PERSISTENCE AND PERIODIC SOLUTIONS IN SYSTEMS OF DELAY DIFFERENTIAL EQUATIONS

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Abstract: We study persistence in the context of systems of delay differential equations and employ guiding functions in order to deduce uniform persistence under appropriate conditions. Moreover, different assumptions allow us to guarantee the existence of T-periodic solutions. These results are part of the paper [2].

Keywords: *delay differential equations, semi-dynamical systems, persistence, guiding functions, periodic solutions, topological degree*

1 INTRODUCTION

With population models in mind [7], we consider the delayed differential system

$$x'(t) = f(t, x(t), x(t - \tau)) \tag{1}$$

where $f : [0, +\infty) \times [0, +\infty)^{2N} \to \mathbb{R}^N$ is continuous and $\tau > 0$ is the delay. An initial condition for (1) can be expressed in the following way

$$x_0 = \varphi, \tag{2}$$

where $\varphi : [-\tau, 0] \to [0, +\infty)^N$ is continuous and $x_t \in C([-\tau, 0], \mathbb{R}^N)$ is defined by $x_t(s) = x(t + s)$. If the solutions are defined over $[0, +\infty)$ and lie in $[0, +\infty)^N$ then the semi-flow associated to the system:

$$\Phi : [0, +\infty) \times C([-\tau, 0], [0, +\infty)^N) \to C([-\tau, 0], [0, +\infty)^N) \tag{3}$$

given by

$$\Phi(t, \varphi) = x_t$$

induces a semi-dynamical system.

Under appropriate conditions, we shall prove that 0 is a uniform repeller of the system. This will imply that all trajectories of the system with positive initial data lie outside a closed ball centered at the origin of $\mathbb{R}^N$ for sufficiently large values of t.

We will employ topological degree methods to find closed orbits of (3). Notice that since the space of initial values is infinite dimensional, the Brouwer degree cannot be applied: we shall use instead Leray-Schauder degree techniques.

2 UNIFORM PERSISTENCE

We say that a semi-flow is *uniformly persistent* if there exists $\varepsilon > 0$ such that

$$\liminf_{t\to+\infty} ||\Phi(t, \varphi)||_\infty > \varepsilon \qquad \forall\, \varphi \in C([-\tau, 0], (0, +\infty)^N).$$

In order to study the persistence of the solutions of (1), we shall consider a C^1 Lyapunov-like mapping

$$V : (0, +\infty)^N \to (0, +\infty)$$

such that

$$\lim_{|x|\to 0} V(x) = 0.$$

The obvious example is $V(x) := |x|^2$, where $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^N$, although many other choices of V could be used in applications. The point is that, in contrast with Lyapunov functions, we shall require that $\dot{V} > 0$ for x close to the origin where, as usual,

$$\dot{V}(t) = \frac{dV \circ x}{dt}.$$

In this sense, V can be considered as a *guiding function* (see e.g. [3], [4]) but, unlike the guiding functions, our conditions shall involve sets of the form $\{V(x) < r\}$ instead of $\{|x| \geq r\}$.

In order to obtain uniform persistence, we will show the existence of an accurate value of $\mu > 0$ such that the set $V^\mu := V^{-1}(0, \mu)$ is a repeller. Indeed, we will prove that V^μ shall contain a set of the form $\{x \in \mathcal{C} : 0 < |x| < \xi\}$ for some positive ξ.

The conditions to ensure that the semi-flow stays away from the origin are:

(H1) $f_j(t, x, y) > 0$ for all x, y such that $x_j = 0$ and $y \neq 0$.

(H2) there exist $t_0, r_0 > 0$ such that

$$\langle \nabla V(x), f(t, x, y) \rangle > 0 \quad \text{for } t > t_0 \text{ and } V(x), V(y) < r_0.$$

(H3) $\langle \nabla V(x), f(t, x, y) \rangle \geq \langle \nabla V(x), f(t, x, x) \rangle, \text{ if } V(x) \leq V(y).$

We can now proceed to state our first result:

Theorem 1 *Assume that* **(H1)**, **(H2)**, **(H3)** *hold, then the system is uniformly persistent. More precisely, all solutions of (1)-(2) with* $\varphi_j(t) \geq 0$ *for all* j *and* $t \in [-\tau, 0]$ *satisfy*

$$\liminf_{t \to +\infty} V(x(t)) \geq r_0,$$

that is, $x(t) \notin V^{r_0}$ *for t sufficiently large.*

Proof. Let us set $v(t) := V(x(t))$ and suppose that $x(t)$ is a positive solution of the system such that

$$\liminf_{t \to +\infty} v(t) = i \in (0, r_0).$$

Then three different situations may be considered: If $v(t) \geq i$ for all $t \gg 0$, we may choose a sequence $t_n \to +\infty$ such that $\lim_{n \to +\infty} v(t_n) = i$, $v'(t_n) \leq 0$ and $v(t_n - \tau) \geq v(t_n)$. Therefore, a contradiction yields under **(H2)-(H3)**. If $v(t) \to i^-$, then also $v(t - \tau) \to i^-$. It follows that there exists c_0 such that $v'(t) > c_0$ for $t \gg 0$, which cannot happen. Thus, we deduce there exists a sequence $s_n \to +\infty$ such that $v(s_n) > i$ and $v(s_n) \to i^+$. Take $t_n \in [s_1, s_n]$ such that $v(t_n) = \min_{t \in [s_1, s_n]} v(t)$, so $v(t_n) \to i$ and, for n large, $v'(t_n) \leq 0$ and $v(t_n) \leq v(t_n - \tau)$. Hence, we are in the first case and the contradiction follows. □

3 Periodic orbits

In order to prove the existence of periodic orbits, inspired by [5], we shall work on the positive cone X of C_T, the Banach space of continuous T-periodic functions and define an appropriate fixed point operator $K : \overline{U} \subset X \to C_T$. Thus, if f is T-periodic in the first coordinate, then the fixed points of K determine T-periodic positive orbits of system (3).

Let us recall the Leray-Schauder degree is defined as follows [1]: Let $U \subseteq C_T$ be open and bounded, and let $K : \overline{U} \to C_T$ be compact with $Kx \neq x$ for $x \in \partial U$. Set $\varepsilon = \inf_{x \in \partial U} ||x - Kx||$. Then define

$$\deg_{L-S}(I - K, U, 0) = \deg_B((I - K_\varepsilon)|_{V_\varepsilon}, U \cap V_\varepsilon, 0),$$

where K_ε is an ε-approximation of K with $\mathrm{Im}(K_\varepsilon) \subseteq V_\varepsilon$ and $\dim(V_\varepsilon) < \infty$.

We will show that the Leray-Schauder degree of the operator $I - K$ is non-zero on an appropriate subset $U \subset X$ and therefore the set of fixed points of the compact operator K is non-empty.

Theorem 2 *If f is T-periodic in the first coordinate,* **(H1)**, **(H2)**, **(H3)** *hold and*

(H4) *there exists $R > r_0$ such that*

$$\langle \nabla V(x), f(t,x,y)\rangle < 0 \text{ for } r_0 \leq V(y) \leq V(x) = R, \text{ with } r_0 \text{ from } \textbf{(H2)}.$$

Then there exists at least one T-periodic positive solution of (1)-(2) in $\Omega = \{x \in [0,+\infty)^N : V(x) \in (r_0, R)\}$ provided that the Euler characteristic of Ω is non-zero.

The proof of the theorem shall be based on the following crucial result (see e.g. [6]):

Theorem 3 (Hopf Theorem) *If ν is the outward normal on a compact, oriented manifold M, then the degree of ν equals the Euler characteristic of M.*

Proof. [Theorem 2] For convenience, a little of extra notation shall be introduced. For a function $x \in C_T$, let us write

$$\mathcal{I}x(t) := \int_0^t x(s)\, ds, \qquad \overline{x} := \frac{1}{T}\mathcal{I}x(T).$$

Moreover, denote by $\mathcal{N}$ the Nemitskii operator associated to the problem, namely

$$\mathcal{N}x(t) := f(t, x(t), x(t-\tau)).$$

Let us consider the open bounded sets $\Omega = \{x \in [0,+\infty)^N : V(x) \in (r_0, R)\} \subseteq \mathbb{R}^N$, $U = \{x \in C_T : x(t) \in \Omega \text{ for all } t > 0\} \subseteq C_T$ and define the compact operator $K : C_T \to C_T$ by

$$Kx(t) := \overline{x} - t\overline{\mathcal{N}x} + \mathcal{I}\mathcal{N}x(t) - \overline{\mathcal{I}\mathcal{N}x}.$$

Via the Lyapunov-Schmidt reduction, if $x \in C_T$ is a fixed point of K then x is a solution of the equation.

Let $K_0 x := \overline{x} - \frac{T}{2}\overline{\mathcal{N}x}$ and consider for $s \in [0,1]$, the homotopy $K_s := sK + (1-s)K_0$. We claim that K_s has no fixed points on ∂U. As mentioned, for $s > 0$ it is clear that $x \in \overline{U}$ is a fixed point of K_s if and only if $x'(t) = s\mathcal{N}x(t)$, that is:

$$x'(t) = sf(t, x(t), x(t-\tau)).$$

Observe that, if we identify $\mathbb{R}^N$ with the set of constant functions of C_T then $U \cap \mathbb{R}^N = \Omega$. Thus the image of K_0 is contained in $\mathbb{R}^N$, whence the Leray-Schauder degree of $I - K_0$ can be computed as the Brouwer degree of its restriction to Ω.

We apply another homotopy,

$$H(s,x) = sK_0(x) - (1-s)\nu(x), \qquad \text{for } (s,x) \in [0,1] \times \overline{\Omega},$$

where ν is the outward normal extended to the interior, which does not have fixed points on $\partial\Omega$.

By the homotopy invariance of the degree and Hopf theorem, we conclude that

$$\deg_{LS}(I - K, U, 0) = \deg_B(I - K_0, \Omega, 0) = \deg_B(-\nu, \Omega, 0) = (-1)^N \chi(\Omega) \neq 0.$$

□

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