Chapter

A Brief Look at the Calderón and Hilbert Operators

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Abstract

The Calderón operator is the sum of the Hardy averaging operator and its adjoint, and plays an important role in the theory of real interpolation. On the other hand, the Hilbert operator arises from the continuous version of Hilbert's inequality. Both operators appear in different contexts and have numerous applications within harmonic analysis. In this chapter we will briefly review the Calderón and Hilbert operators, showing some of the most relevant results within functional analysis and finally we will present recent results on these operators within Fourier analysis.

Keywords: Calderón operator, Hilbert operator, Lebesgue spaces, Lipschitz spaces, BMO spaces, weighted inequalities, Calderón weights

1. Introduction

The Calderón and Hilbert operators are among the most relevant operators in harmonic analysis, arising from Hilbert's double series theorem which is one of the simplest and most beautiful in the theory of double series of positive terms. It was proved by Hilbert, in the course of his investigations in the theory of integral equations, that the series $\sum_{m,n\in\mathbb{N}}\frac{a_m a_n}{a_m+a_n}$, where $a_n \geq 0$ for all $n\in\mathbb{N}$, is convergent whenever $\sum_{n\in\mathbb{N}}a_n^2$ is convergent.

Other proofs of Hilbert's double series theorem and generalizations in different directions were studied and published over time by influential mathematicians such as H. Weyl, F. Wiener, J. Schur, Fejér and F. Riesz, Pólya and Szegö, Francis and Littlewood, G.H. Hardy and M. Riesz, among others.

In [1, 2], G.H. Hardy observed that Hilbert's theorem stated above is an immediate corollary of another theorem which has interest in itself. This theorem is as follows: If $a_n \geq 0$ for all $n\in\mathbb{N}$ and $\sum_{n\in\mathbb{N}}a_n^2$ is convergent, then $\sum_{n\in\mathbb{N}}\left(\frac{1}{n}\sum_{j=1}^{n}a_j\right)^2$ is also convergent.

The first extension of the just stated Hilbert's and Hardy's results in which we are interested is the following (see [3]): Let $1<p<\infty$ and $p'=p/(p-1)$ (i.e. p' is the conjugate of p). If $\sum_{n=1}^{\infty}a_n^p$ and $\sum_{n=1}^{\infty}b_n^{p'}$ are convergent, where a_n and b_n are nonnegative numbers for all $n\in\mathbb{N}$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty} b_n^{p'}\right)^{1/p'} \text{ and} \sum_{n\in\mathbb{N}}\left(\frac{1}{n}\sum_{j=1}^{n} a_j\right)^p \leq (p')^p\sum_{n=1}^{\infty} a_n^p.$$

The constants $\pi/\sin(\pi/p)$ and $(p')^p = (p/(p-1))^p$ are the best possible.

At the same time, the continuous versions of the previous inequalities are the following (see [3, 4]): Let $1 < p < \infty$ and p' the conjugate of p. If $\int_{[0,\infty)}|f|^p$ and $\int_{[0,\infty)}|g|^{p'}$ are finite, then

$$\int_{[0,\infty)}\int_{[0,\infty)}\frac{|f(x)||g(y)|}{x+y}dx\,dy \leq \frac{\pi}{\sin(\pi/p)}\left(\int_{[0,\infty)}|f(x)|^p dx\right)^{1/p}\left(\int_{[0,\infty)}|g(x)|^{p'}dx\right)^{1/p'}$$

and

$$\int_{[0,\infty)}\left(\frac{1}{x}\int_{[0,x]} f(y)dy\right)^p dx \leq \left(\frac{p}{p-1}\right)^p\int_{[0,\infty)}|f(x)|^p dx.$$

Once again, the constants involved are the best possible.

As usual in harmonic analysis, if E is a measurable subset of $\mathbb{R}^n$, then $L^p(E)$, $1 \leq p < \infty$, is the Lebesgue space of all measurable functions f such that $\|f\|^p_{L^p(E)} = \int_E|f(x)|^p dx$ is finite. Recall that $\left(L^p(E), \|\cdot\|_{L^p(E)}\right)$ is a Banach space and in the case $E = \mathbb{R}^n$, it is denoted $\|\cdot\|_p = \|\cdot\|_{L^p(E)}$.

Now, consider the operators H and P defined by

$$Hf(x) = \int_{[0,\infty)}\frac{f(t)}{x+t}dt \qquad \text{and} \qquad Pf(x) = \frac{1}{x}\int_{[0,x]} f(t)dt,$$

which naturally arise from the inequalities presented above. Also consider

$$Qf(x) = \int_{[x,\infty)}\frac{f(t)}{t}dt$$

being the adjoint operator of P and satisfying

$$\int_{[0,\infty)}(Qf(x))^p dx = \int_{[0,\infty)}\left(\int_{[x,\infty)}\frac{f(t)}{t}dt\right)^p dx \leq C\int_{[0,\infty)}(f(x))^p dx,$$

for all $f \in L^p([0,\infty))$, $1 < p < \infty$, where C is a positive constant (see [4]). Therefore, P and Q are bounded operators from $L^p([0,\infty))$ in itself, that is,

$$\|Pf\|_{L^p([0,\infty))} \leq C\|f\|_{L^p([0,\infty))} \text{ and } \|Qf\|_{L^p([0,\infty))} \leq C\|f\|_{L^p([0,\infty))} \quad \text{for all} f \in L^p([0,\infty)).$$

It is immediate that for nonnegative functions f,

$$Hf(x) \leq Pf(x) + Qf(x) \leq 2Hf(x) \qquad \text{for all } x > 0.$$

Consequently H is a bounded operator on $L^p([0,\infty))$, that is,

$$\|Hf\|_{L^p([0,\infty))} \leq C\|f\|_{L^p([0,\infty))} \qquad \text{for all } f \in L^p([0,\infty)).$$

It is well known that P is called the *Hardy averaging operator* and H is called the *Hilbert operator*. Also, the *Calderón operator* S is defined by $S = P + Q$, being then a bounded operator from $L^p([0,\infty))$ in itself.

We end this section with some of the first and most important direct applications obtained from Hilbert's and Hardy's inequalities.

Theorem 1.1 Let E be the interval $(0,1)$ and $f \in L^2(E)$ not null in E. Then

$$\sum_{n=0}^{\infty} \left(\int_E x^n f(x) dx \right)^2 < \pi \int_E f^2(x) dx$$

and the constant π is the best possible. The integrals $\int_E x^n f(x) dx$, $n = 0, 1, \ldots$ are called the *moments of* f *in* E and are important in many theories.

Theorem 1.2 (Carlema's inequalities) Let $\{a_n\}$ be a sequence of positive numbers and $1 < p < \infty$. Then

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} a_k{}^{1/p} \right)^p < \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} a_k \right)^{1/n} < e \sum_{n=1}^{\infty} a_n.$$

The constants involved are the best possible.

The corresponding integral version for the second inequality of Carlema's inequality is: If f is a positive function belonging to $L^1([0,\infty))$, then

$$\int_{[0,\infty)} \exp\left(\frac{1}{x} \int_{[0,x]} \log f(t) dt \right) dx = \int_{[0,\infty)} e^{P(\log f)(x)} dx < e \int_{[0,\infty)} f(x) dx.$$

where the constant e is the best possible.

Theorem 1.3 Let $1 < p \leq 2$ and p' the conjugate of p. If $Lf(s) = \int_0^\infty f(t) e^{-st} dt$, i.e. Lf is the Laplace transform of f, then

$$\int_0^\infty Lf(s)^{p'} ds \leq \frac{2\pi}{p'} \left(\int_0^\infty f(s)^p ds \right)^{p'/p} \qquad \text{for all } f \in L^p([0,\infty)).$$

Therefore L is a bounded operator from $L^p([0,\infty))$ into $L^{p'}([0,\infty))$, $1 < p \leq 2$, and $\|Lf\|_{p'} \leq (2\pi/p')^{1/p'} \|f\|_p$.

The number of applications and results that arise from Hilbert's and Hardy's inequalities is by now very large and it would be impossible to give a detailed survey of all of them in a reasonable amount of text. We have simply made a very brief introduction about them in this section.

2. Calderón weights and L^p-weighted inequalities

A function ω defined on $\mathbb{R}^n$ is called a *weight* if it is locally integrable and positive almost everywhere. For a measurable set $E \subset \mathbb{R}^n$, $|E|$ denote its Lebesgue measure, $\omega(E) = \int_E \omega$, and E^c the complement of E in $\mathbb{R}^n$. Given a ball B, tB is the ball with the same center as B and with radius t times as long, and $f_B = \frac{1}{|B|}\int_B f$. As usual, χ_E denotes the characteristic function of E and $B(x, r)$ denotes a ball centered at x with radius r. Also, C denotes a positive constant.

Let ω be a weight in $\mathbb{R}^n$ and $1 \le p < \infty$. A Lebesgue measurable function f belongs to $L^p(\omega)$ if

$$\|f\|_{L^p(\omega)} = \left(\int_{\mathbb{R}^n} |f|^p \omega\right)^{1/p} < \infty.$$

We say that an oprator T is a bounded operator on $L^p(\omega)$ if

$$\|Tf\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}, \qquad \text{for all} f \in L^p(\omega).$$

Given $1 < p < \infty$, it is said that ω is a Calderón weight of class $\mathcal{C}_p$, that is $\omega \in \mathcal{C}_p$, if the Calderón operator S is bounded on $L^p(\omega)$ (see [5]) or, equivalently, if P and Q are both bounded on $L^p(\omega)$ (see also [6]). It is well known that the class $\mathcal{C}_p$ for $p > 1$ is given by the conditions

$$\begin{aligned} M_p: \quad & \left(\int_{[0,x]} \omega(t)dt\right)^{1/p} \left(\int_{[x,\infty)} \frac{\omega^{1-p'}(t)}{t^{p'}}dt\right)^{1/p'} \le C \quad \text{for all } x > 0; \\ M^p: \quad & \left(\left(\int_{[x,\infty)} \frac{\omega(t)}{t^p}dt\right)^{1/p} \left(\int_{[0,x]} \omega^{1-p'}(t)dt\right)^{1/p'} \le C \quad \text{for all } x > 0. \end{aligned}$$

The Calderón operator plays an important role in the theory of real interpolation and such theory related to Calderón weights is developed in [5]. On the other hand, in [7], the authors considered a maximal operator N on $(0, \infty)$ associated to the basis of open sets of the form $(0, b)$, given by

$$Nf(x) = \sup_{b > x} \frac{1}{b} \int_{[0,b]} |f(t)|dt$$

for measurable functions f. Then, for nonnegative functions f, we have

$$P(x) \le Nf(x) \le Sf(x) \qquad \text{for all } x > 0.$$

The classes of weights ω associated to the boundedness of N on $L^p(\omega)$ are those that satisfy the Muckenhoupt-A_p condition, $1 \le p < \infty$, only for the sets of the form $(0, b)$. These classes are denoted by $A_{p,0}$ and defined as follows:

$$\begin{aligned} A_{1,0}: \quad & N\omega(x) \le C\omega(x) \quad \text{for almost all } x > 0; \\ A_{p,0}: \quad & \left(\frac{1}{x}\int_{[0,x]} \omega\right)\left(\frac{1}{x}\int_{[0,x]} \omega^{1-p'}\right)^{p-1} \le C \quad \text{for all } x > 0, \text{where} 1 < p < \infty. \end{aligned}$$

Then, in [7] is proved that N and S are bounded operators on $L^p(\omega)$ if and only if $\omega \in A_{p,0}$ for $1 < p < \infty$. This result implies, in particular, that the classes of weights $\mathcal{C}_p$ and $A_{p,0}$ coincide for $1 < p < \infty$.

Taking into account these results it is natural to wonder for the action of the Calderón and Hilbert operators over suitable spaces such as BMO or Lipschitz spaces. Also, another interesting question is: which are, in these cases, the Calderón weights in order to obtain weighted inequalities between these spaces?

These problems were treated for instance in the case of the fractional integral operator in [8, 9], which have been the main motivation for the article [10] and for the development of the following sections.

3. The n-dimensional Calderón and Hilbert operators

For $0 \leq \alpha < n$, f a Lebesgue measurable function and $x \in \mathbb{R}^n$, $x \neq 0$, the general n-dimensional Calderón and Hilbert operators are defined by

$$S_\alpha f(x) = P_\alpha f(x) + Q_\alpha f(x) \quad \text{and} \quad H_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{(|x| + |y|)^{n-\alpha}} dy,$$

where $P_\alpha f(x) = \frac{1}{|x|^{n-\alpha}} \int_{|y| \leq |x|} f(y) dy$ and $Q_\alpha f(x) = \int_{|y| > |x|} \frac{f(y)}{|y|^{n-\alpha}} dy$.

Again, it is immediate that for nonnegative functions f, the following pointwise inequalities hold

$$H_\alpha f(x) \leq S_\alpha f(x) \leq 2^{n-\alpha} H_\alpha f(x), \tag{1}$$

and consequently, all weighted-L^p inequalities obtained for S are true for H and reciprocally.

In spite of the punctual comparison (1), we will show in Section 4 that the results obtained for S_α and H_α are not analogous when the BMO^γ and Lipschitz spaces are involved.

Both operators S_α and H_α appear in several different contexts and applications, see for instance [4, 11–17].

Next, we introduce the spaces of functions and the classes of weights which appear in our main results.

Recall that a measurable function f defined on $E \subset \mathbb{R}^n$ is said to be *essentially bounded* provided there is some $M \geq 0$, called an *essential upper bound* for f, for which $|f(x)| \leq M$ for almost all $x \in E$. As usual, the class of all functions that are essentially bounded on E is denoted by $L^\infty(E)$ and $\|f\|_\infty$ is the infimum of the essential upper bounds for $f \in L^\infty(E)$. Then, $(L^\infty(E), \|\cdot\|_\infty)$ is a Banach space.

Now, a Lebesgue measurable function f belongs to $L^\infty(\omega)$ if $\|f\omega\|_\infty < \infty$.

Also recall that $L^1_{loc}(\mathbb{R}^n)$ denotes the space of locally integrable functions f satisfying that $\|f\chi_B\|_1$ is finite for every ball $B \subset \mathbb{R}^n$.

Definition 3.2. Let ω be a weight in $\mathbb{R}^n$ and $0 \leq \gamma < 1/n$. A locally integrable function f belongs to $BMO^\gamma(\omega)$ if there exists a constant C such that for every ball $B \subset \mathbb{R}^n$,

$$\frac{1}{\omega(B)|B|^\gamma} \int_B |f - f_B| \leq C. \tag{2}$$

The seminorm of $f \in BMO^\gamma(\omega)$, $\|f\|_{BMO^\gamma(\omega)}$, is the infimum of all such C.

Definition 3.4. Let ω be a weight in $\mathbb{R}^n$ and $0 \leq \gamma < 1/n$. A locally integrable function f belongs to $BM_0^\gamma(\omega)$ if there exists a constant C such that

$$\frac{1}{\omega(B)|B|^\gamma}\int_B |f| \leq C \tag{3}$$

for every ball $B \subset \mathbb{R}^n$ centered at the origin.

The norm of $f \in BM_0^\gamma(\omega)$, denoted by $\|f\|_{BM_0^\gamma(\omega)}$, is the infimum of all such C. We will denote by $BM_0(\omega) = BM_0^0(\omega)$.

Observe that with these definitions the space $BMO^0(\omega)$ is the weighted version of BMO introduced by Muckenhoupt and Wheeden in [18]. Also, the family of spaces $BMO^\gamma(\omega)$ is contained in the family of weighted Lipschitz spaces $\mathcal{I}_\omega(\gamma)$ defined and studied in [8], and $BMO^\gamma(\omega)$ for $\omega \equiv 1$ is the well known Lipschitz integral space. Furthermore, we note that $L^\infty(\omega^{-1}) \subset BM_0(\omega) \cap BMO(\omega)$.

Given $p > 1$, it is known that a weight ω satisfies the reverse Hölder inequality with exponent p, denoted by $\omega \in RH(p)$, if

$$\left(\frac{1}{|B|}\int_B \omega^p\right)^{1/p} \leq C\frac{1}{|B|}\int_B \omega \tag{4}$$

for all balls $B \subset \mathbb{R}^n$ and some constant C.

Definition 3.7. Given $p > 1$, a weight ω belongs to $RH_0(p)$ if it satisfies (4) but only for balls centered at the origin.

Definition 3.8. A weight ω belongs to D_0 if it satisfies the doubling condition $\omega(2B) \leq C\omega(B)$ for every ball $B \subset \mathbb{R}^n$ centered at the origin and some constant C.

Definition 3.9. Let $\eta \geq 1$, a weight ω belongs to D_η if it satisfies the doubling condition

$$\frac{\omega(2B(x,|x|+r))}{|B(x,|x|+r)|^\eta} \leq C\frac{\omega(B(x,r))}{|B(x,r)|^\eta}$$

every ball $B(x,r) \subset \mathbb{R}^n$ and some constant C.

It is immediate that $D_\eta \subset D_0$ for all η, and D_η is increasing in η. It is well known that each weight in the Muckenhoupt class A_∞ is in $RH(p) \cap D_\eta$ for some p and for some η, see for instance [19]. On the other hand, there exist weights belonging to D_η for some η, such that it is not in A_∞, see [20].

Also, we observe the following property that we will use along this chapter. If $\omega \in D_\eta$ there exists a constant C such that

$$\omega(B) \leq C\omega\left(B\backslash\frac{1}{2}B\right) \tag{5}$$

for every ball $B \subset \mathbb{R}^n$ centered at the origin.

Definition 3.11. Let $0 \leq \alpha < n$ and $1 < p < \infty$. A weight ω belongs to $\mathrm{H}_0(\alpha,p)$ if there exists a constant C such that

$$\left(\int_{B^c}\frac{\omega^{p'}(y)}{|y|^{(n-\alpha+1)}p'}dy\right)^{1/p'} \leq C\frac{\omega(B)}{|B|^{1+1/p-\alpha/n+1/n}} \tag{6}$$

for every ball $B \subset \mathbb{R}^n$ centered at the origin.

A weight ω belongs to $\mathrm{H}_0(\alpha, \infty)$ if there exists a constant C such that

$$\int_{B^c} \frac{\omega(y)}{|y|^{n-\alpha+1}} dy \le C \frac{\omega(B)}{|B|^{1-\alpha/n+1/n}} \tag{7}$$

for every ball $B \subset \mathbb{R}^n$ centered at the origin.

The classes of weights $\mathrm{H}_0(\alpha, p)$ and $\mathrm{H}_0(\alpha, \infty)$ satisfying (6) and (7) respectively but for all ball $B \subset \mathbb{R}^n$, were introduced and studied in [8].

4. Weighted Lebesgue and BMO^γ norm inequalities for S_α and H_α

Before beginning our study of the generalized Calderón operator, we notice that $S_\alpha f$ can be identically infinite for some functions f belonging to $L^p(\omega^{-p})$ or $BM_0^\gamma(\omega)$. For example, for $\omega \equiv 1$ and $\alpha > 0$, if $f(x) = |x|^{-\alpha}\chi_{B^c(0,1)}(x)$ and $n/\alpha < p$, then $f \in L^p(\omega^{-p})$ but $S_\alpha f \equiv \infty$. For the case $n/\alpha = p$, if $g(x) = |x|^{-\alpha}(\log|x|)^{-(1+1/p)/2}\chi_{B^c(0,2)}(x)$, then $g \in L^p(\omega^{-p})$ but $S_\alpha g \equiv \infty$. Also, if $h(x) = \chi_{B^c(0,1)}(x)$, then $h \in BM_0^\gamma(\omega)$ but $S_\alpha h \equiv \infty$ for all $0 \le \alpha < n$. However, in Lemma 4.7 we will show that if f belongs to $L^p(\omega^{-p}) \cup BM_0^\gamma(\omega)$ and $S_\alpha f(x)$ is finite for some $x \ne 0$, then $S_\alpha f$ is finite on $\mathbb{R}^n \setminus \{0\}$. This also happens for the generalized Hilbert operator since the comparison (1).

Therefore, throughout the following sections we shall consider S_α and H_α defined on functions f belonging to $L^p(\omega^{-p})$ or $BM_0^\gamma(\omega)$ such that $S_\alpha f$ and $H_\alpha f$ are finite for some $x \ne 0$.

Also, note that $S_\alpha f$ is finite on $\mathbb{R}^n \setminus \{0\}$ for all compactly supported functions $f \in L^\infty(\omega^{-1})$, and the same holds for $H_\alpha f$. These functions belongs to $L^p(\omega^{-p})$ and those such that zero is not in their support belongs to $BM_0^\gamma(\omega)$.

The operator P is naturally bounded from BM_0 into L^∞ and analogously, Q is naturally bounded from BM_0 into BMO (see Proposition 3.5 in [13]). So, immediately the Calderón operator is bounded from BM_0 into BMO. This natural boundedness is our motivation in order to consider the $BM_0^\gamma(\omega)$ spaces and obtain Theorems 1.5 and 1.7. Likewise, since $L^\infty(\omega^{-1}) \subset BM_0(\omega)$, we get Corollaries 4.1 and 4.2.

We now state the main results of this chapter.

Theorem 1.4 Suppose $\alpha > 0$, $n/\alpha \le p < n/(\alpha-1)^+$, $\eta = 1 + 1/n + 1/p - \alpha/n$ and $\delta = \alpha/n - 1/p$. The operator S_α is bounded from $L^p(\omega^{-p})$ into $BMO^\delta(\omega)$ and $\omega^{p'} \in D_0$ if and only if $\omega \in RH_0(p') \cap D_\eta$.

Theorem 1.5 Suppose $0 \le \alpha < 1$, $0 \le \gamma < 1/n - \alpha/n$, $\eta = 1 + 1/n - \alpha/n - \gamma$ and $\delta = \alpha/n + \gamma$. The operator S_α is bounded from $BM_0^\gamma(\omega)$ into $BMO^\delta(\omega)$ and $\omega \in D_0$ if and only if $\omega \in D_\eta$.

Corollary 4.1. *Let $\eta = 1 + 1/n$. Then S is bounded from $L^\infty(\omega^{-1})$ into $BMO(\omega)$ and $\omega \in D_0$ if and only if $\omega \in D_\eta$.*

Theorem 1.6 Suppose $\alpha > 0$, $n/\alpha \le p < n/(\alpha-1)^+$, $\eta = 1 + 1/n + 1/p - \alpha/n$ and $\delta = \alpha/n - 1/p$. The operator H_α is bounded from $L^p(\omega^{-p})$ into $BMO^\delta(\omega)$ if and only if $\omega \in \mathrm{H}_0(\alpha, p) \cap RH_0(p') \cap D_\eta$.

Theorem 1.7 Suppose $0 \le \alpha < 1$, $0 \le \gamma < 1/n - \alpha/n$, $\eta = 1 + 1/n - \alpha/n - \gamma$ and $\delta = \alpha/n + \gamma$. The operator H_α is bounded from $BM_0^\gamma(\omega)$ into $BMO^\delta(\omega)$ if and only if $\omega \in \mathrm{H}_0(\alpha + n\gamma, \infty) \cap D_\eta$.

Corollary 4.2. *Let $\eta = 1 + 1/n$. Then H is bounded from $L^\infty(\omega^{-1})$ into $BMO(\omega)$ if and only if $\omega \in \mathrm{H}_0(0,\infty) \cap D_\eta$.*

Remark 4.3. It is classic the study of the boundedness of operators between L^∞ and BMO spaces. In [10], the results obtained in Corollaries 4.1 and 4.2 are originals, even in the unweighted case for H. The unweighted case for S is contained in Proposition 3.5 of [13].

Remark 4.4. The limit case $p = \infty$ ($p' = 1$) of Theorem 1.4 is contained in Theorem 1.5 with $\gamma = 0$, since the hypotheses on the weights coincide. This also is true to Theorems 1.6 and 1.7.

Let α, p and η be as in Theorems 1.4 and 1.6. It is not difficult to show that if $\omega^{p'} \in A_{1,0}$ then $\omega \in \mathrm{H}_0(\alpha,p) \cap RH_0(p') \cap D_\eta$. Also, if $\omega(x) = |x|^\beta$ with $\beta \in (0, 1 + n/p - \alpha)$, then $\omega^{p'} \notin A_{1,0}$ but $\omega \in \mathrm{H}_0(\alpha,p) \cap RH_0(p') \cap D_\eta$. Furthermore, if $\omega(x) = |x|^\beta$ with $\beta = 1 + n/p - \alpha$, then $\omega \in RH_0(p') \cap D_\eta$ but $\omega \notin \mathrm{H}_0(\alpha,p)$. Now, if in addition $0 < \alpha < 1$ and $p' > n/(1-\alpha)$, we have that if $\omega^{p'} \in A_{p'+1,0}$ then $\omega \in \mathrm{H}_0(\alpha,p) \cap RH_0(p') \cap D_\eta$. In fact, the $\mathrm{H}_0(\alpha,p)$-condition is obtained directly from the $A_{p'+1,0}$-condition, and by Hölder inequality we have that

$$\begin{aligned}\left(\frac{\omega^{p'}(B(0,2(|x_0|+r)))}{|B(0,2(|x_0|+r))|}\right)^{1/p'} &\le C\frac{|B(0,2(|x_0|+r))|}{\omega^{-1}(B(x_0,r))} \le C\left(\frac{|x_0|+r}{r}\right)^n \frac{\omega(B(x_0,r))}{|B(x_0,r)|} \\ &\le C\left(\frac{|x_0|+r}{r}\right)^{1-\alpha+n/p} \frac{\omega(B(x_0,r))}{|B(x_0,r)|}\end{aligned}$$

for all balls $B(x_0,r) \subset \mathbb{R}^n$. Thus, the $RH_0(p')$ and D_η conditions follow from the last expression.

On the other hand, suppose that α, γ and η be as in Theorems 1.5 and 1.7. If $\omega \in A_{1,0}$ then $\omega \in \mathrm{H}_0(\alpha + n\gamma, \infty) \cap D_\eta$. Also, if $\omega(x) = |x|^\beta$ with $\beta \in (0, 1 - \alpha - n\gamma)$, then $\omega \notin A_{1,0}$ but $\omega \in \mathrm{H}_0(\alpha + n\gamma, \infty) \cap D_\eta$. Finally, if $\omega(x) = |x|^\beta$ with $\beta = 1 - \alpha - n\gamma$, then $\omega \in D_\eta$ but $\omega \notin \mathrm{H}_0(\alpha + n\gamma, \infty)$.

We shall denote by $A(x,r,R)$ with $0 < r < R$ the annulus centered at x with radii r and R, and by C and c positive constants not necessarily the same at each occurrence.

Before proceeding to the proofs of the main theorems we give some previous lemmas.

Suppose that $1 < p < \infty$ and $\omega \in RH_0(p')$, then it is easy to see that there exists C such that

$$\int_B |f| \le C\frac{\omega(B)}{|B|^1/p}\|f\|_{L^p(\omega^{-p})} \tag{8}$$

for all $f \in L^p(\omega^{-p})$and for every ball $B \subset \mathbb{R}^n$ centered at the origin.

Lemma 4.6. (i) *Let $0 < \alpha < n$ and $1 < p < \infty$. If $\omega \in \mathrm{H}_0(\alpha,p)$ then there exists C such that*

$$\int_{B^c} \frac{|f(y)|}{|y|^{n-\alpha+1}}dy \le C\frac{\omega(B)}{|B|^{1+1/p-\alpha/n+1/n}}\|f\|_{L^p(\omega^{-p})}$$

for all $f \in L^p(\omega^{-p})$ and for every ball $B \subset \mathbb{R}^n$ centered at the origin.

(ii) *Let $0 \le \alpha < 1$, $0 \le \gamma < 1/n - \alpha/n$ and $\eta = 1 + 1/n - \alpha/n - \gamma$. If $\omega \in \mathrm{H}_0(\alpha + n\gamma, \infty) \cap D_\eta$ then there exists C such that*

$$\int_{B^c} \frac{|f(y)|}{|y|^{n-\alpha+1}} dy \le C \frac{\omega(B)}{|B|^\eta} \|f\|_{BM_0^\gamma(\omega)}$$

for all $f \in BM_0^\gamma(\omega)$ and for every ball $B \subset \mathbb{R}^n$ centered at the origin.

Proof: The part (i) is immediate from Hölder's inequality and Definition 3.11. For (ii), since the hypothesis on ω and (3.10), for $B = B(0, r)$ we have

$$\begin{aligned}
\int_{B^c} \frac{|f(y)|}{|y|^{n-\alpha+1}} dy &\le C \sum_{k=0}^{\infty} \frac{1}{\left(2^k r\right)^{n-\alpha+1}} \int_{2^k r \le |y| < 2^{k+1} r} |f(y)| dy \\
&\le C \|f\|_{BM_0^\gamma(\omega)} \sum_{k=0}^{\infty} \frac{\omega\left(B\left(0, 2^{k+1} r\right)\right)}{\left(2^k r\right)^{n-\alpha+1-n\gamma}} \\
&\le C \|f\|_{BM_0^\gamma(\omega)} \sum_{k=0}^{\infty} \frac{\omega\left(B\left(0, 2^{k+1} r\right) \backslash B\left(0, 2^k r\right)\right)}{\left(2^k r\right)^{n-(\alpha+n\gamma)+1}} \\
&\le C \|f\|_{BM_0^\gamma(\omega)} \frac{\omega(B)}{|B|^\eta}.
\end{aligned}$$

Lemma 4.7. (i) *Let* $\alpha > 0$, $1 < p < \infty$ *and* $\omega \in RH_0(p')$. *If* $f \in L^p(\omega^{-p})$ *and there exists* $x \neq 0$ *such that* $S_\alpha f(x)$ *is finite, then* $S_\alpha f$ *is finite on* $\mathbb{R}^n \backslash \{0\}$ *and* $S_\alpha f \in L^1_{loc}(\mathbb{R}^n)$. *The claim also holds for* H_α.

(ii) *Let* $\omega \in D_\eta$. *If* $f \in BM_0^\gamma(\omega)$ *and there exists* $x \neq 0$ *such that* $S_\alpha f(x)$ *is finite, then* $S_\alpha f$ *is finite on* $\mathbb{R}^n \backslash \{0\}$ *and* $S_\alpha f \in L^1_{loc}(\mathbb{R}^n)$. *The claim also holds for* H_α.

Proof: Since (3.1) we will only consider the operator S_α. Suppose f is a nonnegative function in $L^1_{loc}(\mathbb{R}^n)$ such that $S_\alpha f(x_0) < \infty$ for some $x_0 \neq 0$. Then $Q_\alpha f(x) < \infty$ for $|x| \ge |x_0|$, and if $0 < |x| < |x_0|$ then

$$Q_\alpha f(x) \le \frac{1}{|x|^{n-\alpha}} \int_{|x| < |y| < |x_0|} f(y) dy + Q_\alpha f(x_0) < \infty.$$

Furthermore, since

$$\int_{B(0,r)} (Q_\alpha f(x) - Q_\alpha f(\nu)) dx \le \int_{B(0,r)} f(y) r^\alpha dy < \infty,$$

where $|\nu| = r$, then $Q_\alpha f \in L^1_{loc}(\mathbb{R}^n)$.

If $\alpha > 0$ it is immediate that $P_\alpha f \in L^1_{loc}(\mathbb{R}^n)$. Therefore, (i) follows from (4.5). For (ii) it remains to show that $P_\alpha f \in L^1_{loc}(\mathbb{R}^n)$ in the case $\alpha = 0$. Let $B_j = B\left(0, 2^{-j} r\right)$, $j = 0, 1, \ldots,$ by (3.10) we have

$$\begin{aligned}
\int_{B_0} \frac{1}{|x|^n} \int_{B(0,|x|)} f(y) dy dx &\le C \|f\|_{BM_0^\gamma(\omega)} \int_{B_0} \frac{\omega(B(0,|x|))}{|x|^{n-n\gamma}} dx \\
&\le C \|f\|_{BM_0^\gamma(\omega)} \sum_{j=0}^{\infty} \frac{r^{n\gamma-n}}{2^{j(n\gamma-n)}} \int_{B_j \backslash B_{j+1}} \omega\left(B_j\right) dx \\
&\le C \|f\|_{BM_0^\gamma(\omega)} r^{n\gamma} \sum_{j=0}^{\infty} \frac{\omega\left(B_j \backslash B_{j+1}\right)}{2^{jn\gamma}} \\
&\le C \|f\|_{BM_0^\gamma(\omega)} r^{n\gamma} \omega(B_0).
\end{aligned}$$

Proof of Theorem 1.4: We begin showing the sufficient condition. Let $B = B(x_0, r)$. If $x_0 = 0$, let $u = re_1/2$ and $v = 3re_1/4$, where $e_1 = (1, \dots, 0)$. If $x_0 \neq 0$, let $u = (|x_0|+r/2)x_0/|x_0|$ and $v = (|x_0|+3r/4)x_0/|x_0|$. Thus, we consider the following two regions

$$\begin{aligned} U &= B(u, r/8) \cap \{u + h : \operatorname{sign}(u_i) = \operatorname{sign}(h_i) \;\; i = 1, \dots, n\}, \\ V &= B(v, r/4) \cap \{v + h : \operatorname{sign}(v_i) = \operatorname{sign}(h_i) \;\; i = 1, \dots, n\}, \end{aligned} \tag{9}$$

where $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$ and $h = (h_1, \dots, h_n)$. In the case $u_i = 0$ for some i, we choose $h_i > 0$. Clearly, we have the estimates $\operatorname{dist}(U, V) = Cr$,

$$|U| = \frac{1}{2^n}|B(u, r/8)| = C|B| \quad \text{and} \quad |V| = \frac{1}{2^n}|B(v, r/4)| = C|B|.$$

Let f a nonnegative function in $L^p(\omega^{-p})$ such that $\operatorname{supp}(f) \subset B(0, |x_0|+r/2)$, where $\operatorname{supp}(f)$ is the closure of the set $\{x : f(x) \neq 0\}$. Then

$$\begin{aligned} \|S_\alpha f\|_{BMO^\delta(\omega)} &\geq \frac{C}{\omega(B)|B|^{1+\delta}} \int_B \int_B |S_\alpha f(x) - S_\alpha f(z)| dz dx \\ &\geq \frac{C}{\omega(B)|B|^{1+\delta}} \int_U \int_V \left| \left(\frac{1}{|x|^{n-\alpha}} - \frac{1}{|z|^{n-\alpha}} \right) \int_{B(0,|x_0|+r/2)} f(y) dy \right| dz dx. \end{aligned}$$

Note that, for $x \in U$ and $z \in V$ we have $\frac{1}{|x|^{n-\alpha}} - \frac{1}{|z|^{n-\alpha}} \geq C \frac{r}{(|x_0|+r)^{n-\alpha+1}}$. Then

$$\|S_\alpha f\|_{BMO^\delta(\omega)} \geq \frac{Cr^{n+1}}{\omega(B)|B|^\delta (|x_0|+r)^{n-\alpha+1}} \int_{B(0,|x_0|+r/2)} f(y) dy. \tag{10}$$

Thus, taking $f(y) = \omega^{p'}(y)\chi_{B(0,|x_0|+r/2)}(y)$ in (10) and since the boundedness of S_α and $\omega^{p'} \in D_0$, we have

$$\left(\frac{\omega^{p'}(B(0, |x_0|+r))}{|B(0, |x_0|+r)|} \right)^{1/p'} \leq C \left(\frac{|x_0| + r}{r} \right)^{1-\alpha+n/p} \frac{\omega(B)}{|B|}.$$

Taking $x_0 = 0$ in the last expression, we have that $\omega \in RH_0(p')$. Then, applying the Hölder's inequality, we obtain that ω satisfies the desired condition D_η.

Now, let us show the necessary condition. Let $f \in L^p(\omega^{-p})$ such that $S_\alpha f(x)$ is finite for some $x \neq 0$ and let $\omega \in RH_0(p') \cap D_\eta$. It is immediate that $\omega^{p'} \in D_0$. Thus $S_\alpha f \in L^1_{loc}(\mathbb{R}^n)$ by (i) of Lemma 4.7. First, we consider $B = B(0, r)$, $x \in B$ and $x \neq 0$. Let ν be such that $|\nu| = r$, and let

$$K_\nu(x, y) = \min\left\{1, \frac{|y|^{n-\alpha}}{|x|^{n-\alpha}}\right\} - \min\left\{1, \frac{|y|^{n-\alpha}}{|\nu|^{n-\alpha}}\right\}.$$

Then, since $K_\nu(x, y) = 0$ for $|y| > |\nu|$, we have

$$S_\alpha f(x) - S_\alpha f(\nu) = \int_{|y| \leq |\nu|} K_\nu(x, y) \frac{f(y)}{|y|^{n-\alpha}} dy. \tag{11}$$

If $|y| \leq |\nu|$ then $K_\nu(x,y) \geq 0$, so

$$\begin{aligned}\frac{1}{\omega(B)}\int_B |S_\alpha f(x) - S_\alpha f(\nu)|dx &\leq \frac{1}{\omega(B)}\int_B\int_B K_\nu(x,y)\frac{|f(y)|}{|y|^{n-\alpha}}dydx \\ &= \frac{1}{\omega(B)}\int_B\int_{|y|\leq|x|} K_\nu(x,y)\frac{|f(y)|}{|y|^{n-\alpha}}dydx + \frac{1}{\omega(B)}\int_B\int_{|x|<|y|\leq r} K_\nu(x,y)\frac{|f(y)|}{|y|^{n-\alpha}}dydx.\end{aligned} \tag{12}$$

Now we estimate each term in (12).

If $|y| \leq |x|$ then $K_\nu(x,y) \leq |y|^{n-\alpha}|x|^{-(n-\alpha)}$. So, by (8) we have

$$\begin{aligned}\frac{1}{\omega(B)}\int_B\int_{|y|\leq|x|} K_\nu(x,y)\frac{|f(y)|}{|y|^{n-\alpha}}dydx &\leq \frac{1}{\omega(B)}\int_B \frac{1}{|x|^{n-\alpha}}\int_B |f(y)|dydx \\ &\leq C\|f\|_{L^p(\omega^{-p})}|B|^\delta.\end{aligned}$$

For the second term, since $0 \leq K_\nu(x,y) \leq 1$ and (8), we have

$$\begin{aligned}\frac{1}{\omega(B)}\int_B\int_{|x|<|y|\leq r} K_\nu(x,y)\frac{|f(y)|}{|y|^{n-\alpha}}dydx &\leq \frac{1}{\omega(B)}\int_B \frac{1}{|x|^{n-\alpha}}\int_{|x|<|y|\leq r} |f(y)|dydx \\ &\leq \frac{C}{\omega(B)}\int_B |f(y)||y|^\alpha dy \\ &\leq C\|f\|_{L^p(\omega^{-p})}|B|^\delta.\end{aligned} \tag{13}$$

Then, by (12) and (13), we have proved

$$\frac{1}{\omega(B)|B|^\delta}\int_B |S_\alpha f(x) - S_\alpha f(\nu)|dx \leq C\|f\|_{L^p(\omega^{-p})}, \tag{14}$$

for every ball B centered at the origin.

We now consider $B = B(x_0, r)$ with $r < |x_0|/8$. By (14) it is enough to consider only these balls B. Let $x \in B$ and $\nu = (|x_0|+r)x_0/|x_0|$. In the same way as (11), we have

$$S_\alpha f(x) - S_\alpha f(\nu) = \int_{|y|\leq|\nu|} K_\nu(x,y)\frac{f(y)}{|y|^{n-\alpha}}dy.$$

Now, we note that if $|y| \leq |\nu|$ then $K_\nu(x,y) \geq 0$. Applying the mean value theorem and using $|\nu| \sim |x|$, then

$$K_\nu(x,y) \leq \frac{|y|^{n-\alpha}}{|x|^{n-\alpha}} - \frac{|y|^{n-\alpha}}{|\nu|^{n-\alpha}} \leq C\frac{r|y|^{n-\alpha}}{|\nu|^{n-\alpha+1}}. \tag{15}$$

Thus, by (8) and $\omega \in D_\eta$, we have

$$\begin{aligned}\frac{1}{\omega(B)}\int_B |S_\alpha f(x) - S_\alpha f(\nu)|dx &\leq C\frac{r}{\omega(B)|\nu|^{n-\alpha+1}}\int_B\int_{|y|\leq|\nu|} |f(y)|dydx \\ &\leq C\|f\|_{L^p(\omega^{-p})}\frac{r^{n+1}}{|\nu|^{n-\alpha+1+n/p}}\frac{\omega(B(0,|\nu|))}{\omega(B)} \\ &\leq C\|f\|_{L^p(\omega^{-p})}|B|^\delta.\end{aligned} \tag{16}$$

Therefore, (14) and (16) complete the proof of the theorem.

Proof of Theorem 1.5: We begin showing the sufficient condition. Let $B = B(x_0, r)$ and let u, v, U and V as in (9) of the proof of Theorem 1.4. Then, we again have

$$\|S_\alpha f\|_{BMO^\delta(\omega)} \geq \frac{Cr^{n+1}}{\omega(B)|B|^\delta(|x_0|+r)^{n-\alpha+1}} \int_{B(0,|x_0|+r/2)} f(y)dy, \tag{17}$$

for every nonnegative function f in $BM_0^\gamma(\omega)$ such that $\operatorname{supp}(f) \subset B(0, |x_0|+r/2)$. Now, if $\gamma = 0$ we take $f(y) = \omega(y)\chi_{B(0,|x_0|+r/2)}(y)$ in (17) and since $\|f\|_{BM_0^\gamma(\omega)} \leq 1$, the boundedness of S_α and $\omega \in D_0$, we have $\omega \in D_\eta$.

If $\gamma > 0$, let $f(y) = P_{n\gamma}\left(\omega\chi_{B(0,|x_0|+r/2)}\right)(y)$, then $\|f\|_{BM_0^\gamma(\omega)} \leq C$ and

$$\begin{aligned}\int_{B(0,|x_0|+r/2)} f(y)dy &= C\int_{B(0,|x_0|+r/2)} \omega(t)((|x_0|+r/2)^{n\gamma} - |t|^{n\gamma})dt \\ &\geq C(|x_0|+r)^{n\gamma}\omega(B(0,(|x_0|+r/2)/2)).\end{aligned} \tag{18}$$

Therefore, using this function f in (17), the boundedness of S_α, (18) and $\omega \in D_0$, we have $\omega \in D_\eta$.

Now, let us show the necessary condition. Let $f \in BM_0^\gamma(\omega)$ such that $S_\alpha f(x)$ is finite for some $x \neq 0$ and let $\omega \in D_\eta$. Thus $S_\alpha f \in L^1_{loc}(\mathbb{R}^n)$ by (ii) of Lemma 4.7. We begin considering $B = B(0, r)$, $x \in B$ and $x \neq 0$. Let ν be such that $|\nu| = r$. In the same way as we did in (12), we have

$$\begin{aligned}\frac{1}{\omega(B)}\int_B |S_\alpha f(x) - S_\alpha f(\nu)|dx &\leq \frac{1}{\omega(B)}\int_B\int_{|y|\leq|x|} K_\nu(x,y)\frac{|f(y)|}{|y|^{n-\alpha}}dydx \\ &+ \frac{1}{\omega(B)}\int_B\int_{|x|<|y|\leq r} K_\nu(x,y)\frac{|f(y)|}{|y|^{n-\alpha}}dydx,\end{aligned} \tag{19}$$

where $K_\nu(x,y) = \min\left\{1, \frac{|y|^{n-\alpha}}{|x|^{n-\alpha}}\right\} - \min\left\{1, \frac{|y|^{n-\alpha}}{|\nu|^{n-\alpha}}\right\}$.

We estimate the first term of (19). Let $B_j = B\left(0, 2^{-j}r\right), j = 0, 1, \ldots$. Thus, since $K_\nu(x,y) \leq |y|^{n-\alpha}|x|^{-(n-\alpha)}$ for $|y| \leq |x|$ and (5), we have

$$\begin{aligned}\frac{1}{\omega(B)}\int_B\int_{|y|\leq|x|} K_\nu(x,y)\frac{|f(y)|}{|y|^{n-\alpha}}dydx &\leq \frac{1}{\omega(B)}\int_B \frac{1}{|x|^{n-\alpha}}\int_{|y|\leq|x|}|f(y)|dydx \\ &\leq C\|f\|_{BM_0^\gamma(\omega)}\frac{1}{\omega(B)}\int_B \frac{\omega(B(0,|x|))}{|x|^{n-\alpha-n\gamma}}dx \\ &\leq C\|f\|_{BM_0^\gamma(\omega)}\frac{r^{n\gamma+\alpha}}{\omega(B)}\sum_{j=0}^\infty \frac{\omega\left(B_j\backslash B_{j+1}\right)}{2^{j(n\gamma+\alpha)}} \\ &\leq C\|f\|_{BM_0^\gamma(\omega)}\frac{|B|^\delta}{\omega(B)}\sum_{j=0}^\infty \omega\left(B_j\backslash B_{j+1}\right) \\ &= C\|f\|_{BM_0^\gamma(\omega)}|B|^\delta.\end{aligned} \tag{20}$$

For the second term of (19), since $0 \leq K_\nu(x,y) \leq 1$, we have

$$\frac{1}{\omega(B)}\int_B \int_{|x| \leq |y| \leq r} K_\nu(x,y)\frac{|f(y)|}{|y|^{n-\alpha}}dydx \leq \frac{1}{\omega(B)}\int_B \frac{|f(y)|}{|y|^{n-\alpha}}\int_{|x| \leq |y|} 1dxdy \\ \leq C\|f\|_{BM_0^\gamma(\omega)}|B|^\delta. \tag{21}$$

Therefore, by (19)-(21) we have proved

$$\frac{1}{\omega(B)|B|^\delta}\int_B |S_\alpha f(x) - S_\alpha f(\nu)|dx \leq C\|f\|_{BM_0^\gamma(\omega)}, \tag{22}$$

for every ball B centered at the origin.

We now consider $B = B(x_0, r)$ with $r < |x_0|/8$. By (22) it is enough to consider only these balls B. Let $x \in B$ and $\nu = (|x_0|+r)x_0/|x_0|$. In the same way as we obtained (11) and (15) in the previous proof, we have

$$S_\alpha f(x) - S_\alpha f(\nu) = \int_{|y| \leq |\nu|} K_\nu(x,y)\frac{f(y)}{|y|^{n-\alpha}}dy$$

and $K_\nu(x,y) \leq Cr|y|^{n-\alpha}|\nu|^{-(n-\alpha+1)}$. By $\omega \in D_\eta$, we have

$$\frac{1}{\omega(B)}\int_B |S_\alpha f(x) - S_\alpha f(\nu)|dx \leq C\frac{r^{n+1}}{\omega(B)|\nu|^{n-\alpha+1}}\int_{|y| \leq |\nu|}|f(y)|dy \\ \leq C\|f\|_{BM_0^\gamma(\omega)}\frac{r^{n+1}}{|\nu|^{n-\alpha+1-n\gamma}}\frac{\omega(B(0,|\nu|))}{\omega(B)} \\ \leq C\|f\|_{BM_0^\gamma(\omega)}|B|^\delta. \tag{23}$$

Therefore, (22) and (23), complete the proof of the theorem.

Let $x, \nu \in \mathbb{R}^n$, $\nu \neq 0$, then

$$|H_\alpha f(x) - H_\alpha f(\nu)| \leq \int_{|y| \leq |\nu|}|f(y)|\left|\frac{1}{(|x| + |y|)^{n-\alpha}} - \frac{1}{(|\nu| + |y|)^{n-\alpha}}\right|dy \\ + \int_{|y| > |\nu|}|f(y)|\left|\frac{1}{(|x| + |y|)^{n-\alpha}} - \frac{1}{(|\nu| + |y|)^{n-\alpha}}\right|dy. \tag{24}$$

Proof of Theorem 1.6: We begin showing the sufficient condition. Let $B = B(x_0, r)$ and let u, v, U and V as in (9) of the proof of Theorem 1.4. Note that if $x \in U$, $z \in V$ then for all $y \in \mathbb{R}^n$,

$$\frac{1}{(|x| + |y|)^{n-\alpha}} - \frac{1}{(|z| + |y|)^{n-\alpha}} \geq C\frac{r}{(|x_0|+r+|y|)^{n-\alpha+1}}. \tag{25}$$

Hence, if f is a nonnegative function in $L^p(\omega^{-p})$ such that $\operatorname{supp}(f) \subset A(0, r, m)$ and taking $x_0 = 0$ in (25), we have

$$\|H_\alpha f\|_{BMO^\delta(\omega)} \geq \frac{Cr^{n+1}}{\omega(B)|B|^\delta}\int_{A(0,r,m)}\frac{f(y)}{|y|^{n-\alpha+1}}dy, \tag{26}$$

for every ball B centered at the origin.

Thus, taking $f_{m,j}(y) = |y|^{-(n-\alpha+1)/(p-1)}\omega^{p'}(y)\chi_{A_{m,j}}(y)$ in (26) where $A_{m,j} = A(0, r, m) \cap \{y : 1/j \le \omega(y) < j\}$, $m, j = 1, 2, \dots$, using the boundedness of H_α and letting $m \to \infty$, $j \to \infty$ we obtain that $\omega \in \mathrm{H}_0(\alpha, p)$.

On the other hand, if f is a nonnegative function in $L^p(\omega^{-p})$ such that $\operatorname{supp}(f) \subset B(0, 2(|x_0|+r))$, then by (25)

$$\|H_\alpha f\|_{BMO^\delta(\omega)} \ge \frac{Cr^{n+1}}{\omega(B)|B|^\delta (|x_0|+r)^{n-\alpha+1}} \int_{B(0,2(|x_0|+r))} f(y)dy. \tag{27}$$

Thus, taking $f_j(y) = \omega^{p'}(y)\chi_{A_j}(y)$ in (27) where $A_j = B(0, 2(|x_0|+r)) \cap \{y : 1/j \le \omega(y) < j\}$, $j = 1, 2, \dots$, and using the boundedness of H_α, we have

$$\left(\int_{A_j} \omega^{p'}(y)dy\right)^{1/p'} \le C\left(\frac{|x_0| + r}{r}\right)^{n-\alpha+1} \frac{\omega(B)}{|B|^{1/p}}.$$

Letting $j \to \infty$ and taking $x_0 = 0$ in the last expression, we can obtain that $\omega \in RH_0(p')$. Then, applying Hölder's inequality, we obtain $\omega \in D_\eta$.

Now, let us show the necessary condition. Let $f \in L^p(\omega^{-p})$ such that $H_\alpha f(x)$ is finite for some $x \ne 0$ and let ω such that $\omega \in \mathrm{H}_0(\alpha, p) \cap RH_0(p') \cap D_\eta$. Hence $H_\alpha f \in L^1_{loc}(\mathbb{R}^n)$ since (i) of Lemma 4.7. We begin considering $B = B(0, r)$, $x \in B$ and $x \ne 0$. Let ν be such that $|\nu| = r$. We estimate the two terms of (24). By (8), we have

$$\begin{aligned}
&\frac{1}{\omega(B)} \int_B \int_{|y| \le |\nu|} \left|\frac{f(y)}{(|x| + |y|)^{n-\alpha}} - \frac{f(y)}{(|\nu| + |y|)^{n-\alpha}}\right| dydx \\
&\qquad \le \frac{C}{\omega(B)} \int_B \int_B \frac{|f(y)|}{|x|^{n-\alpha}} dydx \\
&\qquad \le C\|f\|_{L^p(\omega^{-p})} \int_B \frac{1}{|x|^{n-\alpha+n/p}} dx = C\|f\|_{L^p(\omega^{-p})} |B|^\delta.
\end{aligned} \tag{28}$$

To analyze the second term of (24), we use the mean value theorem, then

$$\left|\frac{1}{(|x| + |y|)^{n-\alpha}} - \frac{1}{(|\nu| + |y|)^{n-\alpha}}\right| \le C\frac{r}{|y|^{n-\alpha+1}}.$$

Thus, by (i) of Lemma 4.6

$$\begin{aligned}
\frac{1}{\omega(B)} \int_B \int_{|y| > |\nu|} \left|\frac{f(y)}{(|x| + |y|)^{n-\alpha}} - \frac{f(y)}{(|\nu| + |y|)^{n-\alpha}}\right| dydx &\le \frac{Cr}{\omega(B)} \int_B \int_{B^c} \frac{|f(y)|}{|y|^{n-\alpha+1}} dydx \\
&\le C\|f\|_{L^p(\omega^{-p})} |B|^\delta.
\end{aligned} \tag{29}$$

Therefore, by (24)–(29), we have proved

$$\frac{1}{\omega(B)|B|^\delta} \int_B |H_\alpha f(x) - H_\alpha f(\nu)| dx \le C\|f\|_{L^p(\omega^{-p})}, \tag{30}$$

for every ball B centered at the origin.

We now consider $B = B(x_0, r)$ with $r < |x_0|/8$. By (28) it is enough to consider only these balls B. Let $x \in B$ and $\nu = (|x_0| + r)x_0/|x_0|$, then $|\nu| \sim |x|$ and $|x| \sim |x_0|$. Using $|y| \le |\nu|$ and the mean value theorem

$$\left| \frac{1}{(|x| + |y|)^{n-\alpha}} - \frac{1}{(|\nu| + |y|)^{n-\alpha}} \right| \le C \frac{r}{|x_0|^{n-\alpha+1}}.$$

Then, by (8) and $\omega \in D_\eta$

$$\begin{aligned} &\frac{1}{\omega(B)} \int_B \int_{|y| \le |\nu|} |f(y)| \left| \frac{1}{(|x| + |y|)^{n-\alpha}} - \frac{1}{(|\nu| + |y|)^{n-\alpha}} \right| dydx \\ &\le \frac{Cr^{n+1}}{\omega(B)|x_0|^{n-\alpha+1}} \int_{|y| \le |\nu|} |f(y)| dy \\ &\le C \|f\|_{L^p(\omega^{-p})} \frac{r^{n+1}}{|x_0|^{n-\alpha+1+n/p}} \omega(B(0, |\nu|)) \\ &= C \|f\|_{L^p(\omega^{-p})} |B|^\delta. \end{aligned} \tag{31}$$

Now, using the mean value theorem

$$\left| \frac{1}{(|x| + |y|)^{n-\alpha}} - \frac{1}{(|\nu| + |y|)^{n-\alpha}} \right| \le C \frac{r}{|y|^{n-\alpha+1}}.$$

Then, by (i) of Lemma 4.6

$$\begin{aligned} \frac{1}{\omega(B)} \int_B \int_{|y| > |\nu|} \left| \frac{f(y)}{(|x| + |y|)^{n-\alpha}} - \frac{f(y)}{(|\nu| + |y|)^{n-\alpha}} \right| dydx &\le \frac{Cr}{\omega(B)} \int_B \int_{B^c} \frac{|f(y)|}{|y|^{n-\alpha+1}} dydx \\ &= C \|f\|_{L^p(\omega^{-p})} |B|^\delta. \end{aligned} \tag{32}$$

Therefore, by (24) with $\nu = (|x_0| + r)x_0/|x_0|$, (31) and (32), we have

$$\frac{1}{\omega(B)|B|^\delta} \int_B |H_\alpha f(x) - H_\alpha f(\nu)| dx \le C \|f\|_{L^p(\omega^{-p})},$$

for every ball $B = B(x_0, r)$ considered. This completes the proof of the theorem.

Proof of Theorem 1.7: We begin showing the sufficient condition. Let $B = B(x_0, r)$ and let u, v, U and V as in (9) of the proof of Theorem 1.4. Then, as in (26) of the proof of Theorem 4 (with $x_0 = 0$), we again have

$$\|H_\alpha f\|_{BMO^\delta(\omega)} \ge \frac{Cr^{n+1}}{\omega(B)|B|^\delta} \int_{A(0,r,m)} \frac{f(y)}{|y|^{n-\alpha+1}} dy \tag{33}$$

for every nonnegative function f in $BM_0^\gamma(\omega)$ such that $\text{supp}(f) \subset A(0, r, m)$ and for every ball B centered at the origin.

Thus, taking $f(y) = |y|^{n\gamma} \omega(y) \chi_{A(0,r,m)}(y)$ in (33), using that $\|f\|_{BM_0^\gamma(\omega)} \le 1$, the boundedness of H_α and letting $m \to \infty$, we have that $\omega \in \mathrm{H}_0(\alpha + n\gamma, \infty)$.

On the other hand, as in (27) of the proof of Theorem 1.6 we again have

$$\|H_\alpha f\|_{BMO^\delta(\omega)} \geq \frac{Cr^{n+1}}{\omega(B)|B|^\delta(|x_0|+r)^{n-\alpha+1}} \int_{B(0,2(|x_0|+r))} f(y)dy, \tag{34}$$

for every nonnegative function f in $BM_0^\gamma(\omega)$ such that $\operatorname{supp}(f) \subset B(0, 2(|x_0|+r))$ and for every ball $B = B(x_0, r)$.

If $\gamma = 0$, we take $f(y) = \omega(y)\chi_{B(0,2(|x_0|+r))}(y)$ in (4.34) and since $\|f\|_{BM_0^\gamma(\omega)} \leq 1$ and the boundedness of H_α, we have $\omega \in D_\eta$.

If $\gamma > 0$, let $f(y) = P_{n\gamma}\left(\omega\chi_{B(0,2(|x_0|+r))}\right)(y)$ then $\|f\|_{BM_0^\gamma(\omega)} \leq C$ and as in (4.18) of the proof of Theorem 1.5, we have

$$\int_{B(0,2(|x_0|+r))} f(y)dy \geq C(|x_0|+r)^{n\gamma}\omega(B(0, |x_0|+r)).$$

Therefore, using this function f in (34) and the boundedness of H_α, we have $\omega \in D_\eta$.

Now, let us show the necessary condition. Let $f \in BM_0^\gamma(\omega)$ such that $H_\alpha f(x)$ is finite for some $x \neq 0$ and let $\omega \in \mathrm{H}_0(\alpha + n\gamma, \infty) \cap D_\eta$. Hence $H_\alpha f \in L^1_{loc}(\mathbb{R}^n)$ by (ii) of Lemma 4.7. We begin considering $B = B(0, r)$, $x \in B$ and $x \neq 0$. Let ν be such that $|\nu| = r$. We estimate the two terms of (24). Then,

$$\begin{aligned}
&\frac{1}{\omega(B)}\int_B\int_{|y|\leq|\nu|}|f(y)|\left|\frac{1}{(|x|+|y|)^{n-\alpha}} - \frac{1}{(|\nu|+|y|)^{n-\alpha}}\right|dydx \\
&\leq \frac{C}{\omega(B)}\int_B \frac{1}{|x|^{n-\alpha}}\int_{|y|\leq|\nu|} f(y)dydx \\
&\leq C\|f\|_{BM_0^\gamma(\omega)}\frac{1}{\omega(B)}\int_B \frac{\omega(B(0,|\nu|))|\nu|^{n\gamma}}{|x|^{n-\alpha}}dx \\
&\leq C\|f\|_{BM_0^\gamma(\omega)}|B|^\delta.
\end{aligned} \tag{35}$$

For the second term of (24), using the mean value theorem

$$\left|\frac{1}{(|x|+|y|)^{n-\alpha}} - \frac{1}{(|\nu|+|y|)^{n-\alpha}}\right| \leq C\frac{r}{|y|^{n-\alpha+1}}. \tag{36}$$

Then, by (ii) of Lemma 4.6

$$\begin{aligned}
&\frac{1}{\omega(B)}\int_B\int_{|y|>|\nu|}|f(y)|\left|\frac{1}{(|x|+|y|)^{n-\alpha}} - \frac{1}{(|\nu|+|y|)^{n-\alpha}}\right|dydx \\
&\leq \frac{Cr}{\omega(B)}\int_B\int_{B^c}\frac{|f(y)|}{|y|^{n-\alpha+1}}dydx \\
&\leq C\|f\|_{BM_0^\gamma(\omega)}|B|^\delta.
\end{aligned} \tag{37}$$

Therefore, by (24) and (35)–(37), we have proved

$$\frac{1}{\omega(B)|B|^\delta}\int_B |H_\alpha f(x) - H_\alpha f(\nu)|dx \leq C\|f\|_{BM_0^\gamma(\omega)}, \tag{38}$$

for every ball B centered at the origin.

We now consider $B = B(x_0, r)$ with $r < |x_0|/8$. By (33) it is enough to consider only these balls B. Let $x \in B$ and $\nu = (|x_0| + r)x_0/|x_0|$, then $|\nu| \sim |x|$ and $|x| \sim |x_0|$. If $|y| \le |\nu|$, by the mean value theorem

$$\left|\frac{1}{(|x| + |y|)^{n-\alpha}} - \frac{1}{(|\nu| + |y|)^{n-\alpha}}\right| \le C\frac{r}{|x_0|^{n-\alpha+1}}.$$

Then, since $\omega \in D_\eta$, we have

$$\begin{aligned}
&\frac{1}{\omega(B)}\int_B\int_{|y|\le|\nu|}|f(y)|\left|\frac{1}{(|x| + |y|)^{n-\alpha}} - \frac{1}{(|\nu| + |y|)^{n-\alpha}}\right|dydx \\
&\le \frac{Cr^{n+1}}{\omega(B)|x_0|^{n-\alpha+1}}\int_{|y|\le|\nu|}|f(y)|dy \\
&\le \|f\|_{BM_0^\gamma(\omega)}|B|^\delta.
\end{aligned} \tag{39}$$

On the other hand, using again the mean value theorem as in (36) and (ii) of Lemma 4.6, we get

$$\begin{aligned}
&\frac{1}{\omega(B)}\int_B\int_{|y|>|\nu|}|f(y)|\left|\frac{1}{(|x| + |y|)^{n-\alpha}} - \frac{1}{(|\nu| + |y|)^{n-\alpha}}\right|dydx \\
&\le \frac{Cr^{n+1}}{\omega(B)}\int_{|y|>|\nu|}\frac{|f(y)|}{|y|^{n-\alpha+1}}dy \\
&\le C\|f\|_{BM_0^\gamma(\omega)}\frac{r^{n+1}}{\omega(B)}\frac{\omega(B(0,|\nu|))}{|\nu|^{n\eta}} \\
&\le C\|f\|_{BM_0^\gamma(\omega)}|B|^\delta.
\end{aligned} \tag{40}$$

Thus, by (24) and (39)–(40), we have proved

$$\frac{1}{\omega(B)|B|^\delta}\int_B|H_\alpha f(x) - H_\alpha f(\nu)|dx \le C\|f\|_{BM_0^\gamma(\omega)},$$

for every ball $B = B(x_0, r)$ considered. This completes the proof of the theorem.

5. Conclusions

As a conclusion to this chapter, we have given necessary and sufficient conditions for the generalized Calderón and Hilbert operators to be bounded from weighted Lesbesgue spaces into suitable weighted BMO and Lipschitz spaces. Then, we have obtained results on the boundedness of these operators from L^∞ into BMO, even in the unweighted case for the Hilbert operator. The class of weights involved are close to the doubling and reverse Hölder conditions related to the Muckenhoupt's classes.

The study of the weighted boundedness for integral operators on function spaces, like the one we develop in this chapter, is one of the main research fields in harmonic analysis. In particular, it has had a profound influence in partial differential equations, several complex variables, and number theory. Evidence of such success and importance is the pioneering work of leading mathematicians Bourgain, Zygmund, Calderón, Muckenhoupt, Wheeden, C. Fefferman, Stein, Ricci, Tao and so on.

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