SUFFICIENT CONDITIONS FOR OSCILLATIONS OF A FIRST-ORDER SYSTEM WITH DELAY.

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Abstract: In this work, we consider a system of first order differential equations with delays which appears in many population biological models ([3] [1]). More specifically we prove, using the topological degree theory, existence of periodic solutions of following system

$$u_i'(t) = a_i u_i(t) + b_i u_i(t-\tau_i) + g_i(u_1(t-\tau_1), u_2(t-\tau_2)) + p_i(t), \quad i = 1, 2. \tag{1}$$

1 INTRODUCTION

Let us consider the system of delay differential equations (1) where $a_i, b_i \in \mathbb{R}$, $\tau_i > 0$ and $g_i : \mathbb{R}^2 \to \mathbb{R}$ are continuous and bounded for $i = 1, 2$. We shall assume that

$$|a_1| < |b_1|, \tag{2}$$

and $p_i \in C(\mathbb{R}, \mathbb{R})$ are $T := 2\pi/\omega$-periodic functions with $\omega := \sqrt{b_1^2 - a_1^2}$. For convenience, we shall denote $u = (u_1, u_2)$, $g = (g_1, g_2)$ and $p = (p_1, p_2)$ and write the problem as a functional equation in the following way.

Consider

$$C_T := \left\{u \in C(\mathbb{R}, \mathbb{R}^2) : u(t) = u(t+T)\right\}, \; C_T^1 := C_T \cap C^1(\mathbb{R}, \mathbb{R}^2)$$

with $L : C_T^1 \to C(\mathbb{R}, \mathbb{R}^2)$ given by

$$L(u_1, u_2)(t) = (L_1(u_1)(t), L_2(u_2)(t))$$

where

$$L_i(u_i)(t) = u_i'(t) - a_i u_i(t) - b_i u_i(t-\tau_i), \; i = 1, 2$$

and

$$N : C_T \to C(\mathbb{R}, \mathbb{R}^2)$$
$$N(u)(t) := g(u_1(t-\tau_1), u_2(t-\tau_2)) + p(t).$$

Then (1) is equivalent to the problem

$$Lu = Nu, \;\; u \in C_T^1.$$

We are interested in the *resonant* case, that is, when the operator L has nontrivial kernel. Specifically, we shall assume that the resonance is produced in the first equation. In more precise terms, consider the characteristic equation of $L_1(u_i)(t) = 0$ given by

$$h(\lambda) = \lambda - a_1 - b_1 e^{-\lambda\tau_1} = 0.$$

Let

$$z = \lambda\tau_1, \;\; \alpha_1 = a_1\tau_1 \;\text{ and }\; \beta_1 = b_1\tau_1$$

and the function $F(z, \alpha_1, \beta_1) := z - \alpha_1 - \beta_1 e^{-z}$, whose zeros are related to the roots of h via the previous change or variables. We know (see [3]) that $z = iy$, $y > 0$, is a purely imaginary root of F if there we take τ_1 such that $(\alpha_1, \beta_1) \in C_k$, where

$$C_k = \left\{(\alpha(y), \beta(y))/\; \alpha(y) := \frac{y\cos y}{\sin y}, \beta(y) := \frac{-y}{\sin y}, y \in (k\pi, (k+1)\pi)\right\},$$

for some $k \in \mathbb{N}$. In this case, $\lambda = i\omega$ is a root of h. Next, consider τ_2, such that $(\alpha_2, \beta_2) \notin C_k$, for any $k \in \mathbb{N}$. It is easy to see that, in this situation, that

$$\text{Ker}(L) = span\{\cos(\omega t), \sin(\omega t)\} \times \{0\}.$$

Let us consider $\mathcal{P} : C^1_T \to \text{Ker}L$ the orthogonal projection, given by

$$\mathcal{P}(u) = (\alpha_{u_1} \cos(\omega t) + \beta_{u_1} \sin(\omega t), 0)$$

where α_v and β_v are the Fourier coefficients given bt

$$\alpha_v := \frac{2}{T} \int_0^T \cos(\omega t) v(t) dt$$

and

$$\beta_v := \frac{2}{T} \int_0^T \sin(\omega t) v(t) dt$$

.

A straightforward computation shows that the adjoint operator of L^1 is given by

$$L_1^* v(t) = -v'(t) - a_1 v(t) - b_1 v(t + \tau_1),$$

where $v \in C^1$. Hence, the characteristic equation of $L_1^* v(t) = 0$ reads $h^*(\lambda) = \lambda + a_1 + b_1 e^{\lambda \tau_1} = 0$. Observe that $\lambda = i\omega$ is a solution of $h(\lambda) = 0$ if and only if $\overline{\lambda}$ is a solution of $h^*(\lambda) = 0$; thus, $\text{Ker}L_1 = \text{Ker}L_1^*$. It follows that $R(L) = (\text{Ker}L^*)^\perp = (\text{Ker}L_1)^\perp \times C$, where

$$(\text{Ker}L_1)^\perp = \left\{ \varphi \in L^2([0,T], \mathbb{R}) : \int_0^T \cos(\omega t) \varphi(t) dt = \int_0^T \sin(\omega t) \varphi(t) dt = 0 \right\}.$$

Then we may define a right inverse $\mathcal{K} : R(L) \to C^1_T$ of the operator L, given by $\mathcal{K}\psi = u$, where u is the unique solution of the problem

$$\begin{cases} Lu = \psi \\ \mathcal{P}(u) = 0. \end{cases}$$

Moreover, due to the Open Mapping Theorem, the following standard estimate is obtained

$$\|u - \mathcal{P}u\|_{H^1} \leq c \|Lu\|_{L^2}, \;\; u \in C^1_T.$$

Hence, from the compactness of the embedding $H^1(0,T) \hookrightarrow C_T$, we conclude that $\mathcal{K}$ is compact.

2 Main result

In the sequel, we shall assume that the limits

$$g_1^{inf}(-\infty) := \liminf_{v \to -\infty} g_1(v, u_2)$$

and

$$g_1^{sup}(+\infty) := \limsup_{v \to +\infty} g_1(v, u_2),$$

exist uniformly for $u_2 \in B_r(0)$, where $r := c(\|g\|_\infty + \|p\|_\infty)$.

Theorem 2.1 *Fix τ_1 as before such that $(\alpha_1, \beta_1) \in C_k$ for some k and assume that (2) and one of the following conditions*

$$\frac{4}{T}\left(g_1^{inf}(-\infty) - g_1^{sup}(+\infty)\right) > \sqrt{\alpha_{p_1}^2 + \beta_{p_1}^2} \tag{3}$$

or

$$\frac{4}{T}\left(g_1^{inf}(+\infty) - g_1^{sup}(-\infty)\right) > \sqrt{\alpha_{p_1}^2 + \beta_{p_1}^2} \tag{4}$$

hold. Then, for almost all $\tau_2 > 0$, system (1) has at least one nontrivial T-periodic solution.

Proof. Choose τ_2 such that $(\alpha_2, \beta_2) \notin C_k$ for any $k \in \mathbb{N}$. For each $\lambda \in [0, 1]$, define the Fredholm operator $F_\lambda : C_T \to C_T$ given by $F_\lambda u = u - T_\lambda u$, where the operator T_λ is defined by

$$T_\lambda u(t) = \mathcal{P}u + \mathcal{P}(Nu) + \lambda\mathcal{K}(Nu - \mathcal{P}(Nu)).$$

We claim that, for $\lambda \in (0, 1]$, $F_\lambda u = 0$ if and only if $u \in C_T^1$ and $Lu = \lambda Nu$.

Indeed, if u is a zero of F_λ, then $u = T_\lambda u \in C_T^1$, since $\mathcal{K} : R(L) \to C_T^1$. Apply $\mathcal{P}$ at both sides, then $\mathcal{P}(Nu) = 0$. So, we have that $u = \mathcal{P}u + \lambda\mathcal{K}(Nu)$. Since $L\mathcal{P} \equiv 0$it is deduced that $Lu = \lambda(Nu)$. Conversely, if $Lu = \lambda(Nu)$ and $u \in C_T^1$, then $Nu \in R(L)$, hence $\mathcal{P}(Nu) = 0$ and $\lambda\mathcal{K}(Nu) = u - \mathcal{P}u$.

In order to verify that F_1 has a zero, we shall firstly prove the existence of $R \gg 0$ such that $F_\lambda u \neq 0$ for $u \in \partial B_R(0)$. Next, by the homotopy invariance of the Leray-Schauder degree, it will suffice to verify that $deg_{LS}(F_0, B_R(0), 0) \neq 0$.

To this end, suppose firstly there exists a sequence $(u^n)_{n\in\mathbb{N}} \subset C_T^1$ and $\lambda_n \in (0, 1]$ such that $F_{\lambda_n} u^n = 0$ and $\|u^n\|_\infty \to \infty$. Then

$$(u_i^n)'(t) = a_i u_i^n(t) + b_i u_i^n(t - \tau_i) + \lambda_n(g_i(u_1^n(t - \tau_1), u_2^n(t - \tau_2)) + p_i(t))$$

which, in turn, implies that $\mathcal{P}(Nu^n) = 0$ for all $n \in \mathbb{N}$ and $\|u^n - \mathcal{P}u^n\|_\infty \leq c(\|g\|_\infty + \|p\|_\infty)$. Hence, $\|\mathcal{P}u^n\|_\infty \to +\infty$. Let us write $\mathcal{P}u^n = (\rho_n \cos(\omega t - \theta_n), 0)$, where $\rho_n \to \infty$, $\theta_n \in [0, 2\pi]$ and

$$u^n(t) = \mathcal{P}u^n(t) + u^n(t) - \mathcal{P}u^n(t) = (\rho_n \cos(\omega t - \theta_n) + \tilde{u}_1^n(t), u_2^n(t))$$

where $\tilde{u}_1^n$ is bounded. Passing to a subsequence if needed, we may assume that θ_n converges to some $\theta \in [0, 2\pi]$. Since $\mathcal{P}(Nu) = 0$, by substitution we obtain:

$$\begin{aligned} -\mathcal{P}(p_1) &= \frac{2}{T}\int_0^T g_1(\rho_n \cos(\omega(t - \tau_1) - \theta_n) + \tilde{u}_1^n(t - \tau_1), u_2^n(t - \tau_2))e^{i\omega t}dt = \\ &= \frac{2}{T}e^{i(\theta_n + \omega\tau_1)}\int_0^T g_1(\rho_n \cos\omega s + \tilde{u}_1(s - \frac{\theta_n}{\omega}), u_2(s + \tau_1 + \frac{\theta_n}{\omega} - \tau_2)e^{i\omega s}ds. \end{aligned} \tag{5}$$

Let us consider the sets $I^+ = \{t \in [0, T] : \cos\omega t > 0\}$ and $I^- = \{t \in [0, T] : \cos\omega t < 0\}$, then by the Dominated Convergence Theorem and the fact that $\int_{I^+} \sin\omega s ds = \int_{I^-} \sin\omega s ds = 0$, we deduce

$$\int_{I^+} g_1(\rho_n \cos\omega s + u_1(s - \frac{\theta_n}{\omega}), u_2(s + \tau_1 + \frac{\theta_n}{\omega} - \tau_2))e^{i\omega s}ds$$
$$\to g_1^{sup}(+\infty)\int_{I^+} \cos\omega s ds = 2g_1^{sup}(+\infty)$$

and

$$\int_{I^-} g_1(\rho_n \cos\omega s + u_1(s - \frac{\theta_n}{\omega}), u_2(s + \tau_1 + \frac{\theta_n}{\omega} - \tau_2))e^{i\omega s}ds$$
$$\to g_1^{inf}(-\infty)\int_{I^-} \cos\omega s ds = -2g_1^{inf}(-\infty)$$

as $n \to +\infty$. From (5), we conclude that

$$\sqrt{\alpha_{p_1}^2 + \beta_{p_1}^2} = \frac{4}{T}\left(g_1^{inf}(-\infty) - g_1^{sup}(+\infty)\right),$$

which contradicts condition (3).

Finally, if $F_0(u) = 0$, then $u = \mathcal{P}(u)$ and a similar argument with $\tilde{u}_1 = u_2 = 0$ shows that $u \notin \partial B_R(0)$ when R is large enough.

This implies that the Leray-Schauder degree of F_λ at 0 is well defined on $B_R(0)$ and that $deg_{LS}(F_1, B_R(0), 0) = deg_{LS}(F_0, B_R(0), 0)$. Moreover, by definition of the degree and the fact that

$$F_0 u = u - \mathcal{P}(u + Nu),$$

we conclude that $deg_{LS}(F_0, B_R(0), 0) = deg_B(F_0|_{\text{Ker}L}, B_R(0) \cap \text{Ker}L, 0)$. Notice that if $u \in \text{Ker}L$, then $F_0 u = -\mathcal{P}(Nu)$; thus, by the product property of the Brouwer degree,

$$\begin{gathered} deg_B(F_0|_{\text{Ker}L}, B_R(0) \cap \text{Ker}L, 0) = \\ = deg_B(\pi_1(-\mathcal{P}Nu), \pi_1(B_R(0)) \cap \text{Ker}L_1, 0)\; deg_B(\pi_2(-\mathcal{P}(Nu)), \pi_2(B_R(0)) \cap \text{Ker}L_2, 0) = \\ = deg_B(\pi_1(-\mathcal{P}(Nu)), \pi_1(B_R(0)) \cap \text{Ker}L_1, 0) \end{gathered}$$

where $\pi_i : \mathbb{R} \to \mathbb{R}$, $\pi_i(x_1, x_2) = x_i$. Hence, the proof is reduced to see that $deg_B(\pi_1(-\mathcal{P}(Nu)), \pi_1(B_R(0)) \cap \text{Ker}L_1, 0) \neq 0$.

Let $u \in B_R(0) \cap \text{Ker}L$, then $u(t) = (\rho\cos(\omega t - \theta), 0)$, with $\theta \in [0, 2\pi]$ and $|\rho| \leq R$ for $R \gg 0$. Via substitution and due to the periodicity of u, we have that

$$\begin{aligned} \pi_1(\mathcal{P}(Nu)) &= \frac{2}{T}\int_0^T g_1(\rho\cos(\omega(t - \tau_1) - \theta), 0)e^{i\omega t}dt + \alpha_{p_1} + i\beta_{p_1} = \\ &= e^{i\theta}\left(\frac{2}{T}e^{i\omega\tau_1}\int_0^T g_1(\rho\cos\omega s, 0)e^{i\omega s}ds\right) + \alpha_{p_1} + i\beta_{p_1}. \end{aligned}$$

Therefore, the degree of the function $F_0 u|_{\text{Ker}L}$ coincides with the index of the curve $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by

$$\gamma(t) = e^{it}\left(\frac{2}{T}e^{i\omega\tau_1}\int_0^T g_1(\rho\cos\omega s, 0)e^{i\omega s}ds\right)$$

around the poing $z_0 := -(\alpha_{p_1} + i\beta_{p_1})$.

Via Dominated Convergence Theorem and taking I^+ and I^- as before, it is seen that

$$\int_0^T g_1(\rho\cos\omega s, 0)e^{i\omega s}ds \to_{\rho\to+\infty} 2\left(g_1^{sup}(+\infty) - g_1^{inf}(-\infty)\right).$$

Hence, for ρ large enough, $|\gamma(t)| \geq \frac{4}{T}\left(g_1^{sup}(+\infty) - g_1^{inf}(-\infty)\right) > \sqrt{\alpha_{p_1}^2 + \beta_{p_1}^2}$, by condition (3).

We conclude that

$$deg_{LS}(F_1, B_R(0), 0) = I(\gamma, z_0) = \pm 1,$$

for R large enough, which proves the existence of a T-periodic solution of problem (1). □

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