Received: 22 June 2022 | Revised: 30 June 2023 | Accepted: 20 July 2023

DOI: 10.1112/plms.12559

RESEARCH ARTICLE

Proceedings of the London Mathematical Society

Pointed Hopf algebras over nonabelian groups with nonsimple standard braidings

Iván Angiono[1] | **Simon Lentner**[2] | **Guillermo Sanmarco**[3]

[1]Facultad de Matemática, Astronomía, Física y Computación, Universidad Nacional de Córdoba, CIEM – CONICET, Córdoba, Argentina

[2]Algebra and Number Theory, University of Hamburg, Hamburg, Germany

[3]Department of Mathematics, Iowa State University, Ames, Iowa, USA

Correspondence
Simon Lentner, Algebra and Number Theory, University of Hamburg, Bundesstraße 55, 20146 Hamburg.
Email: simon.lentner@uni-hamburg.de

Funding information
CONICET; Secyt; Alexander von Humboldt Foundation

Abstract

We construct finite-dimensional Hopf algebras whose coradical is the group algebra of a central extension of an abelian group. They fall into families associated to a semisimple Lie algebra together with a Dynkin diagram automorphism. We show conversely that every finite-dimensional pointed Hopf algebra over a nonabelian group with nonsimple infinitesimal braiding of rank at least 4 is of this form. We follow the steps of the Lifting Method by Andruskiewitsch–Schneider. Our starting point is the classification of finite-dimensional Nichols algebras over nonabelian groups by Heckenberger–Vendramin, which consist of low-rank exceptions and large-rank families. We prove that the large-rank families are cocycle twists of Nichols algebras constructed by the second author as foldings of Nichols algebras of Cartan type over abelian groups by outer automorphisms. This enables us to give uniform Lie-theoretic descriptions of the large-rank families, prove generation in degree 1, and construct liftings. We also show that every lifting is a cocycle deformation of the corresponding coradically graded Hopf algebra using an explicit presentation by generators and relations of the Nichols algebra. On the level of tensor categories, we construct families

of graded extensions of the representation category of a quantum group by a group of diagram automorphism.

MSC 2020
16T05, 17B37 (primary)

Contents

1 | INTRODUCTION

1.1 | Background

Groups and Lie algebras have in common that they both admit a tensor product of representations and a dual for each finite-dimensional representation; in other words, their categories of representations are tensor categories. More generally, the category of representations of a Hopf algebra is also a tensor category. Prominent examples of Hopf algebras are the quantum groups $U_q(\mathfrak{g})$ by Drinfeld–Jimbo [22, 40], which are deformations of the enveloping algebra of a semisimple Lie algebra $\mathfrak{g}$ by a formal parameter q, and the small quantum groups $u_q(\mathfrak{g})$ by Lusztig [46], which are finite-dimensional nonsemisimple quotients of $U_q(\mathfrak{g})$ for q a root of unity. One of the initial motivations for quantum groups was their relation to the monodromy of certain differential equations in conformal field theory [27] and invariants of knots and 3-manifolds [53]. Small quantum groups, their representation categories, and their semisimplification are related to Lie algebras in positive characteristic [1, 45] and affine Lie algebras [28], and they have again applications to topology [41] and conformal field theory [25, 44].

While the classification of finite-dimensional semisimple Hopf algebras is still a very hard problem, we may ask for the classification of Hopf algebras H with a *given* maximal cosemisimple part H_0, called the coradical. For example, if $H_0 = \Bbbk G$ is a group ring, then H is called a pointed Hopf algebra. The so-called *Lifting Method* developed by Andruskiewitsch and Schneider is a program for the classification of pointed Hopf algebras. The present article contributes to the classification of finite-dimensional pointed Hopf algebras over an algebraically closed field $\Bbbk$ of characteristic zero by means of this method for G a nonabelian group. Let us quickly recall the main steps and involved notions.

The Nichols algebra $\mathscr{B}(V)$ of an object V in certain braided tensor category is the smallest braided Hopf algebra generated by V and such that the space of primitive elements is precisely V. It is a difficult problem to determine the structure of the Nichols algebra of a given braided vector space, even to determine whether it is finite-dimensional. A main structural insight [31, 36, 38] is the existence of generalized root systems and Weyl groupoids. It is generalized in the sense that the Weyl groupoid moves between different sets of simple roots for which the sets of positive roots look different since reflections may change the braiding and even the Cartan matrix — an effect that already appears for contragredient Lie superalgebras. Nevertheless, finite Weyl groupoids can be classified [20] and show again a pattern of Lie theory, with some infinite series plus low-rank exceptions.

The entry gate of Nichols algebras into the classification of Hopf algebras is the coradical filtration. Namely, every Hopf algebra H comes with a coalgebra filtration $H_0 \subseteq H_1 \subseteq \dots$ where H_0 is the coradical. If we assume that H_0 is a subalgebra, then the associated graded coalgebra $\operatorname{gr} H$ is a Hopf algebra, which decomposes as bosonization $\operatorname{gr} H \simeq (\bigoplus_{n\geqslant 0} R_n)\#H_0$. Here, $R = \bigoplus_{n\geqslant 0} R_n$ is a (coradically) graded Hopf agebra in a braided category; we pay special attention to the subspace R_1 of primitive elements and the subalgebra of R it generates, which is (isomorphic to) the Nichols algebra $\mathscr{B}(R_1)$. As an example, for the (infinite-dimensional) quantum group $H = U_q(\mathfrak{g})$, the coradical H_0 is the group algebra spanned by the root lattice, the space R_1 is spanned by the simple root vectors E_i, F_i, the Nichols algebra is the tensor product of the positive and negative parts $U_q(\mathfrak{g})^{\pm}$, and in the graded algebra $\operatorname{gr} H$, the relation $[E_i, F_i] = 0$ holds in contrast to the nontrivial relation $[E_i, F_i] = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ in H.

To classify pointed Hopf algebras H with a given coradical $H_0 = \Bbbk G$, the Lifting Method proposes the following three steps.

- First, one classifies $\Bbbk G$-Yetter–Drinfeld modules V with finite-dimensional Nichols algebra $\mathscr{B}(V)$. When G is finite and abelian, V is a braided vector space of diagonal type and Heckenberger has classified all these finite-dimensional Nichols algebras in [32]. Besides the Nichols algebras $u_q(\mathfrak{g})^+$ coming from small quantum groups, his list contains Nichols algebras associated to Lie superalgebras in any characteristic and some exceptions [3].
- Second, one wants to prove that the subspace $R_1 \# H_0$ generates the entire algebra $\operatorname{gr} H$, which is known as the *generation in degree 1* problem. In the abelian group case, this has been proved in [12] using a presentation by generators and relations of the Nichols algebras of diagonal type.
- Third, one determines all possible Hopf algebras H associated to each Nichols algebra in the first step, the so-called liftings. For the Nichols algebra $u_q(\mathfrak{g})^+$ with q of sufficiently large order, the possible liftings are described by deforming relations such as $[E_i, F_i] = 0$ and $E_i^{\ell} = 0$ [10]. For an arbitary Nichols algebra over an abelian group, the liftings were determined in [16] and involves the proof that every lifting is a cocycle deformation of the graded Hopf algebra in the sense of [21, 47].

1.2 | Nichols algebras over nonabelian groups

We now discuss Nichols algebras $\mathscr{B}(M)$ of Yetter–Drinfeld modules M over a finite nonabelian group G, see §2.2. The simple Yetter–Drinfeld modules $M(g^G, \chi)$ are parametrized by a conjugacy class g^G in G and a simple representation χ of the centralizer G^g of g. An arbitrary Yetter–Drinfeld module M is semisimple; we identify the simple summands of M with the simple roots of a generalized root system and call their number the *rank*. The study over nonabelian groups started in [49], where G is a Coxeter group and g is a reflection; two main examples were considered.

- The symmetric group $\mathbb{S}_n$ has a single conjugacy class of reflections with $\binom{n}{2}$ elements, which yields an irreducible Yetter–Drinfeld module. The associated Nichols algebras for $n = 3, 4, 5$ have dimension $12, 576, 8\,294\,400$ and were considered by Fomin and Kirillov [26] in a very different context; for $n \geqslant 6$, they are conjecturally infinite-dimensional.
- The dihedral group $\mathbb{D}_4$ has two conjugacy classes of reflections, each with two elements. These yield a Yetter–Drinfelf module $M = M_1 \oplus M_2$ of dimension $2 + 2$ and rank 2. As it turns out, the generalized root system is of type A_2, indicating roughly that there is a space of braided commutators M_{12}, associated to the third conjugacy class with two elements, and all higher commutators vanish. Since the Nichols algebras of the irreducible modules M_1, M_2, M_{12} have each dimension 4, the Nichols algebra $\mathscr{B}(M)$ has dimension 4^3.

The study now naturally branches into two directions: Nichols algebras of rank 1, meaning of irreducible Yetter–Drinfeld modules, and Nichols algebras of rank > 1 composed of the former via root system theory. In rank 1, more finite-dimensional examples of Nichols algebras were discovered in [7, 29, 33]; later, the research concentrated on successfully ruling out finite-dimensional Nichols algebras over simple groups, see [6] and the references there.

A systematic classification for finite-dimensional Nichols algebras of rank > 1 was achieved by Heckenberger, Schneider, and Vendramin, relying on root system theory. The program was initiated with a study in rank 2 in [34], and after a series of works, it culminated in a full classification for rank $\geqslant 2$ [37]. The surprising observation was that the existence of a finite root system severely

restricts the possible groups G, so that only very few of the (not yet fully classified) Nichols algebras of rank 1 can appear in rank $\geqslant 2$. Next, we summarize the classification over an algebraically closed field of characteristic zero.

Theorem 1.1 [37]. *Let G be a nonabelian group and let $M = M_1 \oplus \cdots \oplus M_n \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, where each M_i is simple and $n \geqslant 2$. Assume that the support of M generates G and that M is braid-indecomposable. If the Nichols algebra $\mathscr{B}(M)$ is finite-dimensional, then M belongs to one of the following.*

(i) *Lie-theoretic families of type α_n ($n \geqslant 2$), δ_n ($n \geqslant 4$), γ_n ($n \geqslant 3$), ϵ_n ($n = 6, 7, 8$), ϕ_4.*
(ii) *Five new exceptional Weyl classes in rank 2 or one of the exceptions β_3', β_3'' in rank 3.*

Thus [37] gives a partial answer for the first step of Lifting Method over nonabelian groups under a mild restriction. This classification is the starting point of our work; in §2.2, we give a precise description of the modules of types α_n, δ_n, γ_n, ϵ_n, and ϕ_4.

Nichols algebras in the families (i) had been previously constructed by Lentner via the folding method [42, 43], which produce central extensions of Hopf algebras and Nichols algebras. In fact, a main result of the present article (Theorem 3.18) is that ultimately, all Nichols algebras in the families (i) can be reduced to this construction. Folding assigns to any Nichols algebra $\mathscr{B}(M)$ over a group Γ with a group Σ of diagram automorphisms, a new Nichols algebra $\mathscr{B}(\widetilde{M})$ over the central extension $\Sigma \to G \to \Gamma$. For example, when applied to the positive part of the small quantum group $u_q(A_2 \times A_2)^+$ and $\Sigma = \mathbb{Z}_2$ for a suitable automorphism switching the two copies of A_2, the folding construction gives a Nichols algebra over the dihedral group. Similarly, folding method can be applied to $\mathscr{B}(M) = u_q(\mathfrak{g})^+$ for $q^2 = -1$ in cases where $\mathfrak{g}$ is simple and has a diagram automorphism, namely, ${}^2A_{2n+1}$ and ${}^2E_6$, and in cases where $\mathfrak{g}$ consists of two copies of the same simply laced Lie algebra interchanged by $\Sigma = \mathbb{Z}_2$, which we denote ${}^2A_n^2$ and ${}^2D_n^2$ and ${}^2E_n^2$, $n = 6, 7, 8$. The root system attached to $\mathscr{B}(M)$ is the folded root system considered in Lie theory, with Σ-orbits of roots becoming the new roots. In all cases, the root system is of Lie type and the Weyl groupoid is again a Weyl group (they are *standard*) and correspond to the families from (i):

[43]	${}^2A_n^2$	${}^2D_n^2$	${}^2E_n^2$	${}^2A_{2n+1}$	${}^2E_6$
[37]	α_n	δ_n	ϵ_n	γ_n	ϕ_4
Root system	A_n	D_n	E_n	C_n	F_4

1.3 | Summary of the main results

The main goal of our article is to determine all pointed Hopf algebra over a finite nonabelian group whose infinitesimal braiding belong to the *Lie-theoretical families* (i) of [37], by solving the remaining two steps of the Lifting Method. On the other hand, the exceptional Nichols algebras in low rank (ii) have no uniform description and need to be treated *by hand*, see, for example, [19].

We now discuss the organization and main results of this paper in more detail. In §2, we review the preliminaries and the classification result by Heckenberger–Vendramin. Thereafter, the paper consists of three main parts, summarized below.

1.3.1 | Reduction via folding

The goal of §3 is to relate the output of Heckenberger–Vendramin classification with the folding construction. Here we follow the construction mainly in the setting of [42]. Namely, for a given Hopf algebra H and a group of biGalois objects, one obtains a Hopf algebra structure on their direct sum. Now we specialize to the case where the Hopf algebra is a smash product $H = \mathscr{B}(M)\#\Bbbk\Gamma$ and the biGalois objects are based on a 2-cocycle σ for Γ and a *twisted symmetry* of M. Then the folding is again a smash product of the centrally extended group with the folded Nichols algebra. Our first main result is that the Nichols algebras of all modules in the family (i) are twists of foldings over central extensions. This opens the door for a uniform and abelian-theoretic treatment of these Nichols algebras.

Theorem 3.18. *Let G be a finite nonabelian group, $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of type α_n, γ_n, δ_n, ϵ_n, or ϕ_4 whose support generates G. Then there exists $\sigma \in H^2(G, \Bbbk)$ such that M^σ is a folding.*

Notice that, by construction, the folding technique produces Nichols algebras with trivial action of the central subgroup $\Sigma = \langle \kappa \rangle$. Thus, to prove this result, we need to somehow *trivialize* the action of that central element. We show in Lemma 2.7 that κ acts trivially on large-rank families in [37]. For small-rank cases $\alpha_2, \alpha_3, \delta_4, \gamma_3, \gamma_4$, and ϕ_4, we show in Proposition 3.15, that there exists a group cocycle σ as above such that κ acts trivially on the twisted module M^σ. This requires a finer analysis on the structure of the group G and significant group cohomology computations, postponed to the Appendix.

1.3.2 | Generation in degree 1

In the brief §4, we see the first application of Theorem 3.18. Namely, we give a positive answer to the generation in degree 1 question by translating it to the respective assertion for certain Nichols algebras of diagonal type.

Theorem 4.1. *Let H be a finite-dimensional pointed Hopf algebra with infinitesimal braiding of type α_n, γ_n, δ_n, ϵ_n, or ϕ_4. Then H is generated by skew-primitive and group-like elements.*

1.3.3 | Computation of relations and liftings

The rest of the paper is devoted to a classification all pointed Hopf algebras with infinitesimal braiding of type $\alpha_n, \gamma_n, \delta_n, \epsilon_n$, or ϕ_4.

The first step toward that goal is to obtain defining relations for these Nichols algebras; this is achieved in §5. Here we present a sketch of our third main result, and refer to the actual Theorem for a precise statement.

Theorem 5.7. *Given $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of type α_n, γ_n, δ_n, ϵ_n, or ϕ_4, we have a presentation by G-homogeneous generators and relations, and a Poincare–Birkhoff–Witt type basis (PBW basis), for the Nichols algebra $\mathscr{B}(M)$.*

To obtain this presentation, we adapt certain constructions and techniques developed in [12, 14] for Nichols algebras of diagonal type. Namely, for each M as above, we construct in §5.2 a

pre-Nichols algebra $\widehat{\mathscr{B}}(M)$. Then in §5.3, we show that the subalgebra of coinvariants under the canonical map $\widehat{\mathscr{B}}(M) \to \mathscr{B}(M)$ is actually a Hopf subalgebra, and describe its algebra structure. We close §5 with Theorem 5.12, where we show that these Nichols algebras are rigid in the sense of [18].

Finally, in §6, we classify all liftings. Following previous experiences, for example, [4, 16, 19], we first construct a big family of them via Hopf–Galois objects, and then show that this family exhausts all liftings. So, in particular, all liftings are cocycle deformation of the associated graded Hopf algebra. For simplicity, here we provide only a sketch of the actual statement.

Theorem 6.3. *Let $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of type α_n, γ_n, δ_n, ϵ_n, or ϕ_4. Let $\mathcal{R}_M$ denote the set of deformation parameters defined in (6.3), and for each $\lambda \in \mathcal{R}_M$, consider the Hopf algebra $\mathcal{L}(\lambda)$ defined explictly in §6.*

Then, for each $\lambda \in \mathcal{R}_M$, the Hopf algebra $\mathcal{L}(\lambda)$ is a lifting of M over $\Bbbk G$ and a cocycle deformation of $\mathscr{B}(M)\#\Bbbk G$.

Conversely, if L is lifting of M over $\Bbbk G$, then there exists $\lambda \in \mathcal{R}_M$ such that $L \simeq \mathcal{L}(\lambda)$.

In §6.3, we discuss how these liftings can be viewed as foldings of liftings of Nichols algebras over the corresponding abelian group.

We close the paper with some open questions and future directions of research, see §7.

2 | PRELIMINARIES

Conventions

We denote $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Given $k < \theta$ in $\mathbb{N}_0$, we put $\mathbb{I}_{k,\theta} = \{n \in \mathbb{N}_0 : k \leqslant n \leqslant \theta\}$ and $\mathbb{I}_\theta = \mathbb{I}_{1,\theta}$. When θ is clear from the context, we just write $\mathbb{I} = \mathbb{I}_\theta$. The canonical basis of $\mathbb{Z}^\theta$ is denoted by $(\alpha_i)_{i\in\mathbb{I}_\theta}$.

We work over an algebraically closed field $\Bbbk$ of characteristic zero and use $\Bbbk^\times$ to denote the group of nonzero elements. If $N \in \mathbb{N}$, we use $\mathbb{G}_N$ to denote the subgroup of Nth roots of unity; the subset of those with order N is $\mathbb{G}'_N$.

Given a group G and an element g, we use g^G and G^g to denote the conjugacy class and the centralizer of g, respectively. By $\widehat{G}$, we mean the group of characters, and $\Bbbk G$ stands for the group algebra. If K is another group, then a pairing (also called a bicharacter) is a map $P : G \times K \to \Bbbk^\times$ such that for all $g, g' \in G$, $k, k' \in K$:

$$P(gg', k) = P(g, k)P(g', k), \qquad P(g, kk') = P(g, k)P(g, k').$$

A skew-polynomial algebra in variables $z_1, \dots, z_k$ is a quotient of the free algebra in these variables by an ideal generated by $z_i z_j - t_{ij} z_j z_i$, $1 \leqslant i, j \leqslant k$, for some $t_{ij} \in \Bbbk^\times$.

We denote Hopf algebras by tuples (H, μ, Δ, S) where μ is the multiplication, Δ the comultiplication, and S the antipode, which we always assume bijective. The subspace of primitive elements is $\mathcal{P}(H)$. The group of group-like elements is $G(H)$. If $\delta : V \to H \otimes V$ is a left H-comodule, we write $\delta(v) = v_{-1} \otimes v_0$; for $g \in G(H)$, we put $V_g := \{v \in V : \delta(v) = g \otimes v\}$. We refer to [51] for any unexplained terminology on Hopf algebras and to [5, §2] for preliminaries on Hopf–Galois objects and cocycle deformations.

2.1 | The Nichols algebra of a braided vector space

A *braided vector space* is a pair (V, c) where V is a vector space and $c \in GL(V \otimes V)$ satisfies

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

By declaring the elements of V to be primitive, the tensor algebra $T(V)$ becomes an $\mathbb{N}_0$-graded *braided* Hopf algebra. There is a largest coideal $\mathcal{J}(V)$ among those that trivially intersect $\Bbbk \oplus V$; it happens to be graded, so we denote $\mathcal{J}(V) = \bigoplus_{n \geqslant 2} \mathcal{J}^n(V)$. The Nichols algebra of (V, c) is defined as the quotient $\mathscr{B}(V) = T(V)/\mathcal{J}(V)$. This is again an $\mathbb{N}_0$-graded braided Hopf algebra, which is strictly graded as a coalgebra and generated by V as an algebra, see [36, §7]. Any intermediate quotient $\mathscr{B} = T(V)/\mathcal{J}$ by an $\mathbb{N}_0$-homogeneous Hopf ideal $\mathcal{J}$ is called a *pre-Nichols algebra* of V.

The *braided commutator* of $T(V)$ is defined by

$$[-,-]_c = \mathrm{mult}(\mathrm{id} - c) : T(V) \otimes T(V) \to T(V).$$

If $u \in V$ and $v \in T(V)$, we denote $(\mathrm{ad}_c\, u)v = [u, v]_c$. In §2.2.1, will define $\mathrm{ad}_c\, u$ for arbitrary $u \in T(V)$. For a fixed basis $(x_i)_{i \in I}$ of V and $k \geqslant 2$, we set

$$x_{i_1 \cdots i_k} := (\mathrm{ad}_c\, x_{i_1}) \cdots (\mathrm{ad}_c\, x_{i_{k-1}}) x_{i_k}, \qquad i_j \in \mathbb{I}. \tag{2.1}$$

Example 2.1. Given $\mathbf{q} = (q_{ij})_{i,j \in \mathbb{I}}$ a matrix of elements of $\Bbbk^\times$, there is a braided vector space $(V, c^{\mathbf{q}})$ where V has basis $(\mathrm{x}_i)_{i \in \mathbb{I}}$ and $c^{\mathbf{q}}$ is given by

$$c^{\mathbf{q}}(\mathrm{x}_i \otimes \mathrm{x}_j) = q_{ij}\, \mathrm{x}_j \otimes \mathrm{x}_i, \qquad i, j \in \mathbb{I}. \tag{2.2}$$

A braided vector space is called *of diagonal type* [9] if (2.2) holds in some basis of V for some $\mathbf{q} = (q_{ij})_{i,j \in \mathbb{I}}$. In this case, we denote the Nichols algebra of (V, c) by $\mathscr{B}_{\mathbf{q}}$, which is now $\mathbb{Z}^{\mathbb{I}}$-graded; we refer to $\mathbf{q}$ as the *braiding matrix*. The Dynkin diagram of $\mathbf{q}$ is a graph with $\mathbb{I}$ as the set of vertices, each vertex i labeled with q_{ii}. There is an edge between $i \neq j$ if and only if $\widetilde{q}_{ij} := q_{ij} q_{ji} \neq 1$; such an edge is labeled with this scalar.

We say that $\mathbf{q}$ is *of Cartan type* [8] if there is a Cartan matrix $\mathbf{a} = (a_{ij})$ such that

$$q_{ij} q_{ji} = q_{ii}^{a_{ij}}, \qquad \text{for all } i, j \in \mathbb{I}.$$

If some q_{ii} is not a root of unity, then the integers a_{ij} are uniquely determined. Otherwise we impose $-\operatorname{ord} q_{ii} < a_{ij} \leqslant 0$ for all $j \neq i$. In this case, we say that $\mathbf{q}$ is of Cartan type $\mathbf{a}$.

Although braided vector spaces of Cartan type seem quite *simple*, the structure of the corresponding Nichols algebra is related either with quantized enveloping algebras (when the entries of $\mathbf{q}$ are not roots of unity), or with Frobenius–Lusztig kernels.

The following example of Cartan type will be particularly relevant in later sections.

Example 2.2. Fix a finite Cartan matrix $\mathbf{a}$ with simply laced Dynkin diagram. Assume that $\mathbf{q} = (q_{ij})$ satisfies the following conditions:

$$q_{ii} = -1; \qquad q_{ij}q_{ji} = \begin{cases} -1, & a_{ij} = -1, \\ 1, & a_{ij} = 0; \end{cases} \qquad i \neq j \in \mathbb{I}. \tag{2.3}$$

Then $\mathfrak{q}$ is of Cartan type $\mathbf{a}$.

Let $\beta_1 < \beta_2 < \cdots < \beta_M$ be a convex order on the set of positive roots Δ_+ of $\mathbf{a}$. In [12], we can find a *root vector* $\mathrm{x}_\beta \in \mathscr{B}_\mathfrak{q}$ for each $\beta \in \Delta_+$, of $\mathbb{Z}^\mathbb{I}$-degree β, obtained recursively as braided commutator of root vectors with smaller degree.

In this case, the Nichols algebra $\mathscr{B}_\mathfrak{q}$ is presented by generators $(\mathrm{x}_i)_{i\in\mathbb{I}}$ and relations

$$\mathrm{x}_\alpha^2 = 0, \qquad \alpha \in \Delta_+; \tag{2.4}$$

$$[\mathrm{x}_{ijk}, \mathrm{x}_j]_c = 0, \qquad a_{ij} = a_{jk} = -1; \tag{2.5}$$

$$\mathrm{x}_{ij} = 0, \qquad a_{ij} = 0. \tag{2.6}$$

Moreover, by [3, 12], a basis for $\mathscr{B}_\mathfrak{q}$ is given by the set

$$\{\mathrm{x}_{\beta_1}^{n_1}\mathrm{x}_{\beta_2}^{n_2}\dots\mathrm{x}_{\beta_M}^{n_M} \mid n_i \in \{0,1\}\}. \tag{2.7}$$

Remark 2.3. Let $\mathfrak{q}$ as above. In §5, we will need the following constructions, due to [14].

(I) The *distinguished pre-Nichols* algebra $\widetilde{\mathscr{B}}_\mathfrak{q}$ is the quotient of $T(V)$ by (2.5), (2.6), and

$$\mathrm{x}_{iij} = 0, \qquad a_{ij} = -1. \tag{2.8}$$

The set $\{\mathrm{x}_{\beta_1}^{n_1}\mathrm{x}_{\beta_2}^{n_2}\dots\mathrm{x}_{\beta_M}^{n_M} \mid n_i \in \mathbb{N}_0\}$ is a basis of $\widetilde{\mathscr{B}}_\mathfrak{q}$.

(II) Let $\pi : \widetilde{\mathscr{B}}_\mathfrak{q} \twoheadrightarrow \mathscr{B}_\mathfrak{q}$ be the canonical projection. By [14], the subalgebra of coinvariants $\mathcal{Z}_\mathfrak{q} := \widetilde{\mathscr{B}}_\mathfrak{q}^{\mathrm{co}\,\pi}$ is a q-polynomial algebra with generators x_β^2, $\beta \in \Delta_+$.

(III) Let $\widehat{\mathscr{B}}_\mathfrak{q}$ be the quotient of $T(V)$ by (2.5), (2.6), and x_i^2, $i \in \mathbb{I}$. Then $\widehat{\mathscr{B}}_\mathfrak{q}$ is a pre-Nichols algebra, which coincides with the quotient of $\widetilde{\mathscr{B}}_\mathfrak{q}$ by x_i^2, $i \in \mathbb{I}$. The set

$$\{\mathrm{x}_{\beta_1}^{n_1}\mathrm{x}_{\beta_2}^{n_2}\dots\mathrm{x}_{\beta_M}^{n_M} \mid n_i \in \{0,1\} \text{ if } \beta_i \text{ is simple}, n_i \in \mathbb{N}_0 \text{ otherwise}\}$$

is a basis of $\widehat{\mathscr{B}}_\mathfrak{q}$, so its Hilbert series is

$$\mathcal{H}_{\widehat{\mathscr{B}}_\mathfrak{q}} = \left(\prod_{\beta\in\Delta_+^\mathfrak{q}-\{\alpha_i\}} \frac{1}{1-t^\beta}\right)\left(\prod_{i\in\mathbb{I}} 1+t_i\right).$$

2.2 | Nichols algebras over nonabelian groups

The goal of this subsection is to introduce the notion of Weyl grupoids. These play a fundamental role in the classification achieved in [37]. We refer to the book [36] for details and unexplained terminology.

2.2.1 | Yetter–Drinfeld modules over groups

For a group G, the category of Yetter–Drinfeld modules ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ consists of G-graded vector spaces $V = \bigoplus_{g\in G} V_g$ endowed with a G-action such that $h \cdot V_g \subset V_{hgh^{-1}}$ for all $h, g \in G$. This is a braided tensor category where the braiding $c_{V,W} : V \otimes W \to W \otimes V$ is given by $c(v \otimes w) = g \cdot w \otimes v$ for $v \in V_g$ and $w \in W$. We recall that a G-grading on a vector space $V = \bigoplus_{g\in G} V_g$ is equivalent to a $\Bbbk G$-comodule structure $\delta : V \to \Bbbk G \otimes V$, declaring $\delta(v) = g \otimes v$ if and only if $v \in V_g$. The *support* of V is

$$\operatorname{supp} V = \{g \in G | \, V_g \neq 0\}.$$

Let (R, μ, Δ, S) be a Hopf algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$. The braided commutator defined above satisfies the following identity: If $u \in R_g$ and $v \in R_h$ for some $g, h \in G$, then for any $w \in R$,

$$[[u, v]_c, w] = [u, [v, w]_c] - (g \cdot v)[u, w]_c + [u, h \cdot w]_c v. \tag{2.9}$$

R admits a *braided* adjoint representation $\operatorname{ad}_c : R \to \operatorname{End}(R)$ given by

$$(\operatorname{ad}_c u)v = \mu(\mu \otimes S)(\mathrm{id} \otimes c)(\Delta \otimes \mathrm{id})(u \otimes v), \qquad u, v \in R.$$

When u is primitive, $\operatorname{ad}_c u$ and $[u, -]_c$ coincide. Notice also that

$$g \cdot ((\operatorname{ad}_c u)v) = (\operatorname{ad}_c g \cdot u)(g \cdot v), \qquad g \in G,\ u \in \mathcal{P}(R),\ v \in R. \tag{2.10}$$

2.2.2 | The Nichols algebra of a Yetter–Drinfeld module

Each $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is a braided vector space, so it has a Nichols algebra $\mathscr{B}(V)$ as discussed in §2.1. In this setting, $T(V)$ and $\mathscr{B}(V) = T(V)/\mathcal{J}(V)$ turn out to be $\mathbb{N}_0$-graded Hopf algebras in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$.

2.2.3 | Skew derivations

There is a criterion, proven, for example, in [49, Proposition 2.4], to decide if any given element of $T(V)$ belongs to $\mathcal{J}(V)$. Fix a basis $x_1, \dots, x_r$ of V with x_i of degree h_i. For each i, we define a skew derivation $\partial_{x_i} \in \operatorname{End}(T(V))$ recursively in $V^{\otimes n}$, $n \geqslant 0$. For $n = 0$, put $\partial_{x_i}(1) = 0$; in $V^{\otimes 1}$, put $\partial_{x_i}(x_j) = \delta_{i,j}$ and in general define

$$\partial_{x_i}(xy) = x\partial_{x_i}(y) + \partial_{x_i}(x)h_i \cdot y, \qquad x, y \in T(V).$$

Then, for any $n \geqslant 2$, we have $x \in \mathcal{J}^n(V)$ if and only if $\partial_{x_i}(x) \in \mathcal{J}^{n-1}(V)$ for all i. The compositions of these derivations with the braided adjoint action satisfy

$$\partial_{x_i}((\operatorname{ad}_c u)v) = u\partial_{x_i}(v) - \partial_{x_i}(g \cdot v)h_i \cdot u, \quad u \in V_g,\ v \in T(V),\ i \in \mathbb{I}_r. \tag{2.11}$$

2.2.4 | Simple Yetter–Drinfeld modules

Fix $g \in G$ and (V, χ) an irreducible representation of the centralizer G^g. We consider the induced G-module $\Bbbk G \otimes_{G^g} V$ endowed with the G-grading determined by declaring the degree of $x \otimes v$ to be gxg^{-1} for all $x \in G$ and $v \in V$. This is a simple Yetter–Drinfeld module over G, which is denoted by $M(g^G, \chi)$ because it depends on the conjugacy class g^G rather than the element g itself. Moreover, all the simple objects of ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ arise in this way, and if G is finite, the category ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is semisimple. There is a concrete description of $M(g^G, \chi)$ in [2, Example 24], which we will use several times to perform computations in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$.

Example 2.4. Let G be a finite group. Assume that $g \in G$ is such that $g^G = \{g, g\kappa\}$ for some $\kappa \neq e \in G$. Then there exists $g_0 \in G$ such that $g_0 g = \kappa g g_0$, $G^g = G^{g\kappa} = G^{g^{-1}}$ is a subgroup of index two since $G = G^g \cup g_0 G^g$, and $\kappa \in Z(G)$, $\kappa^2 = e$.

Let χ be a one-dimensional representation of G^g, that is, $V = \Bbbk$ and $\chi \in \widehat{G^g}$, and $M = M(g^G, \chi)$. Then $\dim M = 2$: we may fix a basis $\{x, y\}$ such that $y = g_0 \cdot x$ and the coaction is given by $\delta(x) = g \otimes x$, $\delta(y) = g\kappa \otimes y$. As $hg_0, g_0^{-1}h \in G^g$ for all $h \notin G^g$, the action satisfies

$$h \cdot x = \begin{cases} \chi(h)x, & h \in G^g, \\ \chi(g_0^{-1}h)y, & g \notin G^g; \end{cases} \qquad h \cdot y = \begin{cases} \chi(g_0^{-1}hg_0)y, & h \in G^g, \\ \chi(hg_0)x, & h \notin G^g; \end{cases} \qquad k \in G.$$

Thus, M is of diagonal type with braiding matrix $\left[\begin{smallmatrix} \chi(g) & \chi(g\kappa) \\ \chi(g\kappa) & \chi(g) \end{smallmatrix}\right]$; from this, we deduce that $\dim \mathscr{B}(M) < \infty$ if and only if $\chi(g) \in \{-1\} \cup \mathbb{G}'_3$.

2.2.5 | Weyl groupoid

Next, we recall the definition of the Weyl groupoid of a nonsimple element $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ such that $\dim \mathscr{B}(M) < \infty$. Let $M = \oplus_{i \in \mathbb{I}} M_i$, where each summand M_i is simple: $M_i = M(g_i^G, \chi_i) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ for some $g_i \in G$, χ_i an irreducible representation of G^{g_i}.

For each $i \neq j \in \mathbb{I}$, set $(\operatorname{ad} M_i)^0 M_j := M_j$, and for $n \in \mathbb{N}$,

$$(\operatorname{ad} M_i)^n M_j := \{(\operatorname{ad}_c m_1) \cdots (\operatorname{ad}_c m_n)m | m_\ell \in M_i, m \in M_j\} \subset \mathscr{B}(M).$$

The *generalized Cartan matrix* of M is $C^M = (c_{ij}^M) \in \mathbb{Z}^{\mathbb{I} \times \mathbb{I}}$, where

$$c_{ii}^M = 2, \qquad c_{ij}^M = -\min\{n \in \mathbb{N}_0 | (\operatorname{ad} M_i)^{n+1} M_j = 0\}, \qquad j \neq i. \tag{2.12}$$

The *i-reflection* of M is $\rho_i M = \sum_{j \in \mathbb{I}} \underline{M}_j$, where

$$\underline{M}_j = \begin{cases} (\operatorname{ad} M_i)^{-c_{ij}^M} M_j, & j \neq i; \\ M_i^*, & j = i. \end{cases}$$

Each $\underline{M}_j \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is simple and $\dim \mathscr{B}(M) = \dim \mathscr{B}(\rho_i M)$.

These reflections generate the *Weyl groupoid* of M, see [36].

2.3 | Heckenberger–Vendramin classification

Let $M = \oplus_{i\in\mathbb{I}_\theta} M_i$ be a finite-dimensional Yetter–Drinfeld module over a nonabelian group G, where each M_i is simple. Here θ is called the *rank* of M.

In [37, Theorem 2.5], the authors classify Yetter–Drinfeld modules as above of rank at least 2, such that the associated Nichols algebra is finite-dimensional. To be precise, one needs to assume that the support of M generates G, and impose a mild nondegeneracy condition on the braiding between different summands of M. Up to a few exceptions in ranks 2 and 3, the classification consists on families α_θ, γ_θ, and δ_θ of arbitrary rank, and types ϵ_θ, $\theta = 6, 7, 8$, ϕ_4, which resemble the classification of finite-dimensional Lie algebras. These Yetter–Drinfeld modules are invariant under the Weyl groupoid action; we say that they are *standard*, adopting the terminology used for diagonal type, see [3]. In this paper, we study this class; next, we give an explicit description of each module.

2.3.1 | Types α_θ, δ_θ, and ϵ_θ

Fix a simply laced indecomposable Cartan matrix $\mathbf{a} = (a_{ij}) \in \mathbb{Z}^{\theta\times\theta}$ of finite type; that is, of type A_θ, $\theta \geqslant 2$, D_θ, $\theta \geqslant 4$, or E_θ for $\theta \in \mathbb{I}_{6,8}$. Let Δ_+ be the set of positive roots. Following [37, Lemma 6.2], we describe a Yetter–Drinfeld module $M = \oplus_{i\in\mathbb{I}} M_i$ over a nonabelian group G with simply laced skeleton and Cartan graph of standard type $\mathbf{a}$. Assume that there exist

- $\kappa \in Z(G)$ such that $\kappa \neq 1$, $\kappa^2 = 1$,
- $g_i \in G$ with $g_i^G = \{g_i, \kappa g_i\}$ for all $i \in \mathbb{I}_\theta$,
- $\chi_i \in \widehat{G^{g_i}}$ such that $\chi_i(g_i) = -1$ for all $i \in \mathbb{I}_\theta$,

satisfying the following:

$$g_i g_j = \kappa g_j g_i, \qquad \chi_i(\kappa g_j^2)\chi_j(\kappa g_i^2) = 1, \qquad a_{ij} = -1; \tag{2.13}$$

$$g_i g_j = g_j g_i, \qquad \chi_i(g_j)\chi_j(g_i) = 1, \qquad a_{ij} = 0; \tag{2.14}$$

$$\chi_i(\kappa)\chi_j(\kappa) = 1, \qquad a_{ij} = 0. \tag{2.15}$$

For $i \in \mathbb{I}_\theta$, let $M_i = M(g_i^G, \chi_i) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, which has a basis $\{x_i, x_{\bar{i}}\}$ with coaction $x_i \mapsto g_i \otimes x_i$, $x_{\bar{i}} \mapsto \kappa g_i \otimes x_{\bar{i}}$. As in [2, Example 37], the braidings c_{M_i,M_j} for $i, j \in \mathbb{I}_\theta$ are determined by

$$\begin{bmatrix} c(x_i \otimes x_i) & c(x_i \otimes x_{\bar{i}}) \\ c(x_{\bar{i}} \otimes x_i) & c(x_{\bar{i}} \otimes x_{\bar{i}}) \end{bmatrix} = \begin{bmatrix} -x_i \otimes x_i & -\chi_i(\kappa)x_{\bar{i}} \otimes x_i \\ -\chi_i(\kappa)x_i \otimes x_{\bar{i}} & -x_{\bar{i}} \otimes x_{\bar{i}} \end{bmatrix}; \tag{2.16}$$

$$\begin{bmatrix} c(x_i \otimes x_j) & c(x_i \otimes x_{\bar{j}}) \\ c(x_{\bar{i}} \otimes x_j) & c(x_{\bar{i}} \otimes x_{\bar{j}}) \end{bmatrix} = \begin{bmatrix} x_{\bar{j}} \otimes x_i & \chi_j(g_i^2)x_j \otimes x_i \\ \chi_j(\kappa)x_{\bar{j}} \otimes x_{\bar{i}} & \chi_j(\kappa g_i^2)x_j \otimes x_{\bar{i}} \end{bmatrix}, \qquad a_{ij} = -1; \tag{2.17}$$

$$\begin{bmatrix} c(x_i \otimes x_j) & c(x_i \otimes x_{\bar{j}}) \\ c(x_{\bar{i}} \otimes x_j) & c(x_{\bar{i}} \otimes x_{\bar{j}}) \end{bmatrix} = \begin{bmatrix} \chi_j(g_i)x_j \otimes x_i & \chi_j(\kappa g_i)x_{\bar{j}} \otimes x_i \\ \chi_j(\kappa g_i)x_j \otimes x_{\bar{i}} & \chi_j(g_i)x_{\bar{j}} \otimes x_{\bar{i}} \end{bmatrix}, \qquad a_{ij} = 0. \tag{2.18}$$

The generalized Cartan matrix of $M = \oplus_{i\in\mathbb{I}} M_i$ is $\mathbf{a}$, and we have $\dim \mathscr{B}(M) = 2^{2|\Delta_+|}$.

Remark 2.5.

- The braiding on M_i is of diagonal type and $c^2_{M_i,M_i} = \mathrm{id}$.
- If $a_{ij} = 0$, then $M_i \oplus M_j$ is of diagonal type and $c_{M_i,M_j} c_{M_j,M_i} = \mathrm{id}_{M_j \otimes M_i}$.
- If $a_{ij} = -1$, then $M_i \oplus M_j$ is of diagonal type if and only if $\chi_i(\kappa) = \chi_j(\kappa) = 1$, but here $c_{M_i,M_j} c_{M_j,M_i} \neq \mathrm{id}_{M_j \otimes M_i}$.

2.3.2 | Type γ_θ, $\theta \geqslant 3$

Following [37, Lemma 7.6], we describe, for each rank $\theta \geqslant 3$, a Yetter–Drinfeld module M of type γ_θ over a nonabelian group G. Assume that there exist

- $\kappa \in Z(G)$ such that $\kappa \neq 1, \kappa^2 = 1$,
- $g_1, \dots, g_\theta \in G$ with $g_i^G = \{g_i, \kappa g_i\}$ for $i \in \mathbb{I}_{\theta-1}$ and $g_\theta^G = \{g_\theta\}$,
- $\chi_i \in \widehat{G^{g_i}}$ such that $\chi_i(g_i) = -1$,

satisfying (2.13), (2.14), and (2.15) for $i, j \in \mathbb{I}_{\theta-1}$, and

$$g_i g_\theta = g_\theta g_i, \qquad \chi_i(g_\theta)\chi_\theta(g_i) = 1, \qquad i < \theta - 1; \tag{2.19}$$

$$\chi_{\theta-1}(g_\theta)\chi_\theta(g_{\theta-1}) = -1. \tag{2.20}$$

Let $M_i = M(g_i^G, \chi_i) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$. Then $M = \oplus_{i \in \mathbb{I}} M_i$ is of type γ_θ and $\dim \mathscr{B}(M) = 2^{2\theta^2-\theta}$. Notice that $M_1 \oplus \cdots \oplus M_{\theta-1}$ is of type $\alpha_{\theta-1}$, so, by §2.3.1, we have a basis $\{x_i, x_{\bar{i}}\}$ of M_i such that for $i, j \in \mathbb{I}_{\theta-1}$, the braiding c_{M_i,M_j} is determined by (2.16), (2.17), and (2.18). On the other hand, $M_\theta = \Bbbk\{x_\theta\}$ is one-dimensional concentrated in degree $g_\theta \in Z(G)$. The braidings $c_{M_\theta,M_\theta}, c_{M_i,M_\theta}, c_{M_\theta,M_i}, i \in \mathbb{I}_{\theta-1}$, are determined by

$$c(x_\theta \otimes x_\theta) = -x_\theta \otimes x_\theta; \tag{2.21}$$

$$c(x_i \otimes x_\theta) = \chi_\theta(g_i) x_\theta \otimes x_i, \qquad c(x_{\bar{i}} \otimes x_\theta) = \chi_\theta(\kappa g_i) x_\theta \otimes x_{\bar{i}}; \tag{2.22}$$

$$c(x_\theta \otimes x_i) = \chi_i(g_\theta) x_i \otimes x_\theta, \qquad c(x_\theta \otimes x_{\bar{i}}) = \chi_i(g_\theta) x_{\bar{i}} \otimes x_\theta. \tag{2.23}$$

2.3.3 | Type ϕ_4

Following [37, Lemma 9.2], we describe a Yetter–Drinfeld module M over a nonabelian group G with Cartan matrix of type F_4. Assume that there exist

- $\kappa \in Z(G)$ such that $\kappa \neq 1, \kappa^2 = 1$,
- $g_1, \dots, g_4 \in G$ with $g_i^G = \{g_i, \kappa g_i\}$ for $i = 1, 2$ and $g_i^G = \{g_i\}$ for $i = 3, 4$,
- $\chi_i \in \widehat{G^{g_i}}$ such that $\chi_i(g_i) = -1$

satisfying the following:

$$\chi_1(g_3)\chi_3(g_1) = \chi_1(g_4)\chi_4(g_1) = \chi_2(g_4)\chi_4(g_2) = 1; \tag{2.24}$$

$$\chi_3(g_4)\chi_4(g_3) = \chi_2(g_3)\chi_3(g_2) = -1; \tag{2.25}$$

$$g_1 g_2 = \kappa g_2 g_1, \qquad \chi_1(\kappa g_2^2)\chi_2(\kappa g_1^2) = 1. \tag{2.26}$$

Let $M_i = M(g_i^G, \chi_i)$. The Cartan matrix of $M = \oplus_{i\in\mathbb{I}} M_i$ is of type F_4 and $\dim \mathscr{B}(M) = 2^{36}$. The braidings c_{M_i,M_j} are given as in types $\alpha_\theta, \gamma_\theta$, depending on i, j.

2.4 | On the structure of the group G

Fix an abelian group $\mathtt{Z}$ and $x, y \in \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle x, y\rangle$. For every $u, v \in \mathtt{Z}$ and every $\kappa \in \mathtt{Z}$ such that $\kappa^2 = e$, there exists a 2-cocycle $\beta \in Z^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathtt{Z})$ such that

$$\beta(x,x) = u, \qquad \beta(x,y) = \kappa, \qquad \beta(y,x) = e, \qquad \beta(y,y) = v.$$

We denote by $\mathtt{Z}_{u,v,\kappa}$ the associated central extension of $\mathtt{Z}$ by $\mathbb{Z}_2 \times \mathbb{Z}_2$:

$$\mathtt{Z} \hookrightarrow \mathtt{Z}_{u,v,\kappa} \twoheadrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle x, y\rangle, \tag{2.27}$$

where $[x, y] = \kappa$, $x^2 = u$, $y^2 = v$.

Next, we describe some general features of a group G realizing the braidings described in §2.3.1, §2.3.2, and §2.3.3. Let $M = \oplus_{i\in\mathbb{I}} M_i$ be of type α_θ, γ_θ, δ_θ, ϵ_θ, or ϕ_4. Here $M_i = M(g_i^G, \chi_i) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, hence $\operatorname{supp} M = \bigcup_{i\in\mathbb{I}} g_i^G$. Note that

- there exists $\kappa \in Z(G)$ such that $\kappa^2 = e$ and $g_i^G = \{g_i, \kappa g_i\}$ for all i such that $g_i \notin Z(G)$,
- for i with $g_i^G = \{g_i, \kappa g_i\}$, there exists $j \neq i$ such that $g_i g_j = \kappa g_j g_i$; hence $g_j^G = \{g_j, \kappa g_j\}$.

The relevance of the central extensions constructed above is explained next.

Lemma 2.6. *Let $i \neq j \in \mathbb{I}$ be such that $g_i g_j = \kappa g_j g_i$, and let $N = G^{g_i} \cap G^{g_j}$. Then*

(a) *The subgroup N is normal, and $G/N \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.*
(b) *If $G = \langle \operatorname{supp} M\rangle$, then N is an abelian subgroup, generated by the elements*

$$g_k, \qquad \text{for all } k \in \mathbb{I} \text{ such that } g_k g_i = g_i g_k,\ g_k g_j = g_j g_k,$$
$$g_k g_i, \qquad \text{for all } k \in \mathbb{I} \text{ such that } g_k g_i = g_i g_k,\ g_k g_j = \kappa g_j g_k, \text{ and}$$
$$g_k g_j, \qquad \text{for all } k \in \mathbb{I} \text{ such that } g_k g_j = g_j g_k,\ g_k g_i = \kappa g_i g_k.$$

(c) *If N is abelian, then $G \simeq N_{g_i^2, g_j^2, \kappa}$.*

Proof. For (a), we note first that $[G : G^{g_i}] = [G : G^{g_j}] = 2$, so both G^{g_i} and G^{g_j} are normal subgroups, and $[G : N] = 4$ since $G = G_i G_j$. Let $g \in G$:

- If $g g_i g^{-1} = g_i$, $g g_j g^{-1} = g_j$, then $gN = N$.
- If $g g_i g^{-1} = g_i$, $g g_j g^{-1} = \kappa g_j$, then $gN = g_i N$.
- If $g g_i g^{-1} = \kappa g_i$, $g g_j g^{-1} = g_j$, then $gN = g_j N$.
- If $g g_i g^{-1} = \kappa g_i$, $g g_j g^{-1} = \kappa g_j$, then $gN = g_i g_j N$.

Hence, $G/N = \{N, g_i N, g_j N, g_i g_j N\}$. Since $g_i^2, g_j^2 \in N$, we get $G/N \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.
(b) and (c) are straightforward. □

A more detailed description of these groups is postponed to the Appendix.

2.4.1 | The parity vector

An important invariant of $M = \oplus_{i\in\mathbb{I}} M_i \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is

$$\mathtt{P} := (\chi_i(\kappa))_{1\leqslant i\leqslant\theta} \in \{\pm 1\}^{\theta}. \tag{2.28}$$

If $\mathtt{P} = (1, \dots, 1)$, then the G-action on M factors to one of $G/\langle\kappa\rangle$. We show next that this happens in most of the cases, in which case M is a braided vector space of diagonal type.

Lemma 2.7. *Let M be either of type α_θ, $\theta \geqslant 4$, γ_θ, $\theta \geqslant 5$, δ_θ, $\theta \geqslant 5$, or ϵ_θ, $\theta = 6, 7, 8$. Then $\mathtt{P} = (1, \dots, 1)$. In other words, κ acts trivially on M.*

Proof. We consider first the case α_4. By (2.14), $g_1, g_2 \in G^{g_4}$, and by (2.13), $\kappa = g_1 g_2 g_1^{-1} g_2^{-1}$; hence, $\chi_4(\kappa) = 1$. By (2.15), $\chi_i(\kappa) = 1$ for all $i \in \mathbb{I}_4$.

Now the proof for α_θ, $\theta \geqslant 5$, δ_θ, $\theta \geqslant 5$, or ϵ_θ, $\theta = 6, 7, 8$ follows because each M_i is contained in a submodule of type α_4; thus, κ acts trivially on M_i by the paragraph above. The same fact says that, for type γ_θ, $\chi_i(\kappa) = 1$ for all $i \in \mathbb{I}_{\theta-1}$. Finally, we use that $g_{\theta-2}, g_{\theta-1} \in G^{g_\theta}$ and $\kappa = g_{\theta-1} g_{\theta-2} g_{\theta-1}^{-1} g_{\theta-2}^{-1}$ to deduce that $\chi_\theta(\kappa) = 1$. □

3 | FOLDINGS OF NICHOLS ALGEBRAS AND TRIVIALIZING THE ACTION OF THE CENTER

Motivated by Lemma 2.7, we pay special attention to Yetter–Drinfeld modules where κ acts trivially. We will show that these examples are related to diagonal braidings (of Cartan type) via the folding construction for Nichols algebras, developed by the second author in [42, 43]. Then we show that for the other cases, the action of the central element can be trivialized via a twist. First, we introduce basic notions needed for the folding construction.

3.1 | Categorical action on Yetter–Drinfeld modules

Given a group Γ and a 2-cocycle $\sigma \in Z^2(\Gamma, \Bbbk^\times)$, we get a pairing $b_\sigma : \Gamma \times \Gamma \to \Bbbk^\times$ given by $b_\sigma(g, h) = \sigma(hgh^{-1}, h)\sigma^{-1}(h, g)$. Each 2-cocycle σ yields a tensor functor $F_\sigma : {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD} \to {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ as follows:

- for an object M, let $F_\sigma(M)$ denote the same $\Bbbk\Gamma$-comodule;
- the $\Bbbk\Gamma$-module structure on $F_\sigma(M)$ is given by

$$g \cdot_\sigma m = b_\sigma(g, m_{-1}) g \cdot m_0, \qquad g \in \Gamma,\ m \in M;$$

- on Hom spaces F_σ is the identity; thus, F_σ is $\Bbbk$-linear, faithful, and exact;

- the monoidal structure $J_\sigma : F_\sigma(M) \otimes F_\sigma(N) \to F_\sigma(M \otimes N)$ is defined by

$$J_\sigma(m \otimes n) = \sigma(m_{-1}, n_{-1}) m_0 \otimes n_0, \qquad m \in M, n \in N.$$

We note that J_σ satisfy the hexagon axiom thanks to the cocycle condition on σ.

Given $V, W \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$, one can see that $F_\sigma \otimes F_\sigma$ commutes with $c_{V,W}$ if and only if $\sigma(ghg^{-1}, g)\sigma(hgh^{-1}, h) = \sigma(g, h)\sigma(h, g)$ for all $g \in \operatorname{supp} V$, $h \in \operatorname{supp} W$. Hence, F_σ is braided if and only if that equality holds for all $g, h \in \Gamma$, which certainly happens if Γ is abelian; indeed, we will only use this construction when Γ is abelian. Anyhow, these functors patch together to an action by tensor autoequivalences of the group $Z^2(\Gamma, \Bbbk^\times)$ on ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$:

- the trivial cocycle acts as the identity, and
- $F_\eta F_\sigma = F_{\eta\sigma}$ for all $\eta, \sigma \in Z^2(\Gamma, \Bbbk^\times)$.

Given a Hopf algebra $\mathscr{B}$ in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$, we set, by abuse of notation,

$$\sigma : \mathscr{B}\#\Bbbk\Gamma \otimes \mathscr{B}\#\Bbbk\Gamma \to \Bbbk, \qquad \sigma(x\#g \otimes y\#h) = \epsilon(x)\epsilon(y)\sigma(g, h). \tag{3.1}$$

Lemma 3.1.

(i) *The map σ is a Hopf 2-cocycle for $\mathscr{B}\#\Bbbk\Gamma$.*
(ii) *For all $M \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$, we have $(\mathscr{B}(M)\#\Bbbk\Gamma)_\sigma \simeq \mathscr{B}(F_\sigma(M))\#\Bbbk\Gamma$.*

Proof. (i) is clear. For (ii), apply [5, 4.14 (a) & (b)]. □

Here is the first notion toward the folding construction.

Definition 3.2. A *folding datum* is a triple $(\sigma, M, \mathbf{u})$ where $\sigma \in Z^2(\Gamma, \Bbbk^\times)$, $M \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ and $\mathbf{u} : F_\sigma(M) \to M$ is an isomorphism in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$. [†]

We will mainly deal with folding data coming from the following source.

Example 3.3. Fix $g_i \in \Gamma$, $\chi_i \in \widehat{\Gamma}$, $\sigma \in Z^2(\Gamma, \Bbbk^\times)$. Let $f : \mathbb{I} \to \mathbb{I}$ be a permutation such that

$$g_{f(i)} = g_i \qquad \text{for all } i \in \mathbb{I}.$$

Consider $M = \oplus_{i\in\mathbb{I}} M(g_i, \chi_i) \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$, and let $0 \neq x_i \in M(g_i, \chi_i)$. Note that M is of diagonal type with braiding matrix $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}}$, $q_{ij} = \chi_j(g_i)$. Then the linear isomorphism $\mathbf{u} : F_\sigma(M) \to M$, $x_i \mapsto x_{f(i)}$ is in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ if and only if

$$\chi_{f(j)} = b_\sigma(-, g_j)\chi_j \qquad \text{for all } j \in \mathbb{I}.$$

In this case, we have that

$$q_{i,f(j)} = b_\sigma(g_i, g_j) q_{ij} \qquad \text{for all } i, j \in \mathbb{I},$$

and f induces an automorphism of the Dynkin diagram of $\mathfrak{q}$ because $b_\sigma(g_i, g_i) = 1$ and $b_\sigma(g_j, g_i)b_\sigma(g_i, g_j) = 1$ for all $i \neq j \in \mathbb{I}$.

[†] In other words, the pair $(M, \mathbf{u})$ is an object in the category $({}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD})^\sigma$ of σ-equivariant objects.

Fixed M, the folding data form a group with unit $(1, M, \mathrm{id})$ and product

$$(\sigma, M, \mathbf{u}) * (\sigma', M, \mathbf{u}') = (\sigma\sigma', M, \mathbf{u} \circ F_\sigma(\mathbf{u}')).$$

The next results are extracted from [42, Part I].

Remark 3.4.

(a) Let $(\sigma, M, \mathbf{u})$ be a folding datum, $H = \mathscr{B}(M)\#\Bbbk\Gamma$. By Lemma 3.1, $\mathbf{u}$ induces a Hopf algebra isomorphism $\mathbf{u} : H_\sigma \to H$.
(b) The map $(1 \otimes \mathbf{u})\Delta_{H_\sigma} : H_\sigma \to H_\sigma \times H$ makes H_σ a right H-Galois object. Moreover, ${}_\sigma H$ is an (H, H)-bi-Galois object.
(c) Given two folding data $(\sigma, M, \mathbf{u})$, $(\sigma', M, \mathbf{u}')$, the map

$$(\mathrm{id} \otimes \mathbf{u}')\Delta_{H_{\sigma\sigma'}} : H_{\sigma\sigma'} \to H_\sigma \square H_{\sigma'}$$

is an isomorphism of bi-Galois objects.
(d) The map in (c) determines a group homomorphism from the group of folding data over $M \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ to the group of bi-Galois objects of $H = \mathscr{B}(M)\#\Bbbk\Gamma$.

3.2 | Folding construction

Let $1 \to \Sigma \to G \to \Gamma \to 1$ be a central extension of a finite abelian group Γ by a finite abelian group Σ. Fix a set-theoretic section $s : \Gamma \to G$, and let

$$\tau \in Z^2(\Gamma, \Sigma), \qquad \tau(g, h) = s(g)s(h)s(gh)^{-1}, \qquad g, h \in \Gamma.$$

For each $t \in \widehat{\Sigma}$, we denote by $\sigma_t \in Z^2(\Gamma, \Bbbk^\times)$ the 2-cocycle $\sigma_t = t \circ \tau$. The assignment $t \mapsto \sigma_t$ is a group homomorphism $\widehat{\Sigma} \to Z^2(\Gamma, \Bbbk^\times)$. Now we fix

- a Yetter–Drinfeld module M over $\Bbbk\Gamma$,
- isomorphisms $\mathbf{u}_t : F_{\sigma_t}(M) \to M$, $t \in \widehat{\Sigma}$, in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ such that the map $t \mapsto (\sigma_t, M, \mathbf{u}_t)$ is a group homomorphism from $\widehat{\Sigma}$ to the group of folding data for M. In particular, we have $\mathbf{u}_0 = \mathrm{id} : {}_{\sigma_0}H = H \to H$.

Note that the above data specify a folding datum in the sense of Definition 3.2.

Remark 3.5. M becomes a $\widehat{\Sigma}$-module, where $t \in \widehat{\Sigma}$ acts by the automorphism $\mathbf{u}_t$. As Σ is finite abelian, the $\widehat{\Sigma}$-action diagonalizes and M decomposes as a direct sum of Σ-eigenspaces:

$$M = \bigoplus_{p \in \Sigma} M^p, \qquad M^p = \{m \in M : \ \mathbf{u}_t(m) = t(p)m \text{ for all } t \in \widehat{\Sigma}\}.$$

Theorem 3.6 ([43, Theorem 3.6]). *Let Γ, G, and M as above. The following structure defines a $\Bbbk G$-Yetter–Drinfeld module $\widetilde{M}$:*

- *as a vector space, $\widetilde{M} = M$,*
- *the G-action is obtained by pulling back the Γ-action (hence Σ acts trivially),*
- *the G-grading is given by $\widetilde{M}_g = M^{gs(g)^{-1}}_{\overline{g}} := M_{\overline{g}} \cap M^{gs(g)^{-1}}$, for each $g \in G$.*

Also, as a braided vector space, $\widetilde{M} = M$.

Next, we introduce the folding construction, which produces a Nichols algebra over G starting from folding data on a Nichols algebra over Γ. The procedure gives a central extension of Hopf algebras and is related with the *Fourier transform* developed in [7].

Theorem 3.7. *Let $H := \mathscr{B}(M)\#\Bbbk\Gamma$. There exists a Hopf algebra structure on $\widetilde{H} := \bigoplus_{t\in\widehat{\Sigma}} H_{\sigma_t}$ given by*

$$\Delta|_{\sigma_t H} = \bigoplus_{t't''=t} (\mathrm{id}\otimes \mathbf{u}_{t''})\Delta_H : H_{\sigma_t} \to \bigoplus_{t't''=t} H_{\sigma_{t'}} \otimes H_{\sigma_{t''}}; \qquad \epsilon_{|H} = \epsilon_H, \qquad \epsilon_{|H_{\sigma_t}} = 0,\ t \neq 0.$$

Moreover, $\widetilde{H} \cong \mathscr{B}(\widetilde{M})\#\Bbbk G$ as Hopf algebras.

Remark 3.8. By the results above, the group of folding data induces a homomorphism of 2-groups $\underline{\widehat{\Sigma}} \to \underline{BiGal}(H)$. This, in turn, defines a homomorphism of 2-groups $\underline{\widehat{\Sigma}} \to \underline{BrPic}(\mathrm{Rep}(H))$ by [50] and thus defines by [23] a Σ-extension of the tensor category $\mathrm{Rep}(H)$. This tensor category coincides with $\mathrm{Rep}(\widetilde{H})$.

A way to see this fact comes from the equivariantization process applied to Hopf algebras because the folding data give a functor from (the category defined from) Σ to the Drinfeld double of $\mathrm{Rep}(H)$. Reciprocally, $\mathrm{Rep}(H)$ is the deequivariantization of $\mathrm{Rep}(\widetilde{H})$ associated to a central extension of Hopf algebras [17], see also [42, Theorem 3.6].

3.3 | Folding data for trivial action of κ

Next, we realize most of the examples of types α_θ, γ_θ, δ_θ, ϵ_θ, and ϕ_4 as foldings of braided vector spaces of diagonal type. In all cases, we can proceed as in Example 3.3 with $\Sigma = \mathbb{Z}_2 = \{e, \kappa\}$.

Example 3.9. Fix a finite Cartan matrix $\mathbf{a} = (a_{ij})_{i,j\in\mathbb{I}}$ with simply laced Dynkin diagram. Assume that Γ is a finite abelian group generated by g_i, $i \in \mathbb{I}$, which admits a 2-cocycle

$$\tau \in Z^2(\Gamma, \Sigma) \text{ such that} \qquad \tau(g_i, g_j) = \begin{cases} \kappa^{a_{ij}}, & i < j \in \mathbb{I}; \\ e & i \geqslant j \in \mathbb{I}. \end{cases} \tag{3.2}$$

Let G be the extension of Γ by Σ associated to τ. Thus, G is generated by g_i and κ; in G, we have $g_i g_j = \kappa^{a_{ij}} g_j g_i$ for $i \neq j \in \mathbb{I}$. Assume further that we have $\chi_i \in \widehat{\Gamma}$, $i \in \mathbb{I}$, satisfying

$$\chi_i(g_i) = -1, \qquad \chi_i(g_j)\chi_j(g_i) = (-1)^{a_{ij}}, \qquad i \neq j \in \mathbb{I}.$$

Then $V = \oplus_{i\in I} \Bbbk^{\chi_i}_{g_i}$ is of Cartan type $\mathbf{a}$ with $q = -1$, as in Example 2.2.

Here, $\widehat{\Sigma} = \{e, \mathtt{t}\}$, with $\mathtt{t}(\kappa) = -1$. Set $\sigma := \mathtt{t}\circ\tau$, and

$$g_{i+\theta} = g_i \qquad \chi_{i+\theta} := b_\sigma(-, g_i)\chi_i \in \widehat{\Gamma}, \qquad i \in \mathbb{I}, \qquad M := \oplus_{i\in I_{2\theta}} \Bbbk^{\chi_i}_{g_i}.$$

Then $M = V \oplus F_\sigma(V)$, and is of Cartan type with Cartan matrix $\widetilde{a} := \left(\begin{smallmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{a} \end{smallmatrix}\right)$.

Set also $f : \mathbb{I}_{2\theta} \to \mathbb{I}_{2\theta}$, $i \mapsto i + \theta$ modulo 2θ. Then $\mathbf{u} : F_\sigma(M) \to M$ as in Example 3.3 is a folding datum, and the map from $\widehat{\Sigma}$ to the group of folding data such that $\mathtt{t} \mapsto (\sigma, M, \mathbf{u})$ is a group

homomorphim. Following [43], if $\mathbf{a}$ is of type $X_\theta \in \{A_\theta|\theta \geqslant 2\} \cup \{D_\theta|\theta \geqslant 4\} \cup \{E_\theta|\theta = 6,7,8\}$, we use ${}^2X_\theta^2$ to denote the corresponding folding of $X_\theta \times X_\theta$ by f, $\mathbf{u}$ as above.

Example 3.10. Fix a finite abelian group Γ generated by g_i, $i \in \mathbb{I}_4$, which admits

$$\tau \in Z^2(\Gamma, \Sigma) \text{ such that} \qquad \tau(g_i, g_j) = \kappa^{\delta_{i3}\delta_{j4}}, \qquad i, j \in \mathbb{I}_4. \tag{3.3}$$

Let G be the extension of Γ by Σ associated to τ. Now G is generated by g_i and κ; the relations $g_3g_4 = \kappa g_4g_3$ and $g_ig_j = g_jg_i$ for $\{i,j\} \neq \{3,4\}$ hold in G.

Assume further that $\chi_i \in \widehat{\Gamma}$, $i \in \mathbb{I}$, satisfy

$$\chi_i(g_i) = -1, \qquad i \in \mathbb{I}_4; \qquad \chi_i(g_j)\chi_j(g_i) = (-1)^{\delta_{i+1,j}}, \qquad i < j \in \mathbb{I}_4.$$

Again, fix $\mathtt{t} \in \widehat{\Sigma}$ such that $\mathtt{t}(\kappa) = -1$. Set $\sigma := \mathtt{t} \circ \tau$, and

$$g_{i+2} = g_i \qquad \chi_{i+2} := b_\sigma(-, g_i)\chi_i \in \widehat{\Gamma}, \qquad i \in \{3,4\}, \qquad M := \oplus_{i \in I_6} \Bbbk_{g_i}^{\chi_i}.$$

Then M is of Cartan type E_6. Let $f : \mathbb{I}_6 \to \mathbb{I}_6$ be the bijection that exchanges $3 \leftrightarrow 5$ and $4 \leftrightarrow 6$. Then $\mathbf{u} : F_\sigma(M) \to M$ as in Example 3.3 is a folding datum, and the map $\mathtt{t} \mapsto (\sigma, M, \mathbf{u})$ is a group homomorphim from $\widehat{\Sigma}$ to the group of folding data. Following [43], ${}^2E_6$ denotes a folding as above.

Example 3.11. Fix a finite abelian group Γ generated by g_i, $i \in \mathbb{I}$, which admits a 2-cocycle

$$\tau \in Z^2(\Gamma, \Sigma) \text{ such that} \qquad \tau(g_i, g_j) = \begin{cases} \kappa, & j = i+1 < \theta, \\ e, & \text{otherwise.} \end{cases} \tag{3.4}$$

Let G be the extension of Γ by Σ associated to τ. Thus, G is generated by g_i and κ; in G, we have the relations $g_ig_{i+1} = \kappa g_{i+1}g_i$ if $i < \theta - 1$, and $g_ig_j = g_jg_i$ otherwise.

Assume further that $\chi_i \in \widehat{\Gamma}$, $i \in \mathbb{I}$, satisfy

$$\chi_i(g_i) = -1, \qquad i \in \mathbb{I}; \qquad \chi_i(g_j)\chi_j(g_i) = (-1)^{\delta_{i+1,j}}, \qquad i < j \in \mathbb{I}.$$

Again, fix $\mathtt{t} \in \widehat{\Sigma}$ such that $\mathtt{t}(\kappa) = -1$. Set $\sigma := \mathtt{t} \circ \tau$, and

$$g_{2\theta-i} = g_i \qquad \chi_{2\theta-i} := b_\sigma(-, g_i)\chi_i \in \widehat{\Gamma}, \qquad i \in \mathbb{I}_{\theta-1}, \qquad M := \oplus_{i \in I_{2\theta-1}} \Bbbk_{g_i}^{\chi_i}.$$

Then M is of Cartan type $A_{2\theta-1}$. Let $f : \mathbb{I}_{2\theta-1} \to \mathbb{I}_{2\theta-1}$, $f(i) = 2\theta - i$. Then $\mathbf{u} : F_\sigma(M) \to M$ as in Example 3.3 is a folding datum, and the map $\mathtt{t} \mapsto (\sigma, M, \mathbf{u})$ is a group homomorphim from $\widehat{\Sigma}$ to the group of folding data. Following [43], ${}^2A_{2n-1}$ denotes the folding above.

Remark 3.12. In the three examples above, we can take $\Gamma = \mathbb{Z}_2^{\mathbb{I}}$, see [43, §5].

Theorem 3.13. *Let M as in §2.3.1, §2.3.2, or §2.3.3. Assume that κ acts trivially on M.*

(a) *If M is of type α_θ, δ_θ, or ϵ_θ, then $\mathscr{B}(M)$ is a folded Nichols algebra as in Example 3.9.*
(b) *If M is of type ϕ_4, then $\mathscr{B}(M)$ is a folded Nichols algebra as in Example 3.10.*
(c) *If M is of type γ_θ, then $\mathscr{B}(M)$ is a folded Nichols algebra as in Example 3.11.*

Proof. Since κ acts trivially, this follows by [43, Theorems 5.6, 5.7, 5.8]. □

Remark 3.14. Fix a braided vector space M of type either $\alpha_\theta, \theta \geqslant 4$, $\delta_\theta, \theta \geqslant 5$, or $\epsilon_\theta, \theta = 6, 7, 8$ (type γ_θ was already considered in general), with Cartan matrix $\mathbf{a} = (a_{ij})_{i,j\in\mathbb{I}_\theta}$. By Lemma 2.7, $\chi_i(\kappa) = 1$ for all $i \in \mathbb{I}$, so M is of diagonal type. We exhibit a basis in which the braiding *is* of diagonal type, and we give the braiding matrix.

(I) Set $q_{ii} = -1$ for all $i \in \mathbb{I}_\theta$, and $q_{ij} = \chi_j(g_i)$ if $a_{ij} = 0$.
(II) Let $1 \leqslant i < j \leqslant \theta$ be such that $a_{ij} = -1$. Let $q_{ij} \in \Bbbk^\times$ be such that $q_{ij}^2 = \chi_j(g_i^2)$, and set $q_{ji} := -q_{ij}^{-1}$. By Step 4 of Proposition 5.2, if $k > i$ also satisfies that $a_{ik} = -1$, we may choose $q_{ik} = q_{ij}$.
(III) We also set $\mathbf{q} = (q_{ij})_{i,j\in\mathbb{I}_{2\theta}}$, where

$$q_{ij} = \begin{cases} -1, & i \leqslant \theta < j \text{ or } j \leqslant \theta < i; \\ -q_{i-\theta,j-\theta}, & i, j > \theta, a_{i-\theta,j-\theta} = -1; \\ q_{i-\theta,j-\theta}, & i, j > \theta, a_{i-\theta,j-\theta} = 0; \\ -1, & i = j > \theta. \end{cases}$$

Given $i \in \mathbb{I}_\theta$, there is $j \neq i$ in $\mathbb{I}_\theta$ such that $a_{ij} \neq 0$. If possible, take $j > i$ such that $a_{ij} \neq 0$; otherwise, take $j < i$ with $a_{ij} \neq 0$. By (II) above, we can define

$$\mathrm{x}_i := x_i + q_{ij}x_{\bar{i}}, \qquad \mathrm{x}_{\bar{i}} := x_i - q_{ij}x_{\bar{i}}, \qquad i \in \mathbb{I}.$$

Using (2.16), (2.17), and (2.18), we verify that

$$c(\mathrm{x}_i \otimes \mathrm{x}_j) = \begin{cases} -1, & i = j; \\ q_{ij}, & i \neq j. \end{cases} \qquad c(\mathrm{x}_{\bar{i}} \otimes \mathrm{x}_{\bar{j}}) = \begin{cases} -1, & i = j \\ -q_{ij}, & a_{ij} = -1; \\ q_{ij}, & a_{ij} = 0. \end{cases}$$

$$c(\mathrm{x}_i \otimes \mathrm{x}_{\bar{j}}) = \begin{cases} -1, & i = j \\ -q_{ij}, & a_{ij} = -1; \\ q_{ij}, & a_{ij} = 0. \end{cases} \qquad c(\mathrm{x}_{\bar{i}} \otimes \mathrm{x}_j) = \begin{cases} -1, & i = j \\ q_{ij}, & a_{ij} = -1; \\ q_{ij}, & a_{ij} = 0. \end{cases}$$

so the braiding matrix of M is $\mathbf{q}$, and M is, respectively, of type $A_\theta \times A_\theta$, $D_\theta \times D_\theta$ or $E_\theta \times E_\theta$, both copies with parameter $q = -1$.

3.4 | Trivializing the action of κ via a twist

Retain the notation introduced in §3.3. Thus, G is a nonabelian group and $M = \oplus_{i\in\mathbb{I}} M_i \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, where $M_i = M(g_i^G, \chi_i)$.

For the cases not covered by Lemma 2.7, we will show the existence of a 2-cocycle σ such that κ acts trivially on $F_\sigma(M)$. Recall the parity vector $\mathtt{P} = (\chi_1(\kappa), \dots, \chi_\theta(\kappa))$ from (2.28).

Let $\sigma \in H^2(G, \Bbbk^\times)$. Following §3.1, the *twisted Yetter–Drinfeld module* associated to σ is $F_\sigma(M) = M^\sigma = \oplus_{i\in\mathbb{I}} M(g_i, \chi_i^\sigma)$, where $\chi_i^\sigma \in \widehat{G^{g_i}}$ is given by the following formula:

$$\chi_i^\sigma(h) = \sigma(hg_ih^{-1}, h)\sigma^{-1}(h, g_i)\chi_i(h), \qquad h \in G.$$

Since κ is central in G, we have $\chi_i^\sigma(\kappa) = \sigma(g_i, \kappa)\sigma^{-1}(\kappa, g_i)\chi_i(\kappa)$.

Proposition 3.15. *Let G be a nonabelian group, $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of type α_2, α_3, δ_4, γ_3, γ_4, or ϕ_4 such that* supp M *generates G. There exists $\sigma \in H^2(G, \Bbbk^\times)$ with $\chi_i^\sigma(\kappa) = 1$ for all i.*

Let us outline the strategy that will be used in the Appendix to prove this statement.

(i) We go through the cases and list the possible $\mathtt{P} = (\chi_i(\kappa))_i \in \{\pm 1\}^n$.
(ii) It is sufficient to consider one $\mathtt{P}$ representing each Weyl groupoid orbit. The ith reflection of M is

$$\rho_i M = \oplus_{j\in\mathbb{I}} M(g_j g_i^{-c_{ij}}, \chi_j \chi_i^{-c_{ij}}),$$

and the parity vector of $\rho_i M$ is $\mathtt{P}' = (\chi_j(\kappa)\chi_i(\kappa)^{-c_{ij}})_{j\in\mathbb{I}}$.
(iii) Next, we introduce an auxiliary *minimal* group $G^{\min}$. Namely, $G^{\min} \subseteq \operatorname{End} M$ is generated by (the action of) g_i. The definition of $G^{\min}$ depends only on the scalars $\chi_i(g_j)$, $\chi_i(\kappa)$. It is enough to prove Proposition 3.15 for this group, since the asserted 2-cocycle σ on $G^{\min}$ can be pulled back to G.
(iv) In the next steps, we show case-wise that there exists $\sigma \in H^2(G^{\min}, \Bbbk^\times)$ such that

$$\sigma(g_i, \kappa)\sigma(\kappa, g_i)^{-1} = \chi_i(\kappa), \qquad \text{for all } i \in \mathbb{I}.$$

(v) For type α_2, there are two Weyl groupoid orbits for $\mathtt{P} = (\chi_1(\kappa), \chi_2(\kappa))$, namely, $\{(1, 1)\}$ and $\{(1, -1), (-1, -1), (-1, 1)\}$, the first one corresponding to $\mathtt{P}$ trivial. For $(-1, 1)$, we get three different types of groups according to the order of $\chi_2(g_1^2)$: we find the desired cocycle using semidirect product decompositions.
(vi) The cases α_3 and γ_3 are treated using spectral sequences arguments for a central extension. Necessary information about the structure of the group (minimal orders of central elements, e.g.) enters conveniently via the existence of a one-dimensional representation, constructed from the structure of M.
(vii) The remaining cases are treated using two simultaneous extensions.

We postpone the proof until §A.3 since we need technical results on group cohomology.
By assumption, G is a central extension of an abelian group Γ by $\langle\kappa\rangle \simeq \mathbb{Z}_2$, say:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow G \longrightarrow \Gamma \longrightarrow 1.$$

To illustrate the proof of Proposition 3.15, we give an example where such central extensions are related to symplectic forms on $\mathbb{Z}_2^\theta$.

Example 3.16. Let $\Gamma = \mathbb{Z}_2^\theta$ with generators g_i, $i \in \mathbb{I}$. The commutator in G defines a symplectic form on Γ such that the radical $\mathbb{Z}_2^r$ is the image of the center $Z(G)$; in particular, g_i, g_j commute if

and only if the symplectic form on them is zero; hence, the size of the conjugacy class of g_i is 1 or 2 depending if g_i is in the radical or not. This symplectic form of type (θ, r) is uniquely determined.

Central extensions $\mathbb{Z}_2 \to G \to \Gamma$ are classified by quadratic forms with fixed symplectic form. For type α_2, we have $(\theta, r) = (2, 0)$ and there are two types of central extensions of order 8 with this commutator structure, namely, the extraspecial groups 2_-^{2+1} (the quaternion group) and 2_+^{2+1} (the dihedral group). The group 2_-^{2+1} has defining relations $g_1^2 = g_2^2 = \kappa$ and trivial second cohomology $H^2(2_-^{2+1}, \Bbbk^\times)$, while the group 2_+^{2+1} has defining relations $g_1^2 = e$, $g_2^2 = \kappa$ (depending on a choice of generators) and nontrivial second cohomology $H^2(2_+^{2+1}, \Bbbk^\times) = \{1, \sigma\}$. Our proof works for the group 2_-^{2+1} because the relations (2.13) defining M imply

$$\chi_1(\kappa) = \chi_1(g_1^2) = 1, \qquad \chi_2(\kappa) = \chi_2(g_2^2) = 1.$$

For 2_+^{2+1}, both $\mathtt{P} = (1, 1)$ and $\mathtt{P} = (-1, 1)$ are possible and we have a 2-cocycle σ with

$$\sigma(g_1, \kappa)\sigma^{-1}(\kappa, g_1) = -1, \qquad \sigma(g_2, \kappa)\sigma^{-1}(\kappa, g_2) = 1.$$

The other choices of generators for 2_+^{2+1} work similarly and also follow from the first choice by using Weyl groupoid reflections.

The cases α_3, δ_4, γ_3, γ_4, ϕ_4 present similar behavior because $\frac{1}{2}(\theta - r) = 1$ in all of them. For each case, there is an underlying extraspecial group $2_\pm^{2+1}$.

Remark 3.17. The proof becomes more involved for an arbitrary abelian group Γ because there are many central extensions by $\mathbb{Z}_2$, parametrized by the powers of the generators, and the existence of a nontrivial group cohomology is very sensitive to these choices.

We are ready to state the main result of this section, which states that the Nichols algebra of a Yetter–Drinfeld module of type α_θ, γ_θ, δ_θ, ϵ_θ, or ϕ_4 is a twist of the corresponding Nichols algebra of diagonal type as in §3.3.

Theorem 3.18. *Let G be a finite nonabelian group, $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of type either α_θ, γ_θ, δ_θ, ϵ_θ, or ϕ_4 whose support generates G. Then there exists $\sigma \in H^2(G, \Bbbk)$ such that $F_\sigma(M)$ is of diagonal type.*

Proof. If M is of type α_θ, $\theta \geqslant 4$, γ_θ, $\theta \geqslant 5$, δ_θ, $\theta \geqslant 5$, or ϵ_θ, $\theta = 6, 7, 8$, then κ acts trivially by Lemma 2.7, so M is naturally a Yetter–Drinfeld module over $\Gamma = G/\langle\kappa\rangle$, a finite abelian group; thus, M is itself of diagonal type. For the other cases, we apply Proposition 3.15. □

Let G and $M = \oplus_{i\in\mathbb{I}} M_i \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ be as above. Consider

$$\ell := \begin{cases} 2 & \text{for type } \phi_4, \\ \theta - 1 & \text{for type } \gamma_\theta, \\ \theta & \text{otherwise.} \end{cases} \tag{3.5}$$

Keeping the notation used for §2.3.1, §2.3.2, and §2.3.3, we have the following.

- If $i \leqslant \ell$, then $\dim M_i = 2$, $g_i^G = \{g_i, \kappa g_i\}$, where $\kappa \in Z(G)$ satisfies $\kappa^2 = 1$. We fix a basis $x_i, x_{\bar{i}}$ of M_i as above.
- If $i > \ell$, then $\dim M_i = 1$, $g_i^G = \{g_i\}$. For a basis, we fix any nonzero element x_i in M_i.

Remark 3.19. Let $\mathtt{G}$ be the group generated by g_i, $i \in \mathbb{I}$ and κ with relations

$$g_i\kappa = \kappa g_i, \qquad \kappa^2 = e, \qquad g_i g_j = \begin{cases} \kappa g_j g_i, & i \neq j \leqslant \ell,\ a_{ij} = -1, \\ g_j g_i & \text{otherwise,} \end{cases}$$

where $\mathbf{a} = (a_{ij})$ denotes the Cartan matrix of M. Then $\mathtt{G}$ is a central extension of $\mathbb{Z}^{\mathbb{I}}$ by $\mathbb{Z}_2$, and the braided vector space M has a realization over $\mathtt{G}$. Moreover, the subgroup of G generated by $\operatorname{supp} M$ is a quotient of $\mathtt{G}$.

The $\mathbb{Z}^{\mathbb{I}}$-grading on $T(M)$ and its homogeneous quotients (in particular, $\mathscr{B}(M)$) is given by the induced coaction of $\Bbbk\mathbb{Z}^{\mathbb{I}} \simeq \Bbbk\mathtt{G}/\langle\kappa\rangle$.

4 | GENERATION IN DEGREE 1

Using Theorem 3.18 and generation-in-degree-one for the diagonal setting [12], we get the following.

Theorem 4.1. *Let H be a finite-dimensional pointed Hopf algebra with infinitesimal braiding M of type α_θ, γ_θ, δ_θ, ϵ_θ, or ϕ_4. Then*

$$\operatorname{gr} H \simeq \mathscr{B}(M)\#\Bbbk G(H).$$

In other words, H is generated by skew-primitive and group-like elements.

Proof. Let R be the diagram of H, so $\operatorname{gr} H \simeq R\#\Bbbk G(H)$. We need to show that R is generated by its degree 1 elements $M = R(1)$. Or, equivalently, we need to show that the canonical map $\mathscr{B}(M) \hookrightarrow R$ is an isomorphism. Put also $\mathscr{B} := R^*$. Notice that $W := M^*$ is of the same type as M; we fix g_i, $i \in \mathbb{I}$, κ as in §2.2. Let G be the subgroup of $G(H)$ generated by g_i, $i \in \mathbb{I}$. Then $\mathscr{B} \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is a finite-dimensional pre-Nichols algebra of W; that is, $\mathscr{B} = \oplus_{n\geqslant 0}\mathscr{B}^n$, where $\mathscr{B}^n = R(n)^*$, is a graded Hopf algebra such that $\mathscr{B}^0 = \Bbbk 1$ and is generated as an algebra by $W = \mathscr{B}^1$.

For $\sigma \in H^2(G, \Bbbk^\times)$ as in Theorem 3.18, set $\sigma : \mathscr{B}\#\Bbbk G \otimes \mathscr{B}\#\Bbbk G \to \Bbbk$ as in (3.1), and let $\mathtt{H} := (\mathscr{B}\#\Bbbk G)_\sigma$, $\mathtt{H}^n := \mathscr{B}^n\#\Bbbk G$. Then $\mathtt{H} = \oplus_{n\geqslant 0}\mathtt{H}^n$ is a graded coalgebra, because twisting by σ leaves the coalgebra structure unchanged. Thus, $\mathtt{H}$ is pointed with coradical $\mathtt{H}^0 \simeq \Bbbk G$ by [51, 5.3.4]. As σ is trivial in degree > 0, $\mathtt{H}$ is a graded Hopf algebra: By [5, 4.14 (a)], $\mathtt{H} \simeq \mathscr{B}'\#\Bbbk G$, where $\mathscr{B}' \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is a pre-Nichols algebra of $F_\sigma(W)$.

As $\kappa \in Z(G)$ and it acts trivially on W^σ, we have that $\kappa \in Z(H)$. Set $Q := \mathtt{H}/\mathtt{H}(\kappa - 1)$, $\Gamma = G/\langle\kappa\rangle$. The G-actions on $F_\sigma(W)$ and on $\mathscr{B}'$ induce respective Γ-actions on them. Also, $F_\sigma(W)$ and $\mathscr{B}'$ become $\Bbbk\Gamma$-comodules via $\pi : G \twoheadrightarrow \Gamma$. Moreover, with these structures both $\mathscr{B}'$ and $F_\sigma(W)$ are in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$, and $\mathscr{B}' \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ is a pre-Nichols algebra of $F_\sigma(W)$. We identify $\mathtt{H} \simeq \mathscr{B}'\#\Bbbk G$ and consider the map $\Phi : \mathtt{H} \to \mathscr{B}'\#\Bbbk\Gamma$, $\Phi(x\#g) = x\#\pi(g)$. Then Φ is a surjective Hopf algebra map such that $\Phi(\kappa - 1) = 0$; hence, Φ induces a surjective Hopf algebra map $\phi : Q \to \mathscr{B}'\#\Bbbk\Gamma$. As $\mathtt{H}$ is a central extension of Q by $\Bbbk\mathbb{Z}_2$ (since $\kappa^2 = 1, \kappa \neq 1$),

$$\dim Q = \frac{1}{2}\dim \mathtt{H} = \frac{1}{2}\dim \mathscr{B}'|G| = \dim \mathscr{B}'|\Gamma|,$$

so ϕ is an isomorphism.

Now Γ is an abelian group since for each $i \neq j$ either $g_ig_j = g_jg_i$ or $g_ig_j = \kappa g_jg_i$ in G, and Γ is generated by the images of the g_i's (see §2.2). Hence, [12] implies that $\mathscr{B}' = \mathscr{B}(F_\sigma(W))$, so $\mathscr{B} = \mathscr{B}(W)$ by [5, 4.14 (b)]. Dualizing, $R = \mathscr{B}(M)$, as desired. □

5 | GENERATORS AND RELATIONS FOR NICHOLS ALGEBRAS

In this section, we exhibit a presentation by generators and relations for the Nichols algebras of the Yetter–Drinfeld modules $M = \oplus_{i\in\mathbb{I}} M_i \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ as in §2.3.1, §2.3.2, and §2.3.3, which are standard, that is, all reflection $\rho_i M$ are of the same type as M. In particular, the root system Δ^M is a classical one. We may assume that G is the group in Remark 3.19.

5.1 | Types α_2 and α_3

We give a presentation of Nichols algebras of types α_2 and α_3. They will be a key step toward the presentation for the general case, since all relations that are not powers of root vectors are supported on smaller submodules of these types.

5.1.1 | Type α_2

Let G be a group, $e \neq \kappa$, $g_1, g_2 \in G$ such that

$$g_1g_2 = \kappa g_2g_1, \qquad \kappa^2 = e, \qquad g_i^G = \{g_i, \kappa g_i\}, \qquad i = 1, 2.$$

Following [34], the subgroup of G generated by g_1 and g_2 is a quotient of

$$G_2 = \langle g_1, g_2, \kappa | \kappa^2 = 1, \kappa g_1 = g_1\kappa, \kappa g_2 = g_2\kappa, g_2g_1 = \kappa g_1g_2\rangle. \tag{5.1}$$

Assume that there are $\chi_i \in \widehat{G^{g_i}}$ such that $\chi_i(g_i) = -1$ and $\chi_1(\kappa g_2^2)\chi_2(\kappa g_1^2) = 1$. For $i \in \mathbb{I}_2$, set $M_i = M(g_i^G, \chi_i) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$; thus, $M = M_1 \oplus M_2$ is of type α_2 by [34, Theorem 4.6].

Proposition 5.1. *Let $M = M_1 \oplus M_2 \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of type α_2 as above. Then $\mathscr{B}(M)$ is presented by generators $x_1, x_{\bar{1}}, x_2, x_{\bar{2}}$ and relations*

$$x_i^2 = x_{\bar{i}}^2 = 0, \qquad (\mathrm{ad}_c\, x_i)x_{\bar{i}} = 0, \qquad i \in \mathbb{I}. \tag{5.2}$$

$$x_{1\bar{2}} = -\chi_2(g_1^2)x_{\bar{1}2}, \qquad x_{\bar{1}\bar{2}} = -\chi_1(\kappa)x_{12}; \tag{5.3}$$

$$x_{12}^2 = 0, \qquad [x_{12}, x_{\bar{1}2}]_c = 0. \tag{5.4}$$

The following set is a PBW basis of $\mathscr{B}(M)$:

$$\left\{x_2^a x_{\bar{2}}^b x_{12}^c x_{\bar{1}2}^d x_1^e x_{\bar{1}}^f : a, b, c, d, e, f \in \{0, 1\}\right\}. \tag{5.5}$$

Proof. We proceed in several steps.

Step 1. By (2.16), the Nichols algebra of M_i for $i = 1, 2$ is a quantum linear space, and (5.6) implies that relations (5.2) hold in $\mathscr{B}(M)$.

Step 2. The inclusion $M_{12} := (\mathrm{ad}_c\, M_1)M_2 \hookrightarrow \mathscr{B}(M)$ extends to a $\mathbb{Z}^2$-graded algebra inclusion $\mathscr{B}(M_{12}) \hookrightarrow \mathscr{B}(M)$ and the multiplication

$$\mathscr{B}(M_2) \otimes \mathscr{B}(M_{12}) \otimes \mathscr{B}(M_1) \longrightarrow \mathscr{B}(M) \tag{5.6}$$

is an isomorphism of $\mathbb{Z}^2$-graded objects in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, where M_1 sits in degree α_1, M_2 in degree α_2, and M_{12} in degree $\alpha_1 + \alpha_2$.

This follows by [34, Theorem 4.6]. In order to find defining relations for $\mathscr{B}(M_{12}) \subset \mathscr{B}(M)$, we need a more explicit description of the structure of M_{12}.

Step 3.

(a) The set $\{x_{12}, x_{\overline{12}}\}$ is a basis of M_{12}, and the braiding in this basis is of diagonal type with matrix

$$\begin{pmatrix} -1 & -\chi_1(\kappa)\chi_2(\kappa) \\ -\chi_1(\kappa)\chi_2(\kappa) & -1 \end{pmatrix}. \tag{5.7}$$

(b) Relations (5.3) and (5.4) hold in $\mathscr{B}(M)$.

Proof of Step 3. Note that $\partial_{x_2}(x_{12}) = x_1$ and $\partial_{x_2}(x_{\overline{12}}) = x_{\overline{1}}$; thus, x_{12} and $x_{\overline{12}}$ are linearly independent in $\mathscr{B}(M)$. Relations (5.3) are verified using the skew derivations of $T(M_1 \oplus M_2)$. The braiding of M_{12} is obtained from a straightforward computation in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$. Now (5.4) follows from (5.6) and (5.7). □

Note that $\mathscr{B}(M_{12})$ is presented by the relations (5.4) and $(x_{\overline{12}})^2 = 0$, which has not been included above because it can be deduced from the previous ones, as we show in Step 4.

With (5.6) in mind, the next step toward exhibiting a presentation of $\mathscr{B}(M)$ should be to find braided commutations between $\mathscr{B}(M_i)$ and $\mathscr{B}(M_{12})$ for $i = 1, 2$. Such relations are known to exist, since $(\mathrm{ad}\, M_i)^2 M_j = 0$ for $i \neq j$ by [34, Lemma 4.2]. However, we show next that these can be deduced from some of the already established relations.

Step 4. Let A denote the quotient of $T(M_1 \oplus M_2)$ by the ideal generated by (5.2) and (5.3). In A, the following relations hold

$$(\mathrm{ad}_c\, x_1)x_{12} = 0, \quad (\mathrm{ad}_c\, x_{\overline{1}})x_{12} = 0, \quad (\mathrm{ad}_c\, x_1)x_{\overline{12}} = 0, \quad (\mathrm{ad}_c\, x_{\overline{1}})x_{\overline{12}} = 0; \tag{5.8}$$

$$[x_{12}, x_2]_c = 0, \quad [x_{12}, x_{\overline{2}}]_c = 0, \quad [x_{\overline{12}}, x_2]_c = 0, \quad [x_{\overline{12}}, x_{\overline{2}}]_c = 0; \tag{5.9}$$

$$x_{12}^2 = 0. \tag{5.10}$$

Proof of Step 4. First, $(\mathrm{ad}_c\, x_1)x_{12} = (\mathrm{ad}_c\, x_1^2)x_2 = 0$ since $x_1^2 = 0$. Analogously $(\mathrm{ad}_c\, x_{\overline{1}})x_{\overline{12}} = 0$ follows from $x_{\overline{1}}^2 = 0$. Next, using that x_{12} is a scalar multiple of $x_{\overline{12}}$, we get that $(\mathrm{ad}_c\, x_{\overline{1}})x_{12} = 0$ follows from $x_{\overline{1}}^2 = 0$. Since $x_{\overline{12}}$ is a scalar multiple of $x_{1\overline{2}}$, we get $(\mathrm{ad}_c\, x_1)x_{\overline{12}} = 0$ from $x_1^2 = 0$. Thus, (5.8) hold.

For (5.9), unpacking the definitions, we see that $[x_{12}, x_2]_c = [x_{1\bar{2}}, x_{\bar{2}}]_c = 0$ follow from $x_2^2 = x_{\bar{2}}^2 = 0$. Now $[x_{12}, x_{\bar{2}}]_c = 0$ follows using the previous argument, since x_{12} is a scalar multiple of $x_{\overline{12}}$. The remaining relation holds similarly.

Finally, we show that (5.10) follows from (5.8), (5.9), and (5.3). In fact,

$$\begin{aligned}(x_{12})^2 &= (x_1x_2 - x_{\bar{2}}x_1)(\mathrm{ad}_c\, x_1)(x_2)\\ &= -(\chi_2(\kappa g_1^2))^{-1}x_1(\mathrm{ad}_c\, x_1)(x_2)x_{\bar{2}} - \chi_2(g_1^2)x_{\bar{2}}(\mathrm{ad}_c\, x_{\bar{1}})(x_2)x_1\\ &= -\chi_2(\kappa)(\mathrm{ad}_c\, x_{\bar{1}})(x_2)x_1x_{\bar{2}} + \chi_2(\kappa g_1^2)(\mathrm{ad}_c\, x_{\bar{1}})(x_2)x_2x_1\\ &= -\chi_2(\kappa)(\mathrm{ad}_c\, x_{\bar{1}})(x_2)(\mathrm{ad}_c\, x_1)(x_{\bar{2}}) = \chi_2(\kappa g_1^2)(x_{\bar{1}2})^2,\end{aligned} \tag{5.11}$$

as claimed. □

We are ready to give a presentation of $\mathscr{B}(M_1 \oplus M_2)$. Let R denote the quotient of $T(M_1 \oplus M_2)$ by the ideal generated by (5.2), (5.3), and (5.4). We already know from (5.6) and Step 3 that the canonical projection $T(M_1 \oplus M_2) \to \mathscr{B}(M)$ factors to a surjective algebra map $R \to \mathscr{B}(M)$. We show that this map is injective. Since $\dim \mathscr{B}(M) = 2^6$, it is enough to verify that the set (5.5) linearly generates R. Let J denote the subspace spanned by (5.5) in R. Since J contains 1, it is enough to show that J is a left ideal, which reduces to verify that $x_iI \subset I$ and $x_{\bar{i}}I \subset I$ for $i = 1, 2$. Clearly, $x_2J \subset J$; as $x_{\bar{2}}x_2$ is a scalar multiple of $x_2x_{\bar{2}}$, it is equally clear that $x_{\bar{2}}J \subset J$. So, we need to verify that $x_1J \subset J$ and $x_{\bar{1}}J \subset J$, which follow since the (5.8) and (5.9) hold in R by Step 4. □

5.1.2 | Type α_3

Let G denote a nonabelian group and let $M = M_1 \oplus M_2 \oplus M_3$ in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of type α_3. By [37, Lemma 5.2] for $i \in \mathbb{I}_2$, the subgroup $\langle \kappa, g_i, g_{i+1}\rangle \subset G$ is a quotient of G_2, see (5.1). Next, we describe $\mathscr{B}(M)$.

Proposition 5.2. *For M of type α_3, the Nichols algebra $\mathscr{B}(M)$ is presented by generators $x_i, x_{\bar{i}}$, $i \in \mathbb{I}_3$ and relations*

$$x_i^2 = x_{\bar{i}}^2 = 0, \qquad (\mathrm{ad}_c\, x_i)x_{\bar{i}} = 0, \qquad i \in \mathbb{I}; \tag{5.12}$$

$$x_{i\bar{j}} = -\chi_j(g_i^2)x_{\bar{i}j}, \qquad x_{\overline{ij}} = -\chi_i(\kappa)x_{ij}, \qquad i < j,\ a_{ij} = -1; \tag{5.13}$$

$$x_{ij}^2 = 0, \qquad [x_{ij}, x_{\bar{i}\bar{j}}]_c = 0, \qquad i < j,\ a_{ij} = -1; \tag{5.14}$$

$$x_{13} = x_{1\bar{3}} = 0, \qquad x_{\bar{1}3} = x_{\overline{13}} = 0; \tag{5.15}$$

$$x_{123}^2 = 0, \qquad [x_{123}, x_{\bar{1}\bar{2}3}]_c = 0; \tag{5.16}$$

$$(\mathrm{ad}_c\, x_2)x_{123} = 0, \qquad (\mathrm{ad}_c\, x_2)x_{\bar{1}23} = 0. \tag{5.17}$$

A PBW basis of $\mathscr{B}(M)$ is given by

$$\left\{x_3^{a_3}x_{\bar{3}}^{b_3}x_{23}^{a_{23}}x_{\bar{2}3}^{b_{23}}x_2^{a_2}x_{\bar{2}}^{b_2}x_{123}^{a_{123}}x_{\bar{1}23}^{b_{123}}x_{12}^{a_{12}}x_{\bar{1}2}^{b_{12}}x_1^{a_1}x_{\bar{1}}^{b_1} :\ a_\beta, b_\beta \in \{0,1\}\right\}. \tag{5.18}$$

Proof. Again, we proceed in several steps.

Step 1. The multiplication map is an isomorphism of $\mathbb{Z}^3$-graded objects in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$:

$$\mathscr{B}(M_3) \otimes \mathscr{B}(M_{23}) \otimes \mathscr{B}(M_2) \otimes \mathscr{B}(M_{123}) \otimes \mathscr{B}(M_{12}) \otimes \mathscr{B}(M_1) \simeq \mathscr{B}(M)$$

This follows by [34, Theorem 2.6]. Next, we give some relations that hold in $\mathscr{B}(M)$.

Step 2. The relations (5.12), (5.13), and (5.15) hold in $\mathscr{B}(M)$.

We know that $M_{13} = 0$, so (5.15) hold, while (5.12) and (5.13) follow since $M_1 \oplus M_2$ and $M_2 \oplus M_3$ are of type α_2. □

Following the treatment in §5.1.1, we describe $\mathscr{B}(M_{123}) \subset \mathscr{B}(M)$. As a first step in this direction, we produce a basis for M_{123}.

Step 3. Let A denote the quotient of $T(M_1 \oplus M_2 \oplus M_3)$ by the ideal generated by (5.15) and (5.13). In A, the following relations hold:

$$x_{1\bar{2}3} = -\chi_2(g_1^2)x_{\bar{1}23}, \qquad x_{\bar{1}23} = -\chi_1(\kappa)x_{123}. \tag{5.19}$$

Proof of Step 3. We only verify the first one. We compute

$$\begin{aligned} x_{1\bar{2}3} &= [x_1, [x_{\bar{2}}, x_3]_c]_c = [[x_1, x_{\bar{2}}]_c, x_3]_c + (g_1 \cdot x_{\bar{2}})[x_1, x_3]_c - \chi_3(\kappa)[x_1, x_{\bar{3}}]x_{\bar{2}} \\ &= [[x_1, x_{\bar{2}}]_c, x_3]_c = -\chi_2(g_1^2)[[x_{\bar{1}}, x_2]_c, x_3]_c = -\chi_2(g_1^2)[x_{\bar{1}}, [x_2, x_3]_c]_c, \end{aligned}$$

where the second equality follows by (2.9), the third from (5.15), the fourth from (5.13), and the fifth one by (2.9) and (5.15). □

Surprisingly, there are further restrictions on the character χ_2:

Step 4. If $M_1 \oplus M_2 \oplus M_3$ is of type α_3, then $\chi_2(g_1^2) = \chi_2(g_3^2)$.

Proof of Step 4. We compute the action of g_2 on $x_{123} \in \mathscr{B}(M)$ following two different approaches. Applying (2.10) first, followed by (5.13), we get

$$\begin{aligned} g_2 \cdot x_{123} &= -(\mathrm{ad}_c\, x_{\bar{1}})(\mathrm{ad}_c\, x_2)x_{\bar{3}} = \chi_3(g_2^2)x_{\bar{1}\bar{2}3} \\ &= \chi_3(g_2^2)(x_{\bar{1}}((\mathrm{ad}_c\, x_{\bar{2}})x_3) - \chi_2(\kappa g_1^2)\chi_3(\kappa g_1)((\mathrm{ad}_c\, x_2)x_3)x_{\bar{1}}). \end{aligned}$$

On the other hand, if we first unpack the definition of $\mathrm{ad}_c\, x_1$ and then let g_2 act, we get

$$\begin{aligned} g_2 \cdot x_{123} &= g_2 \cdot (x_1((\mathrm{ad}_c\, x_2)x_3) - \chi_3(g_1)((\mathrm{ad}_c\, x_{\bar{2}})x_3)x_1) \\ &= -x_{\bar{1}}((\mathrm{ad}_c\, x_2)x_{\bar{3}}) + \chi_2(\kappa)\chi_3(g_1)((\mathrm{ad}_c\, x_{\bar{2}})x_{\bar{3}})x_{\bar{1}} \\ &= \chi_3(g_2^2)x_{\bar{1}}((\mathrm{ad}_c\, x_{\bar{2}})x_3) - \chi_3(g_1)((\mathrm{ad}_c\, x_2)x_3)x_{\bar{1}} \\ &= \chi_3(g_2^2)(x_{\bar{1}}((\mathrm{ad}_c\, x_{\bar{2}})x_3) - \chi_2(\kappa g_3^2)\chi_3(\kappa g_1)((\mathrm{ad}_c\, x_2)x_3)x_{\bar{1}}), \end{aligned}$$

where the second equality follows by (2.10), the third one from (5.13), and the last one from $\chi_2(\kappa g_3^2)\chi_3(\kappa g_2^2) = 1$.

These two equations give $(\chi_2(g_1^2) - \chi_2(g_3^2))((\mathrm{ad}_c\, x_2)x_3)x_{\bar{1}} = 0$, and the claim follows because $\partial_{x_{\bar{1}}}(((\mathrm{ad}_c\, x_2)x_3)x_{\bar{1}}) = (\mathrm{ad}_c\, x_2)x_3 \neq 0$. □

The next result is analog to Step 3 in Proposition 5.1, and its proof follows from similar arguments, so we only give a sketch.

Step 5.

(a) A basis of M_{123} is $\{x_{123}, x_{\bar{1}\bar{2}\bar{3}}\}$, where the braid is diagonal with matrix

$$\begin{pmatrix} -1 & -\chi_1(\kappa)\chi_2(\kappa)\chi_3(\kappa) \\ -\chi_1(\kappa)\chi_2(\kappa)\chi_3(\kappa) & -1 \end{pmatrix}. \tag{5.20}$$

(b) Relations (5.16) hold in $\mathscr{B}(M)$.

Proof of Step 5. (a) The set $\{x_{123}, x_{\bar{1}\bar{2}\bar{3}}\}$ linearly spans M_{123} by Step 3, and it is linearly independent since so are $\partial_{x_3}(x_{123}) = x_{12}$ and $\partial_{x_3}(x_{\bar{1}\bar{2}\bar{3}}) = x_{\bar{1}\bar{2}}$. The braiding is computed using (2.10) and Step 4. Now (b) follows from (a) and Step 1. □

The Nichols algebra of M_{123} is presented by the relations (5.16) and $x_{\bar{1}\bar{2}\bar{3}}^2 = 0$; we will show that this last relation can be deduced from others.

Braided commutations between M_i and M_{123} for $i = 1, 3$ can now be deduced.

Step 6. Let A denote the quotient of $T(M_1 \oplus M_2 \oplus M_3)$ by the ideal generated by (5.2), (5.15), and (5.13). In A, the following relations hold :

$$(\mathrm{ad}_c\, x_1)x_{123} = 0, \quad (\mathrm{ad}_c\, x_{\bar{1}})x_{123} = 0, \quad (\mathrm{ad}_c\, x_1)x_{\bar{1}\bar{2}\bar{3}} = 0, \quad (\mathrm{ad}_c\, x_{\bar{1}})x_{\bar{1}\bar{2}\bar{3}} = 0; \tag{5.21}$$

$$[x_{123}, x_3]_c = 0, \quad [x_{123}, x_{\bar{3}}]_c = 0, \quad [x_{\bar{1}\bar{2}\bar{3}}, x_3]_c = 0, \quad [x_{\bar{1}\bar{2}\bar{3}}, x_{\bar{3}}]_c = 0; \tag{5.22}$$

$$(\mathrm{ad}_c\, x_{\bar{2}})x_{123} = -\chi_1(g_2^{-2})(\mathrm{ad}_c\, x_2)x_{\bar{1}\bar{2}\bar{3}}, \quad (\mathrm{ad}_c\, x_{\bar{2}})x_{\bar{1}\bar{2}\bar{3}} = -\chi_1(g_2^{-2})(\mathrm{ad}_c\, x_2)x_{123}. \tag{5.23}$$

Proof of Step 6. The relations (5.21) can be verified using the argument in the proof of Step 4 of Proposition 5.1. For the first relation in (5.22), use (2.9) to get

$$[x_{123}, x_3]_c = [[x_1, x_{23}]_c, x_3]_c = [x_1, [x_{23}, x_3]_c]_c - x_{\bar{2}}[x_1, x_3]_c + [x_1, x_{\bar{3}}]_c x_2 = 0.$$

The three remaining relations follow similarly. For (5.23), use (2.9), (5.13), and (5.8) to get

$$\begin{aligned}(\mathrm{ad}_c\, x_{\bar{2}})x_{123} &= \left[x_{\bar{2}\bar{1}}, x_{23}\right] + (g_2\kappa \cdot x_1)[x_{\bar{2}}, x_{23}]_c - \chi_3(g_1)[x_{\bar{2}}, x_{\bar{2}\bar{3}}]_c x_1 \\ &= -\chi_1(g_2^{-2})\left[x_{2\bar{1}}, x_{23}\right] = -\chi_1(g_2^{-2})(\mathrm{ad}_c\, x_2)x_{\bar{1}\bar{2}\bar{3}}.\end{aligned}$$

and the other relation follows analogously. □

We employ skew derivations to verify braided commutations between M_2 and M_{123}.

Step 7. Relations (5.17) hold in $\mathscr{B}(M)$.

Proof of Step 7. We focus on the first relation. One can see directly from (2.10) that

$$\partial_{x_i}(M_{123}) = \partial_{x_{\bar{i}}}(M_{123}) = 0, \qquad \partial_{x_3}(x_{123}) = x_{12}, \qquad \partial_{x_{\bar{3}}}(x_{\bar{1}2\bar{3}}) = x_{\bar{1}\bar{2}}.$$

Now, using (2.10) and also Step 4, we get

$$\partial_{x_{\bar{3}}}(x_{123}) = -\chi_2(\kappa)x_{12\bar{3}}, \qquad \partial_{x_3}(x_{\bar{1}2\bar{3}}) = -\chi_3(g_2^2)x_{\bar{1}\bar{2}}.$$

For $i \in \mathbb{I}_2$, both ∂_{x_i} and $\partial_{x_{\bar{i}}}$ annihilate the first relation, and we compute

$$\partial_{x_3}((\mathrm{ad}_c\, x_2)x_{123}) = x_2x_{12} - \chi_3(g_2^2)x_{\bar{1}\bar{2}}x_{\bar{2}} = x_2x_{12} + \chi_3(\kappa g_2^2)x_{12}x_{\bar{2}} = \chi_3(\kappa g_2^2)[x_{12}, x_{\bar{2}}]_c = 0,$$

where the third equality follows by (5.13). Similarly, $\partial_{x_{\bar{3}}}((\mathrm{ad}_c\, x_2)x_{123}) = \chi_2(\kappa)[x_{\bar{1}2}, x_{\bar{2}}]_c = 0$. The other relations follow analogously. □

Next, we deduce that braided brackets between $\mathscr{B}(M_{23})$ and $\mathscr{B}(M_{12})$ can be rewritten in terms of intermediate factors of the decomposition given in Step 1.

Step 8. We have $[\mathscr{B}(M_{12}), \mathscr{B}(M_{23})]_c \subset \mathscr{B}(M_2) \otimes \mathscr{B}(M_{123})$ in $\mathscr{B}(M)$.

Proof of Step 8. This follows from $x_2^2 = 0$, $x_{\bar{2}}^2 = 0$, (5.13), and (5.17). As an illustration:

$$[x_{12}, x_{23}]_c = (\mathrm{ad}_c\, x_1)(\mathrm{ad}_c\, x_2^2)x_3 - x_{\bar{2}}x_{123} - x_{12\bar{3}}x_2,$$

which belongs to $\Bbbk x_{\bar{2}}x_{123} + \Bbbk x_2x_{\bar{1}23}$. □

Step 9. Let A denote the quotient of $T(M_1 \oplus M_2 \oplus M_3)$ by the ideal generated by (5.13), (5.19), (5.21), (5.22), and (5.17). Then in A, we have

$$x_{123}^2 = \chi_2(g_1^4)\chi_3(g_2^2)x_{\bar{1}\bar{2}\bar{3}}^2. \tag{5.24}$$

Proof of Step 9. Use (5.21), (5.22), and (5.17) several times to get explicit braided commutations between x_{123} and each of its monomials:

$$\begin{aligned} x_{123}^2 &= (x_1x_2x_3 - x_1x_{\bar{3}}x_2 - \chi_3(g_1)x_{\bar{2}}x_3x_1 + \chi_3(\kappa g_1)x_{\bar{3}}x_{\bar{2}}x_1)x_{123} \\ &= \chi_2(g_1^2)x_{\bar{1}23}(x_1(x_2x_{\bar{3}} - \chi_3(g_2^2)x_3x_2) - \chi_3(\kappa g_1)(x_{\bar{2}}x_{\bar{3}} - \chi_3(\kappa g_2^2)x_3x_{\bar{2}})x_1) \\ &= \chi_2(g_1^4)\chi_3(g_2^2)x_{\bar{1}\bar{2}\bar{3}}^2, \end{aligned}$$

where the last equality follows from (5.13) and (5.19). □

As in the proof of Proposition 5.1, by Steps 2, 5, and 7, there exists an algebra surjection from $\mathscr{B}$, the algebra presented by relations in Proposition 5.2, onto $\mathscr{B}(M)$. Now we use Steps 1, 3, 6, 8, and 9 together with the fact that $\dim \mathscr{B}(M) = 2^{12}$ to conclude that $\mathscr{B} = \mathscr{B}(M)$. □

5.2 | A kind of distinguished pre-Nichols algebra

For Nichols algebras of diagonal type, a presentation by generators and relations was achieved by the first author in [12]. A fundamental role is played by an intermediate quotient known as the distinguished pre-Nichols algebra. Inspired by that construction, we define a pre-Nichols algebra $\widehat{\mathscr{B}}(M)$ for each one of the braidings M described in §2.3.1, §2.3.2, and §2.3.3. More precisely, our construction resembles the algebra $\widehat{\mathscr{B}}_{\mathbf{q}}$ introduced in Remark 2.3.

This pre-Nichols algebra will also play a key role in §6, where we describe the liftings.

Given $M = \oplus_{i\in\mathbb{I}} M_i$ as in §2.3.1, §2.3.2, and §2.3.3, let σ as in Theorem 3.18, and denote by $\mathbf{q}$ the braiding matrix of $W := F_\sigma(M)$ (see §3.1). Recall the index ℓ defined in (3.5). Consider

$$\bar{\cdot} : \mathbb{I}_\ell \to \mathbb{I}_{\theta+\ell}, \qquad \bar{i} := \begin{cases} i+\theta, & \text{for types } \alpha_\theta, \delta_\theta, \epsilon_\theta, \\ \theta+\ell-i+1, & \text{otherwise,} \end{cases}$$

so $\mathbb{I}_{\theta+\ell} = \mathbb{I}_\theta \cup \{\bar{i} : i \in \mathbb{I}_\ell\}$. We fix a basis x_i, $i \in \mathbb{I}_{\theta+\ell}$, such that

- $\mathrm{x}_i, \mathrm{x}_{\bar{i}}$ is a basis of $F_\sigma(M_i)$ for each $i \in \mathbb{I}_\ell$;
- $\mathrm{x}_i = x_i$ if $\ell < i \leqslant \theta$;
- the braiding in this basis is given by $\mathbf{q}$; that is, $c(\mathrm{x}_i \otimes \mathrm{x}_j) = q_{ij}\mathrm{x}_j \otimes \mathrm{x}_i$.

Let $\mathtt{A} = (\mathtt{a}_{ij})_{i,j\in\mathbb{I}_{\theta+\ell}}$ be the Cartan matrix of $\mathbf{q}$.

Remark 5.3. Let $\Xi : \mathbb{Z}^{\theta+\ell} \to \mathbb{Z}^\theta$ be the group morphism such that

$$\Xi(\alpha_i) = \alpha_i, \quad i \leqslant \theta; \qquad \Xi(\alpha_i) = \alpha_j, \quad i > \theta, i = \bar{j}.$$

This map *identifies* the two elements of the basis above corresponding to each $F_\sigma(M_i)$ when $\dim M_i = 2$. Thus, if $\mathscr{B}$ is an $\mathbb{N}_0^\theta$-graded pre-Nichols algebra such that $F_\sigma(\mathscr{B})$ is $\mathbb{N}_0^{\theta+\ell}$-graded (for the usual grading as pre-Nichols algebra of diagonal type), then

$$\dim \mathscr{B}_\beta = \sum_{\gamma\in\Xi^{-1}(\beta)} \dim F_\sigma(\mathscr{B})_\gamma, \qquad \beta \in \mathbb{N}_0^\theta.$$

Thus, the Hilbert series $\mathcal{H}_{\mathscr{B}}$ is the image of $\mathcal{H}_{F_\sigma(\mathscr{B})}$ under Ξ.

Let $\widehat{\mathscr{B}}(M)$ be the algebra generated x_i, $i \in \mathbb{I}$, $x_{\bar{j}}$, $j \in \mathbb{I}_\ell$, subject to the relations

$$(\mathrm{ad}_c\, x_i)x_{\bar{i}}, \quad x_i^2, \quad x_{\bar{i}}^2, \qquad i \in \mathbb{I}_\ell; \tag{5.25}$$

$$x_i^2, \qquad i > \ell; \tag{5.26}$$

$$x_{i\bar{j}} + \chi_j(g_i^2)x_{\bar{i}j}, \quad x_{\bar{i}\bar{j}} + \chi_i(\kappa)x_{ij}, \qquad i < j \leqslant \ell,\ a_{ij} = -1; \tag{5.27}$$

$$x_{ij}, \quad x_{i\bar{j}}, \quad x_{\bar{i}j}, \quad x_{\bar{i}\bar{j}}, \qquad i < j \leqslant \ell,\ a_{ij} = 0; \tag{5.28}$$

$$x_{ij}, \quad x_{\bar{i}j}, \qquad i \leqslant \ell < j,\ a_{ij} = 0; \tag{5.29}$$

$$(\mathrm{ad}_c\, x_j)x_{ji\bar{i}} = 0, \qquad a_{ij} = -2. \tag{5.30}$$

$$(\mathrm{ad}_c\, x_j)x_{ijk}, \quad (\mathrm{ad}_c\, x_j)x_{\bar{i}jk}, \qquad i < j < k,\ a_{ji} = a_{jk} = -1. \tag{5.31}$$

Proposition 5.4. *$\widehat{\mathscr{B}}(M)$ is a graded pre-Nichols algebra with Hilbert series*

$$\mathcal{H}_{\widehat{\mathscr{B}}(M)} = \prod_{\substack{\beta\in\Delta_+^M-\{\alpha_i\}\\ \dim M_\beta=2}} \left(\frac{1}{1-t^\beta}\right)^2 \prod_{j\in\mathbb{I}_\ell}(1+t_i)^2 \prod_{\substack{\beta\in\Delta_+^M-\{\alpha_i\}\\ \dim M_\beta=1}} \left(\frac{1}{1-t^\beta}\right) \prod_{j>\ell}(1+t_j). \tag{5.32}$$

Proof. First, $\widehat{\mathscr{B}}(M)$ is a quotient of $T(M)$ by an homogeneous ideal, so it is a graded algebra. Also, (5.25)–(5.29) are primitive elements of $T(M)$, while those in (5.31) are primitive modulo the previous relations, see the proofs of Propositions 5.1 and 5.2. Thus, $\widehat{\mathscr{B}}(M)$ is a graded pre-Nichols algebra.

We fix $G = \mathbb{G}$ defined in Remark 3.19, and M has a canonical G-Yetter–Drinfeld module structure. Now $\widehat{\mathscr{B}} := F_\sigma(\widehat{\mathscr{B}}(M)) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is a pre-Nichols algebra of W such that

$$\left(\widehat{\mathscr{B}}(M)\#\Bbbk G\right)_\sigma \simeq \widehat{\mathscr{B}}\#\Bbbk G.$$

We claim that $\widehat{\mathscr{B}} = \widehat{\mathscr{B}}_{\mathtt{q}}$. If so, the statement on the Hilbert series follows from Remarks 2.3 and 5.3. First, note that (2.8) holds in $\widehat{\mathscr{B}}$ for all $i \in \mathbb{I}_{\theta+\ell}$ by (5.25) and (5.26).

To verify (2.6), let $i < j$ be such that $\mathtt{a}_{ij} = 0$. We consider five cases.

- $i \leqslant \ell$, $j = \bar{i}$. The space of primitive elements of $T(M)$ of degree $2\alpha_i \in \mathbb{N}_0^\theta$ is three-dimensional, spanned by x_i^2, $x_{\bar{i}}^2$, and $x_{i\bar{i}}$. Thus, $\widehat{\mathscr{B}}(M)$ has no primitive elements of degree $2\alpha_i$. On the other hand, the spaces of primitive elements of $T(W)$ of degree $\alpha_i + \alpha_j \in \mathbb{N}_0^{\theta+\ell}$ are one-dimensional spanned by x_{ij}, and those of degrees $2\alpha_i$, $2\alpha_j$ are also one-dimensional, spanned by $\mathtt{x}_i^2$ and $\mathtt{x}_j^2$, respectively. As the space of homogeneous primitive elements of $\widehat{\mathscr{B}}$ coincides with that of $\widehat{\mathscr{B}}(M)$, we have that $\mathtt{x}_{ij} = 0$ in $\widehat{\mathscr{B}}$.
- $i \leqslant \ell, \theta < j$, $\mathtt{a}_{ik} = 0$, where $j = \bar{k}$. The space of primitive elements of $T(M)$ of degree $\alpha_i + \alpha_k \in \mathbb{N}_0^\theta$ has dimension 4, spanned by x_{ik}, $x_{i\bar{k}}$, $x_{\bar{i}k}$, $x_{\bar{i}\bar{k}}$, so the space of primitive elements of $\widehat{\mathscr{B}}(M)$ of the same degree is 0. On the other hand, the space of primitive elements of $T(W)$ of degree $\alpha_i + \alpha_k$ is spanned by $\mathtt{x}_{ik}$, $\mathtt{x}_{ij}$, $\mathtt{x}_{k\bar{i}}$, $\mathtt{x}_{j\bar{i}}$. As the space of homogeneous primitive elements of $\widehat{\mathscr{B}}$ coincides with that of $\widehat{\mathscr{B}}(M)$, we have that $\mathtt{x}_{ij} = 0 = \mathtt{x}_{j\bar{i}}$ in $\widehat{\mathscr{B}}$. This also shows that $\mathtt{x}_{ik} = \mathtt{x}_{k\bar{i}} = 0$.
- $i \leqslant \ell, \theta < j$, $\mathtt{a}_{ik} = -1$, where $j = \bar{k}$. The space of primitive elements of $T(M)$ of degree $\alpha_i + \alpha_k \in \mathbb{N}_0^\theta$ is two-dimensional, spanned by (5.27), so $\widehat{\mathscr{B}}(M)$ has no primitive elements of this degree. On the other hand, the spaces of primitive elements of $T(W)$ of degrees $\alpha_i + \alpha_j$ and $\alpha_{\bar{i}} + \alpha_k$ are one-dimensional, spanned by $\mathtt{x}_{ij}$ and $\mathtt{x}_{k\bar{i}}$, respectively, and those of degrees $\alpha_i + \alpha_k$ and $\alpha_j + \alpha_{\bar{i}}$ are 0 since $a_{ik} = a_{\bar{i}j} = -1$. Hence, $\mathtt{x}_{ij} = \mathtt{x}_{k\bar{i}} = 0$ in $\widehat{\mathscr{B}}$.
- $i \leqslant \ell < j \leqslant \theta$. Here, $a_{ij} = 0$, and the space of primitive elements of $T(M)$ of degree $\alpha_i + \alpha_j \in \mathbb{N}_0^\theta$ is two-dimensional, spanned by x_{ij} and $x_{\bar{i}\bar{j}}$, so the space of primitive elements of $\widehat{\mathscr{B}}(M)$ of this degree is 0. On the other hand, the spaces of primitive elements of $T(W)$ of degrees $\alpha_i + \alpha_j$ and $\alpha_{\bar{i}} + \alpha_j$ are one-dimensional, spanned by $\mathtt{x}_{ij}$ and $\mathtt{x}_{j\bar{i}}$, respectively. Hence, $\mathtt{x}_{ij} = \mathtt{x}_{j\bar{i}} = 0$ in $\widehat{\mathscr{B}}$.
- $i, j \leqslant \ell$. Here $a_{ij} = 0$, respectively, $a_{\bar{i}\bar{j}} = 0$, and this case is the second one.

Now we check (2.5) in $T(W)$, modulo (2.4) and (2.6). We have three cases:

♡ $j \leqslant \ell$. The space of primitive elements of $T(M)$, modulo (5.25), (5.26), (5.27), (5.28), and (5.29), of degree $\alpha_i + 2\alpha_j + \alpha_k$ has dimension $\leqslant 2$ and is spanned by (5.31): we can use skew derivations as in type α_3, see the proof of Proposition 5.2. Thus, the space of primitive elements of $\widehat{\mathscr{B}}(M)$ of this degree is 0. On the other hand, the space of primitive elements of $T(W)$, modulo (2.4) and (2.6), of each degree in $\Xi^{-1}(\alpha_i + 2\alpha_j + \alpha_k)$ is either one-dimensional or 0. Indeed, the nonzero cases are spanned by
- $[\mathrm{x}_{ijk}, \mathrm{x}_j]_c$, $[\mathrm{x}_{\overline{ijk}}, \mathrm{x}_{\bar{j}}]_c$ if $j < \ell$;
- $[\mathrm{x}_{\theta-2\,\theta-1\,\theta}, \mathrm{x}_{\theta-1}]_c$, $[\mathrm{x}_{\theta+2\,\theta+1\,\theta}, \mathrm{x}_{\theta+1}]_c$ if $j = \theta - 1$ in type γ_θ;
- $[\mathrm{x}_{123}, \mathrm{x}_2]_c$, $[\mathrm{x}_{653}, \mathrm{x}_5]_c$ if $j = 2$ in type ϕ_4.

As the space of homogeneous primitive elements of $\widehat{\mathscr{B}}$ coincides with that of $\widehat{\mathscr{B}}(M)$, we deduce that (2.5) hold in $\widehat{\mathscr{B}}$.

♡ $j > \ell$. There are three possibilities: $i = \theta - 1$, $j = \theta$, $k = \bar{i}$ in types γ_θ or ϕ_4, and $i \in \{2, 5\}$, $j = 3$, $k = 4$ in type ϕ_4. The proof is analogous, using (5.30) for the first case, and (5.31) for the last one.

From the analysis above, there exists a surjective Hopf algebra map $\widehat{\mathscr{B}} \twoheadrightarrow \widehat{\mathscr{B}}_{\mathfrak{q}}$. In a similar way, checking spaces of homogeneous primitive elements of appropriate degree, each defining relation of $\widehat{\mathscr{B}}(M)$ annihilates in $F_\sigma^{-1}(\widehat{\mathscr{B}}_{\mathfrak{q}}) = F_{\sigma^{-1}}(\widehat{\mathscr{B}}_{\mathfrak{q}})$, so there exists a surjective Hopf algebra map $F_\sigma^{-1}(\widehat{\mathscr{B}}_{\mathfrak{q}}) \twoheadrightarrow \widehat{\mathscr{B}}(M)$. As F_σ preserves the $\mathbb{N}_0$-graduation of the pre-Nichols algebras, both surjective maps are indeed isomorphisms. Hence, $\widehat{\mathscr{B}} = \widehat{\mathscr{B}}_{\mathfrak{q}}$ as we claimed. □

5.3 | The subalgebra of coinvariants

Let $\pi : \widehat{\mathscr{B}}(M) \twoheadrightarrow \mathscr{B}(M)$ be the canonical projection, $\mathcal{Z}(M) := \widehat{\mathscr{B}}(M)^{\mathrm{co}\,\pi}$ the subalgebra of coinvariants. The next step toward the presentation of $\mathscr{B}(M)$ is to describe $\mathcal{Z}(M)$. To uncover the structure of this subalgebra, we will use a cocycle σ as in Theorem 3.18 to translate the problem to the diagonal setting, where the situation is better understood. In particular, we compute the Hilbert series of $\mathcal{Z}(M)$. Since we know that of $\widehat{\mathscr{B}}(M)$, we will thus obtain the Hilbert series of $\mathscr{B}(M)$.

To do so, and also to compute a PBW basis of $\mathscr{B}(M)$ later on, we fix a reduced expression of the element w_0 of maximal length (or equivalently, a convex order on Δ_+) for each type. Using this reduced expression, [36] defines a submodule $M_\beta \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, $\beta \in \Delta_+$. We exhibit a basis $\{x_\beta\}$ or $\{x_\beta, x_{\bar{\beta}}\}$ of the submodule M_β, depending on its dimension.

α_θ The set of positive roots is $\Delta_+ = \{\alpha_{ij} : i \leqslant j \in \mathbb{I}\}$, and

$$\alpha_1 < \alpha_{12} < \alpha_2 < \cdots < \alpha_{\theta-1} < \alpha_{1\theta} < \alpha_{2\theta} < \cdots < \alpha_\theta$$

is a convex order on Δ_+. By [36], the modules M_β can be defined as

$$M_{\alpha_{ij}} = (\mathrm{ad}_c\, M_i) \cdots (\mathrm{ad}_c\, M_{j-1}) M_j. \tag{5.33}$$

A basis of $M_{\alpha_{ij}}$ is given by

$$x_{\alpha_{ij}} = x_{i\,i+1\cdots j}, \qquad x_{\overline{\alpha_{ij}}} = x_{\bar{i}\,i+1\cdots j}. \tag{5.34}$$

δ_θ The positive roots are $\Delta_+ = \{\alpha_{ij} : i \leqslant j \in \mathbb{I}\} - \{\alpha_{\theta-1\theta}\} \cup \{\alpha_{i\theta-2} + \alpha_\theta : i \in \mathbb{I}_{\theta-2}\} \cup \{\alpha_{i\theta-2} + \alpha_{j\theta} : i < j \in \mathbb{I}_{\theta-2}\}$, and a convex order on Δ_+ is given by

$$\begin{aligned}\alpha_1 < \alpha_{12} < \alpha_2 < \cdots < \alpha_{\theta-2} < \alpha_{1\theta-1} < \cdots < \alpha_{\theta-1} < \cdots < \alpha_{\theta-1}\\ < \alpha_{1\theta} + \alpha_{2\theta-2} < \cdots < \alpha_{1\theta} + \alpha_{\theta-2} < \alpha_{2\theta} + \alpha_{3\theta-2} < \cdots < \alpha_{2\theta} + \alpha_{\theta-2}\\ < \cdots < \alpha_{\theta-3\theta} + \alpha_{\theta-2} < \alpha_{1\theta} < \cdots < \alpha_{\theta-2\,\theta} < \alpha_\theta.\end{aligned}$$

The Yetter–Drinfeld modules $M_{\alpha_{ij}}$, $j \neq \theta$, are defined as for type α_θ. For $j = \theta$, let

$$M_{\alpha_{i\theta}} = (\mathrm{ad}_c\, M_i) \cdots (\mathrm{ad}_c\, M_{\theta-3})(\mathrm{ad}_c\, M_{\theta-1})(\mathrm{ad}_c\, M_{\theta-2})M_\theta.$$

For the other roots, we have

$$\begin{aligned}M_{\alpha_{i\theta-2}+\alpha_\theta} &= (\mathrm{ad}_c\, M_i) \cdots (\mathrm{ad}_c\, M_{\theta-2})M_\theta,\\ M_{\alpha_{i\theta}+\alpha_{j\theta-2}} &= [M_{\alpha_{i\theta-2}+\alpha_\theta}, M_{\alpha_{j\theta-1}}]_c.\end{aligned}$$

A basis of M_β for either $\beta = \alpha_{ij}$ or $\beta = \alpha_{i\theta-2} + \alpha_\theta$ is given as in (5.34). For $\beta = \alpha_{i\theta} + \alpha_{j\theta-2}$, a basis of M_β is

$$x_\beta = [x_{i\cdots\theta-2\theta}, x_{j\cdots\theta-1}]_c, \qquad x_{\overline{\beta}} = [x_{\overline{i}i+1\cdots\theta-2\theta}, x_{j\cdots\theta-1}]_c. \tag{5.35}$$

ϵ_θ Here, one can fix a convex order as in [11, §5]. For braidings of diagonal type, a PBW basis is obtained recursively on the height of the roots, starting with $\mathrm{x}_{\alpha_i} = \mathrm{x}_i$ for simple roots, and later $\mathrm{x}_\beta = [\mathrm{x}_{\beta_1}, \mathrm{x}_{\beta_2}]_c$ for some pair (β_1, β_2) such that $\beta_1 + \beta_2 = \beta$, see [13, Corollary 3.17]. For each nonsimple root $\beta \in \Delta_+$, we have, accordingly,

$$M_\beta = [M_{\beta_1}, M_{\beta_2}]_c, \qquad x_\beta = [x_{\beta_1}, x_{\beta_2}]_c, \qquad x_{\overline{\beta}} = [x_{\overline{\beta_1}}, x_{\beta_2}]_c.$$

γ_θ Now $\Delta_+ = \{\alpha_{ij} : i \leqslant j \in \mathbb{I}\} \cup \{\alpha_{i\theta} + \alpha_{k\,\theta-1} : i \leqslant j \in \mathbb{I}_{\theta-1}\}$, and

$$\begin{aligned}\alpha_1 < \alpha_{12} < \alpha_2 < \cdots < \alpha_{\theta-1} < \alpha_{1\theta} < \alpha_{2\theta} < \cdots <\\ \alpha_{1\theta} + \alpha_{1\,\theta-1} < \alpha_{1\theta} + \alpha_{2\,\theta-1} < \cdots < \alpha_{1\theta} + \alpha_{\theta-1} < \cdots <\\ \alpha_{2\theta} + \alpha_{2\,\theta-1} < \cdots < \alpha_{2\theta} + \alpha_{\theta-1} < \cdots < 2\alpha_{\theta-1} + \alpha_\theta < \alpha_{\theta-1} + \alpha_\theta < \alpha_\theta,\end{aligned}$$

is a convex order associated to the following reduced expression of w_0:

$$s_1(s_2s_1)(s_3s_2s_1)\cdots(s_{\theta-1}\cdots s_1)(s_\theta s_{\theta-1}\cdots s_1)(s_\theta\cdots s_2)\cdots s_\theta.$$

The Yetter–Drinfeld modules $M_{\alpha_{ij}}$, $\alpha_{ij} \neq \alpha_\theta$, are as in (5.33), and (5.34) is a basis as well. Now, M_{α_θ} is one-dimensional, spanned by x_θ, and for the other roots, we check that

$$M_{\alpha_{i\theta}+\alpha_{k\theta-1}} = [M_{\alpha_{i\theta}}, M_{\alpha_{k\theta-1}}]_c. \tag{5.36}$$

A basis of $M_{\alpha_{i\theta}+\alpha_{k\theta-1}}$ is given by

$$x_\beta = [x_{i\dots\theta}, x_{k\dots\theta-1}]_c, \qquad x_{\overline{\beta}} = [x_{\overline{ii+1\dots\theta}}, x_{k\dots\theta-1}]_c. \tag{5.37}$$

ϕ_4 The element w_0 of maximal length has a reduced expression

$$s_1s_2s_3s_4s_2s_3s_1s_2s_3s_4s_1s_2s_3s_2s_1s_2s_3s_4s_2s_3s_2s_3s_4s_3, \tag{5.38}$$

which induces the following convex order on the set of positive roots:

$1,$	$12,$	$1^22^23,$	$1^22^234,$	$123,$	$1^22^23^24,$	$12^23,$	$1^22^33^24,$
$1^22^43^24,$	$1^22^43^34,$	$1234,$	$12^234,$	$1^22^43^34^2,$	$12^23^24,$	$12^33^24,$	$2,$
$2^23,$	$2^234,$	$23,$	$2^23^24,$	$234,$	$3,$	$34,$	$4.$

We denote by β_i the ith root according with this order.

Next, we give, for each nonsimple root $\beta = 1^a2^b3^c4^d$ such that $d \neq 0$, a basis for each Yetter–Drinfeld submodule M_β (if $d = 0$, then we choose a basis as for γ_θ):

$$\begin{aligned}
M_{1^22^234} &: \{[x_{12}, x_{\overline{1234}}]_c\}, & M_{1^22^23^24} &: \{[x_{123}, x_{\overline{1234}}]_c\},\\
M_{1^22^33^24} &: \{[x_{12^23}, x_{1234}]_c, [x_{\overline{12^23}}, x_{1234}]_c\}, & M_{1^22^43^24} &: \{[x_{12^23}, x_{12^234}]_c\},\\
M_{1^22^43^34} &: \{[x_{12^23}, x_{12^23^24}]_c\}, & M_{1234} &: \{x_{1234}, x_{\overline{1234}}\},\\
M_{12^234} &: \{[x_{1234}, x_2]_c, [x_{\overline{1234}}, x_2]_c\}, & M_{1^22^43^34^2} &: \{[x_{12^234}, x_{12^23^24}]_c\},\\
M_{12^23^24} &: \{[x_{1234}, x_{23}]_c, [x_{\overline{1234}}, x_{23}]_c\}, & M_{2^234} &: \{x_{\overline{2234}}\},\\
M_{12^33^24} &: \{[x_{12^23^24}, x_2]_c, [x_{\overline{12^23^24}}, x_2]_c\}, & M_{2^23^24} &: \{[x_{\overline{23}}, x_{234}]_c\},\\
M_{234} &: \{x_{234}, x_{\overline{234}}\}, & M_{34} &: \{x_{34}\}.
\end{aligned}$$

where x_β is the first vector fixed for M_β, while for $\dim M_\beta = 2$, we denote by $x_{\overline{\beta}}$ the second vectors in the order fixed above.

Remark 5.5. Let $\beta = 1^{a_1}2^{a_2}\cdots\theta^{a_\theta} \in \Delta_+^M$, $g_\beta := g_1^{a_1}g_2^{a_2}\cdots g_\theta^{a_\theta} \in G$.

(1) If $\dim M_\beta = 1$, then x_β has G-degree g_β.
(2) If $\dim M_\beta = 2$, then x_β has G-degree g_β and $x_{\overline{\beta}}$ has G-degree $g_\beta\kappa$.

We omit the details of the proof that either $\{x_\beta\}$ or $\{x_\beta, x_{\overline{\beta}}\}$ is a basis of M_β. The first step is to check that M_β is spanned by $\{x_\beta\}$, respectively, $\{x_\beta, x_{\overline{\beta}}\}$, using the defining relations of $\widehat{\mathscr{B}}(M)$; this can be done recursively on the convex (total) order. If $\dim M_\beta = 2$, we see that $x_\beta, x_{\overline{\beta}}$ have different G-degree by Remark 5.5, so they are linearly independent.

Proposition 5.6. *The subalgebra $\mathcal{Z}(M) = \widehat{\mathscr{B}}(M)^{\mathrm{co}\,\pi}$ of coinvariants under the canonical projection $\pi : \widehat{\mathscr{B}}(M) \twoheadrightarrow \mathscr{B}(M)$ is a Hopf subalgebra of $\widehat{\mathscr{B}}(M)$. It is a skew-polynomial algebra in variables*

$$x_\beta^2, \qquad \beta \in \Delta_+^M - \{\alpha_i\}, \tag{5.39}$$

$$[x_\beta, x_{\overline{\beta}}]_c, \qquad \beta \in \Delta_+^M - \{\alpha_i\} \textit{ such that } \dim M_\beta = 2. \tag{5.40}$$

Proof. We proceed in several steps. In what follows, $\beta \in \Delta_+$ is not simple.

Step 1. For each $\beta \in \Delta_+$ such that $\dim M_\beta = 1$, we have $x_\beta^2 \in M_\beta^2 \cap \mathcal{Z}(M)$.
For each $\beta \in \Delta_+$ such that $\dim M_\beta = 2$, we have $x_\beta^2, x_{\overline{\beta}}^2, [x_\beta, x_{\overline{\beta}}]_c \in \mathcal{Z}(M)$.

Proof of Step 1. The subalgebra spanned by M_β is isomorphic to the Nichols algebra $\mathscr{B}(M_\beta)$ by [35]. Assume first that $\dim M_\beta = 1$. The braiding of M_β satisfies that $c(x_\beta \otimes x_\beta) = -x_\beta \otimes x_\beta$, so $x_\beta^2 = 0$ in $\mathscr{B}(M)$. Thus, $x_\beta^2 \in \ker\pi \cap M_\beta^2$, and applying F_σ, we get

$$F_\sigma(x_\beta^2) \in F_\sigma\Big(\ker\pi \cap M_\beta^2\Big) = \ker\pi \cap F_\sigma\big(M_\beta\big)^2 = \mathcal{Z}(W).$$

Hence, $x_\beta^2 \in \mathcal{Z}(M)$, since F_σ leaves the coalgebra structure unchanged.
Now, if $\dim M_\beta = 2$, then the braiding of M_β satisfies

$$c(x_\beta \otimes x_\beta) = -x_\beta \otimes x_\beta, \qquad c(x_{\overline{\beta}} \otimes x_{\overline{\beta}}) = -x_{\overline{\beta}} \otimes x_{\overline{\beta}}, \qquad c^2(x_\beta \otimes x_{\overline{\beta}}) = x_\beta \otimes x_{\overline{\beta}}.$$

By a similar argument, $x_\beta^2, x_{\overline{\beta}}^2, [x_\beta, x_{\overline{\beta}}]_c \in \mathcal{Z}(M)$. □

Step 2. For each $\beta \in \Delta_+$ such that $\dim M_\beta = 1$, $\{\mathrm{x}_\beta\}$ is a basis of $F_\sigma(M_\beta)$.
For each $\beta \in \Delta_+$ such that $\dim M_\beta = 2$, $\{\mathrm{x}_\beta, \mathrm{x}_{\overline{\beta}}\}$ is a basis of $F_\sigma(M_\beta)$.

Proof of Step 2. The statement certainly holds for simple roots, so we fix a nonsimple root β. For types $\alpha_\theta, \delta_\theta, \epsilon_\theta$, we always have $\dim M_\beta = 2$ and $M_\beta = [M_{\beta_1}, M_{\beta_2}]_c$ for some $\beta_1, \beta_2 \in \Delta_+$. Notice that $(\mathrm{ad}_c\, x_i)x_{\overline{j}} = 0$ for all $i, j \in \mathbb{I}$ since $i, \overline{j}$ belong to different connected components of the Dynkin diagram of type $X_\theta \times X_\theta$. Hence, $[\mathrm{x}_\gamma, \mathrm{x}_{\overline{\delta}}]_c$ for all $\gamma, \delta \in \Delta_+$, and arguing recursively,

$$F_\sigma(M_\beta) = F_\sigma\Big([M_{\beta_1}, M_{\beta_2}]_c\Big) = \Big[F_\sigma(M_{\beta_1}), F_\sigma(M_{\beta_2})\Big]_c$$
$$= \Big[\Bbbk\mathrm{x}_{\beta_1} + \Bbbk\mathrm{x}_{\overline{\beta_1}}, \Bbbk\mathrm{x}_{\beta_2} + \Bbbk\mathrm{x}_{\overline{\beta_2}}\Big]_c = \Bbbk\Big[\mathrm{x}_{\beta_1}, \mathrm{x}_{\beta_2}\Big]_c + \Bbbk\Big[\mathrm{x}_{\overline{\beta_1}}, \mathrm{x}_{\overline{\beta_2}}\Big]_c = \Bbbk\mathrm{x}_\beta + \Bbbk\mathrm{x}_{\overline{\beta}}.$$

For multiply laced types, the proof follows similarly, case-by-case. □

Step 3. There exist $\mathbb{N}_0^\theta$-homogeneous elements

- $y_\beta \in M_\beta^2$ of G-degree g_β^2 when $\dim M_\beta = 1$,
- $y_\beta, y_{\overline{\beta}} \in M_\beta^2$ of G-degree g_β^2, respectively, $g_\beta^2\kappa$, when $\dim M_\beta = 2$,

whicht q-commute with every G-homogeneous element of $\widehat{\mathscr{B}}$. Moreover, $\mathcal{Z}(M)$ is a skew-polynomial algebra in these variables.

Proof of Step 3. Assume that $\dim M_\beta = 2$. Note that, for all $i \in \mathbb{I}$, the elements x_i^2, $\mathrm{x}_{\overline{i}}^2$ are linear combinations of elements of G-degree g_i^2 and $g_i^2\kappa$ with nontrivial components on each degree. Hence, the elements x_β^2, $\mathrm{x}_{\overline{\beta}}^2$ are written as linear combinations of elements of G-degree g_β^2 and $g_\beta^2\kappa$ with nontrivial components on each degree, since they are obtained applying Lusztig's isomorphisms to appropriate x_i^2, $\mathrm{x}_{\overline{i}}^2$. Thus, there exist y_β and $\mathrm{y}_{\overline{\beta}}$ of G-degrees g_β^2 and $g_\beta^2\kappa$, respectively, that span the same as x_β^2 and $\mathrm{x}_{\overline{\beta}}^2$. When $\dim M_\beta = 1$, we may choose $\mathrm{y}_\beta = \mathrm{x}_\beta^2$.

As $\mathcal{Z}(W)$ is a skew-polynomial algebra in variables x_β^2, $x_{\overline{\beta}}^2$ and each element x_β^2, $x_{\overline{\beta}}^2$ is skew-central, the same holds with respect to y_β, $\mathrm{y}_{\overline{\beta}}$.

Let $\beta \in \Delta_+$ be such that $\dim M_\beta = 2$. We set

$$y_\beta := F_\sigma^{-1}(\mathrm{y}_\beta) \in M_\beta^2, \qquad y_{\overline{\beta}} := F_\sigma^{-1}(\mathrm{y}_{\overline{\beta}}) \in M_\beta^2.$$

Then y_β, $y_{\overline{\beta}} \in \mathcal{Z}(M)$ since F_σ preserves the coalgebra structure. Note that

$$F_\sigma(x_i y_\beta) = \sigma(g_i, g_\beta^2) F_\sigma(x_i)\mathrm{y}_\beta, \qquad F_\sigma(y_\beta x_i) = \sigma(g_\beta^2, g_i)\mathrm{y}_\beta F_\sigma(x_i),$$

and these two elements differ up to a nonzero scalar for all $i \in \mathbb{I}$, thus y_β is skew-central. The same happens for $y_{\overline{\beta}}$, and for y_β when $\dim M_\beta = 1$. In particular, the image under F_σ of a multiplication of various y_β's, $y_{\overline{\beta}}$'s is the multiplication to the corresponding y_β's, $\mathrm{y}_{\overline{\beta}}$'s up to a nonzero scalar, and the Step follows. □

Step 4. For each $\beta \in \Delta_+$ such that $\dim M_\beta = 1$, we have $\Bbbk y_\beta = \Bbbk x_\beta^2$.

For each $\beta \in \Delta_+$ such that $\dim M_\beta = 2$, we have $\Bbbk y_\beta = \Bbbk x_\beta^2$, $\Bbbk y_{\overline{\beta}} = \Bbbk[x_\beta, x_{\overline{\beta}}]_c$.

Proof of Step 4. Assume first that $\dim M_\beta = 1$. Then $\dim M_\beta^2 = 1$ in $\widehat{\mathscr{B}}(M)$, and the claim follows since both y_β and x_β^2 are generators of M_β^2.

Now assume that $\dim M_\beta = 2$. In this case $\dim M_\beta^2 = 3$ in $\widehat{\mathscr{B}}(M)$ since $\dim F_\sigma(M_\beta^2) = 3$ in $\widehat{\mathscr{B}}(W)$. On the other hand, $\dim M_\beta^2 = 1$ in $\mathscr{B}(V)$: it is generated by $x_{\overline{\beta}} x_\beta$ since $x_\beta^2 = x_{\overline{\beta}}^2 = [x_\beta, x_{\overline{\beta}}]_c = 0$. Notice that $\dim F_\sigma(M_\beta)^2 \cap \mathcal{Z}(W) = 2$, and $F_\sigma(M_\beta)^2 \cap \mathcal{Z}(W)$ contains elements with nontrivial components in degrees g_β^2 and $g_\beta^2\kappa$. Hence $\dim M_\beta^2 \cap \mathcal{Z}(M) = 2$, with one-dimensional homogeneous components of degrees g_β^2 and $g_\beta^2\kappa$. Thus $\Bbbk y_\beta = \Bbbk x_\beta^2 = \Bbbk x_{\overline{\beta}}^2$ and $\Bbbk y_{\overline{\beta}} = \Bbbk[x_\beta, x_{\overline{\beta}}]_c$, as claimed. □

Hence, Step 3 shows that $\mathcal{Z}(M)$ is a skew-polynomial algebra, and Step 4 assures that we can choose generating variables as stated. □

5.4 | A presentation of the Nichols algebra

Here we put together the results obtained in §5.2 and §5.3 to get a presentation, a PBW basis, and the Hilbert series for the Nichols algebra.

Theorem 5.7.

(i) *A set of PBW generators for $\mathscr{B}(M)$ is given by*

$$x_\beta,\ \beta \in \Delta_+; \qquad\qquad x_{\overline{\beta}} \text{ when } \dim M_\beta = 2. \tag{5.41}$$

The height of x_β, $x_{\overline{\beta}}$ is 2 for all $\beta \in \Delta_+$.

(ii) *The Nichols algebra $\mathscr{B}(M)$ is presented by generators x_i, $i \in \mathbb{I}$, $x_{\overline{j}}$, $j \in \mathbb{I}_\ell$, and relations (5.25), (5.26), (5.27), (5.28), (5.29), (5.30), (5.31), (5.39), and (5.40).*

Proof.

(i) By [34, Theorem 2.6], the multiplication map

$$\bigotimes_{\beta\in\Delta_+^M} \mathscr{B}(M_\beta) \to \mathscr{B}(M)$$

is an isomorphism of $\mathbb{Z}^\theta$-graded objects in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$. If $\dim M_\beta = 1$, then M_β has braiding $-\mathrm{id}$ and 1, x_β is a basis of $\mathscr{B}(M_\beta)$. If $\dim M_\beta = 2$, then M_β has braiding as in (2.16): that is, $\mathscr{B}(M_\beta)$ is a quantum plane with basis $1, x_\beta, x_{\overline{\beta}}, x_\beta x_{\overline{\beta}}$, and the claim follows.

(ii) By Proposition 5.4, relations (5.25), (5.26), (5.27), (5.28), (5.29), (5.30), and (5.31) hold in $\mathscr{B}(M)$. Also, (5.39) and (5.40) hold in $\mathscr{B}(M)$ because the subalgebra generated by M_β is isomorphic to $\mathscr{B}(M_\beta)$ as an algebra. Therefore, if $\mathscr{B}$ denotes the quotient of $T(M)$ by all these relations, then there exists a canonical projection $\mathscr{B} \twoheadrightarrow \mathscr{B}(M)$ of graded Hopf algebras. Moreover,

$$\mathscr{B} = \widehat{\mathscr{B}}(V)/\langle \mathcal{Z}(V)^+ \rangle,$$

so $\mathcal{H}_{\widehat{\mathscr{B}}(V)} = \mathcal{H}_{\mathcal{Z}(V)}\mathcal{H}_{\mathscr{B}}$ by [15, Lemma 2.4]. By Propositions 5.6 and 5.4,

$$\mathcal{H}_{\mathscr{B}} = \left(\prod_{\beta\in\Delta_+^V\,:\dim M_\beta=2} (1+t^\beta)^2\right)\left(\prod_{\beta\in\Delta_+^V\,:\dim M_\beta=1} (1+t^\beta)\right) = \mathcal{H}_{\mathscr{B}(V)},$$

and we deduce that $\mathscr{B} = \mathscr{B}(M)$. □

Remark 5.8. The order of the elements in the PBW basis in Theorem 5.7 (i) is given by the expression of the element w_0 of maximal length fixed below. For example, for type ϕ_4, we have the following PBW basis:

$$\begin{aligned}
&x_4^{a_1} x_{34}^{a_2} x_{234}^{a_3} x_{234}^{a_4} x_{2^2 3^2 4}^{a_5} x_{23}^{a_6} x_{23}^{a_7} x_{2^2 34}^{a_8} x_{2^2 3}^{a_9} x_2^{a_{10}} x_{12^2 3^2 4}^{a_{11}} x_{12^3 3^2 4}^{a_{12}} x_{12^3 3^2 4}^{a_{13}} x_{12^2 3^2 4}^{a_{14}} \\
&x_{12^2 3^2 4}^{a_{15}} x_{1^2 2^4 3^3 4^2}^{a_{16}} x_{12^2 34}^{a_{17}} x_{12^2 34}^{a_{18}} x_{1234}^{a_{19}} x_{1234}^{a_{20}} x_{34}^{a_{21}} x_{1^2 2^4 3^3 4}^{a_{22}} x_{1^2 2^4 3^2 4}^{a_{23}} x_{1^2 2^3 3^2 4}^{a_{24}} \\
&x_{1^2 2^3 3^2 4}^{a_{25}} x_{12^2 3}^{a_{26}} x_{12^2 3}^{a_{27}} x_{1^2 2^2 3^2 4}^{a_{28}} x_{123}^{a_{29}} x_{123}^{a_{30}} x_{1^2 2^2 34}^{a_{31}} x_{1^2 2^2 3}^{a_{32}} x_{12}^{a_{33}} x_{12}^{a_{34}} x_1^{a_{35}} x_1^{a_{36}},\ a_i \in \mathbb{I}_{0,1}.
\end{aligned} \tag{5.42}$$

5.5 | Rigidity of Nichols algebras

We briefly discuss rigidity for Nichols algebras of Yetter–Drinfeld modules of types α_θ, γ_θ, δ_θ, ϵ_θ, or ϕ_4, inspired by [18] where rigidity for finite-dimensional Nichols algebras over abelian groups is studied. This will come in handy in the next section, where we study the liftings of these Nichols algebras.

Let $\mathcal{R}_M \subset T(M)$ be the set of defining relations as in Theorem 5.7. We start by describing the Yetter–Drinfeld structure of $\Bbbk\mathcal{R}_M$. Recall the index ℓ introduced in (3.5).

Remark 5.9.

(1) The G-degree of $x_{i\bar{i}}$ is $g_i^2\kappa$. Using Example 2.4, it is easy to see that

$$\overline{\chi}_i : G \to \Bbbk^\times, \qquad \overline{\chi}_i(h) := \begin{cases} \chi_i(hg_j^{-1}hg_j) & h \in G^{g_i}, \\ \chi_i(\kappa)\chi_i(h^2) & h \notin G^{g_i}, \end{cases} \tag{5.43}$$

is a character, and the G-action on $x_{i\bar{i}}$ is given by $\overline{\chi}_i$.

(2) The G-degree of x_i^2 and $x_{\bar{i}}^2$ is g_i^2. The G-action when $i \leqslant \ell$ is

$$g \cdot x_i^2 = \begin{cases} \chi_i^2(g)x_i^2, & g \in G^{g_i}, \\ \chi_i^2(g_j^{-1}g)x_i^2, & g \notin G^{g_i}; \end{cases} \qquad g \cdot x_{\bar{i}}^2 = \begin{cases} \chi_i^2(g)x_{\bar{i}}^2, & g \in G^{g_i}, \\ \chi_i^2(gg_j)x_{\bar{i}}^2, & g \notin G^{g_i}. \end{cases}$$

If $i > \ell$, then the action on x_i^2 is given by χ_i^2.

(3) Let $i < j \leqslant \ell$ be such that $a_{ij} = -1$, and set

$$r_1 := x_{i\bar{j}} + \chi_j(g_i^2)x_{\bar{i}j}, \qquad r_2 := x_{\bar{i}\bar{j}} + \chi_i(\kappa)x_{ij}.$$

The G-degrees of r_1 and r_2 are $g_ig_j\kappa$ and g_ig_j. The G-action is given by

$h \in$	$h \cdot r_1$	$h \cdot r_2$
$G^{g_i} \cap G^{g_j}$	$\chi_i(h)\chi_j(h)r_1$	$\chi_i(h)\chi_j(h)r_2$
$G^{g_i} - G^{g_j}$	$\chi_i(\kappa h)\chi_j(g_ih)r_2$	$\chi_i(\kappa h)\chi_j(g_i^{-1}h)r_2$
$G^{g_j} - G^{g_i}$	$\chi_i(g_j^{-1}h)\chi_j(\kappa h)r_2$	$\chi_i(hg_j)\chi_j(\kappa h)r_1$
$G - (G^{g_i} \cup G^{g_j})$	$\chi_i(hg_j)\chi_j(g_ih)r_1$	$\chi_i(hg_j^{-1})\chi_j(g_i^{-1}h)r_2$

(4) Let $i < j \leqslant \ell$ be such that $a_{ij} = 0$. Both x_{ij} and $x_{\bar{i}\bar{j}}$ have G-degree g_ig_j, while the G-degree of $x_{i\bar{j}}$ and $x_{\bar{i}j}$ is $g_ig_j\kappa$. For the action, as G^{g_i} and G^{g_j} are both subgroups of index 2, there are two possibilities. If $G^{g_i} \neq G^{g_j}$[†], then we choose $g_a \in G^{g_i} - G^{g_j}$ and $g_b \in G^{g_j} - G^{g_i}$. By Example 2.4, the G-action is given by:

$h \in$	$h \cdot x_{ij}$	$h \cdot x_{\bar{i}\bar{j}}$	$h \cdot x_{i\bar{j}}$	$h \cdot x_{\bar{i}j}$
$G^{g_i} \cap G^{g_j}$	$\chi_i(h)\chi_j(h)x_{ij}$	$\chi_i(g_b^{-1}hg_b)\chi_j(g_a^{-1}hg_a)x_{\bar{i}\bar{j}}$	$\chi_i(h)\chi_j(g_a^{-1}hg_a)x_{i\bar{j}}$	$\chi_i(g_b^{-1}hg_b)\chi_j(h)x_{\bar{i}j}$
$G^{g_i} - G^{g_j}$	$\chi_i(h)\chi_j(g_a^{-1}h)x_{i\bar{j}}$	$\chi_i(g_b^{-1}hg_b)\chi_j(hg_a)x_{\bar{i}j}$	$\chi_i(h)\chi_j(hg_a)x_{ij}$	$\chi_i(g_b^{-1}hg_b)\chi_j(g_a^{-1}h)x_{\bar{i}\bar{j}}$
$G^{g_j} - G^{g_i}$	$\chi_i(g_b^{-1}h)\chi_j(h)x_{\bar{i}j}$	$\chi_i(hg_b)\chi_j(g_a^{-1}hg_a)x_{i\bar{j}}$	$\chi_i(g_b^{-1}h)\chi_j(g_a^{-1}hg_a)x_{\bar{i}\bar{j}}$	$\chi_i(hg_b)\chi_j(h)x_{ij}$
$G - (G^{g_i} \cup G^{g_j})$	$\chi_i(g_b^{-1}h)\chi_j(g_a^{-1}h)x_{\bar{i}\bar{j}}$	$\chi_i(hg_b)\chi_j(hg_a)x_{ij}$	$\chi_i(g_b^{-1}h)\chi_j(hg_a)x_{\bar{i}j}$	$\chi_i(hg_b)\chi_j(g_a^{-1}h)x_{i\bar{j}}$

[†] This occurs, for example, in type α_θ whenever $j - i \geqslant 3$. We may choose $g_a = g_{j-1} \neq g_b = g_{i+1}$.

Hence, x_{ij}, $x_{\overline{ij}}$, $x_{i\bar{j}}$, $x_{\bar{i}j}$ span a four-dimensional irreducible Yetter–Drinfeld submodule.

In the case $G^{g_i} = G^{g_j}$, we choose $g_a = g_b \notin G^{g_i}$, and the action is described by the second and last rows of the previous table. There are two Yetter–Drinfeld submodules, spanned by $\{x_{ij}, x_{\overline{ij}}\}$, and by $\{x_{i\bar{j}}, x_{\bar{i}j}\}$.

(5) Let $i \leqslant \ell < j$ be such that $a_{ij} = 0$. The G-degree of x_{ij} is $g_i g_j$, while the G-degree of $x_{\bar{i}j}$ is $g_i g_j \kappa$. For the action, notice that $g_j \in Z(G)$: if we pick $\mathbf{g} \notin G^{g_i}$, then

$h \in$	$h \cdot x_{ij}$	$h \cdot x_{\bar{i}j}$
G^{g_i}	$\chi_i(h)\chi_j(h)x_{ij}$	$\chi_i(\mathbf{g}^{-1}h\mathbf{g})\chi_j(h)x_{\bar{i}j}$
$G - G^{g_i}$	$\chi_i(\mathbf{g}^{-1}h)\chi_j(h)x_{\bar{i}j}$	$\chi_i(h\mathbf{g})\chi_j(h)x_{ij}$

(6) For i, j such that $a_{ij} = -2$, the G-degree of $(\mathrm{ad}_c\, x_j)x_{ij\bar{i}}$ is $g_i^2 g_j^2 \kappa$, where G acts via $\overline{\chi}_1\chi_j^2$.

(7) Let $i < j < k$ be such that $a_{ji} = a_{jk} = -1$. Consider

$$\mathbf{r}_1 := (\mathrm{ad}_c\, x_j)x_{ijk}, \qquad \mathbf{r}_2 := (\mathrm{ad}_c\, x_j)x_{\bar{i}jk},$$

which have G-degrees $g_i g_j^2 g_k \kappa$ and $g_i g_j^2 g_k$, respectively. The G-action is given by

$h \in$	$\overline{\chi}_j(h)^{-1}h \cdot \mathbf{r}_1$	$\overline{\chi}_j(h)^{-1}h \cdot \mathbf{r}_2$
$G^{g_i} \cap G^{g_k}$	$\chi_i(h)\chi_k(h)\mathbf{r}_1$	$\chi_i(g_j^{-1}hg_j)\chi_k(h)\mathbf{r}_2$
$G^{g_i} - G^{g_k}$	$\chi_i(h)\chi_k(g_j^{-1}h)\mathbf{r}_2$	$\chi_i(g_j^{-1}hg_j)\chi_k(g_j^{-1}h)\mathbf{r}_1$
$G^{g_k} - G^{g_i}$	$\chi_i(g_j^{-1}h)\chi_k(h)\mathbf{r}_2$	$\chi_i(hg_j)\chi_k(h)\mathbf{r}_1$
$G - (G^{g_i} \cup G^{g_k})$	$\chi_i(g_j^{-1}h)\chi_k(g_j^{-1}h)\mathbf{r}_1$	$\chi_i(hg_j)\chi_k(g_j^{-1}h)\mathbf{r}_2$

(8) Let $\beta \in \Delta_+^M - \{\alpha_i\}$. The G-degrees of x_β^2 and $[x_\beta, x_{\overline{\beta}}]_c$ are g_β^2 and $g_\beta^2\kappa$, respectively, where g_β is as in Remark 5.5. We define accordingly $\overline{\chi}_\beta := \overline{\chi}_1^{a_1} \cdots \overline{\chi}_\theta^{a_\theta}$, and G acts on x_β^2 and $[x_\beta, x_{\overline{\beta}}]_c$ via $\overline{\chi}_\beta$.

Remark 5.10. Let $i, j \in \mathbb{I}$ such that $G^{g_i} \neq G^{g_j}$ and $g_i, g_j \notin Z(G)$. Then

$$G^{g_i g_j} = (G^{g_i} \cap G^{g_j}) \cup (G - (G^{g_i} \cup G^{g_j}))$$

and the following rule defines a character:

$$\chi_{ij} : G^{g_i g_j} \to \Bbbk^\times, \qquad \chi_{ij}(h) := \begin{cases} \chi_i(h)\chi_j(h), & h \in G^{g_i} \cap G^{g_j}, \\ \chi_i(hg_j)\chi_j(hg_i), & h \notin G^{g_i} \cup G^{g_j}. \end{cases} \tag{5.44}$$

Theorem 5.11. *Let $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of type α_θ, γ_θ, δ_θ, ϵ_θ or ϕ_4. Then*

$$\mathrm{Hom}^{\Bbbk G}_{\Bbbk G}(\Bbbk\mathcal{R}_M, M) = 0.$$

Proof. Let $\mathtt{r} \in \mathcal{R}_M$ be G-homogeneous of degree $\mathtt{g} \in G$. By direct computation, $\mathtt{g} \cdot \mathtt{r} = \mathtt{r}$. On the other hand, M has a basis $\{x_i | i \in \mathbb{I}\} \cup \{x_{\overline{j}} | j \in \mathbb{J}\}$, where $\mathbb{J} = \{i \in \mathbb{I} | \dim M_i = 2\}$. Here x_i has degree

g_i, while $x_{\overline{j}}$ has degree $g_j\kappa$, and

$$g_i \cdot x_i = -x_i, \quad i \in \mathbb{I}; \qquad\qquad g_j\kappa \cdot x_{\overline{j}} = -x_{\overline{j}}, \quad j \in \mathbb{J}.$$

Hence, the claim follows. □

Recall that a graded braided bialgebra is *rigid* if it has no nontrivial graded deformations. See [18, §2] and the references therein for details. Next, we address rigidity for $\mathscr{B}(M)$.

Theorem 5.12. *Let $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ be of type either α_θ, γ_θ, δ_θ, ϵ_θ, or ϕ_4. Then $\mathscr{B}(M)$ is rigid.*

Proof. The category ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is semisimple and $\operatorname{Hom}^{\Bbbk G}_{\Bbbk G}(\Bbbk\mathcal{R}_M, M) = 0$ by Theorem 5.11. Hence, [18, Theorem 5.3] applies for $\mathscr{B}(M)$. □

Remark 5.13. The previous notion of rigidity is related to another one introduced in [48] coming from the action of an appropriate algebraic group on the Nichols algebra (viewed as a braided Hopf algebra). In fact, the notion of rigidity in *loc. cit.* is equivalent to generation in degree 1, which holds by Theorem 4.1. This gives a different proof of Theorem 5.12, independent of Theorem 5.11. Anyway, we need Theorem 5.11 to compute liftings.

6 | LIFTINGS OF NICHOLS ALGEBRAS

We describe all *liftings* for Nichols algebras of Yetter–Drinfeld modules of types α_θ, γ_θ, δ_θ, ϵ_θ, and ϕ_4. Even when the braided vector space is of diagonal type (i.e., when κ acts trivially), we cannot invoke [16] since the Yetter–Drinfeld realizations considered here are not *principal*. Nevertheless, we will perform an adaptation of the strategy developed in [5, 16].

We study the lowest rank type α_2 first, with a double purpose. On the one hand, we will only show all the details in this case, with explicit formulas for the defining relations. On the other hand, it will be the starting point to prove the general case, in which we will conclude that all liftings are cocycle deformations of the associated graded Hopf algebras.

Recall that a lifting of M over G is a finite-dimensional Hopf algebra H with coradical $\Bbbk G$ and infinitesimal braiding M. Hence, $\operatorname{gr} H \simeq \mathscr{B}(M)\#\Bbbk G$ by Theorem 4.1.

The family of liftings of M over G will be indexed by a set $\mathcal{R}_M \subseteq \Bbbk^K$ of *deformation parameters*, where K is the number of suitable chosen Yetter–Drinfeld submodules of the subspace spanned by a minimal set $\mathcal{G}$ of generators for the ideal defining $\mathscr{B}(M)$.

For each $\boldsymbol{\lambda} \in \mathcal{R}_M$ and $i \in \mathbb{I}_K$, we define $\boldsymbol{\lambda}^{(i)}, \boldsymbol{\lambda}^{(-i)} \in \Bbbk^K$ by

$$(\boldsymbol{\lambda}^{(i)})_j := \begin{cases} 0 & j \neq i, \\ \lambda_i & j = i; \end{cases} \qquad (\boldsymbol{\lambda}^{(-i)})_j := \begin{cases} \lambda_i & j \neq i, \\ 0 & j = i; \end{cases} \qquad j \in \mathbb{I}_K. \tag{6.1}$$

The aforementioned strategy starts by choosing a *good* stratification $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1 \sqcup \cdots \sqcup \mathcal{G}_l$, meaning that the vector space spanned by $\mathcal{G}_k$ is a Yetter–Drinfeld submodule of $\mathscr{B}(M)$ and the elements of $\mathcal{G}_k$ are primitive in the braided Hopf algebra $\mathscr{B}_k := T(M)/\langle \cup_{j=0}^{k-1} \mathcal{G}_j \rangle$, $k \in \mathbb{I}_{l+1}$, with one possible exception: we do not require primitiveness for the last step.

6.1 | Liftings of type α_2

Let $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of type α_2. Let $\mathcal{R}_M$ be the set of tuples $\boldsymbol{\lambda} = (\mu_1, \mu_2, \lambda_1, \lambda_2, \lambda_{12}, \mu_{12}, \mu'_{12}) \in \Bbbk^7$ that satisfy the constraints

$$\begin{aligned}
\mu_i &= 0 \text{ if either } \chi_i^2 \neq \varepsilon \text{ or } g_i^2 = 1,\ i \in \mathbb{I}_2,\\
\lambda_i &= 0 \text{ if either } \overline{\chi}_i \neq \varepsilon \text{ or } g_i^2 = \kappa,\ i \in \mathbb{I}_2,\\
\lambda_{12} &= 0 \text{ if } \chi_{12} \neq \varepsilon,\\
\mu_{12} &= 0 \text{ if either } \overline{\chi}_1\overline{\chi}_2 \neq \varepsilon \text{ or } (g_1g_2)^2 = 1,\\
\mu'_{12} &= 0 \text{ if either } \overline{\chi}_1\overline{\chi}_2 \neq \varepsilon \text{ or } (g_1g_2)^2 = \kappa.
\end{aligned} \tag{6.2}$$

The definition of $\overline{\chi}_i$ and χ_{ij} was given in (5.43) and (5.44). This subsection is devoted to prove the following.

Theorem 6.1. *Let $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ be of type α_2. For each $\boldsymbol{\lambda} \in \mathcal{R}_M$, let $\mathcal{L}(\boldsymbol{\lambda})$ be the quotient of $T(M)\#\Bbbk G$ by the following set of relations:*

$$\begin{aligned}
&z_i^2 - \mu_i(1 - g_i^2), \qquad\qquad z_{i\bar{i}} - \lambda_i(1 - g_i^2\kappa),\\
&z_{\overline{12}} + \chi_1(\kappa)z_{12} - \lambda_{12}(1 - g_1g_2),\\
&z_{12}^2 + \lambda_1\mu_2(1 - g_1^2\kappa)g_2^2 - \chi_1(\kappa)\mu_1\lambda_2(1 - g_1^2)g_2^2\kappa - \mu_{12}(1 - g_1^2g_2^2\kappa),\\
&[z_{12}, z_{\bar{12}}]_c + 2(1 + \chi_1\chi_2(\kappa))\mu_1\mu_2(1 - g_1^2)g_2^2 - \lambda_1\lambda_2(\kappa - g_1^2)g_2^2 - \mu'_{12}(1 - g_1^2g_2^2),
\end{aligned}$$

where we changed the labels $(x_i, x_{\bar{i}})_{i\in I_2}$ of the generators of $T(M)$ to $(z_i, z_{\bar{i}})_{i\in I_2}$. Then:

(a) $\mathcal{L}(\boldsymbol{\lambda}) \simeq L(\mathcal{A}(\boldsymbol{\lambda}), \mathscr{B}(M)\#\Bbbk G)$.
(b) $\mathcal{L}(\boldsymbol{\lambda})$ *is a lifting of M over $\Bbbk G$.*
(c) $\mathcal{L}(\boldsymbol{\lambda})$ *is a cocycle deformation of $\mathscr{B}(M)\#\Bbbk G$.*

Conversely, if L is lifting of M over $\Bbbk G$, then there exist $\boldsymbol{\lambda} \in \mathcal{R}_M$ such that $L \simeq \mathcal{L}(\boldsymbol{\lambda})$.

Fix a Yetter–Drinfeld module M over G of type α_2. As x_i^2, $x_{\bar{i}}^2$, $x_{i\bar{i}}$, $x_{1\bar{2}} + \chi_2(g_1^2)x_{\bar{1}2}$ and $x_{\overline{12}} + \chi_1(\kappa)x_{12}$ are primitive in $T(M)$ and

$$\begin{aligned}
\Delta(x_{12}^2) &= x_{12}^2 \otimes 1 + x_{1\bar{1}}g_2^2 \otimes x_2^2 - \chi_1(\kappa)x_1^2g_2^2\kappa \otimes x_{2\bar{2}} + g_1^2g_2^2\kappa \otimes x_{12}^2,\\
\Delta([x_{12}, x_{\bar{1}2}]_c) &= [x_{12}, x_{\bar{1}2}]_c \otimes 1 - \chi_1^{-1}(g_2^2)x_{1\bar{1}}g_2^2\kappa \otimes x_{2\bar{2}} + \chi_1(g_2^2)x_1^2g_2^2 \otimes x_2^2\\
&\quad + \chi_1(g_2^2)x_{\bar{1}}^2g_2^2 \otimes x_{\bar{2}}^2 + \chi_1\chi_2(\kappa)x_{\bar{1}}^2g_2^2 \otimes x_2^2 + x_1^2g_2^2 \otimes x_{\bar{2}}^2 + g_1^2g_2^2 \otimes [x_{12}, x_{\bar{1}2}]_c,
\end{aligned}$$

we may choose the following stratification:

$$\mathcal{G}_0 = \{x_i^2,\ x_{\bar{i}}^2,\ x_{i\bar{i}}\}, \qquad \mathcal{G}_1 = \{x_{1\bar{2}} + \chi_2(g_1^2)x_{\bar{1}2},\ x_{\overline{12}} + \chi_1(\kappa)x_{12}\}, \qquad \mathcal{G}_2 = \{x_{12}^2,\ [x_{12}, x_{\bar{1}2}]_c\}.$$

The Yetter–Drinfeld structure for each stratum is given in Remark 5.9.

Let $\mathcal{H}_k := \mathscr{B}_k \# \Bbbk G$. Next, we introduce a family of cleft objects of $\mathcal{H}_k$ parametrized by the set $\mathcal{R}_M$. Given $\boldsymbol{\lambda} \in \mathcal{R}_M$, define $\mathcal{E}_0(\boldsymbol{\lambda}) = \mathscr{B}_0 = T(M)$, $\mathcal{E}_1(\boldsymbol{\lambda}) = \mathscr{B}_1$, but we change the labels of the generators to $(y_i, y_{\bar{i}})_{i \in \mathbb{I}_2}$ in order to differentiate with generators $(x_i, x_{\bar{i}})_{i \in \mathbb{I}_2}$ of the pre-Nichols algebras $\mathscr{B}_k$. Let

$$\mathcal{E}_1(\boldsymbol{\lambda}) := \mathcal{E}_0(\boldsymbol{\lambda}) / \left\langle y_i^2 - \mu_i, y_{\bar{i}}^2 - \mu_i, y_{i\bar{i}} - \lambda_i : i \in \mathbb{I}_2 \right\rangle,$$

$$\mathcal{E}_2(\boldsymbol{\lambda}) := \mathcal{E}_1(\boldsymbol{\lambda}) / \left\langle y_{1\bar{2}} + \chi_2(g_1^2) y_{\bar{1}2} - \lambda_{12},\ y_{\bar{1}2} + \chi_1(\kappa) y_{12} - \lambda_{12} \right\rangle,$$

$$\mathcal{E}_3(\boldsymbol{\lambda}) := \mathcal{E}_2(\boldsymbol{\lambda}) / \left\langle y_{12}^2 - \mu_{12}, [y_{12}, y_{\bar{1}2}]_c - \mu'_{12} \right\rangle.$$

Each $\mathcal{E}_i(\boldsymbol{\lambda})$ is a $\Bbbk G$-module algebra since the ideal is stable under the G-action by (6.2). Thus, we may introduce $\mathcal{A}_i(\boldsymbol{\lambda}) := \mathcal{E}_i(\boldsymbol{\lambda}) \# \Bbbk G$.

Lemma 6.2. *Let $k \in \mathbb{I}_3$. Then $\mathcal{E}_k(\boldsymbol{\lambda}) \neq 0$ and each $\mathcal{A}_k(\boldsymbol{\lambda})$ is an $\mathcal{H}_k$-cleft object. There exists an $\mathcal{H}_k$-colinear section $\gamma_k : \mathcal{H}_k \to \mathcal{A}_k$ that restricts to an algebra map $(\gamma_k)_{|\Bbbk G} \in \mathrm{Alg}(\Bbbk G, \mathcal{A}_k)$.*

Proof. Fix $\boldsymbol{\lambda} \in \mathcal{R}_M$; to simplify the notation, we suppress $\boldsymbol{\lambda}$ and put $\mathcal{E}_k = \mathcal{E}_k(\boldsymbol{\lambda})$, $\mathcal{A}_k = \mathcal{A}_k(\boldsymbol{\lambda})$. We prove the claim recursively on k.

For $k = 1$, we notice that $\mathcal{E}_1 \neq 0$ (and a fortiori $\mathcal{A}_1 \neq 0$) by [5, Lemma 5.16]. Notice that $g_j(y_i^2 - \mu_i)g_j^{-1} = y_{\bar{i}}^2 - \lambda_i$ if $i \neq j$, so in $\mathcal{A}_1$, we have

$$\langle y_i^2 - \mu_i, y_{\bar{i}}^2 - \mu_i : i = 1, 2 \rangle = \langle y_i^2 - \mu_i : i = 1, 2 \rangle.$$

We may refine the stratification and proceed in four steps, quotient out first by x_1^2, then by x_2^2, now by $x_{1\bar{1}}$, and finally by $x_{2\bar{2}}$. At each step, we consider the subalgebra Y' generated by the relation r in the corresponding pre-Nichols algebra, note that Y' is isomorphic to a polynomial ring in one variable since $r \in \mathcal{P}(T(M))_g - 0$, and for this g, we have $g \cdot r = r$. Consider $Y = \mathcal{S}(Y')$. As Y is a polynomial algebra generated by rg^{-1}, there exists an algebra map $\phi : Y \to \mathcal{A}$ such that $\phi(rg^{-1}) = rg^{-1} - \lambda g^{-1}$, $\lambda \in \Bbbk$, which is $\mathcal{H}$-colinear. Applying repeatedly [30, Theorem 8] as in [5, Proposition 5.19], $\mathcal{A}_1$ is a $\mathcal{H}_1$-cleft object and the existence of the desired section γ_1 follows by [5, Proposition 5.8].

For $k = 2$, it is enough to show that $\mathcal{A}_2 \neq 0$. Indeed, in that case, [30, Theorem 8] assures that $\mathcal{A}_2$ is an $\mathcal{H}_2$-cleft object. Now [5, Proposition 5.8] provides a section γ_2 such that $(\gamma_2)_{|\Bbbk G} \in \mathrm{Alg}(\Bbbk G, \mathcal{A}_2)$. As in [16, Lemma 3.4], nonvanishing of $\mathcal{A}_2$ would follow from

$$\mathcal{E}_2(\boldsymbol{\lambda}^{(5)}) = \mathscr{B}_1 / \left\langle y_{1\bar{2}} + \chi_2(g_1^2) y_{\bar{1}2} - \lambda_{12},\ y_{\bar{1}2} + \chi_1(\kappa) y_{12} - \lambda_{12} \right\rangle \neq 0.$$

Indeed, if $\varpi_1 : \mathcal{A}_1(\boldsymbol{\lambda}^{(5)}) = \mathcal{H}_1 \twoheadrightarrow \mathcal{E}_2(\boldsymbol{\lambda}^{(5)}) \# \Bbbk G$ is the canonical projection, then the composition of the (restriction to $\mathcal{E}_1$ of the) coaction $\mathcal{E}_1 \to \mathcal{A}_1 \otimes \mathcal{H}_1$, which is an algebra map, with $\mathrm{id} \otimes \varpi_1$ factors through $\mathcal{E}_2 = \mathcal{E}_2(\boldsymbol{\lambda})$.

To check that $\mathcal{E}_2(\boldsymbol{\lambda}^{(5)}) \neq 0$, we use that $(\mathcal{H}_1)_\sigma$ is the bosonization of a pre-Nichols algebra of diagonal type by G, and that the $(1, g_1 g_2 \kappa)$- and $(1, g_1 g_2)$-primitive elements $y_{1\bar{2}} + \chi_2(g_1^2) y_{\bar{1}2}$ and $y_{\bar{1}2} + \chi_1(\kappa) y_{12}$ span the same subspace as $\mathrm{x}_{1\bar{2}}$ and $\mathrm{x}_{\bar{1}2}$, see the proof of Proposition 5.4. The quotient $(\mathcal{H}_1)_\sigma / \langle \mathrm{x}_{1\bar{2}} - \lambda_{12}, \mathrm{x}_{\bar{1}2} - \lambda_{12} \rangle$ is not zero by [4], and

$$(\mathcal{H}_1)_\sigma / \langle \mathrm{x}_{1\bar{2}} - \lambda_{12}, \mathrm{x}_{\bar{1}2} - \lambda_{12} \rangle \simeq F_\sigma\Big(\mathcal{A}_2(\boldsymbol{\lambda}^{(5)})\Big),$$

which implies $\mathcal{A}_2(\lambda^{(5)}) \neq 0$. Notice that

$$g_1\big(y_{1\bar{2}} + \chi_2(g_1^2)y_{\bar{1}2} - \lambda_{12}\big)g_1^{-1} = -\chi_1(\kappa)\chi_2(g_1^2)\big(y_{\bar{1}2} + \chi_1(\kappa)y_{12} - \lambda_{12}\big),$$

so in $\mathcal{A}_2$,

$$\langle y_{1\bar{2}} + \chi_2(g_1^2)y_{\bar{1}2} - \lambda_{12},\ y_{\bar{1}2} + \chi_1(\kappa)y_{12} - \lambda_{12}\rangle = \langle y_{\bar{1}2} + \chi_1(\kappa)y_{12} - \lambda_{12}\rangle.$$

Finally, as $\mathcal{H}_2 = \widehat{\mathscr{B}}(M)\#\Bbbk G$, $\mathcal{H}_3 = \mathscr{B}(M)\#\Bbbk G$, we have $\mathcal{H}_2^{\mathrm{co}\,\pi_2} = \mathcal{Z}(M)$, a skew-polynomial algebra in variables x_{12}^2, $[y_{12}, y_{\bar{1}2}]_c$ by Proposition 5.6. Hence, [30, Theorem 4] applies and $\mathcal{A}_3$ is $\mathcal{H}_3$-cleft. The claim about γ_3 follows from [5, Proposition 5.8]. □

Proof of Theorem 6.1. Now follows by the same procedure as in [19, Theorem 5.6], using Theorem 5.11. Indeed, if we define $\mathcal{L}_0(\boldsymbol{\lambda}) = \mathcal{H}_0$,

$$\mathcal{L}_1(\boldsymbol{\lambda}) = \mathcal{L}_0(\boldsymbol{\lambda})/\big\langle z_i^2 - \mu_i(1 - g_i^2), z_{i\bar{i}} - \lambda_i(1 - g_i^2\kappa)\big\rangle,$$
$$\mathcal{L}_2(\boldsymbol{\lambda}) = \mathcal{L}_1(\boldsymbol{\lambda})/\big\langle z_{\bar{1}\bar{2}} + \chi_1(\kappa)z_{12} - \lambda_{12}(1 - g_1g_2)\big\rangle,$$

and $\mathcal{L}_3(\boldsymbol{\lambda}) = \mathcal{L}(\boldsymbol{\lambda})$, we can prove recursively that $\mathcal{L}_i(\boldsymbol{\lambda}) \simeq L(\mathcal{A}_i(\boldsymbol{\lambda}), \mathcal{H}_i)$. □

6.2 | The general case

Let $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of type α_θ, γ_θ, δ_θ, ϵ_θ or ϕ_4. Recall the characters $\overline{\chi}_i$ and χ_{ij} defined in (5.43) and (5.44). The set $\mathcal{R}_M$ of deformation parameters contains tuples $\boldsymbol{\lambda}$ satisfying the following constraints:

$$\begin{aligned}
&\mu_i = 0 \text{ if either } \chi_i^2 \neq \varepsilon \text{ or } g_i^2 = 1,\ i \in \mathbb{I}_\theta;\\
&\lambda_i = 0 \text{ if either } \overline{\chi}_i \neq \varepsilon \text{ or } g_i^2 = \kappa,\ i \in \mathbb{I}_\theta;\\
&\lambda_{ij} = 0 \text{ if } i < j \leqslant \ell,\ a_{ij} = -1,\ \chi_{ij} \neq \varepsilon;\\
&\lambda_{ij} = 0 \text{ if } i < j \leqslant \ell,\ a_{ij} = 0,\ \chi_{ij} \neq \varepsilon;\\
&\lambda'_{ij} = 0 \text{ if } i < j \leqslant \ell,\ a_{ij} = 0,\ \chi_{ij} \neq \varepsilon;\\
&\lambda_{ij} = \lambda'_{ij} \text{ if } i < j \leqslant \ell,\ a_{ij} = 0,\ G^{g_i} \neq G^{g_j};\\
&\lambda_{ij} = 0 \text{ if } i \leqslant \ell < j,\ a_{ij} = 0,\ \chi_{ij} \neq \varepsilon;\\
&\lambda_{ijk} = 0 \text{ if } i < j < k,\ a_{ji} = a_{jk} = -1,\ \overline{\chi}_j\chi_{ik} \neq \varepsilon;\\
&\lambda_{ji} = 0 \text{ if } i < j,\ a_{ij} = -2,\ \overline{\chi}_i\chi_j^2 \neq \varepsilon;\\
&\mu_\beta = 0 \text{ if either } \overline{\chi}_\beta \neq \varepsilon \text{ or } g_\beta^2 = 1\,(\beta \in \Delta_+^V - \{\alpha_i\});\\
&\mu'_\beta = 0 \text{ if either } \overline{\chi}_\beta \neq \varepsilon \text{ or } g_\beta^2 = \kappa\,(\beta \in \Delta_+^V - \{\alpha_i\},\ \dim M_\beta = 2).
\end{aligned} \tag{6.3}$$

In this subsection, we prove our last main result:

Theorem 6.3. *Let $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of type α_θ, γ_θ, δ_θ, ϵ_θ, or ϕ_4. For each $\boldsymbol{\lambda} \in \mathcal{R}_M$, see (6.3), let $\mathcal{L}(\boldsymbol{\lambda})$ be the quotient of $T(M)\#\Bbbk G$, where we change the labels of the generators to $(z_i)_{i\in\mathbb{I}_{\theta+\ell}}$ by the following relations:*

$$z_i^2 - \mu_i(1-g_i^2), \qquad i \in \mathbb{I}_\theta;$$

$$z_{i\bar{i}} - \lambda_i(1-g_i^2\kappa), \qquad i \in \mathbb{I}_\ell;$$

$$z_{\bar{i}\bar{j}} + \chi_i(\kappa)z_{ij} - \lambda_{ij}(1-g_ig_j), \qquad i<j\leqslant \ell,\ a_{ij}=-1;$$

$$z_{ij} - \lambda_{ij}(1-g_ig_j), z_{i\bar{j}} - \lambda'_{ij}(1-g_ig_j\kappa), \qquad i<j\leqslant \ell,\ a_{ij}=0;$$

$$z_{ij} - \lambda_{ij}(1-g_ig_j), \qquad i\leqslant \ell<j,\ a_{ij}=0;$$

$$(\mathrm{ad}_c\, z_j)z_{ij\bar{i}} - (2)_{\chi_j(g_i^2\kappa)}\mu_j\lambda_i(1-g_j^2)g_i^2 + \mu_i\lambda_j(1-g_j^2\kappa)g_i^2 - \lambda_{ji}(1-g_i^2g_j^2), \qquad a_{ij}=-2;$$

$$(\mathrm{ad}_c\, z_j)z_{ijk} - (2)_{\chi_j(g_ig_k\kappa)}\mu_j\lambda'_{ik}(1-g_j^2)g_ig_k\kappa + \lambda_{ik}\lambda_j(1-g_j^2\kappa)g_ig_k - \lambda_{ijk}(1-g_ig_j^2g_k\kappa),$$
$$i<j<k,\ a_{ji}=a_{jk}=-1;$$

$$z_\beta^2 - \mathbf{z}_\beta - \mu_\beta(1-g_\beta^2), \qquad \beta \in \Delta_+^M - \{\alpha_i\};$$

$$\left[z_\beta, z_{\bar{\beta}}\right]_c - \mathbf{z}_\beta - \mu'_\beta(1-g_\beta^2\kappa), \qquad \beta \in \Delta_+^M - \{\alpha_i\}, \dim M_\beta = 2,$$

where $\mathbf{z}_\beta$, $\mathbf{z}'_\beta \in T(M)\#\Bbbk G$ are defined recursively on $\beta \in \Delta_+^M - \{\alpha_i\}$ such that $z_\beta^2 - \mathbf{z}_\beta$ is $(g_\beta^2, 1)$-primitive and $[z_\beta, z_{\bar{\beta}}]_c - \mathbf{z}'_\beta$ is $(g_\beta^2\kappa, 1)$-primitive in the quotient of $T(M)\#\Bbbk G$ by the previous relations. Then:

(a) $\mathcal{L}(\boldsymbol{\lambda}) \simeq L(\mathcal{A}(\boldsymbol{\lambda}), \mathscr{B}(M)\#\Bbbk G)$,
(b) $\mathcal{L}(\boldsymbol{\lambda})$ *is a lifting of* M *over* $\Bbbk G$,
(c) $\mathcal{L}(\boldsymbol{\lambda})$ *is a cocycle deformation of* $\mathscr{B}(M)\#\Bbbk G$.

Conversely, if L is lifting of M over $\Bbbk G$, then there exist $\boldsymbol{\lambda} \in \mathcal{R}_M$ such that $L \simeq \mathcal{L}_5(\boldsymbol{\lambda})$.

Let $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of type $\alpha_\theta, \gamma_\theta, \delta_\theta, \epsilon_\theta$, or ϕ_4. We choose first a stratification $\mathcal{G} = \sqcup_{i=0}^4 \mathcal{G}_i$ on the set of defining relations found in Theorem 5.7

$$\mathcal{G}_0 = \{x_i^2,\ x_{\bar{i}}^2,\ x_{i\bar{i}} | i \in \mathbb{I}_\ell\} \cup \{x_i^2 | i > \ell\};$$

$$\mathcal{G}_1 = \{x_{i\bar{j}} + \chi_j(g_i^2)x_{\bar{i}j},\ x_{\bar{i}\bar{j}} + \chi_i(\kappa)x_{ij} | i<j\leqslant\ell,\ a_{ij}=-1\};$$

$$\mathcal{G}_2 = \{x_{ij},\ x_{i\bar{j}},\ x_{\bar{i}j},\ x_{\bar{i}\bar{j}} | i<j\leqslant\ell,\ a_{ij}=0\} \cup \{x_{ij},\ x_{\bar{i}j} | i\leqslant\ell<j,\ a_{ij}=0\};$$

$$\mathcal{G}_3 = \{r_{ijk} := (\mathrm{ad}_c\, x_j)x_{ijk},\ \overline{r}_{ijk} := (\mathrm{ad}_c\, x_j)x_{\bar{i}jk} | i<j<k,\ a_{ji}=a_{jk}=-1\}$$
$$\cup \{(\mathrm{ad}_c\, x_j)x_{ij\bar{i}} | a_{ij}=-2\};$$

$$\mathcal{G}_4 = \{x_\beta^2 | \beta \in \Delta_+^V - \{\alpha_i\}\} \cup \{[x_\beta, x_{\bar{\beta}}]_c | \beta \in \Delta_+^V - \{\alpha_i\},\ \dim M_\beta = 2\}.$$

This stratification is good since

$$\begin{aligned}\Delta(r_{ijk}) &= r_{ijk}\otimes 1 - \chi_j(g_ig_k\kappa)x_j^2 g_ig_k\kappa \otimes x_{i\bar{k}} - \chi_j(g_i^2g_k^2)x_j^2 g_ig_k\kappa\otimes x_{\bar{i}k}\\ &\quad + x_jx_{\bar{j}}g_ig_k\otimes x_{ik} + \chi_j(g_ig_k\kappa)x_{\bar{j}}x_jg_ig_k\otimes x_{\bar{i}\bar{k}} + g_ig_j^2g_k\kappa\otimes r_{ijk},\\ \Delta(\bar{r}_{ijk}) &= \bar{r}_{ijk}\otimes 1 - \chi_j(g_ig_k\kappa)x_j^2 g_ig_k\otimes x_{i\bar{k}} - \chi_j(g_i^2g_k^2)x_j^2 g_ig_k\otimes x_{ik}\\ &\quad + x_jx_{\bar{j}}g_ig_k\kappa\otimes x_{\bar{i}k} + \chi_j(g_ig_k\kappa)x_{\bar{j}}x_jg_ig_k\kappa\otimes x_{i\bar{k}} + g_ig_j^2g_k\otimes \bar{r}_{ijk}.\end{aligned}$$

Set $\mathcal{H}_k := \mathscr{B}_k\#\Bbbk G$. The Yetter–Drinfeld structure of each stratum is given in Remark 5.9.

Let $\boldsymbol{\lambda}\in\mathcal{R}_M$. Define $\mathcal{E}_0(\boldsymbol{\lambda}) = \mathscr{B}_0 = T(M)$, but we change the labels of the generators to $(y_i, y_{\bar{i}})_{i\in\mathbb{I}}$ to differentiate from the generators $(x_i, x_{\bar{i}})_{i\in\mathbb{I}}$ of the pre-Nichols algebras $\mathscr{B}_k$. Let

$$\begin{aligned}\mathcal{E}_1(\boldsymbol{\lambda}) &:= \mathcal{E}_0(\boldsymbol{\lambda})/\Big\langle y_i^2-\mu_i, y_{\bar{i}}^2-\mu_i, y_{i\bar{i}}-\lambda_i : i\in\mathbb{I}_\theta\Big\rangle,\\ \mathcal{E}_2(\boldsymbol{\lambda}) &:= \mathcal{E}_1(\boldsymbol{\lambda})/\Big\langle y_{i\bar{j}}+\chi_j(g_i^2)y_{\bar{i}j}-\lambda_{ij},\ y_{\bar{i}\bar{j}}+\chi_i(\kappa)y_{ij}-\lambda_{ij} | i<j\leqslant\ell,\ a_{ij}=-1\Big\rangle,\\ \mathcal{E}_3(\boldsymbol{\lambda}) &:= \mathcal{E}_2(\boldsymbol{\lambda})/\left\langle \begin{matrix} y_{ij}-\lambda_{ij},\ y_{i\bar{j}}-\lambda'_{ij},\ y_{\bar{i}j}-\lambda'_{ij},\ y_{\bar{i}\bar{j}}-\lambda_{ij},\ i<j\leqslant\ell,\ a_{ij}=0;\\ y_{ij}-\lambda_{ij},\ y_{\bar{i}j}-\lambda_{ij},\ i\leqslant\ell<j,\ a_{ij}=0\end{matrix}\right\rangle,\\ \mathcal{E}_4(\boldsymbol{\lambda}) &:= \mathcal{E}_3(\boldsymbol{\lambda})/\left\langle \begin{matrix}(\mathrm{ad}_c\, y_j)y_{ijk}-\lambda_{ijk},\ (\mathrm{ad}_c\, y_j)y_{\bar{i}jk}-\lambda_{ijk},\ a_{ji}=a_{jk}=-1,\\ (\mathrm{ad}_c\, y_j)y_{ij\bar{i}}-\lambda_{ji},\ a_{ij}=-2\end{matrix}\right\rangle,\\ \mathcal{E}_5(\boldsymbol{\lambda}) &:= \mathcal{E}_4(\boldsymbol{\lambda})/\left\langle \begin{matrix} y_\beta^2-\mu_\beta,\ \beta\in\Delta_+^V-\{\alpha_i\},\\ \big[y_\beta, y_{\bar{\beta}}\big]_c-\mu'_\beta,\ \beta\in\Delta_+^V-\{\alpha_i\},\ \dim M_\beta=2\end{matrix}\right\rangle.\end{aligned}$$

Each $\mathcal{E}_i(\boldsymbol{\lambda})$ is a $\Bbbk G$-module algebra since each defining ideal above is stable under the G-action by (6.3). Thus, we may introduce $\mathcal{A}_i(\boldsymbol{\lambda}) := \mathcal{E}_i(\boldsymbol{\lambda})\#\Bbbk G$.

Lemma 6.4. *Let $k\in\mathbb{I}_5$. Then $\mathcal{E}_k(\boldsymbol{\lambda})\neq 0$ and each $\mathcal{A}_k(\boldsymbol{\lambda})$ is an $\mathcal{H}_k$-cleft object. There exists an $\mathcal{H}_k$-colinear section $\gamma_k : \mathcal{H}_k\to\mathcal{A}_k$ that restricts to an algebra map $(\gamma_k)_{|\Bbbk G}\in\mathrm{Alg}(\Bbbk G, \mathcal{A}_k)$.*

Proof. Fix $\boldsymbol{\lambda}\in\mathcal{R}_M$; again, we simplify the notation and write $\mathcal{E}_k = \mathcal{E}_k(\boldsymbol{\lambda})$, $\mathcal{A}_k = \mathcal{A}_k(\boldsymbol{\lambda})$. The proof is analogous to that of Lemma 6.2, recursively on k.

When $k<5$, the key step is to prove that $\mathcal{E}_k\neq 0$, which implies that $\mathcal{A}_k\neq 0$: if so, then [30, Theorem 8] applies again to conclude that $\mathcal{A}_k$ is $\mathcal{H}_k$-cleft; hence, there exists a section γ_k as in the statement by [5, Proposition 5.8].

To show that $\mathcal{E}_k\neq 0$, it is enough to verify nonvanishing when we *deform just one submodule of relations*; that is, to consider the case $\boldsymbol{\lambda}=\boldsymbol{\lambda}^{(i)}$ for each i and then proceed as in [7, Lemma 3.4]. Indeed, if $\varpi_k : \mathcal{A}_k(\boldsymbol{\lambda}^{(i)}) = \mathcal{H}_{k-1}\twoheadrightarrow\mathcal{E}_k(\boldsymbol{\lambda}^{(I)})\#\Bbbk G$ is the canonical projection, then the composition of the algebra map $\mathcal{E}_{k-1}\to\mathcal{A}_{k-1}\otimes\mathcal{H}_{k-1}$ (given by the coaction) with $(\mathrm{id}\otimes\varpi_k)$ factors through $\mathcal{E}_k = \mathcal{E}_k(\boldsymbol{\lambda})$.

To verify that $\mathcal{E}_k(\boldsymbol{\lambda}^{(i)})\neq 0$ when the submodule of relations to be deformed is neither $\{(\mathrm{ad}_c\, x_j)x_{ijk}, (\mathrm{ad}_c\, x_j)x_{\bar{i}jk}\}$, where $a_{ji}=a_{jk}=-1$, nor $\{(\mathrm{ad}_c\, x_j)x_{ij\bar{i}}\}$ with $a_{ij}=-2$, we may use Lemma 6.2. For these two exceptions, we adapt the argument given in Lemma 6.2 for relations

$y_{12^2} + \chi_2(g_1^2)y_{\overline{12}}$ and $y_{\overline{12}} + \chi_1(\kappa)y_{12}$, and then use the cocycle σ to reduce to deformations of Nichols algebras of diagonal type, so the result follows by [4, Proposition 4.2].

Finally, for $k = 5$, we have that $\mathcal{H}_4^{\mathrm{co}\,\pi_4} = \mathcal{Z}(M)$ is a skew-polynomial algebra in variables x_β^2, $[x_\beta, x_{\overline{\beta}}]_c$ by Proposition 5.6; thus, [30, Theorem 4] assures that $\mathcal{A}_5$ is $\mathcal{H}_5$-cleft. The section γ_5 can be chosen so that $(\gamma_k)_{|\Bbbk G} \in \mathrm{Alg}(\Bbbk G, \mathcal{A}_k)$ by [5, Proposition 5.8]. □

We are ready to prove the main theorem of this section.

Proof of Theorem 6.3. We proceed as in [19, Theorem 5.6], using correspondingly Theorem 5.11 and Lemma 6.4. Indeed, starting with $\mathcal{L}_0(\boldsymbol{\lambda}) = \mathcal{H}_0$, we define successive quotients $\mathcal{L}_i(\boldsymbol{\lambda})$, $i \in \mathbb{I}_5$, where $\mathcal{L}_4$ is the quotient by all the relations of $\mathcal{L}(\boldsymbol{\lambda})$ except the last two sets (parametrized by $\beta \in \Delta_+$). Each $\mathcal{L}_k(\boldsymbol{\lambda})$, $k < 5$, is a Hopf algebra since one if obtained from the previous one by recursively quotient by skew-primitive elements. Working as in [4, Theorem 1.6], $\mathcal{L}_4(\boldsymbol{\lambda}) \simeq L(\mathcal{A}_4(\boldsymbol{\lambda}), \mathcal{H}_4)$, and there exist $\mathbf{z}_\beta \in \mathcal{L}_4(\boldsymbol{\lambda})$ and $\mathbf{z}'_\beta \in \mathcal{L}_4(\boldsymbol{\lambda})$ as stated below; moreover, $\mathcal{L}(\boldsymbol{\lambda}) \simeq L(\mathcal{A}_5(\boldsymbol{\lambda}), \mathcal{H}_5)$. The proof that these are all the liftings follows exactly as in [19, Theorem 5.6] □

6.3 | Foldings of liftings

The folding construction in [42, Part 1] was formulated in the following general setting: Let H be a Hopf algebra and H_σ, $\sigma \in \widehat{\Sigma}$ a group of biGalois objects with coherent choice of isomorphisms $\iota_{\sigma,\tau} : H_{\sigma\tau} \cong H_\sigma \square H_\tau$. By [42, Theorem 1.6], the direct sum of algebras

$$\widetilde{H} := \bigoplus \sum_{\sigma \in \widehat{\Sigma}} H_\sigma$$

can be endowed with the structure of a Hopf algebra with coproduct $\bigoplus_{\sigma,\tau} \iota_{\sigma,\tau}$.

Conversely by [42, Theorem 3.6], any Hopf algebra $\widetilde{H}$ with Σ a central subgroup is a folding of $H = \widetilde{H}/\Sigma^+\widetilde{H}$ by Σ. The biGalois objects are quotients of $\widetilde{H}$ associated to a central character on Σ. The folding data in Section 3 were formulated specifically for the situation $H = \mathscr{B}(M)\#\Bbbk\Gamma$ and for biGalois objects arising from 2-cocycles σ on the group Γ, trivially extended to H, and twisted Yetter–Drinfeld isomorphisms $\mathbf{u} : \mathscr{B}(M)_\sigma \to \mathscr{B}(M)$, extended by the identity on G to H. In Theorem 3.6, we have stated the folding solely in terms of $\sigma, \mathbf{u}$, while in Theorem 3.7, we have stated the folding with these specific choices of biGalois objects as above.

We now discuss the following alternative systematic way to understand the liftings of folded Nichols algebras, which we constructed in the previous section: Let H' be a lifting of $H = \mathscr{B}(M)\#\Bbbk\Gamma$ for a diagonal Nichols algebra, which are classified in [4, 16]. Let again $(H'_\sigma)_{\sigma \in \widehat{\Sigma}}$ be a group of biGalois objects over the lifting, then we have a folding $\widetilde{H'}$, whose graded algebra is the folding $\widetilde{H} = \mathscr{B}(\widetilde{M})\#\Bbbk G$ of H. One source for such biGalois objects could be again folding data $(\sigma, \mathbf{u})$ where in addition $\mathbb{u}$ is compatible with the lifting H', and more precisely, leaves a lifting cocycle invariant. But there are also other possibilities, namely, the 2-cocycle σ over Γ could be nontrivially extended to $\mathscr{B}(M)\#\Bbbk\Gamma$, which would cause a folding of H that is a lifting of $\widetilde{H}$ with values in the new center.

Conversely, we obtain in this way all liftings of $\widetilde{H}$ where Σ is central, acting trivially on $\widetilde{M}$. We have already shown for each Nichols algebra in Theorem 3.18, that this trivial action can always

be achieved by a Doi twist; however, it is not a-priori clear that these Doi twists carry over to the lifting. We will now use this tool to analyze the smallest example:

Example 6.5 (Case ${}^2A_2^2$). We consider the Nichols algebra of a module of type α_2 defined in Section 2.3.1 over a group G generated by g_1, g_2 with $g_1 g_2 = \kappa g_2 g_1$ and κ central of order two, which is a central extension of the abelian group Γ generated by $\bar{g}_1, \bar{g}_2$. We computed all its liftings in Section 6.1. We can conveniently take the group action from Remark 5.9 and the braiding from Section 2.3.1 to replace z_{ij} again by the braided commutator. For example, the first three relations depending on the parameters $\mu_i, \lambda_i, \lambda_{12}$ read

$$z_i^2 = \mu_i(1 - g_i^2), \qquad z_i z_{\bar{i}} + \chi_i(\kappa) z_{\bar{i}} z_i = \lambda_i(1 - g_i^2\kappa),$$
$$z_{\bar{1}} z_{\bar{2}} - \chi_2(\kappa g_1^2) z_2 z_{\bar{1}} + \chi_1(\kappa) z_1 z_2 - \chi_1(\kappa) z_{\bar{2}} z_1 = \lambda_{12}(1 - g_1 g_2).$$

Note that acting with a group element on a relation may produce more relations, as we explained in the proof of Lemma 6.2 for the cleft objects. For example, acting with g_j on the first relation produces the relation $z_{\bar{i}}^2 = \mu_i(1 - g_i^2)$.

The associated Nichols algebra and its liftings are foldings if and only if κ is a central element in the Hopf algebra, that is, $\chi_1(\kappa) = \chi_2(\kappa) = 1$ (which we saw that it is true up to Doi twist). In this case, we saw in Remark 3.14 that the braiding diagonalizes in the basis

$$\mathrm{x}_i = z_i + q_{ij} z_{\bar{i}}, \qquad \mathrm{x}_{\bar{i}} = z_i - q_{ij} z_{\bar{i}}.$$

The elements x_i, $\mathrm{x}_{\bar{i}}$ are not G-homogeneous, but Γ-homogeneous with degrees $\bar{g}_i$, $i \in \mathbb{I}_2$. On the other hand, they are G-eigenvectors with g_i, g_j acting on x_i with eigenvalues $-1, -q_{ji}$ and on $\mathrm{x}_{\bar{i}}$ with eigenvalues $-1, q_{ji}$. The diagonal braiding matrix is of type $A_2 \times A_2$

$$\begin{pmatrix} -1 & -q_{12} & -1 & q_{12} \\ -q_{21} & -1 & q_{21} & -1 \\ -1 & -q_{12} & -1 & q_{12} \\ -q_{21} & -1 & q_{21} & -1 \end{pmatrix}$$

and a twisted symmetry switching the two copies. In the folding construction, $z_i, z_{\bar{i}}$ arise as eigenvalues of this symmetry.

We now rewrite the relations in this basis, starting with those involving just one orbit $\mathrm{x}_i, \mathrm{x}_{\bar{i}}$, which is a diagonal Nichols algebra of type $A_1 \times A_1$:

$$\tfrac{1}{4}(\mathrm{x}_i + \mathrm{x}_{\bar{i}})^2 = \mu_i(1 - g_i^2), \qquad \tfrac{q_{ij}^{-2}}{4}(\mathrm{x}_i - \mathrm{x}_{\bar{i}})^2 = \mu_i(1 - g_i^2),$$
$$\tfrac{q_{ij}^{-1}}{4}((\mathrm{x}_i + \mathrm{x}_{\bar{i}})(\mathrm{x}_i - \mathrm{x}_{\bar{i}}) + (\mathrm{x}_i - \mathrm{x}_{\bar{i}})(\mathrm{x}_i + \mathrm{x}_{\bar{i}})) = \lambda_i(1 - g_i^2\kappa).$$

These relations rewrite to

$$\mathrm{x}_i^2 = (1 + q_{ij}^2)\mu_i(1 - g_i^2) + q_{ij}\lambda_i(1 - g_i^2\kappa), \qquad \mathrm{x}_i \mathrm{x}_{\bar{i}} + \mathrm{x}_{\bar{i}} \mathrm{x}_i = 2(1 - q_{ij}^2)\mu_i(1 - g_i^2) = 0,$$
$$\mathrm{x}_{\bar{i}}^2 = (1 + q_{ij}^2)\mu_i(1 - g_i^2) - q_{ij}\lambda_i(1 - g_i^2\kappa), \qquad \tfrac{1}{2}(\mathrm{x}_i^2 - \mathrm{x}_{\bar{i}}^2) = q_{ij}\lambda_i(1 - g_i^2\kappa),$$

where we have to take into account that Section 6.1 states that $\mu_i \neq 0$, respectively, $\lambda_i \neq 0$, only if $q_{ji}^2 = 1$, so the anticommutator vanishes.

This is consistent with the possible liftings of diagonal $A_1 \times A_1$:

- The anticommutator relation admits a nontrivial lifting if $\chi_i \chi_{\bar{i}} = \epsilon$, but in our case $\chi_i(g_j)\chi_{\bar{i}}(g_j) = -1$.
- The truncation relations admit nontrivial liftings if $\chi_i^2 = \epsilon$, which is the case if and only if $q_{ji}^2 = 1$. If the respective lifting parameters are equal, then this lifting datum is compatible with a folding using the group 2-cocycle. This produces the symmetric lifting depending on μ_i.
- On the other hand, the antisymmetric lifting depending on λ_i requires a lifting cocycle that is nontrivially extended to the Nichols algebra. The corresponding nontrivial biGalois object is determined by plugging the nontrivial central character $\kappa \mapsto -1$.

We now turn to the relation involving λ_{12}:

$$\begin{aligned}\lambda_{12}(1 - g_1g_2) &= \tfrac{q_{12}^{-1}}{2}(\mathrm{x}_1 - \mathrm{x}_{\bar{1}})\tfrac{q_{21}^{-1}}{2}(\mathrm{x}_2 - \mathrm{x}_{\bar{2}}) - q_{12}^2\tfrac{1}{2}(\mathrm{x}_2 + \mathrm{x}_{\bar{2}})\tfrac{q_{12}^{-1}}{2}(\mathrm{x}_1 - \mathrm{x}_{\bar{1}}) \\ &\quad + \tfrac{1}{2}(\mathrm{x}_1 + \mathrm{x}_{\bar{1}})\tfrac{1}{2}(\mathrm{x}_2 + \mathrm{x}_{\bar{2}}) - \tfrac{q_{21}^{-1}}{2}(\mathrm{x}_2 - \mathrm{x}_{\bar{2}})\tfrac{1}{2}(\mathrm{x}_1 + \mathrm{x}_{\bar{1}}) \\ &= \tfrac{1}{2}(\mathrm{x}_1\mathrm{x}_{\bar{2}} - q_{12}\mathrm{x}_{\bar{2}}\mathrm{x}_1) + \tfrac{1}{2}(\mathrm{x}_{\bar{1}}\mathrm{x}_2 + q_{12}\mathrm{x}_2\mathrm{x}_{\bar{1}})\lambda_{12}(1 - g_1g_2\kappa) \\ &= -\tfrac{q_{12}}{2}(\mathrm{x}_1\mathrm{x}_{\bar{2}} - q_{12}\mathrm{x}_{\bar{2}}\mathrm{x}_1) + \tfrac{q_{12}}{2}(\mathrm{x}_{\bar{1}}\mathrm{x}_2 + q_{12}\mathrm{x}_2\mathrm{x}_{\bar{1}}).\end{aligned}$$

Section 6.1 with χ_{12} in (5.44) applied to g_i^2 and g_1g_2 states that $\lambda_{12} \neq 0$ only if $1 = q_{ii}^2 q_{ij}^2$ and $1 = q_{11}q_{21}^2 \cdot q_{22}q_{12}^2$, which is again equivalent to $q_{ij}^2 = q_{ji}^2 = 1$. Possibly reversing 1,2, we may assume that we are in the case $q_{12} = 1, q_{21} = -1$, and then adding and subtracting the previous relations returns:

$$\mathrm{x}_{\bar{1}}\mathrm{x}_2 + \mathrm{x}_2\mathrm{x}_{\bar{1}} = 2\lambda_{12}\Big(1 - g_1g_2\tfrac{\kappa+1}{2}\Big), \qquad \mathrm{x}_1\mathrm{x}_{\bar{2}} - \mathrm{x}_{\bar{2}}\mathrm{x}_1 = 2\lambda_{12}\Big(1 - g_1g_2\tfrac{\kappa-1}{2}\Big).$$

On the other hand, the diagonal Nichols algebra $A_2 \times A_2$ has such liftings of

- $\mathrm{x}_{\bar{1}}\mathrm{x}_2 - q_{ji}\mathrm{x}_2\mathrm{x}_{\bar{1}}$ if $\chi_{\bar{1}}\chi_2 = \epsilon$, which is the case for $q_{12} = 1, q_{21} = -1$.
- $\mathrm{x}_1\mathrm{x}_{\bar{2}} + q_{ij}\mathrm{x}_{\bar{2}}\mathrm{x}_1$ if $\chi_1\chi_{\bar{2}} = \epsilon$, which is the case for $q_{12} = -1, q_{21} = 1$.

Altogether, there is no **u**-symmetric lifting of this type, and the solution we find starts with a lifting H' for one of these relation, again visible at the central character $\kappa \mapsto 1$, and the other of these relations appears in the biGalois object that is nontrivially extended from the group 2-cocycle.

We refrain from discussing the last two relations in a similar manner.

7 | FUTURE DIRECTIONS

We conclude by some outlook questions that naturally arise from our work.

Question 1. Is there a modified folding construction that produces the remaining Nichols algebras in Heckenberger–Vendramin classification?

Question 2. Several folded Nichols algebras in [43] do not appear in [37] because their support is too small. More precisely, these are the cases ${}^2D_n$ and ${}^3D_4$ and ${}^2A_1^2$ familiar from Lie theory, as well as unfamiliar cases ${}^2A_2$ at a third root of unity and several cases involving other diagonal Nichols algebras. We expect that our methods can be applied in these cases.

Question 3. Which modular tensor categories can be constructed from the new pointed Hopf algebras described here?

From the categorical perspective, there is a rather unique Σ-graded extensions of tensor categories with a Σ-crossed braiding [24]. Since the operations of Σ-graded extension and taking Hopf algebra representations commute, this extension could be computed by taking a Σ-symmetric Nichols algebra over an abelian group, which can then be folded to Nichols algebra over the known Σ-extension of the abelian group. To get a braiding, this would require a nontrivial associator (an effect familiar for quantum group of even order root of unity), and for $\Sigma = \mathbb{Z}_2$ conjecturally involve a Tambara–Yamigami category.

Question 4. The Logarithmic Kazhdan–Lusztig Correspondence, see, for example, [25, 44] conjectures the existence of a vertex algebra, realized as subalgebra of a free field algebra, whose tensor category of representations is equivalent to representations of a small quantum group. The folding construction and the previous problem suggests an extension of this construction, where the free field algebra is replaced by an orbifold model.

APPENDIX: PROOF OF THEOREM 3.18

Here, we complete the proof of Theorem 3.18, which states that Nichols algebras of Yetter–Drinfeld modules of types α_θ, γ_θ, δ_θ, ϵ_θ, and ϕ_4 become of diagonal type when an appropriate twist is performed. The cases that remain unsolved are α_2, α_3, γ_3, γ_4, δ_4, and ϕ_4, which will be dealt with in Proposition 3.15. The other cases are diagonal by Lemma 2.7.

A.1 | Group cohomology tools

We start by collecting some useful group extensions and group cohomology statements for later use, following [42, Chapter 7]. Recall the definition of $\mathrm{Z}_{u,v,\kappa}$ in (2.27).

Definition A.1. Let Z be an abelian group and $\mathtt{t}$ a generator of $\mathbb{Z}$. For each $w \in \mathrm{Z}$, $r \in \mathbb{N}$, we consider the abelian group

$$\mathrm{Z}(\sqrt[r]{w}) := \mathrm{Z} \times \mathbb{Z}/\langle (w, \mathtt{t}^{-r}) \rangle.$$

We shall identify $g \in \mathrm{Z}$, $\mathtt{t}^k$, $k \in \mathbb{N}$, with their images $\overline{(g,e)}$, $\overline{(e,\mathtt{t}^k)}$ in $\mathrm{Z}(\sqrt[r]{w})$. The defining relation becomes $w = \mathtt{t}^r$. We think of $\mathrm{Z}(\sqrt[r]{w})$ as the set $\{g\mathtt{t}^k \mid g \in \mathrm{Z}, 0 \leqslant k < r\}$ with product

$$g\mathtt{t}^j \cdot h\mathtt{t}^k = \begin{cases} gh\mathtt{t}^{j+k}, & \text{if } j+k < r, \\ ghw\mathtt{t}^{j+k-r}, & \text{if } j+k \geqslant r, \end{cases} \qquad g,h \in \mathrm{Z}, 0 \leqslant j,k < r.$$

Remark A.2. Fix Z an abelian group, $u, v, w, \kappa \in \mathrm{Z}$, where $\kappa^2 = e$, $r \in \mathbb{N}$.

(i) The inclusion $\mathrm{Z} \hookrightarrow \mathrm{Z}(\sqrt[r]{w})$ extends to an injective map

$$\mathrm{Z}_{u,v,\kappa} \hookrightarrow \mathrm{Z}(\sqrt[r]{w})_{u,v,\kappa}.$$

(ii) Fix also $z \in \mathbb{Z}$, $s \in \mathbb{N}$. There is a canonical isomorphism $\mathbb{Z}(\sqrt[r]{w})(\sqrt[s]{z}) \cong \mathbb{Z}(\sqrt[s]{z})(\sqrt[r]{w})$, which, in turn, induces an isomorphism

$$\mathbb{Z}(\sqrt[r]{w})(\sqrt[s]{z})_{u,v,\varkappa} \cong \mathbb{Z}(\sqrt[s]{z})(\sqrt[r]{w})_{u,v,\varkappa}.$$

Proposition A.3. *Let $\mathbb{Z}$ be an abelian group, $u, v, w, \varkappa \in \mathbb{Z}$, where $\varkappa^2 = e$, $r \in \mathbb{N}$. Given $\sigma \in H^2(\mathbb{Z}_{u,v,\varkappa}, \Bbbk^\times)$, consider $f := f_\sigma : \mathbb{Z} \to \Bbbk^\times$ given by*

$$f(g) = \sigma(g, w)/\sigma(w, g), \qquad g \in \mathbb{Z}.$$

Then σ lifts to a 2-cocycle $\widetilde{\sigma} \in H^2(\mathbb{Z}(\sqrt[r]{w})_{u,v,\varkappa}, \Bbbk^\times)$ if and only if there exists $\mathtt{f} \in \widehat{\mathbb{Z}_{u,v,\varkappa}}$ such that $\mathtt{f}^r = f$, $\mathtt{f}(w) = 1$. In this case,

$$\widetilde{\sigma}(g, \mathtt{t})\widetilde{\sigma}^{-1}(\mathtt{t}, g) = \mathtt{f}(g), \qquad \text{for all } g \in \mathbb{Z}. \tag{A.1}$$

If $w = 1$, then any $\sigma \in H^2(\mathbb{Z}_{u,v,\varkappa}, \Bbbk^\times)$ lifts to $\widetilde{\sigma} \in H^2(\mathbb{Z}(\sqrt[r]{w})_{u,v,\varkappa}, \Bbbk^\times)$.

Proof. Set $\underline{\mathbb{Z}} := \mathbb{Z}_{u,v,\varkappa}$, $G = \mathbb{Z}(\sqrt[r]{w})_{u,v,\varkappa}$. We can write G as the following central extension:

$$1 \to \langle w\mathtt{t}^{-r}\rangle \to \underline{\mathbb{Z}} \times \langle \mathtt{t}\rangle \to G \to 1.$$

As $\Bbbk^\times$ is divisible, the map $H^1(\langle \mathtt{t}\rangle, \Bbbk^\times) \to H^1(\langle w\mathtt{t}^{-r}\rangle, \Bbbk^\times)$ is surjective and $H^2(\langle \mathtt{t}\rangle, \Bbbk^\times) = H^2(\langle w\mathtt{t}^{-r}\rangle, \Bbbk^\times) = 1$. Hence, the exact sequence in [39, §1] associated to the central extension below is

$$\begin{aligned} 1 \to H^1(G, \Bbbk^\times) \to H^1(\underline{\mathbb{Z}} \times \langle \mathtt{t}\rangle, \Bbbk^\times) \to H^1(\langle w\mathtt{t}^{-r}\rangle, \Bbbk^\times) \to \\ \to H^2(G, \Bbbk^\times) \to H^2(\underline{\mathbb{Z}} \times \langle \mathtt{t}\rangle, \Bbbk^\times) \to \operatorname{Pair}(\underline{\mathbb{Z}} \times \langle \mathtt{t}\rangle, \langle w\mathtt{t}^{-r}\rangle). \end{aligned}$$

Using the results above, the Künneth formula for the cohomologies of the direct product, and decomposing the pairings, we get

$$1 \longrightarrow H^2(G, \Bbbk^\times) \longrightarrow H^2(\underline{\mathbb{Z}}, \Bbbk^\times) \times \operatorname{Pair}(\underline{\mathbb{Z}}, \langle \mathtt{t}\rangle) \xrightarrow{\Phi} \operatorname{Pair}(\underline{\mathbb{Z}}, \langle w\mathtt{t}^{-r}\rangle) \times \operatorname{Pair}(\langle \mathtt{t}\rangle, \langle w\mathtt{t}^{-r}\rangle),$$

where Φ is defined as follows:

- for $\sigma \in H^2(\underline{\mathbb{Z}}, \Bbbk^\times)$, we have $\Phi(\sigma) = (B_\sigma, 1)$, where $1 \in \operatorname{Pair}(\langle \mathtt{t}\rangle, \langle w\mathtt{t}^{-r}\rangle)$ is the trivial pairing, and $B_\sigma \in \operatorname{Pair}(\underline{\mathbb{Z}}, \langle w\mathtt{t}^{-r}\rangle)$ is given by

$$B_\sigma(g, w\mathtt{t}^{-r}) = \sigma(g, w)\sigma^{-1}(w, g), \qquad g \in \underline{\mathbb{Z}},$$

- for $P \in \operatorname{Pair}(\underline{\mathbb{Z}}, \langle \mathtt{t}\rangle)$, we have $\Phi(P) = (F'_P, F''_P)$, where

$$F'(g, w\mathtt{t}^{-r}) = P(g, \mathtt{t})^{-r},\ g \in \underline{\mathbb{Z}}; \qquad F''(\mathtt{t}, w\mathtt{t}^{-r}) = P(w, \mathtt{t}).$$

Hence, $(\sigma, P) \in \ker \Phi$ if and only if $\sigma(g, w)\sigma^{-1}(w, g) = P(g, \mathtt{t})^r$ for all $g \in \underline{\mathbb{Z}}$ and $P(w, \mathtt{t}) = 1$. If σ lifts to a 2-cocycle $\widetilde{\sigma} \in H^2(\mathbb{Z}(\sqrt[r]{w})_{u,v,\varkappa}, \Bbbk^\times)$, then set $\mathtt{f}$ as in (A.1). Reciprocally, if there exists

such $\mathtt{f}$, we define $P \in \mathrm{Pair}(\underline{\mathrm{Z}}, \langle \mathtt{t} \rangle)$, by $P(g, \mathtt{t}) = \mathtt{f}(g)$, and get $(\sigma, P) \in \ker \Phi$. By exactness of the sequence, $\ker \Phi$ is the image of the injective map $H^2(G, \Bbbk^\times) \longrightarrow H^2(\underline{\mathrm{Z}}, \Bbbk^\times) \times \mathrm{Pair}(\underline{\mathrm{Z}}, \langle \mathtt{t} \rangle)$; thus, σ lifts to a 2-cocycle $\widetilde{\sigma} \in H^2(\mathrm{Z}(\sqrt[r]{w})_{u,v,\kappa}, \Bbbk^\times)$. Moreover, P describes the values on the additional generator. The last statement is clear. □

Corollary A.4. *Let* Z *be an abelian group,* $u, v, w, z, \kappa \in \mathrm{Z}$*, where* $\kappa^2 = e$*,* $r, s \in \mathbb{N}$*. Then* $\sigma \in H^2(\mathrm{Z}_{u,v,\kappa}, \Bbbk^\times)$ *lifts to* $\widetilde{\sigma} \in H^2(\mathrm{Z}(\sqrt[r]{w})(\sqrt[s]{z})_{u,v,\kappa}, \Bbbk^\times)$ *if and only if there exists* $\mathtt{f}, \mathtt{g} \in \widehat{\mathrm{Z}_{u,v,\kappa}}$ *such that* $\mathtt{f}^r = f = \mathtt{g}^s$*,* $\mathtt{f}(w) = 1 = \mathtt{g}(w)$*.*

Proof. Use the isomorphism in Remark A.2 and Proposition A.3. □

Next we assume that Z splits as $\mathrm{Z} = \Lambda \oplus \Omega$, where $u \in \Lambda$, $v, \kappa \in \Omega$. Recall the extension $\mathrm{Z} \hookrightarrow \mathrm{Z}_{u,v,\kappa} \twoheadrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle x, y \rangle$ from (2.27). Consider

- $\underline{\Lambda}$ the subgroup of $\mathrm{Z}_{u,v,\kappa}$ generated by Λ and x,
- $\underline{\Omega}$ the subgroup of $\mathrm{Z}_{u,v,\kappa}$ generated by Ω and y.

Hence, $\mathrm{Z}_{u,v,\kappa} \simeq \underline{\Lambda} \ltimes \underline{\Omega}$, where x acts on $\underline{\Omega}$ by

$$x \cdot y = \kappa y, \qquad x \cdot h = h, \qquad h \in \Omega.$$

Let $\widetilde{H}^2(\mathrm{Z}_{u,v,\kappa}, \Bbbk^\times)$ denote the kernel of the restriction map

$$H^2(\mathrm{Z}_{u,v,\kappa}, \Bbbk^\times) \to H^2(\underline{\Lambda}, \Bbbk^\times).$$

By [52, Theorem 2 (I)], we have that

$$H^2(\mathrm{Z}_{u,v,\kappa}, \Bbbk^\times) \simeq H^2(\underline{\Lambda}, \Bbbk^\times) \oplus \widetilde{H}^2(\mathrm{Z}_{u,v,\kappa}, \Bbbk^\times).$$

By [52, Theorem 2 (II)], there exists an exact sequence

$$0 \longrightarrow H^1\left(\underline{\Lambda}, \widehat{\underline{\Omega}}\right) \longrightarrow \widetilde{H}^2(\mathrm{Z}_{u,v,\kappa}, \Bbbk^\times) \xrightarrow{res} H^2(\underline{\Omega}, \Bbbk^\times)^{\underline{\Lambda}}. \tag{A.2}$$

The image of the first map is the subspace of $H^2(\mathrm{Z}_{u,v,\kappa}, \Bbbk^\times)$ of 2-cocycles that are cohomologically trivial on $\underline{\Lambda}$ and $\underline{\Omega}$. Next, we will characterize $H^1(\underline{\Lambda}, \widehat{\underline{\Omega}})$ and describe the shape of 2-cocycles coming from this group.

Proposition A.5. *Let* $\mathcal{T}$ *denote the set of triples* $(P, \chi, \psi) \in \mathrm{Pair}(\Lambda, \Omega) \times \widehat{\Lambda} \times \widehat{\Omega}$ *such that*

$$\psi(v)\psi(\kappa) = \chi(u); \quad P(g, \kappa) = 1, \quad \chi(g)^2 = P(g, v), \quad \psi(h)^2 = P(u, h), \quad \text{for all } g \in \Lambda, h \in \Omega.$$

(a) *The map* $H^1(\underline{\Lambda}, \widehat{\underline{\Omega}}) \to \mathcal{T}$*,* $\phi \mapsto (P_\phi, \chi_\phi, \phi(x)_{|\Omega})$*, where*

$$P_\phi(g, h) = \phi(g)(h), \qquad \chi_\phi(g) = \phi(g)(y), \qquad g \in \Lambda, h \in \Omega, \tag{A.3}$$

is bijective.

(b) *The image* $\sigma \in \widetilde{H}^2(\mathbb{Z}_{u,v,\kappa}, \Bbbk^\times) \subseteq H^2(\mathbb{Z}_{u,v,\kappa}, \Bbbk^\times)$ *of a triple* $(P, \chi, \psi) \in \mathcal{T}$ *(viewed as an element of* $H^1(\underline{\Lambda}, \underline{\widehat{\Omega}})$*) under the map* ∂ *in (A.2) satisfies*

$$\frac{\sigma(g,h)}{\sigma(h,g)} = P(g,h), \qquad \frac{\sigma(g,y)}{\sigma(y,g)} = \chi(g), \qquad \frac{\sigma(x,h)}{\sigma(h,x)} = \psi(h),$$

for all $g \in \Lambda$, $h \in \Omega$. *In particular, we have that*

$$\frac{\sigma(g,\kappa)}{\sigma(\kappa,g)} = 1 \text{ for all } g \in \mathbb{Z}, \qquad \frac{\sigma(\kappa,y)}{\sigma(y,\kappa)} = \chi(\kappa), \qquad \frac{\sigma(x,\kappa)}{\sigma(\kappa,x)} = \psi(\kappa).$$

Proof.

(a) As Λ acts trivially on $\underline{\widehat{\Omega}}$, each crossed morphism $\phi \in H^1(\underline{\Lambda}, \underline{\widehat{\Omega}})$ restricts on Λ to a homomorphism, and hence to a pairing $\Lambda \times \underline{\Omega} \to \Bbbk^\times$, which we think as a pair $(P, \chi) \in \mathrm{Pair}(\Lambda, \Omega) \times \widehat{\Omega}$ as in (A.3) such that $\chi(g)^2 = P(g, v)$ for all $g \in \Lambda$ (because $y^2 = v$). We set $\chi_\phi := \phi(x)_{|\Omega} : \Omega \to \Bbbk$. As x acts trivially on Ω, χ is a group homomorphism. Hence, we have an injective map

$$H^1\left(\underline{\Lambda}, \underline{\widehat{\Omega}}\right) \to \{(P, \chi, \psi) \in \mathrm{Pair}(\Lambda, \Omega) \times \widehat{\Lambda} \times \widehat{\Omega} \mid \chi(g)^2 = P(g, v), \text{ for all } g \in \Lambda\}.$$

If $\xi := \phi(x)(y) \in \Bbbk$, then $\xi^2 = \phi(x)(v) = \chi(v)$ and

$$\phi(gx^i)(hy^j) = P(g,h)\psi(g)^j\chi(h)^i\xi^{ij}, \qquad g \in \Lambda, h \in \Omega, i, j \in \{0,1\}. \tag{A.4}$$

Reciprocally, given a triple (P, χ, ψ) as above, set $\phi : \underline{\Lambda} \to \underline{\widehat{\Omega}}$ as in A.4. Then ϕ is a crossed homomorphism if and only if for all $g \in \Lambda$, $h \in \Omega$,

$$\begin{aligned}
\phi(gx)(hy) = \phi(xg)(hy) = \phi(x)(hy)(x \cdot \phi(g)(hy)) &= P(g, \kappa h)\psi(g)\chi(h)\xi, \\
\phi(u)(h) = \phi(x^2)(h) = \phi(x)(h)(x \cdot \phi(x)(h)) &= \chi(h)^2, \\
\phi(u)(hy) = \phi(x^2)(hy) = \phi(x)(hy)(x \cdot \phi(x)(hy)) &= \chi(h)^2\xi^2\chi(\kappa).
\end{aligned}$$

This means $P(g, \kappa) = 1$ for all $g \in \Lambda$, $\chi(h)^2 = P(u, h)$ and $P(u, h)\psi(u) = P(u, h)\chi(v)\chi(\kappa)$ for all $h \in \Omega$.

(b) This follows by explicit computation of the coboundary map ∂, see, for example, the proof of [52, Theorem 2 (II)]. □

The next result will allow us to reduce the question about the existence of a 2-cocycle just for groups of order a power of 2.

Proposition A.6. *Let* $\mathbb{Z} = \mathbb{Z}_2 \times \mathbb{Z}_{odd}$, *where* $|\mathbb{Z}_2| = 2^n$ *for some* $n \in \mathbb{N}$, *and* $|\mathbb{Z}_{odd}|$ *is odd. Let* $u = u_2\mathfrak{u}$, $u = v_2\mathfrak{v}$, *with* $u_2, v_2 \in \mathbb{Z}_2$, $\mathfrak{u}, \mathfrak{v} \in \mathbb{Z}_{odd}$. *Then*

$$\mathbb{Z}_{u,v,\kappa} \simeq (\mathbb{Z}_2)_{u_2,v_2,\kappa} \times \mathbb{Z}_{odd}, \qquad H^2(\mathbb{Z}_{u,v,\kappa}, \Bbbk^\times) \simeq H^2((\mathbb{Z}_2)_{u_2,v_2}, \Bbbk^\times) \times H^2(\mathbb{Z}_{odd}, \Bbbk^\times).$$

Proof. Let $m, n \in \mathbb{N}_0$ be such that $|\mathtt{u}| = 2m+1$, $|\mathtt{v}| = 2n+1$. We write $\widetilde{x}, \widetilde{y}$ for the extra generators of $(\mathtt{Z}_2)_{u_2,v_2,\kappa}$ and keep x, y for those in $\mathtt{Z}_{u,v,\kappa}$. Then

$$\mathtt{Z}_{u,v,\kappa} \to (\mathtt{Z}_2)_{u_2,v_2,\kappa} \times \mathtt{Z}_{odd},$$
$$g_2 \mathtt{g} x^i y^j \mapsto g_2 \mathtt{u}^{i(m-1)} \mathtt{v}^{j(n-1)} \widetilde{x}^i \widetilde{y}^j \mathtt{g}, \qquad g_2 \in \mathtt{Z}_2, \mathtt{g} \in \mathtt{Z}_{odd}, i, j \in \{0,1\},$$

is a group isomorphism. The isomorphism between cohomology groups follows by Künneth's formula. □

A.2 | Nonabelian groups and 2-cocycles

Next, we discuss how to apply Propositions A.3 and A.5 to the main classes of groups $\mathtt{Z}_{u,v,\kappa}$ that will appear in the proof below in order to obtain the desired 2-cocycles.

A.2.1 | Quotient group $Z = \mathbb{Z}_2$

If $\mathtt{Z} = \mathbb{Z}_2 = \langle \mathtt{k} \rangle$, then $\mathtt{Z}_{1,1,\mathtt{k}}$, $\mathtt{Z}_{1,\mathtt{k},\mathtt{k}}$, $\mathtt{Z}_{\mathtt{k},1,\mathtt{k}}$ are isomorphic to the dihedral group of order 8.[†]

(a) We apply Proposition A.5 to $\mathtt{Z}_{1,\mathtt{k},\mathtt{k}}$ with $\Lambda = 1$, $\Omega = \mathtt{Z}$, $P = 1$, $\chi = 1$, $\psi(\mathtt{k}) = -1$; then $H^2(\mathtt{Z}_{1,\mathtt{k},\mathtt{k}}, \Bbbk^\times) \simeq \mathbb{Z}_2 = \langle \sigma \rangle$, where

$$\sigma(x, \mathtt{k})\sigma^{-1}(\mathtt{k}, x) = -1, \qquad \sigma(y, \mathtt{k})\sigma^{-1}(\mathtt{k}, y) = 1.$$

(b) We apply Proposition A.3 to $\mathtt{Z}_{1,\mathtt{k},\mathtt{k}}$, where $f(x) = -1$, $f(y) = 1$. If either $w = 1$ or $2 \nmid r$, then σ can be lifted to $\mathtt{Z}(\sqrt[r]{w})$; but when $w = \mathtt{k}$, $2 \mid r$, the lift does not exist. As a smallest example, $\mathtt{Z}(\sqrt[2]{\mathtt{k}})$ is the almost extraspecial group 2^{3+1}, which has cohomology $\mathbb{Z}_2^2$: all 2-cocycles are lifts of the trivial 2-cocycle on $\mathtt{Z}_{1,\mathtt{k},\mathtt{k}}$ with $f = 1$ and $\mathtt{f} = \pm 1$.

(c) More generally, if there exists a surjective map $\pi : \mathtt{Z} \to \mathbb{Z}_2$ such that $\pi(u) = 1$, $\pi(v) = \mathtt{k} = \pi(\kappa)$, then the pullback of σ from $(\mathbb{Z}_2)_{1,\mathtt{k},\mathtt{k}}$ to $\mathtt{Z}_{u,v,\kappa}$ can be lifted to $\mathtt{Z}(\sqrt[r]{w})$ either when $\pi(w) = 1$ or $2 \nmid r$. For $\pi(w) = \kappa$, $2 \mid r$, it can also be lifted if there is a character $\mathtt{f} : \mathtt{Z} \to \Bbbk^\times$ such that

$$\mathtt{f}(u) \in \mathbb{G}_r, \qquad 2 \nmid r/\mathrm{ord}(\mathtt{f}(u)), \qquad \mathtt{f}(v) \in \mathbb{G}'_{r/2}, \qquad \mathtt{f}(\kappa) = 1.$$

A.2.2 | Quotient group $Z = \mathbb{Z}_2 \times \mathbb{Z}_2$

Let $\mathtt{Z} = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \mathtt{u} \rangle \times \langle \mathtt{k} \rangle$. Reordering generators, the nontrivial possibilities for $\mathtt{Z}_{u,v,\kappa}$ are $\mathtt{Z}_{\mathtt{u},1,\mathtt{k}}$ and $\mathtt{Z}_{\mathtt{u},\mathtt{k},\mathtt{k}}$, which are groups of order 16 with Gap Id 3,4 and Hall-Senior number $\#_{16}9$, $\#_{16}10$, see [42, Chapter 7].

(a) We apply Proposition A.5 to $\mathtt{Z}_{\mathtt{u},1,\mathtt{k}}$ with $\Lambda = \langle \mathtt{u} \rangle$, $\Omega = \langle \mathtt{k} \rangle$, $P \equiv 1$, $\psi(\mathtt{u}) = \chi(\mathtt{k}) = -1$: we get a 2-cocycle σ such that

$$\frac{\sigma(x, \mathtt{k})}{\sigma(\mathtt{k}, x)} = -1, \qquad \frac{\sigma(x, \mathtt{u})}{\sigma(\mathtt{u}, x)} = 1, \qquad \frac{\sigma(y, \mathtt{k})}{\sigma(\mathtt{k}, y)} = 1, \qquad \frac{\sigma(y, \mathtt{u})}{\sigma(\mathtt{u}, y)} = -1;$$

for the nontrivial choice $P(\mathtt{u}, \mathtt{k}) = -1$ there is no suitable ψ.

[†] On the other hand, $\mathtt{Z}_{\mathtt{k},\mathtt{k},\mathtt{k}}$ is the quaternion group, which has trivial cohomology.

For the decomposition $\Lambda = 1$, $\Omega = \mathtt{Z}$, we have the nontrivial choice $P = 1$, $\psi = 1$, $\chi(\mathtt{u}) = \chi(\mathtt{k}) = -1$. We obtain a 2-cocycle σ' such that

$$\frac{\sigma'(x,\mathtt{k})}{\sigma'(\mathtt{k},x)} = -1, \qquad \frac{\sigma'(x,\mathtt{u})}{\sigma'(\mathtt{u},x)} = -1, \qquad \frac{\sigma'(y,\mathtt{k})}{\sigma'(\mathtt{k},y)} = 1, \qquad \frac{\sigma'(y,\mathtt{u})}{\sigma'(\mathtt{u},y)} = 1.$$

Accordingly, it is known that $H^2(\#_{16}9, \Bbbk^\times) = \mathbb{Z}_2 \times \mathbb{Z}_2$.

(b) We apply Proposition A.3 to $\mathtt{Z}_{\mathtt{u},1,\mathtt{k}}$ and the 2-cocycle σ constructed above, where $\mathtt{w} = \mathtt{k}^s\mathtt{u}^t$, $f(x) = (-1)^s$, $f(y) = (-1)^t$, $s, t \in \{0, 1\}$.

(c) Assume that there exists a surjective map $\pi : \mathtt{Z} \to \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $\pi(u) = \mathtt{u}$, $\pi(v) = 1$, $\pi(\kappa) = \mathtt{k}$. The pullback of σ from $(\mathbb{Z}_2 \times \mathbb{Z}_2)_{\mathtt{u},1,\mathtt{k}}$ to $\mathtt{Z}_{u,v,\kappa}$ can be lifted to $\mathtt{Z}(\sqrt[r]{w})$ if either $\pi(w) = 1$ or $2 \nmid r$. If $\pi(w) = \kappa$, $2 \mid r$, then the 2-cocycle can be lifted to $\mathtt{Z}_{u,v,\kappa}$ if there exists $\mathtt{f} \in \widehat{\mathtt{Z}}$ such that $\mathtt{f}(\kappa) = 1$ and

$$\mathtt{f}(u)\,(\text{resp. } \mathtt{f}(v)) = \xi \in \begin{cases} \mathbb{G}_r,\ 2 \nmid r/\mathrm{ord}(\xi), & \text{if } s = 1\,(\text{resp. } t = 1), \\ \mathbb{G}'_{r/2}, & \text{if } s = 0\,(\text{resp. } t = 0). \end{cases}$$

A.2.3 | Quotient group $Z = \mathbb{Z}_k \times \mathbb{Z}_k$ with 4

Fix $t \in \mathbb{N}$. Set $k = 4t$, $\mathtt{Z} = \mathbb{Z}_k \times \mathbb{Z}_k$ with generators $\mathtt{u}, \mathtt{v}$, and consider $\mathtt{k} := \mathtt{v}^{2t}$. The group $\mathtt{Z}_{\mathtt{u},\mathtt{v},\mathtt{k}}$ has order $4k^2$, center $\mathtt{Z}$, and is presented by generators x, y and relations $x^{2k} = y^{2k} = 1$, $[x, y] = y^k$, where $\mathtt{u} = x^2$, $\mathtt{v} = y^2$, $\mathtt{k} = y^k$.

Moreover, $\langle y\rangle \simeq \mathbb{Z}_{2k}$ is a normal subgroup, $\langle x\rangle \simeq \mathbb{Z}_{2k}$ and $\mathtt{Z}_{\mathtt{u},\mathtt{v},\mathtt{k}} \simeq \mathbb{Z}_{2k} \rtimes \mathbb{Z}_{2k}$ with action $x \cdot y = y^{k+1}$ (notice that $x^2 \cdot y = y$).

(a) We apply Proposition A.5 to $\mathtt{Z}_{\mathtt{u},\mathtt{v},\mathtt{k}}$ with $\Lambda = \langle \mathtt{u}\rangle$, $\Omega = \langle \mathtt{v}\rangle$. Fix $\xi \in \mathbb{G}_k$.

- The pairings $P : \Lambda \times \Omega$ such that $P(\mathtt{u}, \mathtt{k}) = 1$ are given by

$$P(\mathtt{u}, \mathtt{v}) = \xi^{2i} \qquad \text{for some } i \in \mathbb{I}_{2t}.$$

- $\chi \in \widehat{\Lambda}$ satisfies $\chi(\mathtt{u})^2 = P(\mathtt{u}, \mathtt{v})$ if and only if $\chi(\mathtt{u}) = p_1\xi^i$ for some $p_1 \in \{\pm 1\}$. Analogously, $\psi \in \widehat{\Omega}$ is given by $\psi(\mathtt{v}) = p_2\xi^i$, $p_2 \in \{\pm 1\}$.
- As $\chi(\mathtt{k}) = (-1)^i$, the condition $\psi(\mathtt{v})\psi(\mathtt{k}) = \chi(\mathtt{u})$ always holds when i is even, and for i odd, there are two choices since we need $p_1p_2 = -1$.

Altogether, when i is even, we obtain $4t$ different 2-cocycles $\sigma_{ip_1p_2}$ with $\chi(\mathtt{k}) = 1$, and for i odd, we have $2t$ different 2-cocycles $\sigma_{ip_1p_2}$ with $\chi(\kappa) = -1$. Set $\sigma := \sigma_{1+-}$.

(b) We apply Proposition A.3 to $\mathtt{Z}_{\mathtt{u},\mathtt{v},\mathtt{k}}$, σ as constructed above and $\mathtt{w} = \mathtt{u}^s\mathtt{v}^t$: here, $f(x) = \psi(\mathtt{v}^t) = (-\xi)^t$ and $f(y) = \chi(\mathtt{u}^s) = \xi^s$.

(c) More generally, fix a surjective map $\pi : \mathtt{Z} \to \mathbb{Z}_k \times \mathbb{Z}_k$, $u, v, \kappa \in \mathtt{Z}$ such that $\pi(u) = \mathtt{u}$, $\pi(v) = \mathtt{v}$, $\pi(\kappa) = \mathtt{k}$, $w = u^sv^t$, $s, t \in \mathbb{I}_k$, and $r \in \mathbb{N}$. The pullback of σ from $(\mathbb{Z}_k \times \mathbb{Z}_k)_{\mathtt{u},\mathtt{v},\mathtt{k}}$ to $\mathtt{Z}_{u,v,\kappa}$ can be lifted to $\mathtt{Z}(\sqrt[r]{w})$ if there exists $\mathtt{f} \in \widehat{\mathtt{Z}}$ such that $\mathtt{f}(x)^r = (-\xi)^t$, $\mathtt{f}(y)^r = \xi^s$ and $\mathtt{f}(\kappa) = 1$.

A.3 | Proof of Proposition 3.15

We proceed case-by-case. We use the representations $M(g, \chi)$ coming from the Yetter–Drinfeld structure for each group in order to get a triple as in Proposition A.5, which, in turn, gives a 2-cocycle, then use Proposition A.3 when we need to extend the group accordingly. For each $M(g, \chi)$,

we choose a basis given by centralizer coset representatives and give the corresponding matrices: When the matrix is a multiple of the identity, we just write the corresponding scalar.

A.3.1 | Type α_2

Here $M = M(g_1, \chi_1) \oplus M(g_2, \chi_2)$, see §5.1.1. The four possibilities for the parity vector $\mathtt{P} = (\chi_1(\kappa), \chi_2(\kappa))$ fall into two orbits under the Weyl groupoid action, namely, (1,1) and $\{(-1,1),(-1,-1),(-1,1)\}$. As the first one corresponds to trivial action of κ, we just need to study $(-1,1)$. Set $\mathbf{a} := \chi_2(g_1^2) = -\chi_1(g_2^2)^{-1}$, $k = \operatorname{ord} \mathbf{a}$. We compute $M(g_1, \chi_1)$, $M(g_2, \chi_2)$, $M(g_1 g_2, \chi_1 \chi_2)$.

	κ	g_1	g_2	$g_1^2 = u$	$g_2^2 = v$
$M(g_1, \chi_1)$	-1	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\mathbf{a}^{-1} \\ 1 & 0 \end{pmatrix}$	1	$-\mathbf{a}^{-1}$
$M(g_2, \chi_2)$	1	$\begin{pmatrix} 0 & \mathbf{a} \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\mathbf{a}$	1
$M(g_1 g_2, \chi_1 \chi_2)$	-1	$\begin{pmatrix} 0 & \mathbf{a} \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -\mathbf{a}^{-1} & 0 \end{pmatrix}$	$\mathbf{a}$	$-\mathbf{a}^{-1}$

From the actions above, we can read off the group $G^{\min}$ explicitly:

$$G^{\min} = \Big\langle g_1, g_2, \kappa \mid [g_1, g_2] = \kappa,\ [g_1, \kappa] = [g_2, \kappa] = (g_1^2\kappa)^k = (g_2^2)^k = \kappa^2 = 1,$$
$$\text{if } 2 \mid k \text{ we add } (g_1^2\kappa)^{k/2} = \kappa \Big\rangle.$$

By Proposition A.6, we reduce to 2-groups, so we have three cases:

- $k = 1$, that is $\mathbf{a} = 1$. Then $x^2 = 1, y^2 = \kappa$ and the group $G^{\min}$ is the dihedral group of order 8, see §A.2.1 (a).
- $k = 2$, that is, $\mathbf{a} = -1$. Here $x^2 = u$ has order 2, $y^2 = v = 1$, $(xy)^2 = \kappa u$. Then $G^{\min}$ is the group $\#_{16}9$ and such 2-cocycle exists, see §A.2.2 (a).
- $k = 2^n$, $n \geqslant 2$. In this case, $x^{2^n} = y^{2^n} = 1$ and $y^{2^{n-1}} = \kappa$. Then $|G^{\min}| = 2^{2+2n}$, the center is $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$, and moreover, $G^{\min} \simeq \mathbb{Z}_{2^{n+1}} \ltimes \mathbb{Z}_{2^{n+1}}$ as in A.2.3 (a), so there exists such 2-cocycle.

A.3.2 | Type α_3

Here $M = M(g_1, \chi_1) \oplus M(g_2, \chi_2) \oplus M(g_3, \chi_3)$, see §5.1.2. As $\chi_1(\kappa) = \chi_3(\kappa)$, there are four choices of $\mathtt{P} = (\chi_i(\kappa))_{i \in \mathbb{I}_3}$, which fall into two orbits under the Weyl groupoid action, namely,

$$\{(1,1,1)\} \qquad \text{and} \qquad \{(-1,1,-1),(-1,-1,-1),(1,-1,1)\}.$$

Thus, we just need to study $\mathtt{P} = (-1,1,-1)$. We compute the representations $M(g_i, \chi_i)$, which contains the previous case A_2. Set

$$\mathbf{a} := \chi_2(g_1^2) = -\chi_1(g_2^2)^{-1}, \qquad \mathbf{b} := \chi_3(g_1) = \chi_1(g_3)^{-1}, \qquad \mathbf{c} = \chi_2(g_1 g_3^{-1}).$$

Hence, the action of G on M is given by:

	κ	g_1	g_2	g_3
$M(g_1, \chi_1)$	-1	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\mathbf{a}^{-1} \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \mathbf{b}^{-1} & 0 \\ 0 & -\mathbf{b}^{-1} \end{pmatrix}$
$M(g_2, \chi_2)$	1	$\begin{pmatrix} 0 & \mathbf{a} \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & \mathbf{ca} \\ \mathbf{c} & 0 \end{pmatrix}$
$M(g_3, \chi_3)$	-1	$\begin{pmatrix} \mathbf{b} & 0 \\ 0 & -\mathbf{b} \end{pmatrix}$	$\begin{pmatrix} 0 & -(\mathbf{c}^2\mathbf{a})^{-1} \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Using that $g_1g_2 = \kappa g_2 g_1$, $g_3g_2 = \kappa g_2 g_3$, $g_1g_3 = g_3g_1$, κ is central and $\kappa^2 = 1$, cf. §5.1.2, the subgroup $\widetilde{\mathsf{Z}}$ generated by [†]

$$u := g_1^2, \qquad \upsilon := g_2^2\kappa, \qquad \mathtt{t} := g_3 g_1^{-1}\kappa, \qquad \kappa,$$

is contained in $Z(G^{\min})$, and $G^{\min}/\widetilde{\mathsf{Z}}$ has four elements: $\widetilde{\mathsf{Z}}$, $g_1\widetilde{\mathsf{Z}}$, $g_2\widetilde{\mathsf{Z}}$, $g_1g_2\widetilde{\mathsf{Z}}$. From here, we check that $G^{\min} \simeq \widetilde{\mathsf{Z}}_{u,\upsilon,\kappa}$, where $x \mapsto g_1$, $y \mapsto g_2$.

Let Z be the subgroup generated by u, υ, κ. Set $N_1 = \operatorname{lcm}(\operatorname{ord}\mathbf{a}, \operatorname{ord}\mathbf{b}^2)$, $N_2 = \operatorname{lcm}(\operatorname{ord}\mathbf{a}, \operatorname{ord}\mathbf{c}^2)$, $N_3 = \operatorname{lcm}(\operatorname{ord}\mathbf{b}, \operatorname{ord}\mathbf{c})$. As $\mathtt{t}^{N_3} = \mathrm{id}$, we can define

$$\begin{aligned} r &:= \min\{s \in \mathbb{I}_{N_3} | \mathtt{t}^s \in \mathsf{Z}\} \\ &= \min\{s \in \mathbb{I}_{N_3} | \exists m \in \mathbb{I}_{N_1}, n \in \mathbb{I}_{N_2} : \mathbf{c}^{2n} = \mathbf{b}^{2m+2r}, \mathbf{a}^n\mathbf{b}^r = 1, \mathbf{a}^m = \mathbf{c}^r\}. \end{aligned}$$

We have that $\widetilde{\mathsf{Z}} \simeq \mathsf{Z}(\sqrt[r]{w})$ for $w = \mathtt{t}^r$. The action of $u, \upsilon, \mathtt{t}$ on M_1, M_2, M_3 is given, respectively, by the following scalars:

$$(1, \mathbf{a}, \mathbf{b}^{-2}), \qquad (\mathbf{a}^{-1}, 1, \mathbf{a}^{-1}\mathbf{c}^{-2}), \qquad (\mathbf{b}^{-1}, \mathbf{c}, \mathbf{b}^{-1}).$$

We also set $k_1 = \operatorname{ord}\mathbf{a}$, $k_2 = \operatorname{ord}\mathbf{b}$, $k_3 = \operatorname{ord}\mathbf{c}$. Then,

$$\mathsf{Z} \simeq \left\langle \kappa, u, \upsilon \mid \kappa^2 = 1, u^{N_1} = 1, \upsilon^{N_2} = 1, \kappa = (u^i\upsilon)^{k_1/2} \text{ if } 2 \mid k_1, \mathbf{b}^{2i} = \mathbf{c}^{-k_1} \right\rangle.$$

As $k_1 | N_1, N_2$, there exists a surjective map $\mathsf{Z} \twoheadrightarrow \mathsf{Z}'$, where Z' is an abelian group as in §A.3.1, so there exists a 2-cocycle σ' for Z' such that $\frac{\sigma'(g_2,\kappa)}{\sigma'(\kappa,g_2)} = 1$, $\frac{\sigma'(g_1,\kappa)}{\sigma'(\kappa,g_1)} = -1 = \frac{\sigma'(g_3,\kappa)}{\sigma'(\kappa,g_3)}$. Let σ be the pullback of σ' on Z: we look for a lift on $G^{\min} \simeq \mathsf{Z}(\sqrt[r]{w})_{u,\upsilon,\kappa}$, so we look for a character $\mathtt{f}$ as in Proposition A.3.

By Proposition A.6, it is enough to solve the case in which the three k_i are powers of 2. We split in three cases as in §A.3.1.

- $k = 1$, that is, $\mathbf{a} = 1$. Either $w = 1$ or else $k_2 > k_3$, $w = \kappa$ and $r = k_2/2 > 1$. In the second case, we construct $\mathtt{f} \in \widehat{\mathsf{Z}}$ in $M(g_1, \chi_1)^* \otimes M(g_3, \chi_1)$: that is, $\mathtt{f}(u) = \mathbf{b}^2 \in \mathbb{G}'_r$, $\mathtt{f}(\upsilon) = \mathbf{c}^{-2} \in \mathbb{G}'_{r/2}$, $\mathtt{f}(\kappa) = 1$. Then we apply §A.2.1 (c).
- $k = 2$, that is, $\mathbf{a} = -1$. If $k_2 > k_3$, then $r = k_2/2$, $w = \kappa$; if $k_2 < k_3$, then $r = k_3/2$, $w = u$; otherwise, $k_2 = k_3$, $r = k_2/2$ and $w = u\kappa$. In any case, we construct $\mathtt{f} \in \widehat{\mathsf{Z}}$ in $M(g_1, \chi_1)^* \otimes M(g_3, \chi_1)$ as in §A.2.2 (c), and there exists such a lift.

[†] We set the generators of $\widetilde{\mathsf{Z}}$ according to generators for the symplectic root system $n = 3$, $r = 1$ in [43, Thm. 4.5].

- $k = 2^n$, $n \geqslant 2$. Here $r = \max(k_2/k_1,\ k_3/k_1,\ 1)$, and we construct $\mathtt{f} \in \widehat{\mathtt{Z}}$, again in $M(g_1, \chi_1)^* \otimes M(g_3, \chi_1)$:

$$\mathtt{f}(\kappa) = 1, \qquad \mathtt{f}(u) = \mathbf{b}^2, \qquad \mathtt{f}(v) = \mathbf{c}^2.$$

This $\mathtt{f}$ satisfies the conditions in §A.2.3 (c), and there exists such a lift.

A.3.3 | Type δ_4

As $\chi_1(\kappa) = \chi_3(\kappa) = \chi_4(\kappa)$, the choices of $\mathtt{P} = (\chi_i(\kappa))_{i\in\mathbb{I}_4}$ fall into two orbits under the Weyl groupoid action, namely,

$$\{(1,1,1,1)\} \qquad \text{and} \qquad \{(-1,1,-1,-1),(-1,-1,-1,-1),(1,-1,1,1)\},$$

where the second entry denotes the center node in the Dynkin diagram. Now we study $\mathtt{P} = (-1,1,-1,-1)$. We fix the central elements[†] $z = g_3 g_1^{-1}, z' = g_4 g_1^{-1}$. Then this case can be achieved by combining the previous result of extending α_2 to α_3 by z and by z', see Corollary A.4.

A.3.4 | Type γ_3

Here $M = M(g_1, \chi_1) \oplus M(g_2, \chi_2) \oplus M(g_3, \chi_3)$, with $g_3 \in Z(G)$, $\chi_3 \in \widehat{G}$. As $\kappa = [g_1, g_2]$, we have that $\chi_3(\kappa) = 1$. The possible $\mathtt{P} = (\chi_i(\kappa))_{i\in\mathbb{I}_3}$ fall into two orbits under the Weyl groupoid action, namely, $\{(1,1,1)\}$ and $\{(-1,1,1),(-1,-1,1),(1,-1,1)\}$. Fix $\mathtt{P} = (-1,1,1)$ and set

$$\mathbf{a} := \chi_2(g_1^2) = -\chi_1(g_2^{-2}),\ \mathbf{b} := \chi_3(g_1) = \chi_1(g_3^{-1}),\ \mathbf{c} := \chi_3(g_2) = -\chi_2(g_3^{-1}).$$

Now we compute the representations $M(g_i, \chi_i)$:

	κ	g_1	g_2	g_3
$M(g_1,\chi_1)$	-1	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\mathbf{a}^{-1} \\ 1 & 0 \end{pmatrix}$	$\mathbf{b}^{-1}$
$M(g_2,\chi_2)$	$+1$	$\begin{pmatrix} 0 & \mathbf{a} \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$-\mathbf{c}^{-1}$
$M(g_3,\chi_3)$	$+1$	$\mathbf{b}$	$\mathbf{c}$	-1

If $u = g_1^2$, $v = g_2^2$, $\mathtt{t} = g_3$, then $G^{\min} \simeq \mathtt{Z}(\sqrt[r]{w})_{u,v,\kappa}$ for $\mathtt{Z} = \langle u, v, \kappa\rangle$ and appropriate $w \in \mathtt{Z}$, $r \in \mathbb{N}$. Moreover, $G^{\min}$ is isomorphic to the one in §A.3.2; hence, there exists a 2-cocycle as in Proposition 3.15.

A.3.5 | Type γ_4

Here we have central elements $z = g_3 g_1^{-1}$, $z' = g_4$, and this case is solved by combining the previous result of extending A_2 to A_3 by z and by z', see Corollary A.4.

[†] According to generators for the symplectic root system, $n = 4$, $r = 2$ in [43].

A.3.6 | Type ϕ_4

Here $M = M(g_1, \chi_1) \oplus M(g_2, \chi_2) \oplus M(g_3, \chi_3) \oplus M(g_4, \chi_4)$, with $g_3, g_4 \in Z(G)$, $\chi_3, \chi_4 \in \widehat{G}$, and $\chi_3(\kappa) = \chi_4(\kappa) = 1$. The possible $\mathtt{P} = (\chi_i(\kappa))_{i \in \mathbb{I}_4}$ fall into two orbits under the Weyl groupoid action: $\{(1,1,1,1)\}$ and $\{(-1,1,1,1), (-1,-1,1,1), (1,-1,1,1)\}$. Fix $\mathtt{P} = (-1,1,1,1)$ and set

$$\mathbf{a} := \chi_2(g_1^2) = -\chi_1(g_2^{-2}), \qquad \mathbf{b} := \chi_3(g_1) = \chi_1(g_3^{-1}), \qquad \mathbf{c} := \chi_3(g_2) = -\chi_2(g_3^{-1}),$$
$$\mathbf{b}' := \chi_4(g_1) = \chi_1(g_4^{-1}), \qquad \mathbf{c}' := \chi_4(g_2) = \chi_2(g_4^{-1}), \qquad \mathbf{d} := \chi_4(g_3) = -\chi_3(g_2^{-1}).$$

Now we compute the representations $M_i := M(g_i, \chi_i)$:

	κ	g_1	g_2	g_3	g_4
$M(g_1, \chi_1)$	-1	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\mathbf{a}^{-1} \\ 1 & 0 \end{pmatrix}$	$\mathbf{b}^{-1}$	$\mathbf{b}'^{-1}$
$M(g_2, \chi_2)$	1	$\begin{pmatrix} 0 & \mathbf{a} \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$-\mathbf{c}^{-1}$	$\mathbf{c}'^{-1}$
$M(g_3, \chi_3)$	1	$\mathbf{b}$	$\mathbf{c}$	-1	$-\mathbf{d}^{-1}$
$M(g_4, \chi_4)$	1	$\mathbf{b}'$	$\mathbf{c}'$	$\mathbf{d}$	-1

Set $u = g_1^2$, $v = g_2^2$. We will construct a 2-cocycle σ on $G^{\min}$ by using appropriate 2-cocycles from the previous cases:

- Set $G_{12} = \langle g_1, g_2 \rangle$, $\mathtt{Z} = \langle \kappa, u, v \rangle$; let $\widetilde{G}_{12}, \mathtt{Z}_{12} \subset \mathrm{End}(M_1 \oplus M_2)$ be the subgroups obtained by restriction. Then $\mathtt{Z}$, $\mathtt{Z}_{12}$ are central subgroups, $G_{12} = \mathtt{Z}_{u,v,\kappa}$, $\widetilde{G}_{12} = (\mathtt{Z}_{12})_{u,v,\kappa}$, with canonical projections $G_{12} \twoheadrightarrow \widetilde{G}_{12}$, $\mathtt{Z} \twoheadrightarrow \mathtt{Z}_{12}$, and $\widetilde{G}_{12}$, $\mathtt{Z}_{12}$ are as in §A.3.1. Hence, there exists a 2-cocycle as we need: the pullback σ_{12} on G_{12} satisfies $\frac{\sigma_{12}(g_1,\kappa)}{\sigma_{12}(\kappa,g_1)} = -1$, $\frac{\sigma_{12}(g_2,\kappa)}{\sigma_{12}(\kappa,g_2)} = 1$.
- For $j = 3, 4$, we set $G_{12j} = \langle g_1, g_2, g_j \rangle$, $\mathtt{Z}_{12j} = \langle \kappa, u, v, g_j \rangle$. Then $\mathtt{Z}_{12j}$ is a central subgroup of the form $\mathtt{Z}_{12j} \simeq \mathtt{Z}(\sqrt[r_j]{w_j})$ for appropriate $r_j \in \mathbb{N}$, $w_j \in \mathtt{Z}$: the proof for $j = 3$ is the same as in §A.3.4 since $M_1 \oplus M_2 \oplus M_3$ is of type C_3, and for $j = 4$, we have the same structure (luckily). Using the same argument as in §A.3.4, we check the existence of 2-cocycle σ_{12j} on G_{12j} such that $\frac{\sigma_{12j}(g_1,\kappa)}{\sigma_{12j}(\kappa,g_1)} = -1$, $\frac{\sigma_{12j}(g_2,\kappa)}{\sigma_{12j}(\kappa,g_2)} = \frac{\sigma_{12j}(g_j,\kappa)}{\sigma_{12j}(\kappa,g_j)} = 1$.
- Finally, $G^{\min} \simeq \mathtt{Z}(\sqrt[r_3]{w_3})(\sqrt[r_4]{w_4})_{u,v,\kappa} \simeq \mathtt{Z}(\sqrt[r_4]{w_4})\mathtt{Z}(\sqrt[r_3]{w_3})_{u,v,\kappa}$. The existence of a 2-cocycle σ on $G^{\min}$ such that

$$\frac{\sigma(g_1,\kappa)}{\sigma(\kappa,g_1)} = -1, \qquad \frac{\sigma(g_2,\kappa)}{\sigma(\kappa,g_2)} = \frac{\sigma(g_3,\kappa)}{\sigma(\kappa,g_3)} = \frac{\sigma(g_4,\kappa)}{\sigma(\kappa,g_4)} = 1$$

follows from the 2-cocycles σ_{12j} on G_{12j}, $j = 3, 4$ and Corollary A.4.

This concludes the proof of Proposition 3.15. □

ACKNOWLEDGMENTS

The authors thank Istvan Heckenberger for many useful discussions, suggestions, and hospitality. In particular, for the discussion of Lemma 2.7. The work of I. A. was partially supported by CONICET and Secyt (UNC). The authors were partially supported by the Alexander von Humboldt Foundation through the Research Group Linkage Programme.

Open access funding enabled and organized by Projekt DEAL.

JOURNAL INFORMATION

The *Proceedings of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

ORCID

Simon Lentner https://orcid.org/0000-0003-2847-1114

REFERENCES

1. H. H. Andersen, J. C. Jantzen, and W. Soergel, *Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p*, Astérisque **220** (1994), 321.
2. N. Andruskiewitsch, *An introduction to Nichols algebras*, Quantization, geometry and noncommutative structures in mathematics and physics, A. Cardona, P. Morales, H. Ocampo, S. Paycha, A. Reyes, eds., Springer Nature, Switzerland, AG, 2017, pp. 135–195.
3. N. Andruskiewitsch and I. Angiono, *On finite-dimensional Nichols algebras of diagonal type*, Bull. Math. Sci. **7** (2017), 353–573.
4. N. Andruskiewitsch, I. Angiono, and A. G. Iglesias, *Liftings of Nichols algebras of diagonal type I. Cartan type A*, Int. Math. Res. Not. **2017** (2017), no. 9, 2793–2884.
5. N. Andruskiewitsch, I. Angiono, A. G. Iglesias, A. Masuoka, and C. Vay, *Lifting via cocycle deformation*, J. Pure Appl. Alg. **218** (2014), 684–703.
6. N. Andruskiewitsch, G. Carnovale, and G. García, *Finite-dimensional pointed Hopf algebras over finite simple groups of Lie type V. Mixed classes in Chevalley and Steinberg groups*, Manuscripta Math. **166** (2021), 605–647.
7. N. Andruskiewitsch and M. Graña, *From racks to pointed Hopf algebras*, Adv. Math. **178** (2003), 177–243.
8. N. Andruskiewitsch and H.-J. Schneider, *Finite quantum groups and Cartan matrices*, Adv. Math. **154** (2000), 1–45.
9. N. Andruskiewitsch and H.-J. Schneider, *Pointed Hopf algebras*, Recent developments in Hopf algebras theory, MSRI Publ., vol. 43, Cambridge University Press, Cambridge, 2002, pp. 1–68.
10. N. Andruskiewitsch and H.-J. Schneider, *On the classification of finite-dimensional pointed Hopf algebras*, Ann. of Math. **171** (2010), 375–417.
11. I. Angiono, *On Nichols algebras with standard braiding*, Algebra Number Theory **1** (2009), 35–106.
12. I. Angiono, *On Nichols algebras of diagonal type*, J. Reine Angew. Math. **683** (2013), 189–251.
13. I. Angiono, *A presentation by generators and relations of Nichols algebras of diagonal type and convex orders on root systems*, J. Eur. Math. Soc. **17** (2015), 2643–2671.
14. I. Angiono, *Distinguished pre-Nichols algebras*, Transform. Groups **21** (2016), 1–33.
15. I. Angiono, E. Campagnolo, and G. Sanmarco, *Finite GK-dimensional pre-Nichols algebras of super and standard type*, `arXiv:2009.04863`.
16. I. Angiono and A. G. Iglesias, *Liftings of Nichols algebras of diagonal type II. All liftings are cocycle deformations*, Selecta Math. **25** (2019), Paper No. 5, 95 pp.
17. I. Angiono, C. Galindo, and M. Pereira, *De-equivariantization of Hopf algebras*, Algebr. Represent. Theory **17** (2014), 161–180.
18. I. Angiono, M. Kochetov, and M. Mastnak, *On rigidity of Nichols algebras*, J. Pure Appl. Algebra **219** (2015), 5539–5559.
19. I. Angiono and G. Sanmarco, *Pointed Hopf algebras over non-abelian groups with decomposable braidings. I*, J. Algebra **549** (2020), 78–111.
20. M. Cuntz and I. Heckenberger, *Finite Weyl groupoids of rank three*, Trans. Amer. Math. Soc. **364** (2012), 1369–1393.

21. Y. Doi and M. Takeuchi, *Multiplication alteration by two-cocycles - the quantum version*, Comm. Algebra **22** (1994), 5715–5732.
22. V. Drinfeld, *Quantum groups*, Proc. Int. Congr. Math., Berkeley, 1986, vol. 1, 1987, pp. 798–820.
23. P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor categories*, Mathematical Surveys and Monographs, vol. 205, AMS, Providence, RI, 2015, xvi+343pp.
24. P. Etingof, D. Nikshych, V. Ostrik, and E. Meir, *Fusion categories and homotopy theory*, Quantum Topol. **1** (2010), 209–273.
25. B. Feigin, A. Gainutdinov, A. Semikhatov, and I. Tipunin, *Kazhdan-Lusztig correspondence for the representation category of the triplet W-algebra in logarithmic CFT*, Theor. Math. Phys. **148** (2006), 1210.
26. S. Fomin and K. N. Kirillov, *Quadratic algebras, Dunkl elements, and Schubert calculus*, Progr. Math. **172** (1999), 146–182.
27. E. Frenkel and D. Ben-Zvi, *Vertex algebras and algebraic curves*, Mathematical Surveys and Monographs, vol. 88, American Mathematical Society, Providence, RI, 2004.
28. J. Fuchs, *Affine Lie algebras and quantum groups*, Cambridge Monographs Math. Phys., Cambridge University Press, Cambridge, 1994.
29. M. Graña, *On Nichols algebras of low dimension*, Contemp. Math. **267** (2000), 111.
30. R. Günther, *Crossed products for pointed Hopf algebras*, Comm. Algebra **27** (1999), 4389–4410.
31. I. Heckenberger, *The Weyl groupoid of a Nichols algebra of diagonal type*, Invent. Math. **164** (2006), 175–188.
32. I. Heckenberger, *Classification of arithmetic root systems*, Adv. Math. **220** (2009), 59–124.
33. I. Heckenberger, A. Lochmann, and L. Vendramin, *Braided racks, Hurwitz actions and Nicholsalgebras with many cubic relations*, Transform. Groups **17** (2012), 157–194.
34. I. Heckenberger and H.-J. Schneider, *Nichols algebras over groups with finite root system of rank two I*, J. Algebra **324** (2010), 3090–3114.
35. I. Heckenberger and H.-J. Schneider, *Right coideal subalgebras of Nichols algebras and the Duflo order on the Weyl groupoid*, Israel J. Math. **197** (2013), 139–187.
36. I. Heckenberger and H.-J. Schneider, *Hopf algebras and root systems*, Mathematical Surveys and Monographs, vol. 247, American Mathematical Society (AMS), Providence, RI, 2020 (ISBN 978-1-4704-5232-2/hbk; 978-1-4704-5680-1/ebook). xix, 582 p.
37. I. Heckenberger and L. Vendramin, *A classification of Nichols algebras of semi-simple Yetter-Drinfeld modules over non-abelian groups*, J. Eur. Math. Soc. **19** (2017), 299–356.
38. I. Heckenberger and H. Yamane, *A generalization of Coxeter groups, root systems, and Matsumoto's theorem*, Math. Z. **259** (2008), 255–276.
39. N. Iwahori and H. Matsumoto, *Several remarks on projective representations of finite groups*, J. Faculty of Science Univ. Tokyo **10** (1964), 129–146.
40. M. Jimbo, *A q-difference analogue of $U(g)$ and the Yang Baxter equation*, Lett. Math. Phys. **10** (1985), 63–69.
41. T. Kerler and V. V. Lyubashenko, *Non-semisimple topological quantum field theories for 3-manifolds with corners*, Springer Lect. Notes Math., vol. 1765, Springer, New York, 2001.
42. S. Lentner, *Orbifoldizing Hopf and Nichols algebras*, Dissertation, LMU München, 2012, https://edoc.ub.uni-muenchen.de/15363/.
43. S. Lentner, *New large-rank Nichols algebras over non-abelian groups with commutator subgroup $\mathbb{Z}_2$*, J. Algebra **419** (2014), 1–33.
44. S. D. Lentner, *Quantum groups and Nichols algebras acting on conformal field theories*, Adv. Math. **378** (2021), 107517. https://doi.org/10.1016/j.aim.2020.107517
45. G. Lusztig, *Modular representations and quantum groups*, Contemp. Math., vol. 82, Amer. Math. Soc., Providence, RI, 1989, pp. 59–77.
46. G. Lusztig, *Introduction to quantum groups*, Birkhäuser, Boston, MA, 1993.
47. A. Masuoka, *Abelian and non-abelian second cohomologies of quantized enveloping algebras*, J. Algebra **320** (2008), 1–47.
48. E. Meir, *Geometric perspective on Nichols algebras*, J. Algebra **601** (2022), 390–422.
49. A. Milinski and H.-J. Schneider, *Pointed indecomposable hopf algebras over coxeter groups*, Contemp. Math. **267** (2000), 215–236.

50. M. Mombelli, *The Brauer-Picard group of the representation category of finite supergroup algebras*, Rev. Unión Mat. Argent. **55** (2014), 83–117.
51. S. Montgomery, *Hopf algebras and their actions on rings*, CMBS, vol. 82, American Mathematical Society, Providence, RI, 1993.
52. K. Tahara, *On the second cohomology groups of semidirect products*, Math. Z. **129** (1972), 365–379.
53. V. G. Turaev, *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Mathematics, vol. 18, Walter de Gruyter & Co, Berlin, 1994.