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GAUGED LIFSHITZ MODEL WITH CHERN–SIMONS TERM

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We present a gauged Lifshitz Lagrangian including second- and fourth-order spatial derivatives of the scalar field and a Chern–Simons term, and study nontrivial solutions of the classical equations of motion. While the coefficient β of the fourth-order term should be positive in order to guarantee positivity of the energy, the coefficient α of the quadratic one need not be. We investigate the parameter domains and find significant differences in the field behaviors. Apart from the usual vortex field behavior of the ordinary relativistic Chern–Simons–Higgs model, we find in certain parameter domains oscillatory solutions reminiscent of the modulated phases of Lifshitz systems.

Keywords: Chern–Simons; vortex solutions; Lifshitz theories.

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1. Introduction

In their well-honored proposal to describe dual strings,[1] Nielsen and Olesen stressed the connection between the Abelian Higgs model and the Ginzburg–Landau theory of superconductivity, relating the free energy in the latter with the action for static configurations of the former. In this way, the vortex filaments of type-II superconductors were identified with string-like classical solutions in a gauge theory with spontaneous symmetry breaking.

More than 30 years ago, Ginzburg proposed[2] a generalization of the Ginzburg–Landau functional by including higher derivative terms, this implying an anisotropic coordinate scaling, in order to describe superdiamagnets — a class of materials with strong diamagnetism but differing from conventional superconductors. Such generalization was then used to analyze[3] the properties of superconductors near a

tricritical Lifshitz point, a point in the phase diagram at which a disordered phase, a spatially homogeneous ordered phase and a spatially modulated ordered phase meet.

The study of Lifshitz critical points has recently attracted much attention, not only in connection with condensed matter systems (see Ref. 4 and references therein) but also in the analysis of gravitational models in which anisotropic scaling leads to improved short-distance behavior (see Ref. 5 and references therein). A link between these two issues was established in Ref. 6 within the framework of the gauge/gravity correspondence by searching gravity duals of nonrelativistic quantum field theories with anisotropic scaling, dubbed in Refs. 7 and 8 as "Lifshitz field theories."

The question that we address in this work is whether one can find Nielsen–Olesen-like solutions when anisotropic scaling is introduced in the Abelian Higgs model through the addition of higher-order spatial derivatives. As a laboratory, we consider a $(2+1)$-dimensional model with a complex Higgs scalar coupled to a $U(1)$ gauge field with a Chern–Simons (CS) action.[9,10] The topological character of the CS term avoids the possibility of including higher-order derivatives for the gauge field action (as it would be the case for the Maxwell action).

When higher-order derivative terms in the scalar Lagrangian are absent, the CS–Higgs model has vortex-like finite energy solutions carrying both quantized magnetic flux Φ and nontrivial electric charge $Q = -\kappa\Phi$ with κ the CS coefficient.[11–13] Moreover, for an appropriate sixth-order symmetry breaking potential, first-order BPS equations[14–16] exist, which can be easily found by analyzing the supersymmetric extension of the model.[17] Our goal will be to determine whether this kind of solutions also exists in a "Lifshitz Abelian Higgs" model and, in the affirmative, how they depend on the parameters associated to the Lagrangian scaling anisotropy.

The plan of the paper is the following: we introduce, in Sec. 2, a $(2 + 1)$-dimensional Lifshitz–Higgs model with the gauge field dynamics governed by a CS term. In order to solve the classical equations of motion, we make the same ansatz leading to vortex solutions in the ordinary (relativistic) case. Then, in Sec. 3, we analyze the asymptotic behavior of the gauge and scalar fields resulting from the equations of motion, showing the existence of four regions according to the values of the parameters of the model. We discuss in Sec. 4, the properties of the solutions obtained numerically. Finally, we give a summary of the results and a discussion on possible extensions of our work in Sec. 5. We briefly describe in an appendix the linearized approximation we employed to determine the asymptotic behavior of the solutions in different parameter regions.

2. The Lagrangian

We consider a $(2 + 1)$-dimensional model with CS–Higgs Lagrangian

$$L = \gamma|D_0[A]\phi|^2 - \alpha|D_i[A]\phi|^2 - \beta|D_i[A]D_i[A]\phi|^2 + V[|\phi|] + \frac{\kappa}{2}\varepsilon^{\mu\nu\alpha}A_\mu\partial_\nu A_\alpha \quad (1)$$

with $\mu = 0, 1, 2$ and $i = 1, 2$. The metric signature is $(1, -1, -1)$. We consider space and time coordinate units so that

$$[x]^2 = [t] \,. \tag{2}$$

Accordingly, γ, β and κ are dimensionless and α has length dimensions $[\alpha] = -2$. Concerning the dimensions of the complex scalar ϕ and $U(1)$ gauge field A_μ, one has $[\phi] = 0$, $[A_i] = -1$, $[A_0] = -2$.

The Lagrangian (1) is a generalization of the one considered in Refs. 14 and 15 incorporating higher (fourth)-order covariant derivative terms for the scalar fields. For vanishing potential and at the "Lifshitz point" $\alpha = 0$, the Lagrangian is invariant under anisotropic scaling with "dynamical critical exponent" $z = 2$

$$x \rightarrow \lambda x \,, \quad t \rightarrow \lambda^2 t \,. \tag{3}$$

Note that the choice of a CS term ensures that the scale invariance is preserved even in the presence of gauge fields (as opposed to what would happen with a standard Maxwell term).

The covariant derivative D_μ acts on the scalar field ϕ according to

$$D_\mu[A]\phi = (\partial_\mu + ieA_\mu)\phi \tag{4}$$

with $[e] = 0$. The potential $V[\phi]$ is to be specified below.

Given the Lagrangian (1), one gets Gauss's law by differentiating with respect to A_0,

$$\kappa \varepsilon^{0ij} \partial_i A_j = j^0 \,, \tag{5}$$

where

$$j_0 = ie\gamma(\phi^* D_0 \phi - \phi D_0 \phi^*) = -2e^2 \gamma A_0 |\phi|^2 \,. \tag{6}$$

Defining

$$B = -\varepsilon^{ij} \partial_i A_j \tag{7}$$

one then has, using Eq. (5),

$$A_0 = \frac{\kappa}{2e^2\gamma} \frac{B}{|\phi|^2} \,. \tag{8}$$

Inserting this result in Eq. (6), one gets

$$j_0 = -\kappa B \tag{9}$$

so that the usual CS–Higgs model relation between charge Q and magnetic flux Φ holds

$$Q = \int d^2x \, j^0 = -\kappa \int d^2x \, B \equiv -\kappa\Phi \,. \tag{10}$$

The energy density $\bar{\mathcal{E}}$ associated to Lagrangian (1) is

$$\bar{\mathcal{E}} = \alpha |D_i[A]\phi|^2 + \beta |D_i[A]D_i[A]\phi|^2 + \frac{1}{4\gamma e^2}\frac{\kappa^2 B^2}{|\phi|^2} + V[|\phi|]\,. \tag{11}$$

A lower bound for the energy requires β to be positive while α can have any sign.

As stated before, in the $\beta = 0$, $\gamma = 1$ relativistic case and for a sixth-order symmetry breaking potential this theory is known to have, at the classical level, self-dual vortex solutions both in the Abelian case[14,15] and in its non-Abelian extension.[16]

In order to solve the Euler–Lagrange equations deriving from Lagrangian (1), we consider the static axially symmetric ansatz

$$\phi = f(r)\exp(-in\varphi)\,, \tag{12}$$

$$A_\varphi = -\frac{A(r)}{r}\,, \tag{13}$$

$$A_0 = A_0(r)\,, \tag{14}$$

with $n \in \mathbb{Z}$. Given this ansatz, the magnetic and electric fields read

$$B(r) = \frac{1}{r}\frac{dA(r)}{dr}\,, \quad E(r) = -\frac{dA_0}{dr}\,. \tag{15}$$

The equations of motion take the form

$$-\frac{\kappa}{r}\frac{dA(r)}{dr} + 2\gamma e^2 A_0(r)f^2(r) = 0\,, \tag{16}$$

$$\kappa\frac{dA_0}{dr} + \frac{4e^2\beta}{r}\left(\frac{n}{e}+A\right)f\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{e^2}{r^2}\left(\frac{n}{e}+A\right)^2\right)f$$
$$-\alpha\frac{2e^2}{r}\left(\frac{n}{e}+A\right)f^2 = 0\,, \tag{17}$$

$$\beta\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{e^2}{r^2}\left(\frac{n}{e}+A\right)^2\right)\left(\frac{d^2 f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{e^2}{r^2}\left(\frac{n}{e}+A\right)^2 f\right)$$
$$-\alpha\left(\frac{d^2 f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{1}{r^2}(n+eA)^2 f\right) - \gamma e^2 A_0^2(r)f = \frac{1}{2}\frac{\partial V}{\partial f}\,. \tag{18}$$

The potential V is in general chosen so as to allow for spontaneous symmetry breaking. In the relativistic $(2+1)$-dimensional case, the most general renormalizable self-interacting scalar potential is sixth-order and in fact to find the first-order BPS equations it should be of this order and take the form[14,15]

$$V = \frac{e^4\tau}{8\kappa^2}f^2(f^2 - v^2)^2 \tag{19}$$

with v the Higgs field vacuum expectation value (VEV) and the coupling constant τ has length dimensions $[\tau] = -4$. In the relativistic model, the first-order self-dual

equations exist at a certain value $\tau = \tau_{\text{BPS}}$ which would correspond in the present Lifshitz case to $\tau_{\text{BPS}} = 8\alpha^2\gamma$. From here on, unless otherwise stated, we shall take V as given in (19) and $\tau = \tau_{\text{BPS}}$ in order to compare the Lifshitz model results with those arising in the relativistic case.

3. Asymptotic Behavior

We start by discussing the conditions that we shall impose at the origin and at the boundary. We choose as conditions at the origin those leading to regular solutions in the relativistic case (see for example Ref. 14):

$$f(r) = f_0 r^{|n|}, \quad A_0(r) = a_0 + c_0 r^{2|n|}, \quad r \to 0, \quad A(r) = d_0 r^{2|n|+2}. \quad (20)$$

Note that a constant term a_0 in the $A_0(r)$ expansion is included in order to achieve consistency of Eq. (16) at the origin. Coefficients a_0 and d_0 are related according to

$$d_0 = \frac{e^2}{\kappa(|n|+1)} a_0 f_0^2. \quad (21)$$

Concerning large r, we write

$$f(r) \approx v + h(r), \quad (22)$$

$$A(r) \approx -\frac{n}{e} + a(r), \quad r \to \infty, \quad (23)$$

$$A_0(r) \approx a_0(r) \quad (24)$$

with $h(r)$, $a(r)$ and $a_0(r)$ small fluctuations. We then linearize the equations of motion which reduce to

$$-\beta \nabla_r^2 \nabla_r^2 h(r) + \alpha \nabla_r^2 h(r) - \sigma h(r) = 0, \quad (25)$$

$$-\frac{1}{r}\frac{da(r)}{dr} + \gamma\mu a_0(r) = 0, \quad (26)$$

$$\frac{da_0(r)}{dr} - \frac{\alpha\mu}{r} a(r) = 0, \quad (27)$$

where

$$\nabla_r^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}, \quad (28)$$

$$\sigma = \frac{e^4 \tau v^4}{2\kappa^2}, \quad \mu = \frac{2e^2 v^2}{\kappa}. \quad (29)$$

Equations (26) and (27) can be written as two decoupled second-order equations

$$\frac{d^2 a_0}{dr^2} + \frac{1}{r}\frac{da_0}{dr} - \alpha\gamma\mu^2 a_0 = 0, \quad (30)$$

$$\frac{d^2 a}{dr^2} - \frac{1}{r}\frac{da}{dr} - \alpha\gamma\mu^2 a = 0. \quad (31)$$

First, we deal with the scalar field behavior. After writing

$$h(r) = \frac{h^0}{\sqrt{r}} \exp(qr) \,, \tag{32}$$

with h^0 a constant, the solutions are determined from the equation

$$q_{\pm}^2 = \frac{1}{2\beta} \left(\alpha \pm \sqrt{\alpha^2 - 4\beta\sigma} \right) . \tag{33}$$

The asymptotic behavior of the scalar field is then given by

$$f(r) \approx v + \frac{h^0}{\sqrt{r}} \exp(-q_{\pm}r) \,. \tag{34}$$

From the results, one can see that there is a critical value for β

$$\beta_{\text{crit}} = \frac{\alpha^2}{4\sigma} \tag{35}$$

above which $q_{\pm}^2$ become imaginary.

In the region $\alpha > 0$ and $\beta < \beta_{\text{crit}}$, the solutions for $q_{\pm}$ are real. In particular, for $\beta\sigma \ll \alpha^2$

$$q_+^2 \approx \frac{\alpha}{\beta} \,, \quad q_-^2 \approx \frac{\sigma}{\alpha} \,. \tag{36}$$

Note that q_- coincides with the standard relativistic case solution where it plays the role of the Higgs field mass.[14]

Concerning the region $\alpha \geq 0$ and $\beta > \beta_{\text{crit}}$, one has

$$q_{\pm}^2 = \frac{1}{2\beta} \left(\alpha \pm 2i\sqrt{\beta\sigma}\sqrt{1 - \frac{\alpha^2}{4\beta\sigma}} \right) \tag{37}$$

which gives a complex solution. This region corresponds to underdamped oscillations of the Higgs field. We can write this as

$$q_{\pm}^2 = \frac{\sigma}{\beta} e^{\pm i\chi} \,, \tag{38}$$

where

$$\tan(\chi) = \frac{\sqrt{4\beta\sigma - \alpha^2}}{\alpha} \,. \tag{39}$$

The solution is therefore

$$h = \frac{h^0 \exp\left(-\lambda r\right)}{\sqrt{r}} \cos(kr + \delta) \,, \tag{40}$$

where

$$\lambda = \sqrt{\frac{\sigma}{\beta}} \left| \cos\left(\frac{\chi}{2}\right) \right| \,, \quad k = \sqrt{\frac{\sigma}{\beta}} \left| \sin\left(\frac{\chi}{2}\right) \right| \,, \tag{41}$$

where δ is a constant phase.

We now consider the case of $\alpha < 0$. In this case for $\beta < \beta_{\text{crit}}$, we have that

$$q_\pm^2 = \frac{1}{2\beta}\left(-|\alpha| \pm \sqrt{\alpha^2 - 4\sigma\beta}\right) \tag{42}$$

which is always negative leading to oscillatory solutions with wave numbers $|q_\pm|$.

Finally, let us consider the $\beta > \beta_{\text{crit}}$ region where the solutions become

$$q_\pm^2 = \frac{\sigma}{\beta} e^{\mp i\chi} \tag{43}$$

leading for the scalar field behavior to a situation similar to the case of $\alpha > 0$ with $\beta > \beta_{\text{crit}}$.

Let us now study the asymptotic behavior of the gauge fields. For $\alpha > 0$, the consistent asymptotic behavior is

$$a_0(r) \approx \frac{a_{0\infty}}{\sqrt{r}} \exp(-\bar{k}r), \quad a(r) \approx a_\infty \sqrt{r} \exp(-\bar{k}r). \tag{44}$$

Notice that in this region the asymptotic field behavior ensures finite energy and quantized magnetic flux as in the relativistic case

$$\Phi = \frac{2\pi}{e} n, \quad n \in \mathbb{Z}. \tag{45}$$

In the $\alpha = 0$ case, linearization leading to Eqs. (30) and (31) is no longer valid. Instead, writing $a = \sqrt{r}g(r)$ and using the gauge field equations of motion one gets a second-order nonlinear equation for g compatible with bounded solutions at infinity. As will be discussed in the next section, we do find a bounded numerical solution for $\alpha = 0$.

Concerning the $\alpha < 0$ region, one has

$$a_0(r) \approx \frac{a_{0\infty}}{\sqrt{r}} \sin(\bar{k}r + \bar{\varphi}), \quad a(r) \approx a_\infty \sqrt{r} \cos(\bar{k}r + \bar{\varphi}) \tag{46}$$

with

$$\bar{k} = \sqrt{|\alpha|\gamma}\mu, \quad a_\infty = -\sqrt{\frac{\gamma}{|\alpha|}}\, a_{0\infty}. \tag{47}$$

The oscillatory behavior of configurations satisfying (46) will require the introduction of appropriate boundary conditions at a finite radius R.

4. Solutions

We shall present in this section numerical solutions of Eq. (18) satisfying the asymptotic condition discussed above. For definiteness, we take $n = 1$ and we shall fix $\gamma = 1$ (since we are considering static solutions, changing gamma amounts to a redefinition of the scalar field coupling with A_0). In order to ensure positivity of the energy, we shall take $\beta > 0$. We shall separately consider $\alpha \geq 0$ and $\alpha < 0$ regions. Following the discussion in the previous section, we shall distinguish regions with $\beta \lesseqgtr \beta_{\text{crit}}$. The numerical procedure is based on a fourth-order finite differences method applied in the interval (ϵ, R) with ϵ close to the origin and R large, in combination with the behavior of fields close to the origin given by Eq. (20).
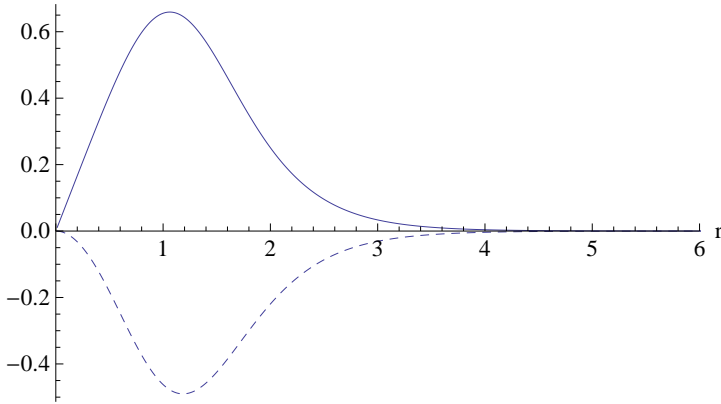
Fig. 1.   The electric (solid line) and magnetic (dashed line) fields in the region $\alpha > 0$, $\beta < \beta_{\rm crit}$, with $\alpha = 1$, $\beta_{\rm crit} = 0.0625$ and $\beta = 0.04$. As in the relativistic CS–Higgs model, the magnetic and electric fields form a ring surrounding the vortex core.

4.1. *The $\alpha \geq 0$ region*

We start by studying the $\alpha > 0$, $\beta < \beta_{\rm crit}$ region. We give the results of our numerical calculation for E and B in Fig. 1 and the scalar field in Fig. 2.

One can see that the profile of the fields in this region exhibit slight deviations to the relativistic case, originated by the fourth-order derivative terms. It should be noted that as β grows, we found numerically that the maximum magnitude of the electric and magnetic fields decrease. Concerning the Higgs field, it reaches its vacuum value exponentially according to Eq. (34), as can be seen in Fig. 2 with a similar profile as that corresponding to the relativistic case as shown in Fig. 2.
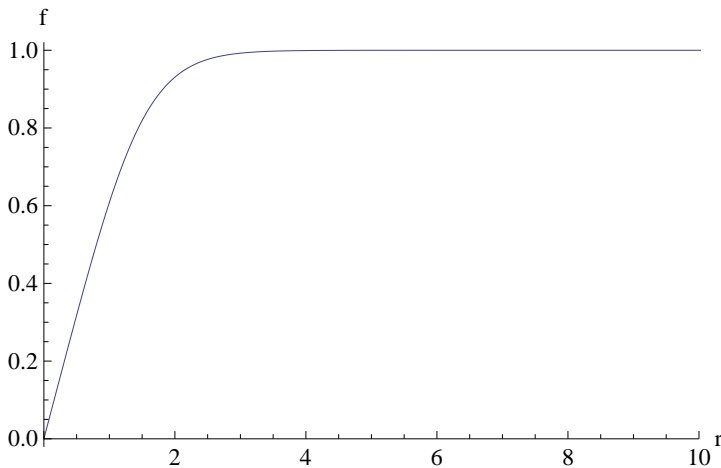

Fig. 2.   The Higgs field profile in the region corresponding to $\alpha > 0$, $\beta < \beta_{\rm crit}$ ($\alpha = 1$, $\beta_{\rm crit} = 0.0625$ and $\beta = 0.04$).
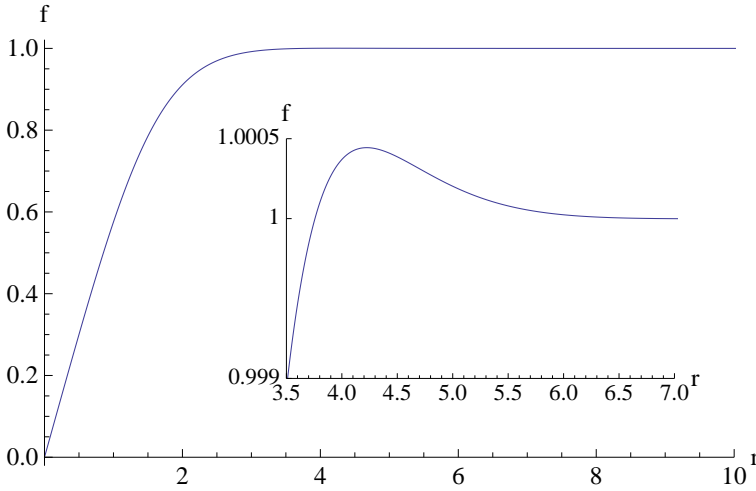
Fig. 3. The Higgs field profile in the region $\alpha > 0$, $\beta > \beta_{\text{crit}}$. We have chosen $\alpha = 1$, $\beta_{\text{crit}} = 0.0625$ and $\beta = 0.2$. The inset shows a zoom of the region where f overshoots its VEV and comes back to it, as is characteristic of an underdamped behavior.

Let us now consider $\beta > \beta_{\text{crit}}$ range where the roots $q_{\pm}$ are complex (37), thus giving rise to underdamped oscillations in the Higgs profile, as shown in Fig. 3.

For a given value of α, the magnetic and electric field solutions for $\beta > \beta_{\text{crit}}$ are qualitatively the same as those shown in Fig. 1 for $\beta < \beta_{\text{crit}}$.

We then conclude that in the $\alpha > 0$ region the electric and magnetic field behavior is very similar to the ordinary relativistic CS–Higgs model. Concerning the scalar field, as one crosses from $\beta < \beta_{\text{crit}}$ to $\beta > \beta_{\text{crit}}$, it changes from the usual to an underdamped approach to its VEV.

We have studied the β-dependence of the energy in this region finding a linear behavior for small β. As an example, we show in Fig. 4, a numerical calculation of the energy $\mathcal{E}$ as a function of β for $\alpha = 1$, $\beta_{\text{crit}} = 0.0625$. We find that $\mathcal{E}$ behaves approximately as $\mathcal{E} \approx \mathcal{E}_0 + 0.25\beta$.

We end this subsection by discussing the $\alpha = 0$ case for which, for vanishing potential ($\sigma = 0$), the Lagrangian is invariant under anisotropic scaling with "dynamical critical exponent" $z = 2$. As in this case there is no length scale, we do not expect a localized solution. On the other hand, if we consider $\alpha = 0$ but $\sigma \neq 0$ (and then $\tau \neq \tau_{\text{BPS}}$), one has $\beta_{\text{crit}} = 0$. So, for any $\beta > 0$, the Higgs field shows an underdamped behavior. We have numerically confirmed this result and also found bounded solutions for the gauge fields. The field profiles are qualitatively similar to those found for $\alpha > 0$, $\beta > \beta_{\text{crit}}$.

4.2. *The $\alpha < 0$ region*

One expects in this region a clearly different behavior compared to the relativistic CS–Higgs system since the negative sign of α in the $|D_i\phi|^2$ energy term implies not
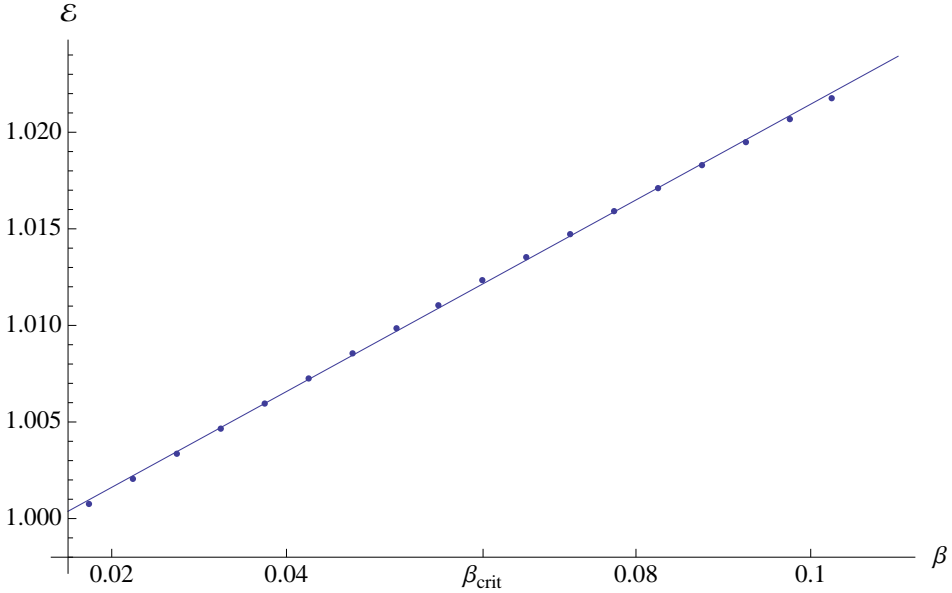
Fig. 4. The energy as a function of β for $\alpha = 1$, $\beta_{\text{crit}} = 0.0625$.

only a change of sign in the $|\nabla\phi|^2$ term but also in the gauge field "mass term" that now has the "wrong" sign.

We start by studying the $\beta > \beta_{\text{crit}}$ region where the fields asymptotic behavior is given by Eqs. (43)–(47). This behavior leads to an oscillatory energy density (and consequently to an, in general, unbounded energy). For example, the third term in expression (11) for the energy density takes the asymptotic form

$$\mathcal{E}_3 = \frac{1}{4\gamma e^2} \frac{\kappa^2 B^2}{|\phi|^2} \approx |\alpha| a_{0\infty}^2 e^2 v^2 \frac{\sin^2(\bar{k}r + \bar{\varphi})}{r} \,. \tag{48}$$

We show in Fig. 5 the electric and magnetic fields in the $\alpha < 0$, $\beta > \beta_{\text{crit}}$ region. Their profiles show the asymptotic oscillatory damped behavior consistent with Eq. (46). The behavior of the scalar field is presented in Fig. 6. A zoom outside the vortex core shows damped oscillations consistent with Eqs. (34)–(43).

In the $\beta < \beta_{\text{crit}}$ region, the roots we found in Sec. 3, Eq. (42), lead to pure oscillatory solutions with no damping. The assumption of h in Eq. (22) being asymptotically a small perturbation to the scalar VEV v is then not self-consistent. We have not been able to find stable solutions of our $2 + 1$ model with the ansatz (12) and (13). We indeed know that in the absence of dynamical gauge fields this range of parameters corresponds to the modulated ordered Lifshitz phase associated to spontaneous breaking of translations.[19,20] We then conclude that in this region a more detailed numerical study allowing the implementation of more general ansatz would be necessary.
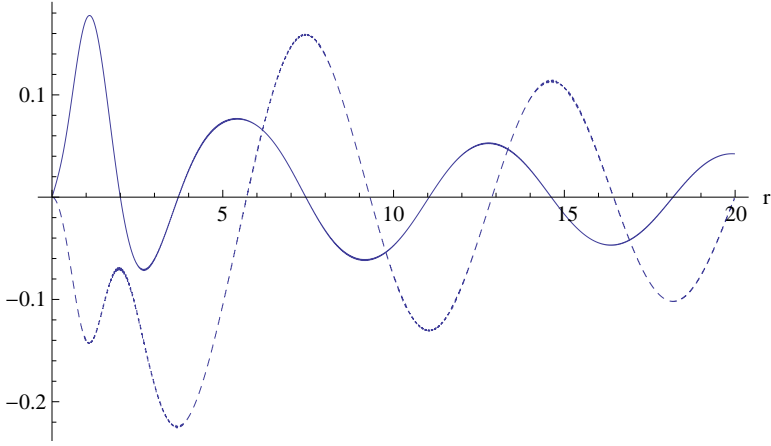
Fig. 5. The electric (solid line) and magnetic (dashed line) fields in the region $\alpha < 0$, $\beta > \beta_{\mathrm{crit}}$, with $\alpha = -0.2$, $\beta_{\mathrm{crit}} = 0.0025$ and $\beta = 0.25$.
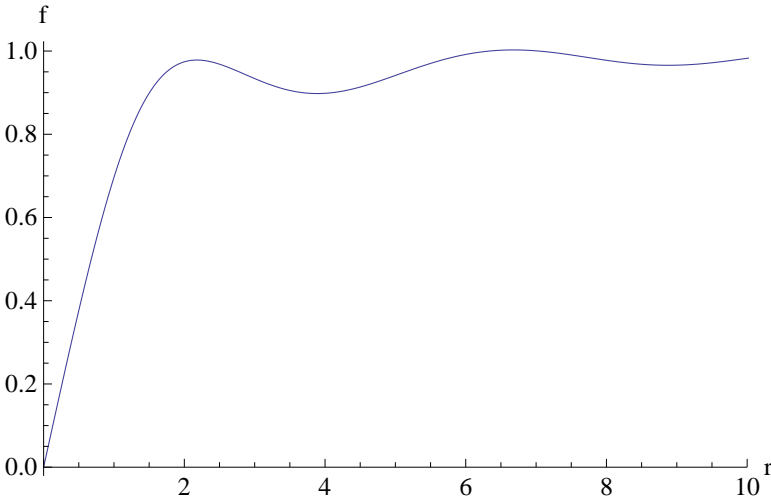

Fig. 6. The Higgs field profile in the region $\alpha < 0$, $\beta > \beta_{\mathrm{crit}}$, with $\alpha = -0.2$, $\beta_{\mathrm{crit}} = 0.0025$ and $\beta = 0.25$.

5. Summary and Discussion

We have proposed a gauged Lifshitz Lagrangian with higher (fourth)-order spatial derivatives of the scalar field and a CS term, and studied numerically nontrivial solutions of the classical equations of motion. Notice that contrary to previous analysis of Lifshitz theories with CS term[18] with $z = 2$, we considered higher derivatives for the scalar field rather than for the gauge fields. As a consequence, the classical solutions of our model have a different character of the ones resulting from such model.[22]

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Coming back to the model we analyzed, let us recall that β, the coefficient of the fourth-order derivatives term, was taken positive in order to ensure positivity of the energy. In contrast, the α coefficient multiplying the ordinary second-order derivative term could take both positive and negative values being $\alpha = 0$ the Lifshitz point at which the model exhibits $z = 2$ anisotropic scaling in the absence of a potential term.

In order to solve the equations of motion, we have made the static axially symmetric ansatz that leads to vortex solutions in the relativistic case. For $\alpha > 0$, we have found solutions with magnetic and electric fields qualitatively similar to those of the ordinary relativistic model. The magnetic flux is quantized and the usual relation between electric charge and magnetic field in CS systems holds. The difference with the standard relativistic case manifests more pronouncedly in the Higgs field behavior which for $\beta > \beta_{\text{crit}}$ approaches its VEV with underdamped oscillations. The critical value is given by formula (35), $\beta_{\text{crit}} = \alpha^2/4\sigma^2$, showing a dependence on the coefficient of the quadratic derivative coefficient and on the parameters of the model (the value v of the Higgs field at the minimum, the gauge coupling e, the CS coefficient κ and the Higgs field self-interaction coupling constant τ). For $\alpha = 0$ and $\sigma \neq 0$, the numerical solutions that we found are qualitatively similar to those found for $\alpha > 0$, $\beta > \beta_{\text{crit}}$.

The situation for the $\alpha < 0$ region radically changes basically because of the change in sign of the gauge field mass term. The ansatz led to pure oscillatory solutions for the gauge fields with no damping. Concerning the scalar field, one can again distinguish two situations depending on whether β is larger or smaller than β_{crit}. In the former case, we were able to find solutions exhibiting electric and magnetic field profiles with an asymptotic oscillatory behavior while the Higgs field profile shows damped oscillations. This behavior leads, in general, to an oscillatory energy density and an unbounded energy. In the $\beta < \beta_{\text{crit}}$ region, the proposed ansatz led to pure oscillatory solutions with no damping.

We think that in the region $\alpha < 0$, other terms in the Lagrangians as those considered by Ginzburg for the free energy of superdiamagnets and superconductors[2] might become relevant. Also, more general ansatz, not purely relying in cylindrical symmetry should be considered in order to incorporate the possibility of asymptotic breaking of translational symmetry which is characteristic of modulated Lifshitz phases. We hope to come back to this problem in a future work.

Acknowledgments

Appendix. The Fröbenius Method

In this section, we wish to apply Fröbenius's method to the linearized Higgs field equation of motion in order to determine its behavior close to $r = 0$, where f is

assumed to be small (see Eq. (20)) and the equation has a regular singular point. Following Ref. 21, we recast the equation of motion (17) for the Higgs field close to $r = 0$ in simplified form as

$$-\beta f'''' - \frac{2\beta}{r}f''' + \frac{(3\beta + \alpha r^2)}{r^2}f'' + \frac{(-3\beta + \alpha r^2)}{r^3}f'$$
$$+ \left(\frac{\sigma}{4}v^4 + \gamma e^2 a_0^2\right)f + \frac{(3\beta - \alpha r^2)}{r^4}f = 0\,, \tag{A.1}$$

where we take the vorticity $n = 1$ and ignore the contribution from $A(r)$ given that this vanishes at the origin. Note that the higher-order terms in f coming from the potential are to be ignored in the linearized analysis. We proceed to make a Fröbenius ansatz for the behavior close to the origin of the form

$$f_{(\lambda)} = \sum_{m=0}^{\infty} \mathcal{F}_m(\lambda) r^{m+\lambda}\,. \tag{A.2}$$

Upon substituting this ansatz in Eq. (A.1) and looking at the lowest order in r, one obtains the indicial equation of the system, hence we look at the equation at order $r^{\lambda-4}$ where we obtain

$$(\lambda - 3)(\lambda - 1)^2(\lambda + 1) = 0\,. \tag{A.3}$$

Therefore, we have three distinct roots $\lambda = 3, 1, -1$ with multiplicities $1, 2, 1$, respectively. We proceed to determine the coefficients a_m by looking at higher-orders in r. The equation at order $r^{\lambda-3}$ implies that $\mathcal{F}_1 = 0$. The order $r^{\lambda-2}$ equation leads to

$$\mathcal{F}_2(\lambda) = \frac{-\alpha \mathcal{F}_0}{\beta(1 + \lambda)(3 + \lambda)} \tag{A.4}$$

which gives solutions for both roots $\lambda = 1$ and $\lambda = 3$ as

$$f_{(1)} = r \sum_{m=0}^{\infty} \mathcal{F}_m(1)r^m\,, \quad f_{(3)} = r^3 \sum_{m=0}^{\infty} \mathcal{F}_m(3)r^m\,, \tag{A.5}$$

where a_2 and b_2 are coefficients extracted from Eq. (A.4) with the appropriate choice for λ, and an ill-defined solution for $\lambda = -1$ which we will return to later. The solution f_2 corresponds to the behavior used in Eq. (20) at $n = 1$. Both these solutions and their derivatives are well behaved at the origin. The next order coefficients can be extracted from the order r^λ equation as

$$\mathcal{F}_4(\lambda) = -\frac{\mathcal{F}_0\big(\alpha^2(1 + 3\lambda + \lambda^2) - \gamma e^2 a_0^2 \beta(3 + 4\lambda + \lambda^2)\big)}{\beta^2(1 + \lambda)(3 + \lambda)^2(7 + 17\lambda + 8\lambda^2 + \lambda^3)} \tag{A.6}$$

with higher-order $\mathcal{F}_m$'s for odd m vanishing. Being $\lambda = 1$ a multiplicity 2 root, we know that the λ derivative of this solution is also a solution of the equations of motion. In general, if $f_{(\lambda)}$ is a solution of the form Eq. (A.2), then

$$\frac{df_{(\lambda)}}{d\lambda} = \ln r f_{(\lambda)} + r^\lambda \sum_{m=0}^{\infty} \frac{d\mathcal{F}_m(\lambda)}{d\lambda} r^m \tag{A.7}$$

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which means that an independent solution is of the form

$$\bar{f}_{(1)} = \frac{df_{(1)}}{d\lambda} = r \sum_{m=0}^{\infty} \mathcal{F}_m(1) r^m \ln r + r \sum_{m=0}^{\infty} \frac{d\mathcal{F}_m(1)}{d\lambda} r^m \,. \tag{A.8}$$

This solution has a singular derivative at the origin.

The solution of the linearized problem for $\lambda = -1$ takes the form

$$f_{(-1)} = \frac{1}{r} \sum_{m=0}^{\infty} \mathcal{B}_m r^m + r \sum_{m=0}^{\infty} \mathcal{C}_m r^m \ln r \,, \tag{A.9}$$

where as before the sum extends over even m and one finds that $\mathcal{B}_0$ and $\mathcal{C}_0$ are nonvanishing. This solution of the linearized problem diverges at $r = 0$ and, hence, should not be taken into account for searching physically acceptable solutions.

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