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Inertial effects, mass separation and rectification power in Lévy ratchets

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ABSTRACT

We study the dynamics of particles in an external anisotropic periodic potential under the influence of additive white Lévy noise, in a general not overdamped situation. Different quantities characterizing directionality, coherence and dispersion are analyzed as functions of the mass and other systems parameters. We show that, while the current decreases monotonously with the stability index of the Lévy noise, there exists a particular intermediate value of such parameter (slightly dependent on the mass) that minimizes the time required to form a coherent particle package advancing in the preferred direction. Moreover, we show the possibility of observing mass separation. This means that particles of different masses may advance in opposite directions when influenced by the same ratchet potential and the same Lévy noise. Finally, we show that the ratio of the advanced distance to the total distance travelled constitutes a relevant measure for the rectification power, useful not only for Lévy ratchets but also for general ratchets systems. In particular, we find that it behaves quite similar to the rectification efficiency for standard models of rocking and flashing ratchets found in the literature.

1. Introduction

Unbiased fluctuations can induce directed transport of particles in anisotropic spatially periodic systems. Such an effect, often referred to as rectification of fluctuations or noise induced transport, is of relevance for many problems in physics [1,2] and biology [3], and also, it is an interesting starting point for the development of technological devices [2].

The basic models consider a particle in one dimension under the influence of an external anisotropic periodic potential (generically called *ratchet potential*) plus thermal noise, and different kinds of input signals or fluctuations generating non equilibrium conditions [2]. The more common input signals considered in the literature [1,2] are time periodic or time correlated random processes, that may enter as multiplicative modulations of the ratchet potential (*flashing ratchets*) or as additive forces (*rocked ratchets*).

Recent papers have analyzed the overdamped dynamics of a particle in a static ratchet potential driven by additive white Lévy noises [4–6]. The general problem of transport in ratchets under the influence of Lévy noises is of relevance from various perspectives. Firstly, there is an intrinsic theoretical interest in generalizing previous ratchet models to include cases where fluctuations present long tailed probability distribution functions (pdf), giving rise to anomalously large particle displacements or accelerations. Among non Gaussian noises, Lévy noises occupy a notable place because of the properties of stability and self-affinity of Lévy pdf [7], that make Lévy type statistics ubiquitous, and of interest for many fields of science and technology [8]. A remarkable characteristic of the dynamics of particles driven by Lévy noises in ratchets is that, in a generic situation, the mean square velocity of particles is divergent. Moreover, depending on the parameters defining the associated Lévy distribution, even the mean velocity may result divergent. These facts give way to the additional

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challenge of finding appropriate quantities for characterizing transport, as was discussed in Ref. [4]. From another point of view, as was pointed out in some detail in Ref. [5], Lévy ratchets may be of direct interest for problems in magnetically confined fusion plasmas. Recently, a model based on the ratchet effect was proposed for describing transport of impurities in fusion plasmas [9], while the relevant role of Lévy statistics in such systems is very well established from theoretical and experimental observations [10].

Interestingly, results in Refs. [4,5] have shown that a single additive symmetric white Lévy noise is enough to produce directed transport of particles in a static ratchet potential, with no extra forcing. The preferred direction of motion is towards the steepest slope of the potential. These facts were explained in Ref. [4] as consequences of the large Lévy jumps, which lead the particles to the flatter zones of the potential with higher probability than to the steeper zones. Studies in Ref. [4] focus on the characterization of transport in situations where not only the mean square velocity but even the mean velocity is divergent. New quantities such as the *group velocity*, measuring the directionality of transport, and the *interquantile distances* measuring the spread of particles in random directions are introduced. The work in Ref. [5] provides not only Langevin but also Fokker Planck results for the overdamped Lévy rocked ratchet. The authors analyze in great detail the dependence of the emerging current on the external (load) force, on the degree of asymmetry of the ratchet potential, and on the parameters defining the Lévy noise.

In this paper we go beyond the overdamped limit and study the influence of the mass on the dynamics of particles in the Lévy rocked ratchet. In particular, we show the possibility of observing mass separation. This means that particles of different masses may advance in opposite directions when influenced by the same ratchet potential and the same Lévy noise. Our results also provide complementary information on the velocity distributions and on the evolution of the coherence of the particle distributions that was not given in previous studies (not even for the overdamped limit). In order to characterize transport we analyze the time evolution of an ensemble of particles following an approach similar to that on [4], and we study different quantities measuring directionality and dispersion. In the last part of our work we study the ratio of the advanced distance to the total distance travelled by a single particle as a measure of the rectification power. By this we mean, the quality or *efficiency* of the transport mechanism, although we prefer to book the word efficiency for quantities that take into account the energetics of the system. We find the mentioned quantifier especially appropriate for Lévy ratchets, for which commonly used rectification measures may result divergent, but also useful for general ratchet systems. In fact, we show that for standard ratchet models it behaves quite similar to the *rectification efficiency*. Our studies on this item are therefore of relevance beyond the particular line on Lévy ratchets.

The organization of the paper is as follows. In Section 2 we present the model. In Section 3 we discuss the results for the velocity distributions. Section 4 contains the main analysis of the dependence of the transport properties on the mass and on the parameters defining the Lévy noise. In Section 5 we analyze the mass separation phenomenon. Finally, in Section 6 we study the ratio of the advanced distance to the total distance travelled as a measure of the rectification power. Section 7 is devoted to our conclusions.

2. Model

We consider a system described by the Langevin equation

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -U'(x) - \frac{\mathrm{d}x}{\mathrm{d}t} - L + \xi_{\alpha,\sigma}(t), \tag{1}$$

where $-U'(x)$ is the ratchet force derived from the periodic asymmetric potential

$$U(x) = \frac{1}{2\pi}\sin 2\pi x + \frac{1}{8\pi}\sin 4\pi x, \tag{2}$$

and $\xi_{\alpha,\sigma}(t)$ is the white Lévy noise, whose precise definition in terms of the parameters α and σ we give below. The spatial coordinate x and the time variable t can be considered as non dimensional, and scaled in order to fix the values of the period of the potential and of the friction coefficient equal to one. The free parameters of the system are the scaled mass m, the scaled load force L, and the parameters α and σ defining the Lévy noise. Note that the amplitude and shape of the potential has been fixed. Our choice is such that, for $L = 0$, in the limit $m \to 0$ we recover the system studied in Ref. [4].

The white Lévy noise $\xi_{\alpha,\sigma}(t)$ can be defined in terms of the properties of its integral $W(t) = \int_{t_0}^{t} \xi_{\alpha,\sigma}(t')\mathrm{d}t'$, which is a generalized Wiener process with non Gaussian, stationary and independent increments. Given a time interval $[t_0, t_{max}]$ and a time discretization $t_0, t_1, \ldots, t_N = t_{max}$ with $t_j = j\delta t$, in the limit of high N (i.e. infinitesimal δt), the increments of the Wiener process satisfies

$$W(t_{j+1}) - W(t_j) = (\delta t)^{1/\alpha} y_j. \tag{3}$$

Here, the y_j $(j = 0, \ldots N-1)$ are independent random numbers distributed with the Lévy probability distribution function $L_{\alpha,\sigma}(y)$, which is defined through its Fourier transform

$$\phi(k) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky} L_{\alpha,\sigma}(y)\mathrm{d}y = \exp\left[-\sigma^{\alpha}|k|^{\alpha}\right]. \tag{4}$$

The parameters α $(0 < \alpha \leqq 2)$ and σ $(0 < \sigma)$ are called respectively stability index and length scale. The first defines the decaying power law of the Lévy distribution $L_{\alpha,\sigma}(y) \sim |y|^{-(\alpha+1)}$ for $y \to \pm\infty$ $(\alpha < 2)$, while the second acts for our

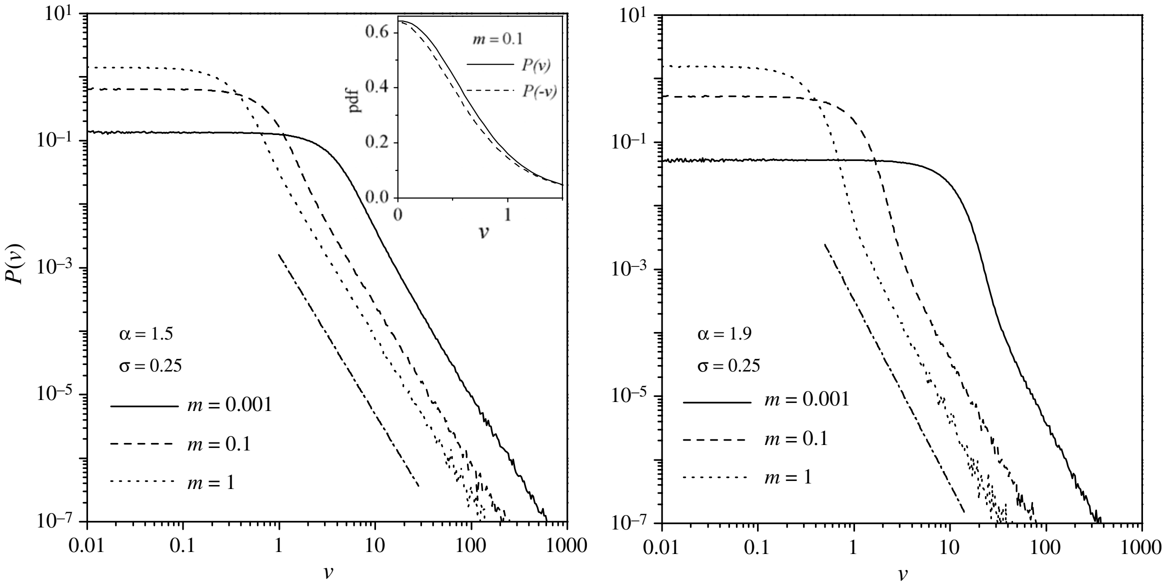

Fig. 1. Velocity distributions for different values of α and m. In each panel, the dash–dotted segment below the curves indicates the corresponding $v^{-(1+\alpha)}$ decay law. The inset in the left panel shows the asymmetry of the distribution. Results were obtained from 10^4 realizations of running time equal to 100 with steps of 10^{-3}.

purpose as a noise intensity. In the limit $\alpha = 2$ the Lévy distribution is the normalized Gaussian distribution with zero mean and variance σ^2. It is worth mentioning that more general white Lévy noises can be defined in terms of the general Lévy distributions $L_{\alpha,\mu,\beta,\sigma}(y)$ that include the location parameter (μ) and the skewness parameter (β) [5,7]. Eq. (4) corresponds to the choice $\mu = 0$ and $\beta = 0$, which implies a symmetric (pair) distribution $L_{\alpha,\sigma}(y)$.

Numerical solutions of Eq. (1) were obtained by using a standard algorithm (see Ref. [4] and references in Refs. [4,5]). Numbers distributed according to Lévy laws were obtained by the method explained in Ref. [11]. We have carefully checked both the Lévy generator – by comparing the obtained pdfs to numerical distributions computed with a high precision integrator –, and the solutions of the dynamical equation (1) by analyzing convergence for varying time steps and reproducing the main results in Refs. [4,5]. The algorithm for the second order differential equation correctly reproduces the results of the overdamped dynamics when considering small enough values of m (typically $m \lesssim 10^{-3}$). Most of the results shown in this paper were obtained using time steps equal to 10^{-3}.

3. Velocity distributions

Similarly to what occurs in the overdamped regime, results for the general case $m \neq 0$ show that the preferred direction of motion for vanishing or small load forces is towards the steepest slope of the potential, which is to the right when considering the potential of Eq. (2).

The distribution for instantaneous velocities is found to decay as $|v|^{-(1+\alpha)}$ for $v \to \pm\infty$ independently of the value of m, meaning that its second moment diverges for $\alpha < 2$ while its mean value diverges for $\alpha < 1$. In this work we will focus our analysis mainly on the region $1 < \alpha \leq 2$. However, some results and a discussion on the behavior for $\alpha < 1$ will also be considered as relevant. In Fig. 1 we show results for the distribution of instantaneous velocities for different values of α and m. It can be seen that, as m is increased at a fixed value of α, the pdf becomes narrower and the asymptotic behavior is reached at lower values of $|v|$. This could be expected, since the asymptotic behavior is determined by the contribution of very large fluctuations of the Lévy forcing, that make the effect of the ratchet potential temporarily negligible. Such large fluctuations, however, produce smaller accelerations and, ultimately, smaller velocities, when acting on particles of larger masses. The inset in the top panel shows that the distribution is perfectly symmetric for high values of $|v|$. This is because a very large Lévy fluctuation – no matter its sign – makes the particle behave effectively as free, until its velocity is reduced by damping, or changed by the action of subsequent fluctuations. The asymmetry in the velocity distribution responsible for the directed motion occurs at intermediate values of $|v|$, which are normally found in the absence of large fluctuations, when the ratchet force is relevant.

4. Advance, dispersion and coherence of particle concentration

Ratchet systems always entail an interplay between directed and diffusive-like motion. The quality or power of the rectification mechanisms depends on the relative weight of these components of transport. Hence, it is always important to

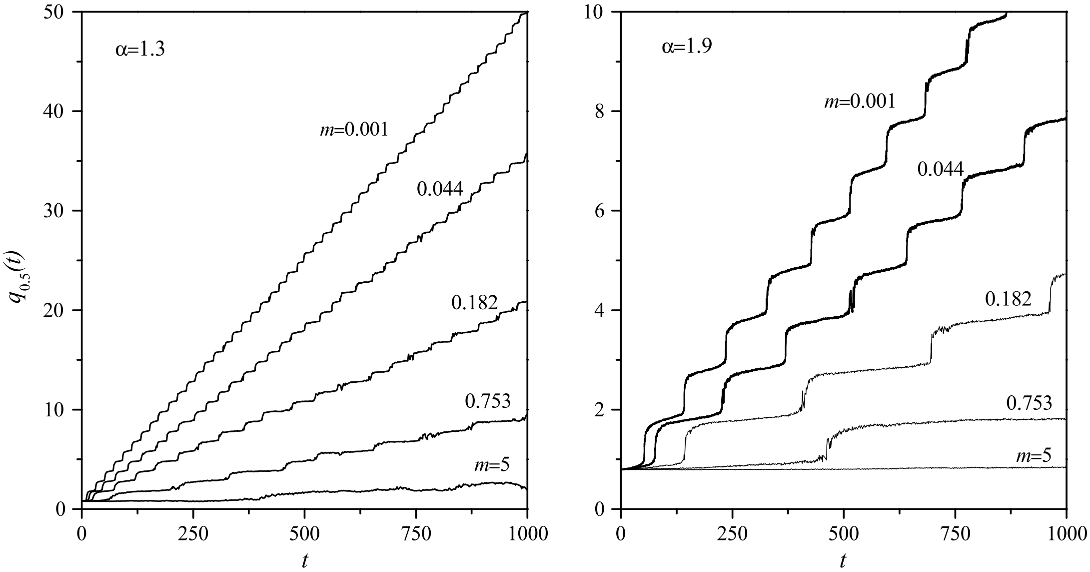

Fig. 2. Evolution of $q_{0.5}(t)$. Results correspond to simulations of 2×10^4 particles for the case $\alpha = 1.3$ and 5×10^3 particles for the case $\alpha = 1.9$.

characterize not only the advance of particles in the preferred direction (typically done in terms of the mean velocity $\langle v \rangle$), but also the way in which particles spread in random directions. The most natural quantity characterizing the latter effect is the variance of the velocity distribution. For Lévy forced ratchets, however, alternative quantities have to be considered due to the mentioned divergences in the momenta of the velocity pdf.

In order to study the transport properties of the Lévy ratchet of Eq. (1) we follow an approach similar to that in Ref. [4], which is appropriate for introducing transport quantifiers other than mean velocity and mean square velocity. We study the evolution of an ensemble of particles departing all from a fixed position $x_0 \simeq 0.809$, coincident with a minimum of the potential.

We call $P(x, t)$ the normalized pdf describing particle concentration, satisfying the initial condition $P(x, 0) = \delta(x - x_0)$. Following [4], let us define the *quantiles* $q_s(t)$ $(0 < s < 1)$ through the relation $\int_{-\infty}^{q_s(t)} P(x, t)\mathrm{d}x = s$.

The overall movement of particles toward the preferred direction can be characterized in terms of the advance of $q_{0.5}(t)$, which is just the median of $P(x, t)$. The quantity of interest is the group velocity [4], that we call v_g, and is defined as

$$v_g = \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\mathrm{d}q_{0.5}}{\mathrm{d}t}(t)\mathrm{d}t = \lim_{T\to\infty} \frac{(q_{0.5}(T) - x_0)}{T}. \tag{5}$$

In our simulations we consider a number of particles that typically ranges from 5×10^3 for $\alpha = 1.9$, to 8×10^4 for $\alpha = 1.1$. Except when otherwise indicated, we consider $\sigma = 0.25$.

In Fig. 2 we show the time evolution of $q_{0.5}(t)$ for two different values of α and several values of m. The advance occurs through periodic jumps of length $\Delta q_{0.5} = 1$, coincident with the period of the ratchet potential, as was observed in Ref. [4] for the overdamped regime. The average advance is clearly linear in time, meaning that the definition of the group velocity as the average slope of the $q_{0.5}(t)$ curve makes perfect sense. By analyzing curves in both panels of Fig. 2 it can be inferred that the group velocity decreases both with α and m. These tendencies are clearly confirmed by the results in Fig. 3, where we show v_g as function of m for several values of α (left panel), and v_g as function of α for three values of m (right panel). The dependence of v_g on α at fixed m is analogous to that described in Ref. [4] for the overdamped regime. In particular, it exhibits the well known behavior of vanishing net transport for $\alpha \to 2$ (thermal noise limit).

Note that, as we are focusing on the region $\alpha > 1$, for which the mean velocity $\langle v \rangle$ is well defined, we could have considered $\langle v \rangle$ instead of v_g for quantifying the advance of particles in the preferred direction. However, we find that, while results from simulations for both quantities are statistically indistinguishable, v_g has better properties of convergence (specially for values of α close to $\alpha = 1$). Thus, it results in a better and more robust quantifier.

The dispersion of particles in random directions is directly related to the width of the distribution. Being the variance of $P(x, t)$ generically divergent, the analysis of the growth of the width can be made in terms of the quantiles q_s. In Fig. 4 we show the evolution of $q_{0.1}$, $q_{0.2}$, $q_{0.4}$, $q_{0.5}$, $q_{0.6}$, $q_{0.8}$ and $q_{0.9}$ for four different values of the mass at a fixed value of α. Let us first analyze the results for $m = 0.1$ shown in the top right panel. It can be seen that, at short times, the width of $P(x, t)$ rapidly enhances, with the consequence that $q_{0.1}$ and $q_{0.2}$ decrease to negative values. This means that about 20% of the particles moves effectively backwards. However, at longer times, $q_{0.2}$ and $q_{0.1}$ attain minimal values ($q_{0.2}$ does it first), and they start to increase towards positive values following the general tendency of forward propagation.

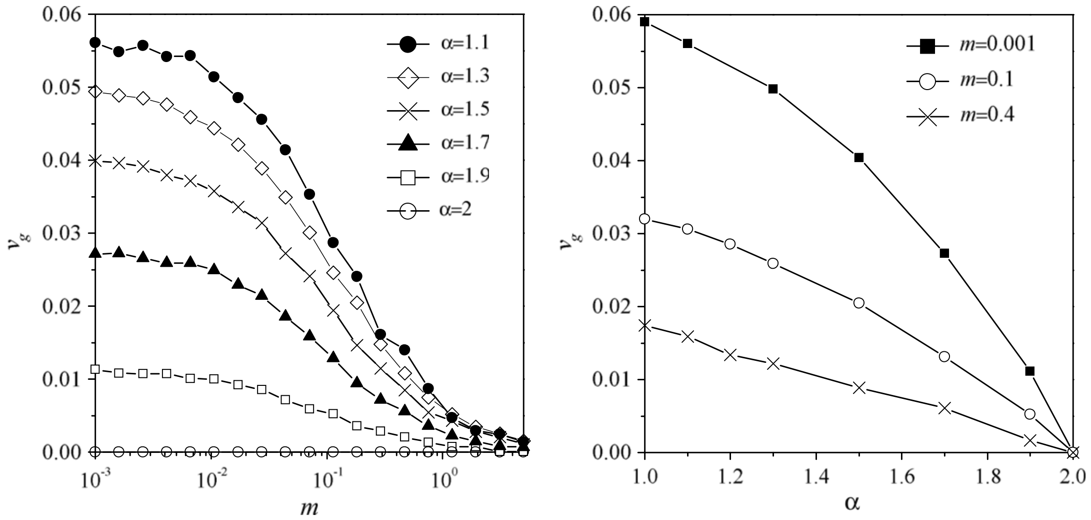

Fig. 3. Group velocity as function of m (left) and group velocity as function of α (right) for different values of the parameters. Results for $m = 0.001$ are very close (indistinguishable in the scale of the figure) to those for the overdamped limit. Compare the $m = 0.001$ curve in right panel with results in Figure 7 of Ref. [4] for $\sigma = 0.25$.

The results for different values of the mass are completely analogous to those just described. However, the dynamics becomes slower for increasing m, in the sense that the times needed for $q_{0.2}$ and $q_{0.1}$ to reach positive values increase with m. This can be seen by comparing the curves on the four panels of Fig. 4, which are all in the same time scale. Note that the initial back propagation can be of several periods of the potential. For instance, it is of about 10 periods for $q_{0.2}$ and 30 periods for $q_{0.1}$ for the case in the bottom left panel.

Provided that we consider $\alpha > 1$, the quantile evolution found for different values of α is also similar to that just described. When varying α in such domain the only appreciable effect is that the times needed for $q_{0.2}$ and $q_{0.1}$ to get to positive values increase with decreasing α (in a similar fashion as when m is increased). This effect can be appreciated in Fig. 5.

For $\alpha < 1$ there is an important qualitative change in the dynamics that we will shortly explain. In Ref. [4] it has been show that, in the overdamped regime, the interquantile distances defined as $d_s(t) = q_{1-s}(t) - q_s(t)$ $(0 < s < 0.5)$ – for instance $d_{0.1} = q_{0.9}(t) - q_{0.1}(t)$, $d_{0.2} = q_{0.8}(t) - q_{0.2}(t)$, etc. –, grow asymptotically as $t^{1/\alpha}$, independently of s. We find that this is also true for $m \neq 0$ (at least in a very good approximation), as we show in Fig. 6 for the case of d_1. This result has the following consequences. For $\alpha > 1$, the advance of the median is faster than the growth of the width of $P(x, t)$ measured in terms of interquantile distances. Thus, at long times, essentially all the density is located at $x > x_0$. More precisely, we have that for an arbitrary small s it is always possible to find a long enough time τ_s such that $q_s(t) > x_0$ holds for every $t > \tau_s$. In contrast, for $\alpha < 1$, the advance of the median $(q_{0.5}(t) - x_0)$ found at long times results negligible compared to the width of the distribution. Thus, $P(x, t)$ results essentially symmetric around $x = x_0$, so that the transport mechanism becomes highly inefficient for $\alpha < 1$.

It is worth mentioning that the interquantile distances are found to satisfy very well the $t^{1/\alpha}$ growing law only for $\alpha \lesssim 1.5$. For higher values of α, a small departure from this law (to a slower one) is observed. Look closely at the long time behavior of d_1 for cases $\alpha = 1.7$ and $\alpha = 1.9$ in Fig. 6. Nevertheless, these differences found at relatively high values of α do not affect the discussion on the behavior around $\alpha = 1$.

As mentioned at the beginning of this section, the search for optimal rectification mechanisms will always involve the commitment between maximizing directed motion and minimizing dispersion. The results in terms of v_g and interquantile distances in the domain $1 < \alpha < 2$ show that both, directed motion and dispersion, increase with decreasing α. Thus, the possibility of finding intermediate values of α satisfying different optimization criteria is open. We will shortly do so when analyzing the coherence of particle advance and also in Section 6, when studying the ratio of the advanced distance to the total distance travelled. Note that the limits $\alpha \to 1$ and $\alpha \to 2$ should be recognized as extremely inefficient mechanisms by any measure of rectification. The first due to the huge dispersion of particles before indicated, while the second due to the absence of directed motion.

The time τ_s before defined as that at which q_s $(s < 0.5)$ surpasses the position x_0 is an interesting parameter to analyze, since it provides an additional time scale for the dispersive processes. For instance, consider the case $s = 0.2$. The corresponding time $\tau_{0.2}$ is such that, for $t > \tau_{0.2}$, more than 80% of the particles are advancing in the forward direction. Conversely, for $t < \tau_{0.2}$, more than 20% of the particles are propagating backward. Hence, τ_s represents a sort of *coherence time* that takes the system to organize a package containing the $(1 - s)$ fraction of the particles in the ensemble, to move in the preferred direction.

In Fig. 7 we show results for $\tau_{0.2}$ as function of α for different values of m. It can be seen that $\tau_{0.2}$ attains a minimum at a value $\alpha \equiv \alpha_0$, which is close to 1.7 for $m = 10^{-3}$ (i.e. near the overdamped situation), and increases slightly with m.

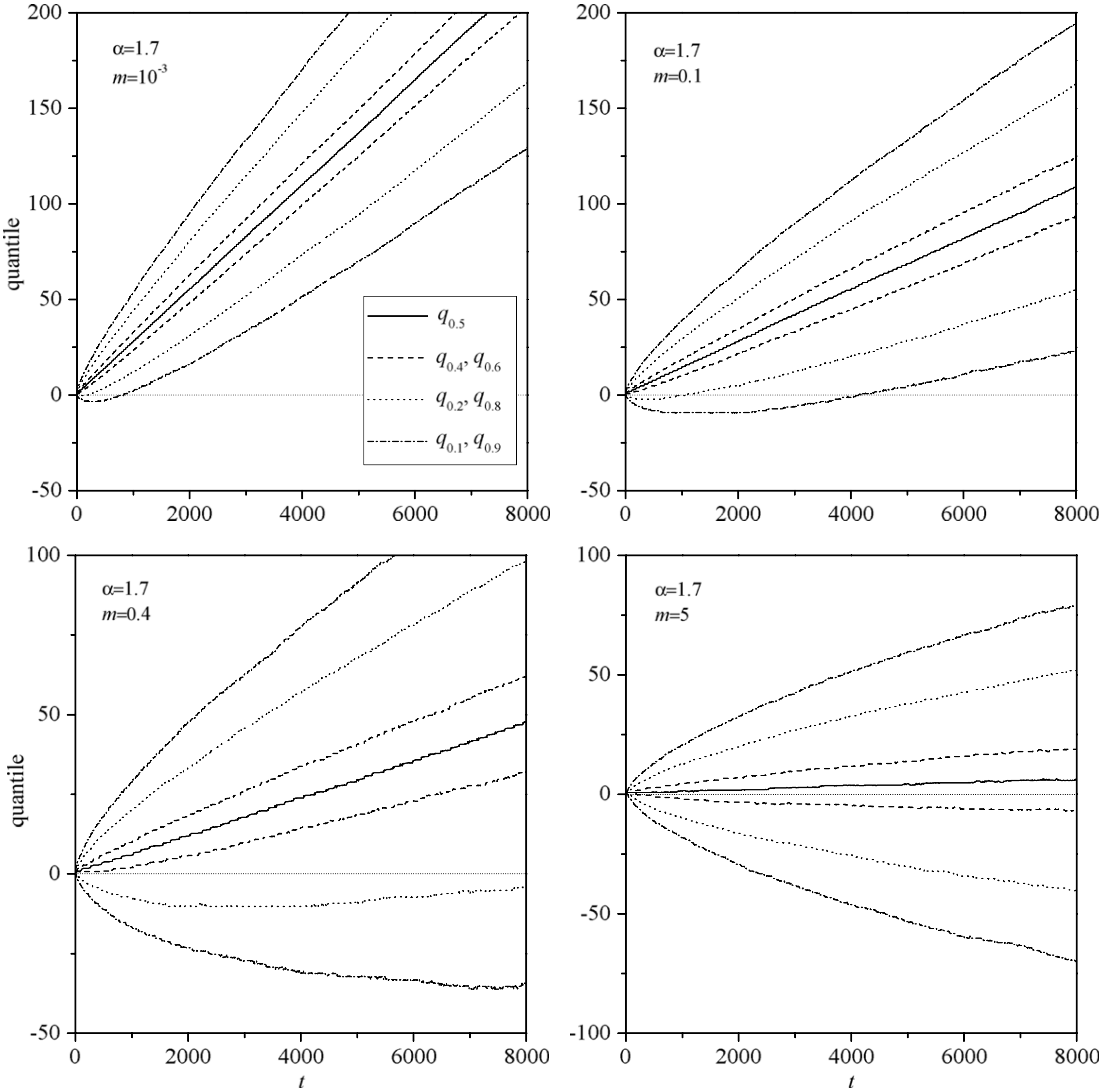

Fig. 4. Evolution of $q_{0.1}$, $q_{0.2}$, $q_{0.4}$, $q_{0.5}$, $q_{0.6}$, $q_{0.8}$ and $q_{0.9}$ for different values of m for $\alpha = 1.7$. Results for 1.5×10^4 particles.

Note that we have only analyzed relatively small values of m since $\tau_{0.2}$ grows very fast for increasing m and simulations become very time consuming. The results are found to be robust to variations of s. For instance, we have verified that $\tau_{0.1}$ has its minimum essentially at the same values of α as $\tau_{0.2}$. Thus, we see that, for relatively small values of m, the found value $\alpha_0 \sim 1.7$ is optimal concerning the minimization of the time required to form a coherent package that advances in the preferred direction. Note that the values of τ_s are expected to diverge for $\alpha \to 1$ and for $\alpha \to 2$ since in such limits q_s decreases monotonously with t and never reaches values greater than x_0.

The value of τ_s gives us a different (and complementary) characterization of the dispersive properties of the system to that provided by the asymptotic growing rates of the interquantile distances. In fact, it focus on the behavior at relative short times. Interestingly, the value of τ_s enables us to distinguish between the dispersive processes for different values of m, while the asymptotic growing rates of the interquantile distances are independent of m at fixed α. Concerning the behavior at fixed m we see that, while the growing rates of the interquantile distances decrease with α in the whole range $1 < \alpha < 2$, τ_s exhibit a non monotonous behavior. In particular, for $\alpha > \alpha_0$, τ_s increases with α, thus, although the transport mechanism becomes less dispersive in terms of interquantile distances, it becomes worse in what concerns the formation of a coherent package. Recalling that v_g always decreases with α, we see that the transport mechanism becomes considerably poor when increasing α in the region $\alpha_0 < \alpha < 2$. In contrast, for $1 < \alpha < \alpha_0$, dispersion in terms of interquantile distances and dispersion in terms of τ_s work together, thus, the opposition (and interplay) between enhancement of directed motion and decreasing of dispersion is clearly present in such region of parameters.

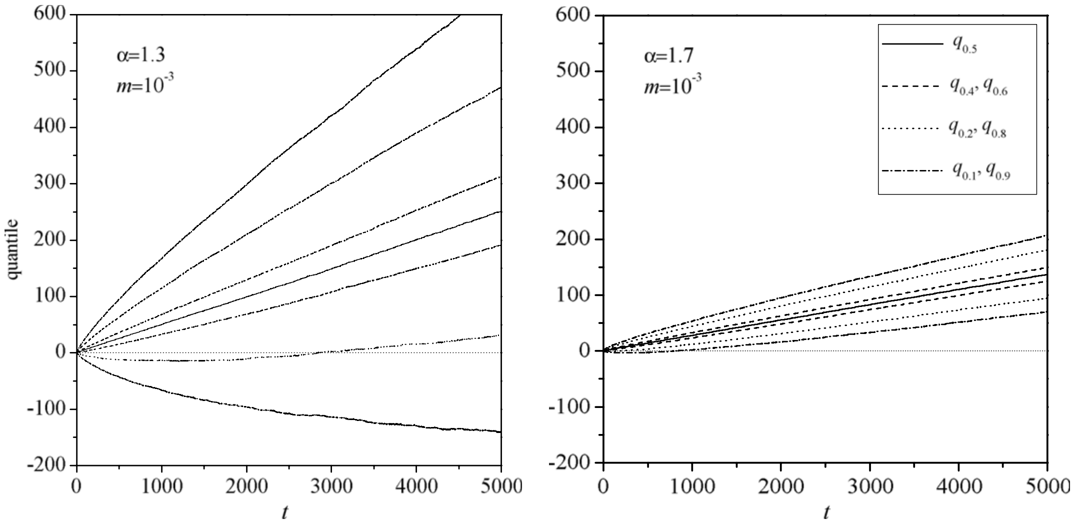

Fig. 5. Quantile evolution for different values of α. Results for $\alpha = 1.3$ and $\alpha = 1.7$ correspond respectively to simulations of 3×10^4 and 1.5×10^4 particles.

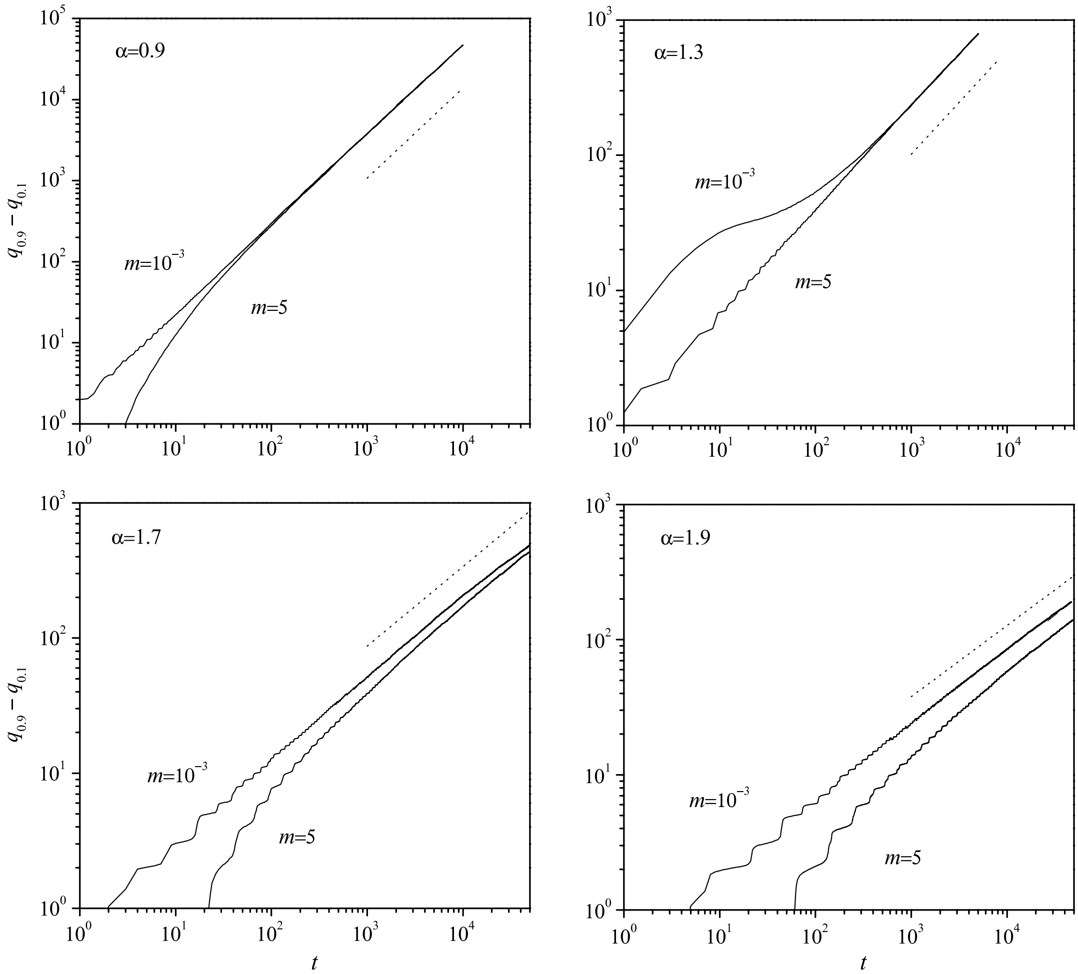

Fig. 6. Long time behavior of the interquantile distances for different values of α and m. In all the cases, the results for $10^{-3} < m < 5$ (not shown) lie between the shown curves for $m = 10^{-3}$ and $m = 5$. In each panel, the dotted segment indicates the corresponding $t^{1/\alpha}$ growing law. Note that the results shown for $\alpha = 0.9$ are exceptional for this work in which we focus on the region $\alpha > 1$.

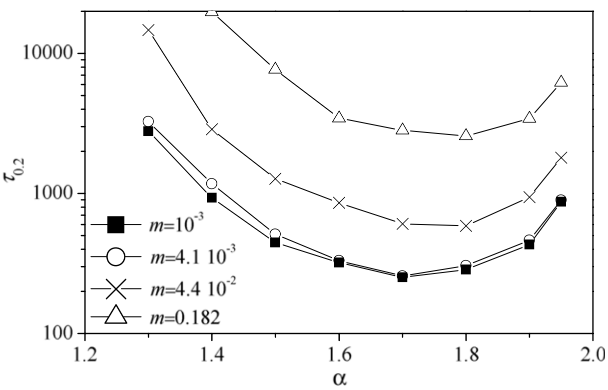

Fig. 7. Coherence time $\tau_{0.2}$ as function of α for different values of m.

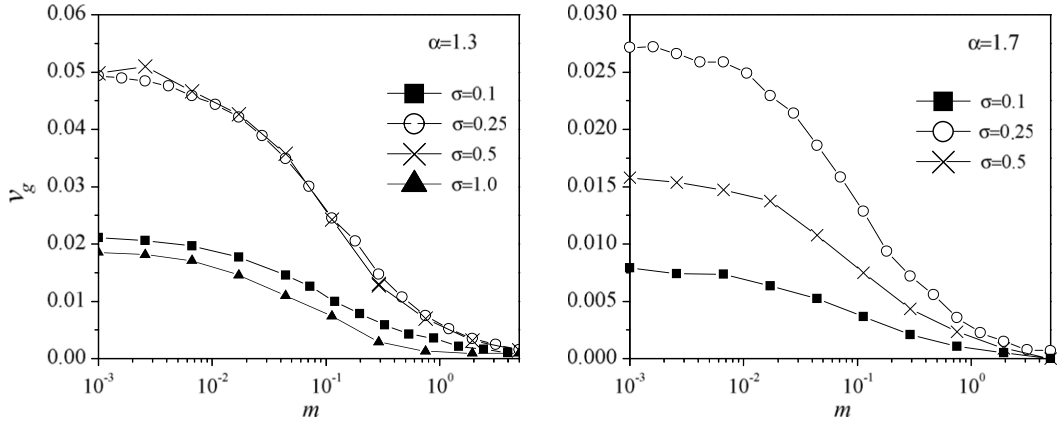

Fig. 8. Group velocity as function of m for different values of σ.

4.1. Dependence on σ

Up to now we have considered $\sigma = 0.25$ in all our calculations. As σ acts as a noise intensity, it should be expected that v_g attains a maximum for an intermediate value of such parameter and vanishes for $\sigma \to 0$ and for $\sigma \to \infty$. This was clearly confirmed in Ref. [5] for the overdamped situation. Here we present results that confirm it for the general case $m \neq 0$, and also show that the value $\sigma = 0.25$ is generic, or even of special interest. A complete analysis of the dependence of v_g, the interquantile distances, and τ_s on σ will however remain undone.

Fig. 8 shows v_g as a function of m for different values of σ and α. It can be seen that for the case $\alpha = 1.3$ (left panel) the results for $\sigma = 0.25$ and $\sigma = 0.5$ are quite similar, while v_g shows smaller vales for $\sigma = 0.1$ and $\sigma = 1$. For $\alpha = 1.7$ (right panel) the decay of the group velocity occurs at smaller values of σ, since $\sigma = 0.5$ leads us to considerably lower v_g than $\sigma = 0.25$. Thus, at least for the two generic values of α here considered, the value $\sigma = 0.25$ used throughout this work is close to an optimum value that maximizes v_g for the whole range of m.

5. Results for non zero load forces: mass separation

Here we analyze the dynamics of Eq. (1) for no vanishing load forces. As we have seen, for $L = 0$ the group velocity is always positive. It decreases monotonously with m and tends to zero for very high values of m. The inclusion of a positive load force always causes a reduction of the current. In particular, here we show that a relatively small load force $L > 0$ causes an inversion of current for high values of m. Hence, as we will see, it is possible to find situations in which particles of small masses advance preferably to the right while particles of large masses do it to the left, when considering the same ratchet system with the same Lévy noise and load force. A similar effect was reported before for a ratchet system with time correlated non Gaussian noises of a different type from Lévy ones [12]. Mass separation in ratchets was introduced first in Ref. [13], for a system with Gaussian noise.

In Fig. 9 we show v_g as function of m for different values of L and α. Panels (a) and (b) show how the v_g vs. m curve moves towards negative values when a load force $L > 0$ is applied. Curves for different values of L are found to be almost parallel each other. It can be seen that the limiting value of the mass, m_0, defined such that $v_g > 0$ for $m < m_0$ and $v_g < 0$ for $m > m_0$,

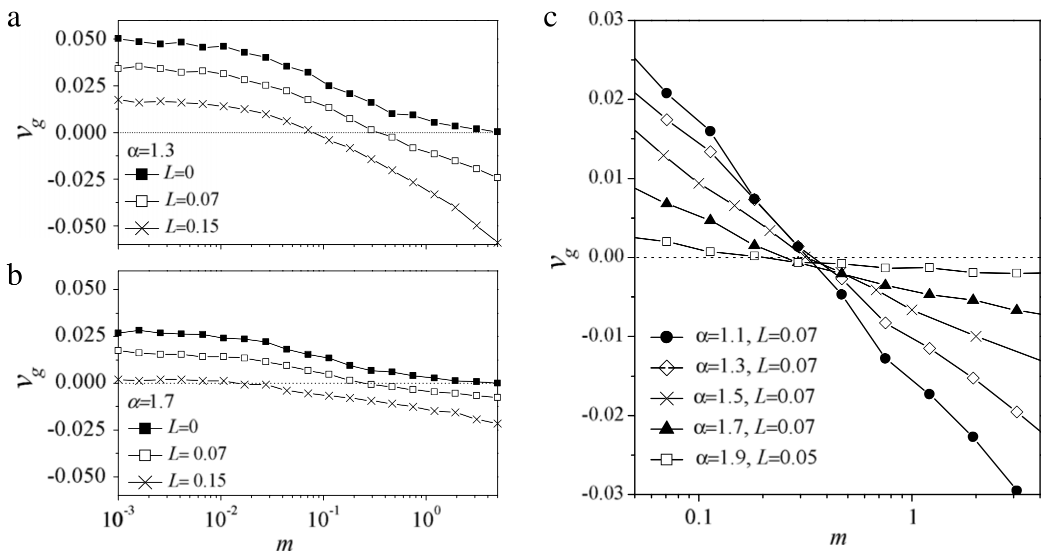

Fig. 9. v_g vs. m curves for different values of L and α showing the possibility of mass separation for $L > 0$.

decreases with increasing L. Results in Fig. 9(c) shows that for very different values of α, similar values of L ($L = 0.07$ or $L = 0.05$) produce separation of masses with almost the same limiting value of the mass ($m_0 \sim 0.35$ for the case studied). Moreover, it can be seen that the differences between the group velocities for systems with $m < m_0$ and $m > m_0$ increase for decreasing α. So, if we were interested on finding a mechanism for separating particles of two types, one with $m < m_0$ and the other with $m > m_0$, we would be tempted to think that smaller values of α would give the best results. However, it should be noted that the dispersion of particles plays a fundamental role in the process of separation of masses. In Fig. 10 we show the evolution of $q_{0.2}$, $q_{0.4}$, $q_{0.6}$ and $q_{0.8}$ for $m = 0.1$ and $m = 0.8$, both with $\alpha = 1.3$ (left panel) and $\alpha = 1.7$ (right panel). The rest of the parameters are the same as those in Fig. 9(c), with $m_0 \sim 0.35$. Clearly, when considering equal run times, the $P(x, t)$ distributions for the two values of masses result much better separated, i.e. with less overlap, for $\alpha = 1.7$ than for $\alpha = 1.3$. In general, for two given values of the mass $m_1 < m_2$, the search for an optimal mechanism for mass separation could be performed by maximizing the growing rate of, for instance, the distance $q_{0.4}(t, m_1) - q_{0.6}(t, m_2)$, in terms of the free parameters L, α and σ. Here, $q_s(t, m)$ simply represents the quantiles $q_s(t)$ for the corresponding value of the mass. Note that, as we assumed $m_1 < m_2$, $q_{0.4}(t, m_1)$ is expected to be positive and time increasing at long times, while $q_{0.6}(t, m_2)$ should be negative and time decreasing, provided an appropriate value of L is considered. A complete analysis of these features for varying values of all the parameters is out of the scope of this work, in which we aim at presenting simple results indicating the possibility of mass separation.

6. Advanced distance vs. total distance travelled as a measure of the rectification power

Up to now we have analyzed transport in terms of quantities that measure either the velocity of directed transport or the dispersive properties of the system. We now aim at introducing a single quantifier capable of weighting both effects in order to characterize the rectification power of the ratchet system.

Quantities measuring the rectification power usually found in the literature [2] such as the Peclet number [14,15] cannot be defined for the system in Eq. (1) because they depend on the variance of the velocity pdf. Moreover, the calculation (and even the definition) of rectification efficiencies taking into account energetic considerations, such as those introduced in Refs. [16,17], could involve important difficulties for the Lévy ratchet system, and is out of the scope of the present paper.

Here we propose considering the ratio of the advanced distance to the total distance travelled by a single particle as a simple measure of the rectification power. We call it f, and we define it by considering the corresponding time averages for the trajectory of a particle in an infinite time interval:

$$f = \lim_{T\to\infty} \frac{\int_0^T \dot{x}(t)\mathrm{d}t}{\int_0^T |\dot{x}(t)|\mathrm{d}t} = \lim_{T\to\infty} \frac{x(T) - x_0}{\int_0^T |\dot{x}(t)|\mathrm{d}t}. \tag{6}$$

The quantity could be perhaps interpreted as a *geometrical* or *kinematical* efficiency, however, as stated in the introduction, we prefer to book the name efficiency only for quantities with clear energetic meaning as those analyzed in Refs. [16,17].

In Fig. 11 we show results for f as a function of m for different values of α (left panel) and f as a function of α for three values of m (right panel). Optimal values of m and α maximizing f can be clearly identified. In particular, it can be seen that the optimal value of m for fixed α increases with α, and the optimal value of α for fixed m increases with m. Moreover, it can

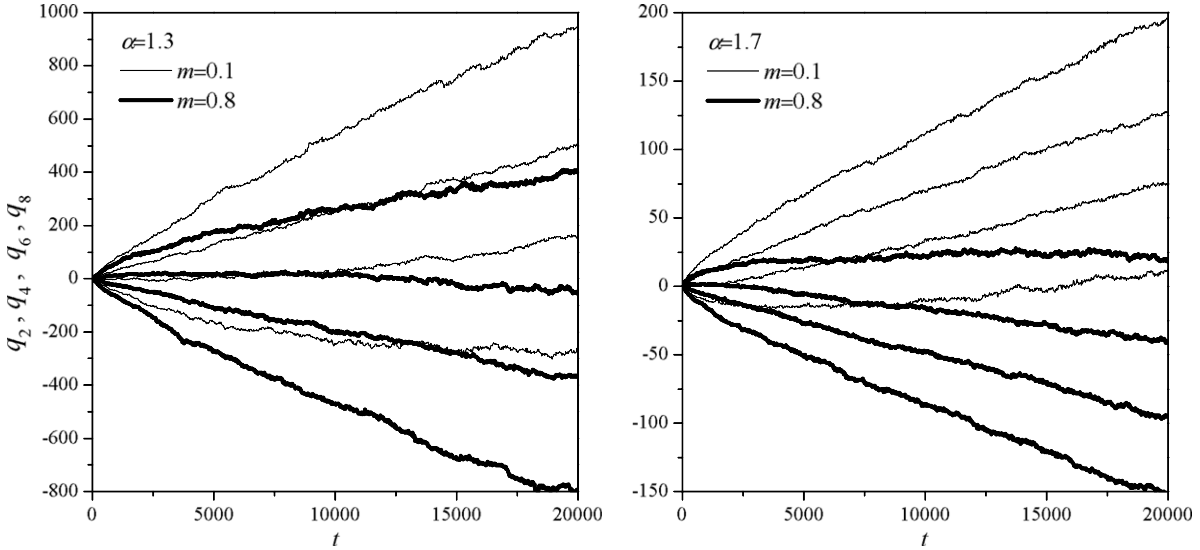

Fig. 10. Left: Evolution of $q_{0.2}$, $q_{0.4}$, $q_{0.6}$ and $q_{0.8}$ for $\alpha = 1.3$ computed for 1000 particles for two different values of m. For each value of the mass, from bottom to top the four curves correspond to $q_{0.2}$, $q_{0.4}$, $q_{0.6}$ and $q_{0.8}$. Right: Ibid for $\alpha = 1.7$.

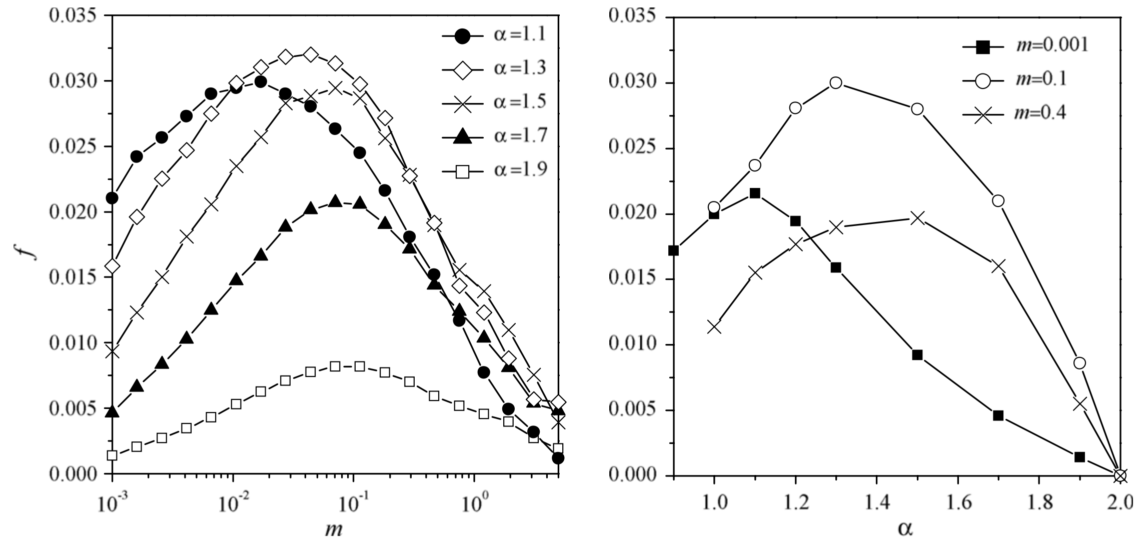

Fig. 11. Averaged advanced distance vs. averaged total distance travelled as function of m and α. In order to make easy the comparison of the results for f and v_g, the symbols in both panels were chosen to match with those in Fig. 3 for equal values of the parameters.

be inferred that an absolute maximization of f as a function of the two parameters is attained for $m \sim 0.04$, $\alpha \sim 1.3$. Note that the maximum of f as function of α for fixed m is always found at α lower than the value α_0 minimizing τ_s (at least for small enough m), and f results considerably small for $\alpha > \alpha_0$. Thus, the transport mechanism is found to be of poor quality in such a region, as was anticipated in Section 4 when analyzing the results on τ_s.

Although the meaning of f is quite intuitive, and its relevance as a measure of the rectification power is clear from its definition, it is interesting to test f in other ratchet systems for which other quantities characterizing efficiency can be defined. We have chosen as testing examples the systems studied in Refs. [16,17], for which authors have provided clear definitions of the rectification efficiency. They correspond respectively to a rocked ratchet and a flashing ratchet. Both models can be described by the single general equation $\ddot{x}+\gamma\dot{x} = -\Theta(t)V'(x) - L + a\cos(\omega t) + \sqrt{2\gamma T}\zeta(t)$, with appropriate redefinitions of the parameters. For the system in Ref. [16] we have constant $\Theta(t) = 1$ and $L = 0$. For the system in Ref. [17] we have $a = 0$, while $\Theta(t)$ is the periodic function with period $t_{on} + t_{off}$ which is equal to 1 for $0 \leq t < t_{on}$ and equal to 0 for $t_{on} \leq t < t_{on} + t_{off}$. For both systems $\zeta(t)$ is a zero mean Gaussian white noise, while $V(x)$ corresponds to a different periodic ratchet potential for each model [16,17]. The rectification efficiency, here referred to as η, is essentially defined as the dissipated energy associated with the directional movement against the frictional force, plus the work against the load

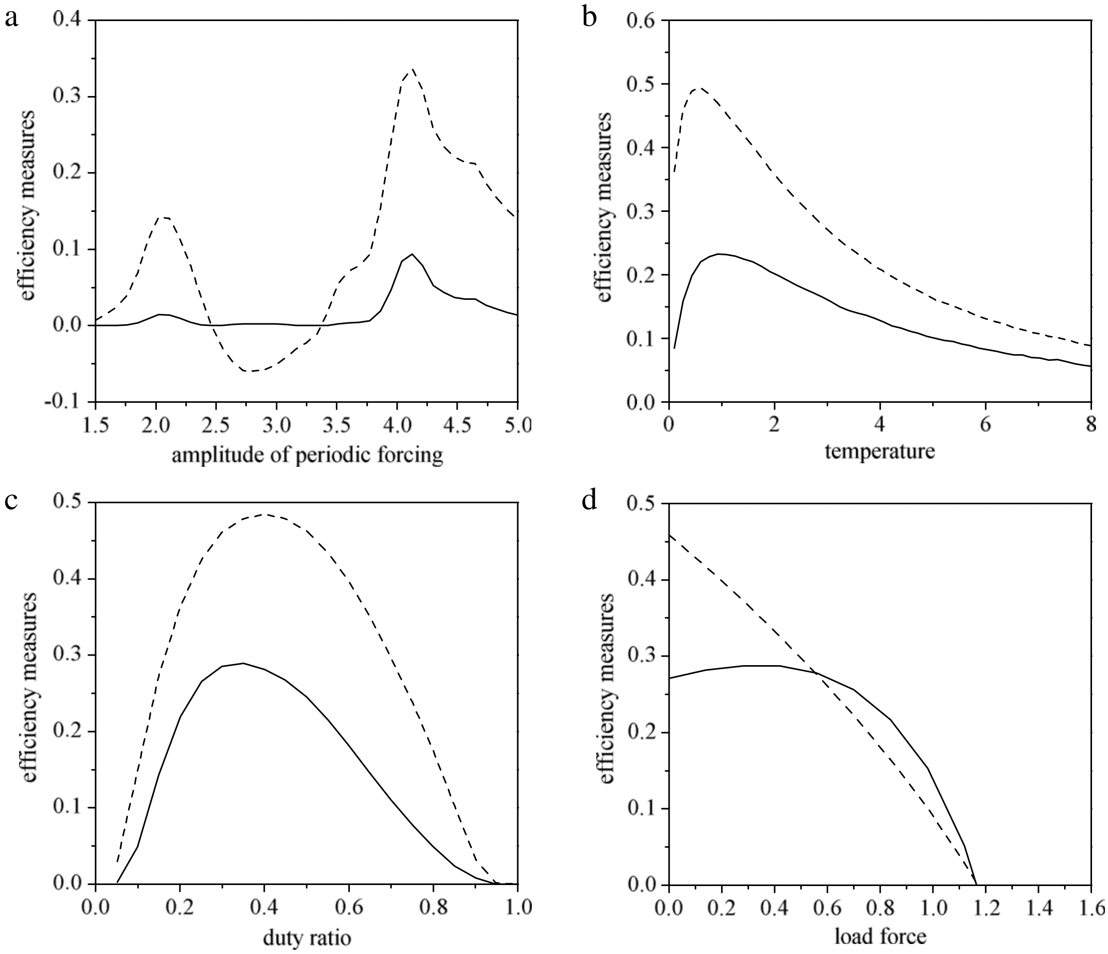

Fig. 12. Results for rectification efficiency (solid lines) together with results for f (dashed lines) as function of different control parameters for the systems in Refs. [16] (panel (a)) and [17] (panels (b)–(d)).

force, all divided by the input energy. Precise definitions suitable for each system are given in the corresponding references accompanied by detailed and interesting discussions on the energetics of the systems.

In Fig. 12 we show results for the rectification efficiency together with results for the function f defined in Eq. (6) for the models in Refs. [16,17]. Panel a shows η and f as functions of the amplitude of the periodic forcing a for the system in Ref. [16]. The curve for η is our reproduction of that on Figure 1(d) of Ref. [16]. It can be seen that η and f share the same behavior as functions of the parameter a. In particular, the identification of the optimal values of a can be done by analyzing any of the two functions. The only relevant difference between both quantities is that f becomes negative in the region where $\langle v \rangle$ is negative, while η remains positive. However, in such a region, the local maximum of η corresponds to a minimum of f. Fig. 12(b), (c) and (d) correspond to the system in Ref. [17]. They show η and f as functions of the temperature T, the duty ratio $t_{on}/(t_{on} + t_{of})$ and the load force L respectively. The results for η correspond to those in Figures 2, 3 and 4 of Ref. [17]. For the cases in panels b and c we find again that f behaves quite similar to η as a function of the corresponding parameters. In contrast, the dependence of both quantities on L shown in panel d are qualitatively different. This means that the behaviors of η and f seem to match only for $L = 0$, which was the value of the load force considered in panels a, b and c. In other words, f mimics solely the behavior of the work-against-friction contribution to the efficiency. It has to be noted, however, that finding measures of efficiency in the absence of a load force is quite relevant, as was particularly stressed in Ref. [16]. Moreover, for non vanishing load forces, the definition of f is still meaningful as a measure of the quality of transport, regardless of the work against the load force.

7. Conclusions and final remarks

We have studied the dynamics of particles in a rocked ratchet system forced by white Lévy noise beyond the overdamped limit. The special properties of the Lévy noises require a special treatment enabling the definition of suitable quantities for the characterization of transport. In the main part of our work we followed an approach similar to the one proposed in

Ref. [4] for the analysis of the overdamped situation. By means of numerical solutions of the Langevin equations, we studied the evolution of the particle concentration for an ensemble of particles departing all from the same position. Within such an approach, the overall advance of particles in the preferred direction is naturally characterized in terms of the *group velocity* [4] (even for cases of divergent mean velocity), while the dispersive processes can be characterized in terms of *interquantile distances* [4].

Firstly, we have found that the group velocity decreases monotonously with the particle mass for fixed stability index (α) of the Lévy noise. Conversely, for a fixed value of the mass, the group velocity decreases with α. An interesting finding is that, at short times, a non negligible fraction of the particles in the ensemble moves effectively backwards. This could also happen in other ratchet systems, but in the case of Lévy noises the initial back propagation can be of many periods of the ratchet potential and it can involve a relatively high fraction of the particles. However, at long enough times (excepting for the case $\alpha < 1$ for which transport is highly inefficient) forward propagation prevails, and an arbitrarily high fraction of the particles is found to advance in the preferred direction. These observations lead us to the definition of the *coherence time* τ_s $(0 < s < 0.5)$, as the time that takes the system to form a coherent package that advances in the preferred direction containing a fraction $(1 - s)$ of the particles in the ensemble. Such coherence time provides us with an additional characterization of the dispersive processes that complement the one given by the interquantile distances. In particular, it allows us to distinguish the dispersive phenomena occurring for different values of the mass.

We have also shown that, when considering non vanishing load forces, an inversion of current occurs for high values of m. It is therefore possible to find situations in which particles of small masses advance preferably to the right while particles of large masses do it to the left, when considering the same ratchet system with the same Lévy noise and load force. The capability of the system to effectively separate particles of different masses depends not only on the group velocity found for each value of the mass, but also on the dispersive phenomena.

Finally, in Section 6 we have proposed to consider the ratio of the advanced distance to the total distance travelled by a single particle as a measure of the rectification power. The quantifier is especially suitable for Lévy ratchets, for which commonly used measures of rectification may result divergent. Moreover, we have shown that for standard models of rocking and flashing ratchets with periodic forcing (and no Lévy noises), it behaves quite similar to the *rectification efficiency* [16,17] as a function of the different parameters of the systems. Hence, the relevance of the quantity as a measure of the quality of transport goes clearly beyond the specific case of Lévy ratchets.

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