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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A SUPERLINEAR SECOND ORDER EQUATION ARISING IN A TWO-ION ELECTRODIFFUSION MODEL

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Abstract: We study a second order ordinary superlinear differential equation under Robin-type boundary conditions. Using variational methods, we prove existence and multiplicity results.

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1 INTRODUCTION

Let us consider the equation

$$u''(x) = g(x,u) + A(x), \tag{1}$$

with radiation boundary conditions

$$u'(0) = a_0 u(0),\ u'(1) = a_1 u(1), \quad \text{with } a_0, a_1 > 0, \tag{2}$$

where $A \in L^2(0,1)$ and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ continuous and superlinear, that is:

$$\lim_{|u|\to+\infty} \frac{g(x,u)}{u} = +\infty$$

uniformly in x. We shall study existence, uniqueness and multiplicity of solutions using variational methods. A particular case of interest is $g(x,u) = \frac{1}{2}u^3 + (a+bx)u$ for a, b, A, a_0 and a_1 some specific constants. This Painlevé II model in two-ion electrodiffusion was derived independently by Grafov and Chernenko in [4] and Bass in [2]. In [3], Bracken *et al* investigated novel flux quantization aspects associated with the iteration of the Bäcklund transformations. Due to that connection, the previous Robin-type boundary value conditions were derived for the Painlevé II equation.

The following results will be established:

Theorem 1 *Problem (1)-(2) admits at least one classical solution.*

Theorem 2 *Let $G(x,u) = \int_0^u g(x,s)ds$ and define Φ as the unique solution of the linear problem*

$$\begin{cases} -\Phi''(x) + 2\frac{\partial g}{\partial u}(x,0)\Phi(x) = 0 \\ a_0\Phi(0) = \Phi'(0) = a_0. \end{cases} \tag{3}$$

Assume that

1. $\Phi'(1) < a_1\Phi(1)$,
2. $g(x,0) = 0$ *and* $\frac{\partial g}{\partial u}(x,0) < 0$, *for all* $x \in (0,1)$,
3. $G(x,u) \geq 0$, *for all* $u \in \mathbb{R}$ *and* $x \in (0,1)$,
4. $\exists\theta \in (0, \frac{1}{2})$ *and* $u_0 > 0$ *such that* $G(x,u) \leq \theta u g(x,u)$ *for all* $|u| > u_0$ *and* $x \in (0,1)$.

Then, there exists $\tilde{A} > 0$ such that if $\|A\|_{L^2} < \tilde{A}$, problem (1)-(2) has at least two classical solutions.

Note 1 *In the special case $g(x,u) = \frac{1}{2}u^3 + (a+bx)u$, mentioned in the introduction, it can be proven that condition 1. holds if and only if $b > 0$.*

2 Variational setting

Let us introduce a variational formulation for the boundary problem (1)-(2). Let $J : H^1(0,1) \to \mathbb{R}$ by

$$J(u) = \int_0^1 \left(\frac{1}{2}(u')^2 + G(x,u) + A(x)u\right) dx + \frac{a_0}{2}u(0)^2 - \frac{a_1}{2}u(1)^2.$$

It is readily shown that $J \in C^1(H^1(0,1), \mathbb{R})$, with

$$DJ(u)(v) = \int_0^1 \big(u'v' + g(x,u)v + Av\big)\, dx + a_0 u(0)v(0) - a_1 u(1)v(1).$$

Moreover, $u \in H^1(0,1)$ is a critical point of J if and only if u is a classical solution of (1)-(2). Indeed, let u be a critical point of J. As $DJ(u)(v) = 0$ for all $v \in C_0^1(0,1)$, we deduce that u is a weak solution of (1). From the embedding $H^1(0,1) \hookrightarrow C^1([0,1])$ we conclude that u has a continuous second order weak derivative and hence it is a classical solution. It remains to prove that u verifies the boundary condition. For arbitrary $v \in H^1(0,1)$, integrate by parts equality $DJ(u)(v) = 0$ to obtain:

$$u'(1)v(1) - u'(0)v(0) = a_1 u(1)v(1) - a_0 u(0)v(0).$$

Take $v \in H^1(0,1)$ such that $0 = v(0) \neq v(1)$, then $u'(1) = a_1 u(1)$; in the same way, taking $0 = v(1) \neq v(0)$, we deduce that $u'(0) = a_0 u(0)$. The converse is trivial.

3 Proof of Theorem 1

We shall prove that J achieves a global minimum in $H^1(0,1)$, which is a critical point of J and hence, a classical solution to our problem. By standard results (see, e. g. [5]), J is weakly lower semi-continuous, so it suffices to prove that J is coercive. By contradiction, suppose $(J(u_n))_{n\in\mathbb{N}}$ is bounded for some u_n such that $\|u_n\|_{H^1} \to +\infty$ as $n \to +\infty$. A simple computation shows that

$$J(u) \geq \|u_n\|_{L^2}^2 \left(\int_0^1 \left(\frac{(u_n')^2}{2\|u_n\|_{L^2}^2} + \frac{G(x,u_n)}{\|u_n\|_{L^2}^2} + \frac{Au_n}{\|u_n\|_{L^2}^2}\right) dx - \frac{a_1 u(1)^2}{2\|u_n\|_{L^2}^2}\right). \tag{4}$$

Moreover, for given $u \in H^1(0,1)$, fix $x_0 \in [0,1]$ such that $|u(x_0)| = \min_{x\in[0,1]} |u(x)|$, then

$$u(1) = u(x_0) + \int_{x_0}^1 u'(x)dx,$$

$$|u(1)| \leq |u(x_0)| + \int_0^1 \big|u'(x)\big|\, dx \leq \|u\|_{L^2} + \|u'\|_{L^2},$$

and thus, for arbitrary $\delta > 0$,

$$u(1)^2 \leq \left(1 + \frac{1}{\delta}\right)\|u\|_{L^2}^2 + (1+\delta)\|u'\|_{L^2}^2. \tag{5}$$

On the other hand, since g is superlinear, for each $M > 0$ there exists $R_0 > 0$ such that

$$\frac{g(x,u)}{|u|} > 2M,$$

for all x and all u such that $|u| > R_0$. Hence

$$G(x,u) > Mu^2 + K \tag{6}$$

for some constant K. Suppose firstly that $\|u_n'\|_{L^2}$ is bounded, then $\|u_n\|_{L^2} \to +\infty$. Replacing (6) in (4), we get

$$J(u) \geq \|u_n\|_{L^2}^2 \left(\int_0^1 \left(\frac{(u_n')^2}{2\|u_n\|_{L^2}^2} + \frac{Mu_n^2}{\|u_n\|_{L^2}^2} + \frac{K}{\|u_n\|_{L^2}^2} + \frac{Au_n}{\|u_n\|_{L^2}^2}\right) dx - \frac{a_1 u_n(1)^2}{\|u_n\|_{L^2}^2}\right).$$

As

$$\frac{a_1 u_n(1)^2}{\|u_n\|_{L^2}^2} \le \left(1+\frac{1}{\delta}\right) a_1 + (1+\delta)\frac{\|u_n'\|_{L^2}^2}{\|u_n\|_{L^2}^2} \longrightarrow \left(1+\frac{1}{\delta}\right) a_1,$$

taking M large enough we deduce that $J(u_n) \longrightarrow +\infty$, a contradiction. So, we may assume that $\|u_n'\|_{L^2}^2 \to +\infty$.

Moreover, observe that

$$J(u_n) \ge \|u_n'\|_{L^2}^2 \left(\frac{1}{2} + \int_0^1 \left(\frac{G(x,u_n)}{\|u_n'\|_{L^2}^2} + \frac{Au_n}{\|u_n'\|_{L^2}^2}\right) dx - \frac{a_1 u_n(1)^2}{\|u_n'\|_{L^2}^2}\right). \tag{7}$$

If $\|u_n\|_{L^2}$ is bounded, then from (7) we deduce again that $J(u_n) \to +\infty$, a contradiction. Then, we may suppose that $\|u_n\|_{L^2} \to +\infty$. Due to (5),

$$J(u_n) \ge \|u_n'\|_{L^2}^2 \left(\frac{1}{2} - a_1(1+\delta) + \frac{1}{\|u_n'\|_{L^2}^2}\left(\int_0^1 (G(x,u_n) + Au_n)\, dx - a_1\left(1+\frac{1}{\delta}\right)\|u_n\|_{L^2}^2\right)\right).$$

From (6),

$$\int_0^1 G(x,u_n)dx \ge M\|u_n\|_{L^2}^2 + K,$$

and, since $\|u_n\|_{L^2} \to +\infty$, we can assume that $\|u_n\|_{L^2} \ge 1$, then

$$\int_0^1 Au_n dx \ge -\|A\|_{L^2}\|u_n\|_{L^2} \ge -\|A\|_{L^2}\|u_n\|_{L^2}^2.$$

So we have

$$J(u_n) \ge \|u_n'\|_{L^2}^2 \left\{\frac{1}{2} - (1+\delta)a_1 + \frac{K}{\|u_n'\|_{L^2}^2} + \left(M - \|A\|_{L^2} - a_1\left(1+\frac{1}{\delta}\right)\right)\frac{\|u_n\|_{L^2}^2}{\|u_n'\|_{L^2}^2}\right\}. \tag{8}$$

If $\liminf \frac{\|u_n\|_{L^2}}{\|u_n'\|_{L^2}} > 0$, taking limit in (8) with M large enough we get $J(u_n) \to +\infty$, a contradiction. Otherwise, passing to a subsequence we may assume that $\frac{\|u_n\|_{L^2}}{\|u_n'\|_{L^2}} \to 0$. Let $v_n := \frac{u_n}{\|u_n'\|_{L^2}}$. As $\|v_n'\|_{L^2} = 1$, then taking a subsequence we may assume that $v_n \to 0$ uniformly and the contradiction $J(u_n) \to +\infty$ is now deduced directly from (7).

This completes the proof of the coerciveness of J and, hence, the existence of a solution is proven.

4 Sketch of the proof of Theorem 2

From Theorem 1 we know that J achieves a global minimum at some $u_1 \in H^1(0,1)$, hence, the next linking theorem by Rabinowitz will provide another solution $u_0 \in H^1(0,1)$. Firstly, let us recall that if X is a Banach space and $J \in C^1(X,\mathbb{R})$, a sequence $(u_n) \subset X$ is called a *Palais-Smale sequence* if $|J(u_n)| \le c$ for some constant c and $DJ(u_n) \to 0$; and J is said to satisfy condition (PS) if any Palais-Smale sequence has a convergent subsequence in X.

Theorem 3 (Rabinowitz) *Let X be a Banach space and $J \in C^1(X,\mathbb{R})$. Assume $X = X_1 \oplus X_2$, with $dim(X_1) < \infty$, and*

$$\max_{u\in X_1:\|u\|_X=r} J(u) < \rho := \inf_{u\in X_2} J(u) \tag{9}$$

for some $r > 0$ and J satisfy (PS). Then J has at least one critical point u_0 such that $J(u_0) = \rho$.

Solutions u_0, u_1 are such that $J(u_0) \geq \rho > J(u_1)$, hence, they are different.

In our case, we consider $X = H^1(0,1)$ and the sets $X_1 = span\{\Phi\}$, where Φ is the solution of (3), and $X_2 = \{u \in H^1(0,1) : u(1) = 0\}$.

Firstly, let us prove that $X = X_1 \oplus X_2$. As $\frac{\partial g}{\partial u}(x,0) > 0$, Φ is strictly increasing; thus $\Phi(1) > 0$ and $X_1 \cap X_2 = \{0\}$. Moreover, $u \in X$ can be written $u = a\Phi + u - a\Phi$, where $a = \frac{u(1)}{\Phi(1)}$.

To prove (9), we find a suitable $r > 0$ such that

$$\max_{u \in X_1, \|u\| = r} J(u) \leq C\|A\|_{L^2},$$

where C is a negative constant only depending on g and Φ.

For the right-hand side of the inequality, a simple computation shows that

$$\inf_{u \in X_2} J(u) \geq -\|A\|_{L^2}^2 \left(\frac{1}{2} + \frac{1}{2a_0}\right).$$

Thus, (9) is true, provided that $\|A\|_{L^2} < \frac{-2Ca_0}{a_0+1}$.

To conclude the proof, let us verify that J satisfies the (PS) condition. Let $(u_n)_{n\in\mathbb{N}} \subset H^1(0,1)$ such that $|J(u_n)| \leq c$ y $DJ(u_n) \to 0$. If there exists $K > 0$ such that $\|u_n\|_{H^1} \leq K$ for all $n \in \mathbb{N}$ then, taking a subsequence, we may assume that there exists $u \in H^1(0,1)$ such that $u_n \to u$ weakly in $H^1(0,1)$ and uniformly. Since $J \in C^1$ and $DJ(u_n)(u) \to 0$, we deduce

$$0 = DJ(u)(u) = \int_0^1 \left((u')^2 + g(x,u)u + Au\right) dx + a_0 u(0)^2 - a_1 u(1)^2.$$

As $DJ(u_n)(u_n) \to 0$, it is seen that $\|u_n\|_{L^2} \to \|u\|_{L^2}$, then $u_n \to u$ strongly.

If $\|u_n\|_{H^1} \to +\infty$, let $v_n = \frac{u_n}{\|u_n\|_{H^1}}$. Since $DJ(u_n)(v_n) \to 0$, we can prove that

$$\left| DJ(rv_n)(\phi) - \int_0^1 A\phi dx \right| \longrightarrow 0,$$

for all ϕ, that is: $DJ(rv_n) \to A$. Since v_n is bounded in $H^1(0,1)$, we may assume that taking a subsequence, v_n converges weakly in $H^1(0,1)$ and uniformly to some v. As $DJ(rv_n)(\phi) \to DJ(rv)(\phi)$ for all ϕ, it follows that $DJ(rv)(\phi) = \int_0^1 A(x)\phi(x)dx$. Let us consider the functional given by $\tilde{J}(u) = J(u) - \int_0^1 Au$. Then $\tilde{J}(rv_n) \to 0$ and it is easy to verify that $\tilde{J}(rv_n)$ is bounded. Thus, $(rv_n)_{n\in\mathbb{N}}$ is a bounded Palais-Smale sequence for $\tilde{J}$, and we deduce that it has a convergent subsequence as before. Hence, we may assume that $rv_n \to rv$ strongly and in particular $\|v\|_{H^1} = 1$. Moreover, rv is a solution of (1)-(2) with $A = 0$. As g is superlinear and r is arbitrary, it follows that $v = 0$, a contradiction.

Then the linking theorem holds and the proof is complete.

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