma_{\phi_m} f(x) \leq C\, M^\gamma_{t^r} f(x) = \left(M^{\gamma r}_{Q_i}|f|^r(x)\right)^{1/r} \leq C\left(M^{\gamma r}_{\eta}|f|^r(x)\right)^{1/r} = C\, M^{\gamma,r}_{\eta} f(x),$$

for every $x \in Q_i$. Finally, by Lemma 4.2 and Lemma 4.7

$$\begin{aligned}\int_\Omega |T^{\gamma,m}_{\beta,b} f(x)|^q\, u(x)\, d\mu(x) &\le \sum_i \int_{P_i} |T^{\gamma,m}_{\beta,b}(f\chi_{Q_i})(x)|^q\, u(x)\, d\mu(x)\\ &\le C \sum_i \|b\|^{mq}_{BMO(Q_i)} \int_{Q_i} |M^{\gamma,r}_\eta f(x)|^q\, u_i(x)\, d\mu(x)\\ &\le C\, \|b\|^{mq}_{BMO_\beta(\Omega)} \int_\Omega |M^{\gamma,r}_\eta f(x)|^q\, u(x)\, d\mu(x),\end{aligned}$$

where $u_i = u|_{Q_i}$, as we wanted to prove. □

Now, as a consequence of the latter proposition we can obtain Theorem 1.30. In its proof we will start by considering β such that $3\beta < 1/14$. The larger values of β will be studied in a different way. For this, following again the reasoning applied in [7], we split the operator into two terms. To this aim, we take a smooth cut function η, defined on $(0,\infty)$, such that $0 \le \eta \le 1$, $\eta(t) = 1$ when $0 \le t \le 1$ and $\eta(t) = 0$ for $t \ge 2$. We denote by $\mathrm{d}(x) = \mathrm{d}(x,\Omega^c)$ and for α fixed such that $0 < 3\,(2\alpha) < 1/14$, we define

$$T^{\gamma}_{\beta,0} f(x) = \int_\Omega K(x,y)\,\Big(1-\eta\Big(\frac{\mathrm{d}(x,y)}{\alpha\,\mathrm{d}(x)}\Big)\Big)\, f(y)\, d\mu(y)\,, \tag{5.1}$$

and its commutator

$$T^{\gamma,m}_{\beta,b,0} f(x) = \int_\Omega |b(x)-b(y)|^m\, K(x,y)\,\Big(1-\eta\Big(\frac{\mathrm{d}(x,y)}{\alpha\,\mathrm{d}(x)}\Big)\Big)\, f(y)\, d\mu(y)\,. \tag{5.2}$$

The technique we will apply involves the following pointwise estimate.

Lemma 5.3. *Let $0<\delta<1$, $0 \le \gamma < 1$ and Q_i as in Lemma 4.2. Then, for each $x \in Q_i$, there exist constants C and r with $0 < \gamma r < 1$ such that*

$$\begin{aligned}&\Big(M^\sharp_{Q_i}\big(T^{\gamma,m}_{\beta,b,0}(f\chi_{\mathcal{N}_\beta(P_i)})\chi_{P_i}\big)^\delta(x)\Big)^{1/\delta}\\ &\qquad \le C\;\|b\|^m_{\mathrm{BMO}_\beta(\Omega)}\left(M^{\gamma,r}_{Q_i} f(x) + \left(\frac{1}{\mu(P_i)^{1-\gamma r}}\int_{\mathcal{N}_\beta(\mathcal{N}_\beta(P_i))} |f|^r d\mu\right)^{1/r}\right),\end{aligned}$$

for all $0<\beta<1/14$, where the last maximal is defined as in (1.21).

Proof. We assume that $0<\gamma<1$ (the case $\gamma = 0$ can be obtained follwing a quite similar reasoning). We consider $x \in Q_i = B(x_i, R_i)$ with $R_i = \theta\, r_i$ (see Lemma 4.2) and let $x_B \in Q_i$ and $B = B(x_B, r_B)$ containing x. To simplify the notation, we will denote $g = f\chi_{\mathcal{N}_\beta(P_i)}$.

First, we assume that $2B \in \mathcal{F}_\beta$. In this case we take $g_1 = g\chi_{(2B)_{Q_i}}$ and $g_2 = g - g_1$. Observe that for each $y \in B_{Q_i} = B \cap Q_i$

$$\begin{aligned}T^{\gamma,m}_{\beta,b,0} g(y) &\le |b(y) - b_{(2B)_{Q_i}}|^m\, T^\gamma_{\beta,0} g(y)\\ &\quad + T^\gamma_{\beta,0}\big((b-b_{(2B)_{Q_i}})^m g_1\big)(y) \;+\; T^\gamma_{\beta,0}\big((b-b_{(2B)_{Q_i}})^m g_2\big)(y)\,.\end{aligned}$$

So, we have

$$\left(\frac{1}{\mu(B_{Q_i})}\int_{B_{Q_i}} \big|\big(T^{\gamma,m}_{\beta,b,0} g(y)\big)^\delta - c\,\big|\, d\mu(y)\right)^{1/\delta}$$

$$\leq \left(\frac{1}{\mu(B_{Q_i})}\int_{B_{Q_i}} \left(|b(y)-b_{(2B)_{Q_i}}|^m\, T^{\gamma}_{\beta,0}g(y)\right)^{\delta} d\mu(y)\right)^{1/\delta}$$
$$+\left(\frac{1}{\mu(B_{Q_i})}\int_{B_{Q_i}} \left(T^{\gamma}_{\beta,0}\big((b-b_{(2B)_{Q_i}})^m g_1\big)(y)\right)^{\delta} d\mu(y)\right)^{1/\delta}$$
$$+\left(\frac{1}{\mu(B_{Q_i})}\int_{B_{Q_i}} \left|T^{\gamma}_{\beta,0}\big((b-b_{(2B)_{Q_i}})^m g_2\big)(y)-c\,\right|^{\delta} d\mu(y)\right)^{1/\delta}$$
$$= I_1(x)+I_2(x)+I_3(x)\,.$$

To estimate $I_1(x)$ we observe first that $|T^{\gamma}_{\beta,0}f(x)|\leq C\,M^{\gamma}_{\beta}f(x)$ for each $x\in B_{Q_i}$ (see in the proof of Theorem 4.1 in [7]).

By using Hölder's inequality with exponent $s>1$ such that $\delta s<1$, then by the equivalence between norms in BMO and since M^{γ}_{β} is of weak type $(1,1/(1-\gamma))$, (see [7]), Kolmogorov's inequality allows us to get

$$\begin{aligned} I_1(x) &\leq C\,\left(\frac{1}{\mu(B_{Q_i})}\int_{B_{Q_i}} |b-b_{(2B)_{Q_i}}|^{m\delta s'}\,d\mu\right)^{1/\delta s'}\left(\frac{1}{\mu(B)}\int_B |T^{\gamma}_{\beta,0}f|^{\delta s}\,d\mu\right)^{1/\delta s}\\ &\leq C\,\|b\|^m_{\mathrm{BMO}(Q_i)}\left(\frac{1}{\mu(B)}\int_B |M^{\gamma}_{\beta}f|^{\delta s}\,d\mu\right)^{1/\delta s}\\ &\leq C\,\|b\|^m_{\mathrm{BMO}(Q_i)}\,\frac{1}{\mu(B_{Q_i})^{1-\gamma}}\int_{B_{Q_i}}|f|\,d\mu\\ &\leq C\,\|b\|^m_{\mathrm{BMO}_\beta(\Omega)}\,M^{\gamma}_{Q_i}f(x)\\ &\leq C\,\|b\|^m_{\mathrm{BMO}_\beta(\Omega)}\,M^{\gamma,r}_{Q_i}f(x)\,, \end{aligned}$$

where we applied Lemma 4.7 and the last maximal is defined as in (1.21).

Now, in order to estimate $I_2(x)$, we apply Kolmogorov's inequality again and Hölder inequality with $r>1$ such that $\gamma r<1$. Thus

$$\begin{aligned} I_2(x) &\leq C\,\mu(B_{Q_i})^{\gamma-1}\int_{B_{Q_i}} |b-b_{(2B)_{Q_i}}|^m|f_1|\,d\mu\\ &\leq C\,\mu(B_{Q_i})^{\gamma-1}\left(\int_{B_{Q_i}}|b-b_{(2B)_{Q_i}}|^{mr'}\,d\mu\right)^{1/r'}\left(\int_{B_{Q_i}}|f|^r d\mu\right)^{1/r}\\ &= C\,\left(\frac{1}{\mu(B_{Q_i})}\int_{B_{Q_i}}|b-b_{(2B)_{Q_i}}|^{mr'}\,d\mu\right)^{1/r'}\left(\frac{1}{\mu(B_{Q_i})^{1-\gamma r}}\int_{B_{Q_i}}|f|^r d\mu\right)^{1/r}\\ &\leq C\,\|b\|^m_{\mathrm{BMO}_\beta(\Omega)}\,M^{\gamma,r}_{Q_i}f(x)\,. \end{aligned}$$

Finally, for $I_3(x)$, we take $c=T^{\gamma}_{\beta,0}\big((b-b_{(2B)_{Q_i}})^m f_2\big)(x_B)$. Then, taking into account the support of K, for each $y\in B_{Q_i}$ we estimate the integrand in $I_3(x)$ as follows

$$\begin{aligned} &\left|T^{\gamma}_{\beta,0}\big((b-b_{(2B)_{Q_i}})^m g_2\big)(y)-T^{\gamma}_{\beta,0}\big((b-b_{(2B)_{Q_i}})^m g_2\big)(x_B)\right|\\ &\qquad\leq \int_{(2B)^c\cap\Omega}|b(z)-b_{(2B)_{Q_i}}|^m|K(y,z)-K(x_B,z)|\,|g(z)|\,d\mu(z) \end{aligned}$$

$$= \int_{(2B)^c \cap \mathcal{N}_\beta(P_i)} |b(z) - b_{(2B)_{Q_i}}|^m |K(y,z) - K(x_B,z)|\,|f(z)|\,d\mu(z)$$
$$\leq \sum_{j: P_j \in \mathcal{W}_{a,Q_i}} \int_{(2B)^c \cap P_j \cap Q_i} |b(z) - b_{(2B)_{Q_i}}|^m |K(y,z) - K(x_B,z)|\,|f(z)|\,d\mu(z)\,,$$

where the last inequality is obtained by using the fact that $5Q_i \notin \mathcal{F}_\beta$, Lemma 4.2, and Lemma 3.4. Moreover, the cardinality of $\mathcal{W}_a(Q_i)$ is finite, let us say M, and independent of i. Thus, choosing s, q and r such that $\frac{1}{s} + \frac{1}{q} + \frac{1}{r} = 1$, by Hölder's inequality we get

$$\left| T^\gamma_{\beta,0}\big((b - b_{(2B)_{Q_i}})^m f_2\big)(y) - T^\gamma_{\beta,0}\big((b - b_{(2B)_{Q_i}})^m f_2\big)(x_B) \right|$$
$$\leq \sum_{j=1}^{M} \mu(P_j) \Bigg[\left(\frac{1}{\mu(P_j)} \int_{B^c \cap P_j} |b(z) - b_{(2B)_{Q_i}}|^{ms} d\mu(z) \right)^{1/s}$$
$$\times \left(\frac{1}{\mu(P_j)} \int_{(2B)^c \cap P_j \cap Q_i} |K(y,z) - K(x_B,z)|^q d\mu(z) \right)^{1/q}$$
$$\times \left(\frac{1}{\mu(P_j)} \int_{P_j} |f(z)|^r d\mu(z) \right)^{1/r} \Bigg].$$

If we show that for each $t > 0$

$$\left(\frac{1}{\mu(P_j)} \int_{B^c \cap P_j} |b(z) - b_{(2B)_{Q_i}}|^t d\mu(z) \right)^{1/t} \leq C\ \|b\|_{\mathrm{BMO}_\beta(\Omega)}\,, \tag{5.4}$$

and each $q > 1$

$$\left(\frac{1}{\mu(P_j)} \int_{(2B)^c \cap P_j \cap Q_i} |K(y,z) - K(x_B,z)|^q\, d\mu(z) \right)^{1/q} \leq C\ \mu(P_j)^{\gamma - 1}\,, \tag{5.5}$$

then, since $\mu(P_j) \geq C\,\mu(P_i)$ (see Lemma 3.5) we have

$$\left| T^\gamma_{\beta,0}\big((b - b_{(2B)_{Q_i}})^m\, f_2\big)(y) - T^\gamma_{\beta,0}\big((b - b_{(2B)_{Q_i}})^m f_2\big)(x_B) \right|$$
$$\leq C\ \|b\|^m_{\mathrm{BMO}_\beta(\Omega)} \sum_{j=1}^{M} \mu(P_j)^\gamma \left(\frac{1}{\mu(P_j)} \int_{P_j} |f|^r d\mu \right)^{1/r}$$
$$\leq C\ \|b\|^m_{\mathrm{BMO}_\beta(\Omega)} \left(\frac{1}{\mu(P_i)^{1-\gamma r}} \int_{\mathcal{N}_\beta(\mathcal{N}_\beta(P_i))} |f|^r d\mu \right)^{1/r}.$$

In summary, we have proven that

$$\left(M^\sharp_{Q_i} \big(T^{\gamma,m}_{\beta,b,0}(f \chi_{\mathcal{N}_\beta(P_i)}) \chi_{P_i} \big)^\delta (x) \right)^{1/\delta}$$
$$\leq C\ \|b\|^m_{\mathrm{BMO}_\beta(\Omega)} \left(M^{\gamma,r}_{Q_i} f(x) + \left(\frac{1}{\mu(P_i)^{1-\gamma r}} \int_{\mathcal{N}_\beta(\mathcal{N}_\beta(P_i))} |f|^r d\mu \right)^{1/r} \right),$$

for every ball B such that $2B \in \mathcal{F}_\beta$.

Now, we consider the case $2B \notin \mathcal{F}_\beta$ and we decompose the function f as $f_1 = f\chi_{\mathcal{N}_\beta(B)}$ and $f_2 = f - f_1$. In this situation, we can deduce that $\mu(B_{Q_i}) \ge C\ \mu(Q_i)$, because it is not difficult to see that $r > C\, R_i$, then in the same way as before

$$\left(\frac{1}{\mu(Q_i)}\int_{Q_i}\left|\left(T^{\gamma,m}_{\beta,b,0}f(y)\right)\chi_{Q_i}(y) - c\right|^\delta d\mu(y)\right)^{1/\delta} \le I_1(x) + I_2(x) + I_3(x)\,.$$

For the terms $I_1(x)$ and $I_2(x)$ we proceed as in the case above. However, the support of f_2 is in $\Omega\setminus\mathcal{N}_\beta(B)$ and $B(y,\beta \mathrm{d}(y,\Omega^c)) \subset \mathcal{N}_\beta(B)$ then $K(y,w) = 0$. Similarly, as $B(z,\beta\mathrm{d}(z,\Omega^c)) \subset \mathcal{N}_\beta(B)$ we have that $K(z,w) = 0$ and so $I_3(x) = 0$.

Finally, proceeding as above we also have the required estimate for the average

$$\begin{aligned}\frac{1}{\mu(Q_i)}\int_{Q_i}&\left|T^{\gamma,m}_{\beta,b,0}f(y)\right| d\mu(y)\\ &\le C\ \|b\|^m_{\mathrm{BMO}_\beta(\Omega)}\left(M^{\gamma,r}_{Q_i}f(x) + \left(\frac{1}{\mu(P_i)^{1-\gamma r}}\int_{\mathcal{N}_\beta(\mathcal{N}_\beta(P_i))}|f|^r d\mu\right)^{1/r}\right),\end{aligned}$$

and the proof of lemma will be complete provided that we will show (5.4) and (5.5).

For (5.4) it is sufficient to prove it in the case $B \in \mathcal{F}_{\beta/2}$ and $P \in \mathcal{W}_a(Q_i)$ such that $P \cap B \neq \emptyset$ (if the latter does not occur, we can proceed by considering a chain of balls as in the end of the Lemma 4.4). So,

$$\begin{aligned}\left(\frac{1}{\mu(P)}\int_{B^c\cap P}|b - b_{(2B)_{Q_i}}|^t d\mu\right)^{1/t} &\le \left(\frac{1}{\mu(P)}\int_P |b - b_P|^t d\mu\right)^{1/t} + |b_P - b_{(2B)_{Q_i}}|\\ &\le C\ \|b\|_{\mathrm{BMO}_\beta(\Omega)} + |b_P - b_{(2B)_{Q_i}}|\,.\end{aligned}$$

If $r_B > 2r_P$ there is nothing to do since $P \subset 2B$. On the other hand, it is easy to see that $B \subset 5P \in \mathcal{F}_\beta$ whenever $r_B \le 2r_P$. Then

$$|b_P - b_{(2B)_{Q_i}}| \le \frac{C}{\mu(5P)}\int_{5P}|b - b_{5P}|\, d\mu \le C\ \|b\|_{\mathrm{BMO}_\beta(\Omega)}\,.$$

Now, for (5.5), since $y \in B$ and $z \in Q_i \cap (2B)^c$ we have that $B(y, 3\mathrm{d}(y,z)) \supset B(x_B, \mathrm{d}(x_B,z))$. So, in a similar way as in Lemma 4.7 we get $B(y,3\mathrm{d}(y,z)) \in \mathcal{F}_{3\tau/(1-\tau)}$ and $\mu(B(y,\mathrm{d}(z,y))) \ge C\,\mu(B(x_B,\mathrm{d}(z,x_B)))$. Then, from the size condition of the kernel we can estimate as follows

$$\begin{aligned}\int_{(2B)^c\cap P_j\cap Q_i}&|K(y,z) - K(x_B,z)|^q\, d\mu(z)\\ &\le C\int_{(2B)^c\cap P_j\cap Q_i}\frac{1}{\mu(B(y,\mathrm{d}(y,z)))^{(1-\gamma)q}}\left(\frac{\mathrm{d}(y,x_B)}{\mathrm{d}(y,z)}\right)^{\varepsilon q} d\mu(z)\\ &\le C\sum_{k=1}^{k_j-1}2^{-j\varepsilon q}\int_{(B_{k+1}\setminus B_k)\cap P_j\cap Q_i}\mu(B(x_B,2^k r_B))^{(\gamma-1)q}\, d\mu(z)\,,\end{aligned}$$

where $B_k = B(x_B, 2^k r_B)$ and k_j is the smallest index such that B_{k_j} totally contains P_j, that is, $B_{k_j} \supset P_j$ but $B_{k_j-1} \not\supset P_j$. Recalling that μ is doubling on $\mathcal{F}$ again we proceed as follows

$$\int_{(2B)^c\cap P_j\cap Q_i}|K(y,z) - K(x_B,z)|^q\, d\mu(z)$$

$$\leq C \int_{(2B)^c \cap P_j \cap Q_i} \frac{1}{\mu(B(y, \mathrm{d}(y,z)))^{(1-\gamma)q}} \left(\frac{\mathrm{d}(y,x_B)}{\mathrm{d}(y,z)}\right)^{\varepsilon q} d\mu(z)$$

$$\leq C \sum_{k=1}^{k_j-1} 2^{-j\varepsilon q} \mu(B(x_B, 2^{k_j} r_B))^{(\gamma-1)q}\, \mu(P_j)$$

$$\leq C\ \mu(P_j)^{(\gamma-1)q+1},$$

which prove (5.5) and the lemma in the case $0 < \gamma < 1$.

Finally, as we said before, taking $s = 1/\delta$ in the estimation of $I(x)$, using the $(1,1)$ boundedness of M_β and the corresponding estimates for the kernel when $\gamma = 0$, we obtain the lemma for this case as well. Then the proof is complete. □

With all this, we are in position to prove Theorem 1.30.

Proof of Theorem 1.30. On the one hand, if β is such that $3\beta < 1/14$, since (5.9) with p, q and r as in the hypothesis implies (1.17) for p/r and q/r, by Proposition 1.33 and Theorem 1.15 we obtain the desired boundedness result, that is (1.30).

On the other hand, for $3\beta \geq 1/14$, we will adapt the idea followed in the proof of Theorem 4.1 in [7]. Thus, we consider the operator $T^{\gamma,m}_{\beta,b,0}$ defined as in (5.2). The operator $T^{\gamma,m}_{\beta,b,1} = T^{\gamma,m}_{\beta,b} - T^{\gamma,m}_{\beta,b,0}$ is the commutator of a $(2\alpha,\gamma)$-local operator, (the proof is similar to the one for the case $\gamma = 0$, which can be seen in [7]). Then, since $0 < 3\,(2\alpha) < 1/14$, in an analogous way as before we obtain the result for $T^{\gamma,m}_{\beta,b,1}$. So, it just remains to see $T^{\gamma,m}_{\beta,b,0}$. For this, we consider $\beta' = \beta/50$ and take $\{P'_i\}$ the covering associated with the family $\mathcal{F}_{\beta'}$ for a such that $\beta'/160 < a$ given by Lemma 3.4. Moreover, since $3\beta' < 1/14$, we get that the associated balls $\{Q'_i\}$ are spaces of homogeneous type as in Lemma 4.2. Since $u_i = u|_{Q_i} \in A_\infty(Q_i)$ whenever $u \in A^\beta_\infty$ (see [7]), we be able to take $0 < \delta < 1$ such that $u_i \in A_{q/\delta}(Q_i)$ and apply Lemma 2.5 (where the hypothesis hold by the known continuity of the maximal operator in weighted Lebesgue spaces). Taking into account the support of $T^{\gamma,m}_{\beta,b,0}$ allow us to get

$$\int_\Omega \left|T^{\gamma,m}_{\beta,b,0} f(x)\right|^q u(x)\, d\mu(x)$$

$$\leq \sum_i \int_{P'_i} \left|T^{\gamma,m}_{\beta,b,0}(f\chi_{\mathcal{N}_\beta(P'_i)})(x)\right|^q u(x)\, d\mu(x)$$

$$\leq C\ \sum_i \int_{Q'_i} \left|\Big(\big(T^{\gamma,m}_{\beta,b,0}(f\chi_{\mathcal{N}_\beta(P'_i)})\big)\chi_{P'_i}\Big)^\delta(x)\right|^{q/\delta} u_i(x)\, d\mu(x)$$

$$\leq C\ \sum_i \int_{Q'_i} \left|M\Big(\big(T^{\gamma,m}_{\beta,b,0}(f\chi_{\mathcal{N}_\beta(P'_i)})\big)^\delta \chi_{P'_i}\Big)(x)\right|^{q/\delta} u_i(x)\, d\mu(x)$$

$$\leq C\ \sum_i \int_{Q'_i} \Big(M^\sharp_{Q'_i}\big(\big(T^{\gamma,m}_{\beta,b,0}(f\chi_{\mathcal{N}_\beta(P'_i)})\big)^\delta \chi_{P'_i}\big)(x)\Big)^{q/\delta} u_i(x)\, d\mu(x).$$

Then, from Lemma 5.3, we obtain

$$\int_\Omega \left|T^{\gamma,m}_{\beta,b,0} f(x)\right|^q u(x)\, d\mu(x)$$

$$\leq C \ \|b\|^m_{\mathrm{BMO}_\beta(\Omega)} \left(\sum_i \int_{Q'_i} \left(M^{\gamma,r}_{Q'_i} f(x) \right)^q u_i(x)\, d\mu(x) \right.$$
$$\left. + \ \sum_i \int_{Q'_i} \left(\frac{1}{\mu(P'_i)^{1-\gamma r}} \int_{\mathcal{N}_\beta(\mathcal{N}_\beta(P'_i))} |f|^r d\mu \right)^{q/r} u_i(x)\, d\mu(x) \right).$$

Our hypothesis on the weights, as we said at the beginning of the proof, implies (1.17) for p/r and q/r. In [7] the authors proved that the classes of weights are invariant under change of equivalent metrics. Then, without loss of generality, we can suppose that the $\mathcal{P}$ property is valid in our space. So, as we have seen before, for each ball $B = B(x_B, r_B)$ with $x_B \in Q'_i$ and $r_B < 2R'_i$ we have that $\mu(B) \leq C\, \mu(B_{Q_i})$.

In addition $\|\cdot\|_{\psi,A} \leq C\, \|\cdot\|_{\psi,B}$ for any $A \subset B$ such that $\mu(B) \leq C\, \mu(A)$. In fact, since ψ is a Young function it satisfies that $\psi(st) \geq s\psi(t)$, for any $s \geq 1$ and $t > 0$. Then, if we take $\lambda = \|\cdot\|_{\psi,B}$

$$\frac{1}{\mu(A)} \int_A \psi\Big(\frac{|f|}{C\lambda}\Big)\, d\mu \leq \frac{C}{\mu(B)} \int_B \psi\Big(\frac{|f|}{C\lambda}\Big)\, d\mu \leq \frac{1}{\mu(B)} \int_B \psi\Big(\frac{|f|}{\lambda}\Big)\, d\mu \leq 1\,,$$

which proves our statement. Then it is clear that the pair of weights (u_i, v_i) satisfies (1.23) with $\phi(t) = t^{q/r}$, for every ball $B \in F(Q'_i)$ and constant independent of i. Then, by Theorem 1.22 and the bounded overlapping of $\{Q'_i\}$ we can deduce

$$\begin{aligned}\sum_i \int_{Q'_i} \left(M^{\gamma,r}_{Q'_i} f(x) \right)^q u_i(x)\, d\mu(x) &= \sum_i \int_{Q'_i} \left(M^{\gamma r}_{Q'_i}(|f|^r)(x) \right)^{q/r} u_i(x)\, d\mu(x) \\ &\leq C \ \sum_i \left(\int_{Q'_i} |f(x)|^p\, v_i(x)\, d\mu(x) \right)^{q/p} \\ &\leq C \ \left(\int_\Omega |f(x)|^p\, v(x)\, d\mu(x) \right)^{q/p} .\end{aligned}$$

For the second term we proceed as in the proof of Theorem 1.15. Since $\sigma_r \in \mathcal{D}(\mathcal{F}_\beta \backslash \mathcal{F}_{\tilde{\beta}})$, combining Remark 3.6 and Remark 3.7 and using the fact that $(u, v) \in A^{\beta,\gamma}_{p,q}$, that is (1.11). we can apply Hölder inequality with $p/r > 1$ and obtain

$$\begin{aligned}&\sum_i \int_{Q'_i} \left(\frac{1}{\mu(P'_i)^{1-\gamma r}} \int_{\mathcal{N}_\beta(\mathcal{N}_\beta(P'_i))} |f|^r d\mu \right)^{q/r} u_i(x)\, d\mu(x) \\ &\quad\leq C \ \sum_i \mu(P'_i)^{(\gamma r-1)q/r}\, u_i(Q'_i) \Big(\int_{\mathcal{N}_\beta(\mathcal{N}_\beta(P'_i))} |f|^r d\mu \Big)^{q/r} \\ &\quad\leq C \ \sum_i \Big(\mu(P'_i)^{(\gamma r-1)}\, u(P'_i)^{1/(q/r)} \left(\sigma_r(\mathcal{N}_\beta(\mathcal{N}_\beta(P'_i))) \right)^{1/(p/r)'} \Big)^{q/r} \\ &\qquad \times \ \left(\int_{\mathcal{N}_\beta(\mathcal{N}_\beta(P'_i))} |f|^p v_i d\mu \right)^{q/p} \\ &\quad\leq C \ \left(\int_\Omega \chi_{\mathcal{N}_\beta(\mathcal{N}_\beta(P'_i))}\, |f|^p\, v\, d\mu \right)^{q/p} \\ &\quad\leq C \ \left(\int_\Omega |f|^p\, v\, d\mu \right)^{q/p},\end{aligned}$$

where the Remark 3.10 was applied in the last step. So the proof of the theorem is complete. □

Now, in the context of partial differential equations, we consider $X = \mathbb{R}^n$, endowed with the Lebesgue measure, Ω an open subset of $\mathbb{R}^n$ and the m-Laplacian operator, that is Δ^m, where the notation means that we compose m times the Laplacian operator in Ω.

For U a solution of the problem $\Delta^m U = f$, many estimates are known in the context of classical weighted Sobolev spaces. Particularly, in the context of local weights, we refer to [6], where the authors consider a version of weighted Sobolev spaces that take into account the distance to the boundary. More precisely, with $\mathrm{d}(x)$ as the distance from x to Ω^c we denote

$$W^{k,p}_{\mathrm{d},\omega}(\Omega) = \left\{ f \in L^1_{loc}(\Omega) : \|f\|_{W^{k,p}_{\mathrm{d},\omega}(\Omega)} = \sum_{|\alpha|\le k} \|\mathrm{d}^{|\alpha|} D^\alpha f\|_{L^p(\Omega,\omega)} < \infty \right\}.$$

For U as above they proved (see [6, Theorem 4.2]) that

$$\|U\|_{W^{2m,p}_{\mathrm{d},\omega}(\Omega)} \le C \left(\|U\|_{L^p(\Omega,\omega)} + \|\mathrm{d}^{2m} f\|_{L^p(\Omega,\omega)} \right), \tag{5.6}$$

for any weight $\omega \in A^\beta_p$.

We now consider a similar result in the case of two different weights. We can give an answer using (5.6) and Theorem 1.15.

Theorem 5.7. *Let $1 < p \le q < \infty$. For a pair of weights (u, v) satisfying the hypothesis of Theorem 1.15, with $u \in \mathrm{A}^\beta_q$ and U a solution of the problem $\Delta^m U = f$ in Ω we have*

$$\|U\|_{W^{2m,q}_{d,u}(\Omega)} \le C \left(\|U\|_{L^p(\Omega,v)} + \|d^{2m} f\|_{L^p(\Omega,v)} \right). \tag{5.8}$$

Proof. Since $u \in \mathrm{A}^\beta_q$ we have by (5.6)

$$\|U\|_{W^{2m,q}_{\mathrm{d},u}(\Omega)} \le C \left(\|U\|_{L^q(\Omega,u)} + \|\mathrm{d}^{2m} f\|_{L^q(\Omega,u)} \right).$$

By the Lebesgue's differentiation Theorem $f(x) \le M_\beta f(x)$ for every locally integrable function f. Then, by Theorem 1.15 we have

$$\|U\|_{L^q(\Omega,u)} \le C \, \|M_\beta U\|_{L^q(\Omega,u)} \le C \, \|U\|_{L^p(\Omega,v)}$$

and

$$\|\mathrm{d}^{2m} f\|_{L^q(\Omega,u)} = \|f\|_{L^q(\Omega,u\mathrm{d}^{2mq})} \le C \, \|M_\beta f\|_{L^q(\Omega,u\mathrm{d}^{2mq})} \le C \, \|\mathrm{d}^{2m} f\|_{L^p(\Omega,v)},$$

where we use the Theorem 1.15 for the pair of weights $(u\,\mathrm{d}^{2mq}, v\,\mathrm{d}^{2mp})$. The proof will be complete if we show that this pair satisfies the hypothesis. But this is true because for every ball $B \in \mathcal{F}_\beta$ with center x_0, we get that $\mathrm{d}(x) \simeq \mathrm{d}(x_0)$ for each $x \in B$. Then, it is clear that $u \in \mathcal{D}(\mathcal{F}_\beta)$ and $\sigma = v^{1-p'} \in \mathcal{D}(\mathcal{F}_\beta \backslash \mathcal{F}_{\tilde{\beta}})$ implies that $u\mathrm{d}^{2mq} \in \mathcal{D}(\mathcal{F}_\beta)$ and $\sigma = (v\mathrm{d}^{2mp})^{1-p'} \in \mathcal{D}(\mathcal{F}_\beta \backslash \mathcal{F}_{\tilde{\beta}})$ respectively. Moreover, since

$$\left(\int_B u(x)\mathrm{d}(x)^{2mq} d\mu(x) \right)^{1/q} \left\| \left(v\mathrm{d}^{2mp}\right)^{-1/p} \right\|_{\psi,B} \le C\, \mathrm{d}(x_0)^{2m} \mathrm{d}(x_0)^{-2m} \left(\int_B u \, d\mu \right)^{1/q} \left\| v^{-1/p} \right\|_{\psi,B},$$

we get (1.17) for $(u\,\mathrm{d}^{2mq}, v\,\mathrm{d}^{2mp})$. □

Also we can obtain the following embedding result.

Theorem 5.9. *Let $0 < \beta < 1$ and $1 < p \le q < \infty$. Let u and v be weights such that $u \in A_\infty^\beta$, $\sigma_r = v^{1-(p/r)'} \in \mathcal{D}(\mathcal{F}_\beta \backslash \mathcal{F}_{\tilde{\beta}})$, with $\tilde{\beta} < \beta/4000$, and*

$$\mu(B)^{1/n-1/p+1/q} \left(\frac{1}{\mu(B)} \int_B u \, d\mu \right)^{1/q} \|v^{-r/p}\|_{\psi,B}^{1/r} \le C \,,$$

for all $B \in \mathcal{F}_\beta$, for some $r > 1$ such that $r < n$, where ψ be a doubling Young function such that $\tilde{\psi}$ belongs to B_p. Then for $g \in W_{d,v}^{1,p}(\Omega)$ we have

$$\|d\, g\|_{L_u^q(\Omega)} \le C \, \|g\|_{W_{d,v}^{1,p}(\Omega)}.$$

Proof. The key is in the proof of Theorem 5.3 of [7], where the authors proved that, for any $x \in \Omega$, we have

$$|g(x)| \le C \left(\mathrm{d}(x)^{-1} I_\beta^{1/n}(|g|(x)) + I_\beta^{1/n}(|\nabla(g)|(x)) \right), \tag{5.10}$$

where $I_\beta^{1/n}$ is as in (1.27). Then, taking into account that $(u\mathrm{d}^q, v\mathrm{d}^p)$ satisfies the same hypothesis as (u, v) (see proof of Theorem 5.7), applying the case $m = 0$ and $\gamma = 1/n$ of Theorem 1.30 we get

$$\begin{aligned} \|\mathrm{d}\, g\|_{L_u^q}^q &\le C \left(\|I_\beta^{1/n}(|g|)\|_{L_u^q}^q + \|\mathrm{d}\, I_\beta^{1/n}(|\nabla g|)\|_{L_u^q}^q \right) \\ &= C \left(\|I_\beta^{1/n}(|g|)\|_{L_u^q}^q + \|I_\beta^{1/n}(|\nabla g|)\|_{L_{u\mathrm{d}^q}^q}^q \right) \\ &= \|g\|_{W_{\mathrm{d},v}^{1,p}(\Omega)}. \end{aligned}$$

□

Remark 5.11. The inequality (5.10) is proved in [7] (see proof of Theorem 5.3) and it does not require the condition $\frac{1}{p} = \frac{1}{q} + \frac{1}{n}$ since the weights are not involved in it. The necessity of this condition appears when the boundedness of the fractional maximal is applied. Here, we use another proof, which only requires $1 < p \le q < \infty$.

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