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Anisotropic estimates for H(curl)- and H(div)-conforming elements on prisms and applications.

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Abstract: We prove anisotropic local interpolation error estimates for two different operators and use this result to estimate the global approximation error for the mixed formulation of the Poisson problem with Dirichlet conditions in a non-convex polyhedral domain with edge singularities.

Keywords: *Anisotropic, mixed, Raviart-Thomas, edge elements.*

1 Introduction

We consider the edge elements of Nédélec and the $3D-$generalization of Raviart-Thomas elements [6] in arbitrarily anisotropic prisms, and their corresponding k-th order interpolation operators π_k and r_k respectively, and prove the following anisotropic and uniform local interpolation error estimates:

$$\|\mathbf{u}-\pi_k\mathbf{u}\|_{L^p(K)} \leqslant C\left(\sum_{|\alpha|=m+1} h^\alpha\|D^\alpha\mathbf{u}\|_{L^p(K)} + h_K^{m+1}\|D^m(\mathbf{curl}\,\mathbf{u})\|_{L^p(K)}\right)$$

for $p>2$, $\mathbf{u}\in W^{m+1,p}(K)$, $1<m\leqslant k-1$ and $k\geqslant 1$, and

$$\|\mathbf{u}-r_k\mathbf{u}\|_{L^p(K)} \leqslant C\left(\sum_{|\alpha|=m+1} h^\alpha\|D^\alpha\mathbf{u}\|_{L^p(K)} + h_K^{m+1}\|D^m(\operatorname{div}\mathbf{u})\|_{L^p(K)}\right)$$

for $p\geqslant 1$, $\mathbf{u}\in W^{m+1,p}(K)$, $m\leqslant k$.

This is a generalization of what is known for tetra and hexahedra [3, 4, 5] and, in practice, it allows to reduce the number of degrees of freedom as well as to avoid the use of a kind of narrow tetrahedra for which anisotropic error estimates are not valid.
Edge elements are one example of H(**curl**)−conforming elements, and they were defined to determine a natural interpolation operator for fields with continuous tangential components. Similarly, the other elements considered are H(div)−conforming, and we will use them to interpolate fields with continuous tangential components. These are the cases of the solutions of Stokes' equations, the solutions of time harmonic Maxwell's equations, and the vectorial variable of the following problem, which will be our application.
We work in the mixed formulation of the Poisson problem with data in L^2:

$$\begin{aligned} \nabla u &= -\boldsymbol{\sigma} && \text{in } \Omega \\ \operatorname{div}\boldsymbol{\sigma} &= f && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega, \end{aligned}$$

on a polyhedral product domain Ω with an edge e with an interior angle ω_e greater than π. We proved the following optimal approximation error estimates

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^2(\Omega)} \leqslant C\,h\,\|f\|_{L^2(\Omega)}$$

$$\|u-u_h\|_{L^2(\Omega)} \leqslant C\,h\,\|f\|_{L^2(\Omega)}.$$

The solution of the Poisson problem with Dirichlet boundary conditions on non convex polyhedral domains, as the one being considered, may show singularities on edges which degrade the convergence of the FEMs [2]; in fact, it not necessarily belongs to $H^2(\Omega)$. Although, we have a characterization for the regularity of the solution in terms of weighted Sobolev norms. For a positive $\delta > 1 - \frac{\pi}{\omega_e}$,

$$\|\partial_{x_i} u\|_{V_\delta^{1,2}(\Omega)} \leqslant \|f\|_{L^2(\Omega)}$$
$$\|\partial_{x_3} u\|_{V_0^{1,2}(\Omega)} \leqslant \|f\|_{L^2(\Omega)}$$

where the weight is given by the distance to the singular edge

$$\|v\|^2_{V_\delta^{1,2}(\Omega)} = \sum_{|\alpha|\leqslant 1} \int_\Omega |r(\mathbf{x})^{\delta-1+|\alpha|} D^\alpha v(\mathbf{x})|^2 \, d\mathbf{x}.$$

The way by which we recovered the optimal approximation order consists of using the well known family of graded meshes [1], which are inevitably anisotropic, and therefore we needed this kind of local interpolation estimate in the first place. Similar behaviors exhibit the solutions of the other problems mentioned so far [7].

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