

SKEW BRACES OF SIZE p^2q II: NON-ABELIAN TYPE

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ABSTRACT. In this paper we enumerate the skew braces of non-abelian type of size p^2q for p, q primes with $q > 2$ by the classification of regular subgroups of the holomorph of the non-abelian groups of the same order. Since Crespo dealt with the case $q = 2$, this paper completes the enumeration of skew braces of size p^2q started in a previous work by the authors. In some cases, we provide also a structural description of the skew braces. As an application, we prove a conjecture posed by V. Bardakov, M. Neshchadim and M. Yadav.

1. INTRODUCTION

The set-theoretical Yang–Baxter equation (YBE) was introduced in [13] as a discrete version of the braid equation. A pair (X, r) where X is a set and $r : X \times X \rightarrow X \times X$ is a map, is a *set-theoretical solution to the Yang–Baxter equation* if

$$(1) \quad (id_X \times r)(r \times id_X)(id_X \times r) = (r \times id_X)(id_X \times r)(r \times id_X)$$

holds. A solution (X, r) is *non-degenerate* if the map r is defined as

$$(2) \quad r : X \times X \longrightarrow X \times X, \quad (x, y) \mapsto (\sigma_x(y), \tau_y(x)),$$

where σ_x, τ_x are permutations of X for every $x \in X$. Non-degenerate solutions have been studied intensively over the past years [14, 15, 18, 25].

In order to describe *non-degenerate involutive solutions*, that is solutions satisfying $r^2 = id_{X \times X}$, Rump introduced the ring-like binary algebraic structures called *(left) braces* in [21]. Later *skew (left) braces* have been defined by Guarnieri and Vendramin in [16], to capture also non-involutive solutions.

A *skew (left) brace* is a triple $(B, +, \circ)$ where $(B, +)$ and (B, \circ) are groups (not necessarily abelian) such that

$$a \circ (b + c) = a \circ b - a + a \circ c$$

holds for every $a, b, c \in B$. Braces are skew braces with abelian additive group.

The problem of finding non-degenerate solutions to (1) is closely related to the classification problem for skew braces. Indeed in [5], a canonical structure of skew brace over a permutation group related to a non-degenerate solution has been described and it has been shown how to recover all the non-degenerate solutions with a given associated skew brace. So, in a sense, the study of non-degenerate solutions to (1) can be reduced to the classification of skew braces.

Some recent results on the classification problem for (skew) braces are, for instance: the classification of braces with cyclic additive group [22, 23], skew braces of size pq for p, q different primes [2], braces of size p^2q for p, q primes with $q > p + 1$ [12], skew braces of size p^2q with cyclic p -Sylow subgroup and $p > 2$ [8], skew braces of size $2p^2$ [11], braces of order p^2, p^3 where p is a prime [4], skew braces of order p^3 [19] and skew braces of squarefree size [3].

This paper is the second part of the enumeration and classification of skew braces of size p^2q where p, q are different primes. In [1], we dealt with the left braces. In this paper, we complete the classification focusing on the skew braces with non-abelian additive group.

In [8] the authors partially obtained the same results, through the connection between skew braces and Hopf–Galois extensions explained in [24] (and they also claim that the missing cases are the subject of a second paper in preparation). The common results of [8] and [12] agree with ours. In fact, our methods are different but we are covering and extending their results related to the enumeration of skew braces of order p^2q .

Key words and phrases. Yang–Baxter equation, set-theoretical solution, skew brace, Hopf–Galois extensions.

The main tool in this paper is the algorithm for the construction of skew braces with a given additive group developed in [16] (we used the same approach in [2]). We obtain all the skew braces with additive group B from regular subgroups of its holomorph $\text{Hol}(B) = B \rtimes \text{Aut}(B)$. The isomorphism classes of skew braces are parametrized by the orbits of such subgroups under the action by conjugation of the automorphism group of B in $\text{Hol}(B)$ [16, Section 4].

We enumerate the skew braces according to their additive group, and the structure of the paper is displayed in the following table. The classification of the groups of size p^2q can be found in [7] and the description of their automorphism groups in [9].

	Groups	Sections
$p = 1 \pmod{q}$	$\mathbb{Z}_{p^2} \rtimes_t \mathbb{Z}_q$	3.1
	\mathcal{G}_k	3.2, 3.3, 3.4, 3.5
$p = -1 \pmod{q}$	\mathcal{G}_F	4
$q = 1 \pmod{p}$,	$\mathbb{Z}_q \rtimes_r \mathbb{Z}_{p^2}$	5.1
$q \neq 1 \pmod{p^2}$	$\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_r \mathbb{Z}_p)$	5.2
$q = 1 \pmod{p^2}$	$\mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_{p^2}$	6.1
	$\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_p)$	6.2
	$\mathbb{Z}_q \rtimes_h \mathbb{Z}_{p^2}$	6.3

The enumeration results are collected in suitable tables in each section. The skew braces of non-abelian type of size $2p^2$ have been enumerated by Crespo, so we do not include tables for this case and we refer the reader to [11, Section 5]. If $q = 1 \pmod{p}$ and $q \neq 1 \pmod{p^2}$ there are 2 non-abelian groups if $p^2q \neq 12$. If $p^2q = 12$, there are 3 groups. Since the skew braces of size 12 are included in the GAP library *YangBaxter*, [27], we omit them.

In [16, Table 5.3] we have the number of skew braces of order $n \leq 120$ with some exceptions. Due to a computational improvement of the algorithm calculating skew braces, in [6] we have many tables collecting the number of skew braces of order $n \leq 868$ with some exceptions. We highlight that none of these exceptions are of the form p^2q for different primes p, q . All this data is available in [27]. This information was of invaluable help when working on the present paper and all of our results agree with those tables.

In [6, Conjecture 4.1], the authors conjecture the number of skew braces of size p^2q based on their tables. In Section 7, as an application of the results obtained in [1] and in the present work, we can give a positive answer to that conjecture.

2. PRELIMINARIES

A *skew (left) brace* is a triple $(B, +, \circ)$ where $(B, +)$ and (B, \circ) are groups and

$$a \circ (b + c) = a \circ b - a + a \circ c$$

holds for every $a, b, c \in B$. We say that a skew brace $(B, +, \circ)$ is of χ -type if $(B, +)$ has the group theoretical property χ .

A *bi-skew brace* is a skew brace $(B, +, \circ)$ such that $(B, \circ, +)$ is also a skew brace (see [10]). Equivalently

$$x + (y \circ z) = (x + y) \circ x' \circ (x + z)$$

holds for every $x, y, z \in B$, where x' denotes the inverse of x in (B, \circ) .

Given a skew brace $(B, +, \circ)$, the mapping

$$\lambda : (B, \circ) \rightarrow \text{Aut}(B, +), \quad \lambda_a(b) = -a + a \circ b$$

is a homomorphism of groups.

A subgroup I of $(B, +)$ is a *left ideal* of B if $\lambda_a(I) \leq I$ for every $a \in B$. Every left ideal is a subgroup of (I, \circ) . In particular,

$$\text{Fix}(B) = \{a \in B : \lambda_b(a) = a, \text{ for every } b \in B\}$$

is a left ideal. If I is also a normal subgroup of both $(B, +)$ and (B, \circ) then I is called an *ideal*.

2.1. Skew braces and regular subgroups. Given a group A we denote by π_1 and π_2 the canonical mappings

$$\begin{aligned}\pi_1 : \text{Hol}(A) &\longrightarrow A, & (x, f) &\mapsto x, \\ \pi_2 : \text{Hol}(A) &\longrightarrow A, & (x, f) &\mapsto f.\end{aligned}$$

In order to classify all skew braces of non-abelian type of size p^2q where p, q are primes, we are going to apply the same ideas as in [1], where we obtained a classification of skew braces of abelian type (also called braces).

A subgroup G of $\text{Hol}(A)$ is said to be *regular* if the corresponding set of permutations is a regular subgroup of \mathbb{S}_A , the symmetric group over A (i.e. the natural action is transitive and it has trivial stabilizers). We are going to exploit the connection between skew braces with given additive group A and regular subgroups of the holomorph of A , as explained in [16]. This connection is summarized in the following theorem.

Theorem 2.1. [16, Theorem 4.2, Proposition 4.3] *Let $(A, +)$ be a group. If \circ is an operation such that $(A, +, \circ)$ is a skew brace, then $\{(a, \lambda_a) : a \in A\}$ is a regular subgroup of $\text{Hol}(A, +)$. Conversely, if G is a regular subgroup of $\text{Hol}(A, +)$, then A is a skew brace with*

$$a \circ b = a + f(b)$$

where $(\pi_1|_G)^{-1}(a) = (a, f) \in G$ and $(A, \circ) \cong G$.

Moreover, isomorphism classes of skew braces over A are in bijective correspondence with the orbits of regular subgroups of $\text{Hol}(A)$ under the action of $\text{Aut}(A)$ by conjugation.

We can take advantage of the classification of groups of size p^2q for odd primes provided in [7, Proposition 21.17] and of size $4q$ in [17] and of the description of their automorphism groups given in [9] to compute the orbits of regular subgroups under the action by conjugation of $\text{Aut}(A)$ on $\text{Hol}(A)$. The enumeration of such regular subgroups will provide the enumeration of skew braces of the same size, according to Theorem 2.1.

Remark 2.2. Let $(A, +)$ be a group and G a regular subgroup of $\text{Hol}(A)$. According to Theorem 2.1, $(A, +, \circ)$ where

$$(3) \quad a \circ b = a + \pi_2((\pi_1|_G)^{-1}(a))(b)$$

for every $a, b \in A$ is a skew brace. In other words, $\lambda_a = \pi_2((\pi_1|_G)^{-1}(a))$ and so $|\ker \lambda| = \frac{|G|}{|\pi_2(G)|}$.

As in [1], in some cases, we will use (3) to explicitly compute the operations of the skew braces.

We identify an element of the holomorph $(a, f) \in \text{Hol}(A, +)$ with a permutation over the underlying set A acting as $(a, f) \cdot x = a + f(x)$ for every $x \in A$.

Notice that if $(a, f), (b, g) \in G$ then

$$\pi_1(a, f) = \pi_1(b, g) \text{ if and only if } (a, f)^{-1}(b, g) \in H = G \cap (\{1\} \times \text{Aut}(A)).$$

Therefore the mapping π_1 restricted to G factors through the canonical projection onto the set of cosets with respect to the subgroup H . If G is finite, we have that $|G| = |H||\pi_1(G)|$ and $|\pi_1(G)|$ divides the size of G .

Lemma 2.3. [1, Lemma 2.5] *Let A be a finite group and $G \leq \text{Hol}(A)$. The following are equivalent:*

- (i) G is regular.
- (ii) $|A| = |G|$ and $\pi_1(G) = A$.
- (iii) $|A| = |G|$ and $G \cap (\{1\} \times \text{Aut}(A)) = 1$.

Lemma 2.4. *Let A be a group and $G = \langle u_i \alpha_i, i \in I \rangle \leq \text{Hol}(A)$ where $u_i \in A$, $\alpha_i \in \text{Aut}(A)$ for $i \in I$. Then:*

- (1) $\pi_1(G) \subseteq \langle h(g) \mid g \in U, h \in \pi_2(G) \rangle$, where $U = \langle u_i, i \in I \rangle$.
- (2) If $\alpha_i(u_i) = u_i$ then $\langle u_i \rangle \subseteq \pi_1(G)$.

Proof. (1) We have that

$$(4) \quad u_i \alpha_i x f = u_i \alpha_i(x) \alpha_i f \quad \text{and} \quad (u_i \alpha_i)^{-1} = \alpha_i^{-1}(u_i)^{-1} \alpha_i^{-1},$$

for every $i \in I$, $x \in A$ and $f \in \text{Aut}(A)$, thus the claim follows by induction.

(2) According to (4) we have that

$$(u_i \alpha_i)^n = u_i^n \alpha_i^n$$

and so $\pi_1((u_i \alpha_i)^n) = u_i^n$ for every $n \in \mathbb{Z}$. □

In order to show that a subgroup of the holomorph is regular or not we will make use of Lemma 2.3 and Lemma 2.4. In most of the cases we will omit the explicit computations.

In [1] we described a technique, inspired by [20, Section 2.2], to obtain the conjugacy classes we are looking for. Let us briefly recall it.

As a general strategy, we show a list of representatives of regular subgroups and then we prove that any regular subgroup is conjugate to one of the groups in the list. Given such list, the isomorphism class of the representatives of conjugacy classes can be easily obtained using the list of groups of size p^2q and so we omit a proof of such isomorphisms.

To compute the list of representatives, first we need to identify some properties of regular subgroups that are invariant under conjugation by elements of the subgroup $\{1\} \times \text{Aut}(A)$.

- The map π_2 is a group homomorphism and, if G is a regular subgroup of $\text{Hol}(A)$, the size of $\pi_2(G)$ divides both $|(A, +)|$ and $|\text{Aut}(A, +)|$ and it is invariant under conjugation. So we can compute the regular subgroups according to the size of their image under π_2 . The unique trivial skew brace over A is clearly associated to the unique regular subgroup with trivial image under π_2 (i.e. $A \times \{1\}$), so we assume that $|\pi_2(G)| > 1$.
- Let us assume that the image under π_2 has size $k > 1$. The conjugacy class of the image under π_2 in $\text{Aut}(A)$ is invariant under conjugation. So we can provide a list of representatives of conjugacy classes of subgroups of $\text{Aut}(A)$ of size k and we assume that $\pi_2(G)$ is one of the subgroups in the list.
- Let G and G' be two regular subgroups such that $\pi_2(G) = \pi_2(G')$. If $hGh^{-1} = G'$ then h is an element of the normalizer of their image under π_2 and moreover $h \ker \pi_2|_G h^{-1} = h(\ker \pi_2|_G) = \ker \pi_2|_{G'}$.

Sometimes the invariants in the list above do not identify a conjugacy class of regular subgroups in the sense of Theorem 2.1, and so we need also to employ some further invariant to provide a list of non-conjugate regular subgroups. Assume that G and G' are regular subgroups of $\text{Hol}(A)$ conjugate by $h \in \text{Aut}(A)$, then:

- the p -Sylow subgroups of G are conjugate by h to the p -Sylow subgroups of G' .
- Let \mathfrak{H} be a normal subgroup of $\text{Hol}(A)$. The following diagram is commutative:

$$(5) \quad \begin{array}{ccc} \text{Hol}(A) & \xrightarrow{\widehat{h}} & \text{Hol}(A) \\ \downarrow & & \downarrow \\ \text{Hol}(A)/\mathfrak{H} & \xrightarrow{\widehat{h\mathfrak{H}}} & \text{Hol}(A)/\mathfrak{H} \end{array}$$

where \widehat{h} , $\widehat{h\mathfrak{H}}$ are the inner automorphisms and the unlabeled arrows are the canonical homomorphisms. Then their images in the quotient $\text{Hol}(A)/\mathfrak{H}$ are conjugate by $h\mathfrak{H}$. So the conjugacy class of the image in the factor $\text{Hol}(A)/\mathfrak{H}$ is invariant up to conjugation.

Using Lemma 2.3 and the invariants above we can provide a list of regular subgroups for each admissible value of k and claim that they are not conjugate. Usually we start by computing the values of the main invariants we already mentioned, namely:

- Find a set of representatives of the conjugacy classes of subgroups of size k of $\text{Aut}(A, +)$.
- The kernel of π_2 is a subgroup of $(A, +)$. So we need to find a set of representatives of the orbits of subgroups of size $\frac{|A|}{k}$ of A , under the action of the normalizer of the image of π_2 on A .

The next step is to prove that such list is actually a set of representatives, namely that any other regular subgroup is conjugate to one of them. First we need to construct the regular subgroups.

Let G be a regular subgroup. If the set $\{\alpha_i : 1 \leq i \leq n\}$ generates $\pi_2(G)$ and $\{k_j : 1 \leq j \leq m\}$ generates the kernel of $\pi_2|_G$ then G has the following *standard presentation*:

$$G = \langle k_1, \dots, k_m, u_1\alpha_1, \dots, u_n\alpha_n \rangle,$$

for some $u_i \in A$. In particular, $u_i \neq 1$, otherwise $\alpha_i \in G \cap (\{1\} \times \text{Aut}(A))$ and so G is not regular. We can replace any generator g of G by the product of g with any other generator of G , in order to obtain a nicer presentation of the same group G . In particular, we can choose any representative of the coset of u_i with respect to the kernel, without changing the group G .

The group G has to satisfy the following necessary conditions, that together provide constraints over the choice of the elements u_i :

- (K) The kernel of $\pi_2|_G$ is normal in G .
- (R) The generators $\{u_i\alpha_i : 1 \leq i \leq n\}$ satisfy the same relations as $\{\alpha_i : 1 \leq i \leq n\}$ modulo $\ker \pi_2|_G$ (e.g. if $\alpha_i^n = 1$ then $(u_i\alpha_i)^n \in \ker \pi_2|_G$).

Given one such group, we can conjugate by the normalizer of $\pi_2(G)$ in $\text{Aut}(A)$ that stabilizes the kernel of $\pi_2|_G$ in order to show that G is conjugate to one of the groups in the list of the chosen representatives.

2.2. Notation. If C is a cyclic group acting on a group G by $\rho : C \rightarrow \text{Aut}(G)$ and $\rho(1) = f$, then $G \rtimes_f C$ denotes the semidirect product determined by the action ρ . In this case the group operation on $G \rtimes_f C$ is

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 f^{y_1}(x_2) \\ y_1 y_2 \end{pmatrix},$$

for every $x_1, x_2 \in G$ and $y_1, y_2 \in C$.

Remark 2.5. [1, Remark 3.5] Let p, q be primes such that $p \equiv 1 \pmod{q}$ and let g be an element of order q of \mathbb{Z}_p^\times . As in [1] we denote by \mathfrak{B} the subset of \mathbb{Z}_q that contains $0, 1, -1$ and one out of k and k^{-1} for $k \neq 0, 1, -1$ and by $\mathcal{D}_{a,b}$ the diagonal matrix with diagonal entries g^a and g^b . We are using that, up to conjugation, the subgroups of order q of $GL_2(p)$ are generated by one of the matrices

$$\mathcal{D}_{1,s} = \begin{bmatrix} g & 0 \\ 0 & g^s \end{bmatrix}, \quad \mathcal{D}_{0,1} = \begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix},$$

for $0 \leq s \leq q-1$ and a set of representatives of conjugacy classes of such groups in $GL_2(p)$ is given by the groups generated by $\mathcal{D}_{1,s}$ for $s \in \mathfrak{B}$.

Up to conjugation, the unique subgroup of order p of $GL_2(p)$ is generated by

$$C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

A set of representatives of conjugacy classes of subgroups of order pq of $GL_2(p)$ is given by $H_s = \langle C, \mathcal{D}_{1,s} \rangle$ and $\tilde{H} = \langle C, \mathcal{D}_{0,1} \rangle$, where $0 \leq s \leq q-1$.

3. SKEW BRACES OF SIZE p^2q WITH $p \equiv 1 \pmod{q}$

Let p, q be primes such that $p \equiv 1 \pmod{q}$ and $p, q > 2$. Recall that we omit the case $q = 2$ since it was covered by Crespo in [11]. Following the same notation as in [1] we denote by g a fixed element of order q in \mathbb{Z}_p^\times and by t a fixed element of order q in $\mathbb{Z}_{p^2}^\times$. The non-abelian groups of size p^2q are the following:

- (i) $\mathbb{Z}_{p^2} \rtimes_t \mathbb{Z}_q = \langle \sigma, \tau \mid \sigma^{p^2} = \tau^q = 1, \tau\sigma\tau^{-1} = \sigma^t \rangle$.
- (ii) $\mathcal{G}_k = \langle \sigma, \tau, \epsilon \mid \sigma^p = \tau^p = \epsilon^q = 1, \epsilon\sigma\epsilon^{-1} = \sigma^g, \epsilon\tau\epsilon^{-1} = \tau^{g^k} \rangle \cong \mathbb{Z}_p^2 \rtimes_{\mathcal{D}_{1,k}} \mathbb{Z}_q$, for $k \in \mathfrak{B}$.

Tables 1 and 2 collect the enumeration of skew braces according to their additive and multiplicative groups, taking into account that the groups \mathcal{G}_k and $\mathcal{G}_{k^{-1}}$ are isomorphic for $k \neq 0$.

Note that for $q = 3$ we have $\mathfrak{B} = \{0, 1, -1\}$ and $2^{-1} = 2 = -1$ in \mathbb{Z}_3^\times . The enumeration of skew braces according to the isomorphism class of their multiplicative group is slightly effected by these facts and so we have a separate table for this case.

$+\backslash\circ$	\mathbb{Z}_{p^2q}	$\mathbb{Z}_{p^2} \rtimes_t \mathbb{Z}_q$	$\mathbb{Z}_p^2 \times \mathbb{Z}_q$	$\mathcal{G}_s, s \neq 0, \pm 1, 2$	\mathcal{G}_0	\mathcal{G}_{-1}	\mathcal{G}_1	\mathcal{G}_2
$\mathbb{Z}_{p^2} \rtimes_t \mathbb{Z}_q$	4	$2(q-1)$	-	-	-	-	-	-
$\mathcal{G}_k, k \neq 0, \pm 1$	-	-	4	$8(q+1)$	$8(q+1)$	$4(q+1)$	$4(q-1)$	$8(q+1)$
\mathcal{G}_0	-	-	2	$4q$	$4q$	$2q$	$2(q-1)$	$4q$
\mathcal{G}_{-1}	-	-	3	$4q+p+2$	$4q+p+2$	$3q+p-1$	$2(q-1)$	$4q+p+2$
\mathcal{G}_1	-	-	5	$3(q+2)$	$4(q+1)$	$2(q+1)$	$3q-1$	$6q$

TABLE 1. Enumeration of skew braces of size p^2q according to the additive and multiplicative isomorphism class of groups where $p \equiv 1 \pmod{q}$ and $q > 3$.

$+\backslash\circ$	\mathbb{Z}_{3p^2}	$\mathbb{Z}_{p^2} \rtimes_t \mathbb{Z}_3$	$\mathbb{Z}_p^2 \times \mathbb{Z}_3$	\mathcal{G}_0	\mathcal{G}_{-1}	\mathcal{G}_1
$\mathbb{Z}_{p^2} \rtimes_t \mathbb{Z}_3$	4	4	-	-	-	-
\mathcal{G}_0	-	-	2	12	6	4
\mathcal{G}_{-1}	-	-	3	$p+14$	$p+8$	4
\mathcal{G}_1	-	-	5	16	10	8

TABLE 2. Enumeration of skew braces of size $3p^2$ according to the additive and multiplicative isomorphism class of groups where $p \equiv 1 \pmod{3}$.

3.1. Skew braces of $\mathbb{Z}_{p^2} \rtimes_t \mathbb{Z}_q$ -type. In this section we denote by A the group $\mathbb{Z}_{p^2} \rtimes_t \mathbb{Z}_q$. According to the description of the automorphism group of groups of order p^2q given in [9, Theorem 3.4], we have that

$$\phi : \text{Hol}(\mathbb{Z}_{p^2}) = \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_{p^2}^\times \longrightarrow \text{Aut}(A), \quad (i, j) \mapsto \varphi_{i,j} = \begin{cases} \tau \mapsto \sigma^i \tau, \\ \sigma \mapsto \sigma^j \end{cases}$$

is an isomorphism of groups. In particular $|\text{Aut}(A)| = p^3(p-1)$. Thus if G is a regular subgroup of $\text{Hol}(A)$ then $|\pi_2(G)| \in \{p, q, p^2, pq, p^2q\}$. Let us show that $|\pi_2(G)|$ can not equal p .

Lemma 3.1. *Let G be a regular subgroup of $\text{Hol}(A)$. Then $|\pi_2(G)| \neq p$.*

Proof. Assume that G is a subgroup of size p^2q of $\text{Hol}(A)$ and $|\pi_2(G)| = p$. Up to conjugation, we can choose the generators of $\ker \pi_2|_G$ to be τ and σ^p . Hence the standard presentation of G is

$$G = \langle \tau, \sigma^p, \sigma^a \alpha \rangle$$

where $\alpha \in \text{Aut}(A)$ is an automorphism of order p that satisfies $\alpha(\tau) = \sigma^{pb} \tau$ for some $0 \leq b \leq p-1$. By condition (K) we have that

$$\sigma^a \alpha \tau \alpha^{-1} \sigma^{-a} = \sigma^{a(1-t)+pb} \tau \in \langle \sigma^p, \tau \rangle.$$

Then $a \equiv 0 \pmod{p}$, i.e. $a = a'p$. Hence $(\sigma^p)^{-a'} \sigma^{a'p} \alpha = \alpha \in G \cap (\{1\} \times \text{Aut}(A))$ and then G is not regular by Lemma 2.3. \square

Lemma 3.2. *There exists a unique conjugacy class of regular subgroups G of $\text{Hol}(A)$ with $|\pi_2(G)| = pq$. A representative is*

$$H = \langle \sigma^p, \sigma \varphi_{0,p+1}, \tau^{-1} \varphi_{0,t} \rangle \cong \mathbb{Z}_{p^2q}.$$

Proof. According to Lemma 2.4(2), $\langle \sigma^p, \tau \rangle \subseteq \pi_1(H)$ and clearly $\sigma \in \pi_1(H)$. So $|\pi_1(H)| > pq$ and so $|\pi_1(H)| = |H| = p^2q$. According to Lemma 2.3, H is regular.

Let G be a regular subgroup of $\text{Hol}(A)$ with $|\pi_2(G)| = pq$. Then $K = \ker \pi_2|_G = \langle \sigma^p \rangle$. Up to conjugation, the subgroups of size pq of $\text{Aut}(A)$ are $\langle \varphi_{p,1}, \varphi_{0,t} \rangle$ and $\langle \varphi_{0,t}, \varphi_{0,p+1} \rangle$.

In the first case, we have

$$G = \langle \sigma^p, \sigma^a \tau^b \varphi_{p,1}, \sigma^c \tau^d \varphi_{0,t} \rangle.$$

We need to check the conditions (R). The generators of $\pi_2(G)$ satisfy the relations $(\varphi_{p,1})^p = 1$ and $\varphi_{0,t} \varphi_{p,1} \varphi_{0,t}^{-1} = \varphi_{p,1}^t$. Thus we have

$$(\sigma^a \tau^b \varphi_{p,1})^p K = \sigma^a \sum_{j=0}^{p-1} t^{bj} \tau^{bp} K \stackrel{(R)}{=} K$$

and so $b = 0$. Accordingly, $d \neq 0$, otherwise by Lemma 2.4(1) we have that $\pi_1(G) \subseteq \langle \sigma \rangle$. Moreover

$$(\sigma^c \tau^d \varphi_{0,t}) \sigma^a \varphi_{p,1} (\sigma^c \tau^d \varphi_{0,t})^{-1} K = \sigma^{t^{d+1}a} \varphi_{p,1}^t K \stackrel{(R)}{=} (\sigma^a \varphi_{p,1})^t K = \sigma^{ta} \varphi_{p,1}^t K$$

and then $a = pa'$. Therefore, $\sigma^{-a'p}\sigma^{a'p}\varphi_{p,1} = \varphi_{p,1} \in G$ and so G is not regular, contradiction.

Let $\pi_2(G) = \langle \varphi_{0,t}, \varphi_{0,p+1} \rangle$. Then,

$$G = \langle \sigma^p, \sigma^a \tau^b \varphi_{0,p+1}, \sigma^c \tau^d \varphi_{0,t} \rangle.$$

The relations satisfied by the generators of $\pi_2(G)$ are $\varphi_{0,t}^q = \varphi_{0,p+1}^p = [\varphi_{0,t}, \varphi_{0,p+1}] = 1$. By the conditions (R) we have that $\sigma^a \tau^b \varphi_{0,p+1}$ and $\sigma^c \tau^d \varphi_{0,t}$ satisfy the same relations modulo K . Accordingly, since

$$\begin{aligned} (\sigma^a \tau^b \varphi_{0,p+1})^p K &= \sigma^a \sum_{j=0}^{p-1} t^{bj} \tau^{bp} K \stackrel{(R)}{=} K, \\ (\sigma^a \varphi_{0,p+1}) \sigma^c \tau^d \varphi_{0,t} K &= \sigma^{a+c} \tau^d \varphi_{0,p+1} \varphi_{0,t} K \stackrel{(R)}{=} (\sigma^c \tau^d \varphi_{0,t}) \sigma^a \varphi_{0,p+1} K = \sigma^{t^{d+1} a + c} \tau^d \varphi_{0,p+1} \varphi_{0,t} K, \end{aligned}$$

then $b = 0$ and $d = -1$. Finally,

$$(\sigma^c \tau^{-1} \varphi_{0,t})^q K = \sigma^{qc} K \stackrel{(R)}{=} K$$

and so $c = 0 \pmod{p}$. If $a = 0 \pmod{p}$, then according to Lemma 2.4(2), $\pi_1(G) \subseteq \langle \sigma^p, \tau \rangle$ and so $a \neq 0 \pmod{p}$. Then G is conjugate to H by $\varphi_{0,a-1}$. \square

Lemma 3.3. *A set of representatives of regular subgroups G of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2$ is*

$$G_s = \langle \tau, \sigma^{\frac{1}{t-1}} \varphi_{1,(p+1)^s} \rangle \cong \mathbb{Z}_{p^2q}$$

for $s = 0, 1$.

Proof. The groups G_0 and G_1 are not conjugate since their images under π_2 are not. Applying Lemma 2.3 and Lemma 2.4 as we did in Lemma 3.2, we can show that the groups G_s are regular. Indeed, $(\sigma^{\frac{1}{t-1}} \varphi_{1,(p+1)^s})^p = \sigma^{\frac{p}{t-1}} \varphi_{p,1}$ and $\varphi_{p,1} (\sigma^{\frac{p}{t-1}})^p = \sigma^{\frac{p}{t-1}}$. So we have that $\langle \tau, \sigma^p \rangle \subseteq \pi_1(G_s)$ and $\sigma^{\frac{1}{t-1}} \in \pi_1(G_s)$. Thus $|\pi_1(H)| = p^2q$ and so H is regular.

Let G be a regular subgroup of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2$. According to [1, Lemma 2.1], the p -Sylow subgroup of the multiplicative group of the skew brace associated to G is cyclic and then so is $\pi_2(G)$. The unique cyclic subgroups of order p^2 of $\text{Aut}(A)$ up to conjugation are $\langle \varphi_{1,(p+1)^s} \rangle$ for $s = 0, 1$. The size of the kernel of π_2 is q and so, in both cases, we can assume that $\ker \pi_2|_G$ is generated by τ up to conjugation by an automorphism in the normalizer of $\pi_2(G)$. Therefore

$$G = \langle \tau, \sigma^b \varphi_{1,(p+1)^s} \rangle$$

for some $b \neq 0$. The group G has a normal q -Sylow subgroup and then it is abelian, and thus $b = \frac{1}{t-1}$. \square

The group

$$(6) \quad \mathfrak{H}_1 = \langle \sigma, \varphi_{1,1} \rangle$$

is normal in $\text{Hol}(A)$. Let u be a generator of $\mathbb{Z}_{p^2}^\times$, then $\text{Hol}(A)/\mathfrak{H}_1 = \langle \tau, \varphi_{0,u} \rangle \cong \mathbb{Z}_q \times \mathbb{Z}_{p^2}^\times$ and in particular it is abelian.

Lemma 3.4. *The skew braces of A -type with $|\ker \lambda| = p^2$ are $B_s = (A, +, \circ)$ where $(B_s, +) = \mathbb{Z}_{p^2} \rtimes_t \mathbb{Z}_q$ and $(B_s, \circ) = \mathbb{Z}_{p^2} \rtimes_{t \frac{s+1}{s}} \mathbb{Z}_q$ for $1 \leq s \leq q-1$. In particular, B_s is a bi-skew brace and*

$$(B_s, \circ) \cong \begin{cases} \mathbb{Z}_{p^2q}, & \text{if } s = q-1, \\ A, & \text{otherwise.} \end{cases}$$

Proof. Let us consider the groups

$$G_s = \langle \sigma, \tau^s \varphi_{0,t} \rangle \cong \begin{cases} \mathbb{Z}_{p^2q}, & \text{if } s = q-1, \\ A, & \text{otherwise} \end{cases}$$

for $1 \leq s \leq q-1$. The subset $\pi_1(G_s)$ contains $\langle \sigma \rangle$ and τ^s and then $|\pi_1(G)| > p^2$ and it divides p^2q . So $\pi_1(G) = A$ and according to Lemma 2.3 we have that G_s is regular. Let \mathfrak{H}_1 be the group defined in (6). If G_s and $G_{s'}$ are conjugate, then so are their images in $\text{Hol}(A)/\mathfrak{H}_1$ which is abelian. Therefore $\langle \tau^s \varphi_{0,t} \rangle = \langle \tau^{s'} \varphi_{0,t} \rangle$, and so it follows that $s = s'$.

Let G be a regular subgroup of $\text{Hol}(A)$ with $|\pi_2(G)| = q$. The unique subgroup of order p^2 of A is the unique p -Sylow subgroup generated by σ and, up to conjugation, the unique subgroup of order q in $\text{Aut}(A)$ is generated by $\varphi_{0,t}$. Then

$$G = \langle \sigma, \tau^s \varphi_{0,t} \rangle = G_s$$

for some $s \neq 0$.

For the skew brace B_s associated to G_s we have that $\langle \sigma \rangle \leq \ker \lambda$ and $\tau \in \text{Fix}(B_s)$. So, using [1, Lemma 2.2] we have that $\lambda_{\sigma^n \tau^m} = \lambda_{\sigma^n \tau^{\frac{sm}{s}}} = \lambda_{\tau^m}^n$ and then the multiplicative group structure of B_s is given by the formula

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \circ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + t^{\frac{s+1}{s}} x_2 y_1 \\ x_2 + y_2 \end{pmatrix}$$

for every $0 \leq x_1, y_1 \leq p^2 - 1$, $0 \leq x_2, y_2 \leq q - 1$. So $(B_s, +) = \mathbb{Z}_{p^2} \rtimes_t \mathbb{Z}_q$ and $(B_s, \circ) = \mathbb{Z}_{p^2} \rtimes_{t^{\frac{s+1}{s}}} \mathbb{Z}_q$ and according to [2, Corollary 1.2] B_s is a bi-skew brace. To compute the isomorphism class of (B_s, \circ) note that the group is abelian if and only if $s = q - 1$. \square

Lemma 3.5. *A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2 q$ is*

$$G_d = \langle \sigma^{\frac{1}{t-1}} \varphi_{1,1}, \tau^d \varphi_{0,t} \rangle \cong A$$

where $1 \leq d \leq q - 1$.

Proof. According to Lemma 2.4(2), $\langle \sigma, \tau \rangle \subseteq \pi_1(G_d)$ and so $\pi_1(G_d) = A$. The same argument used in Lemma 3.4 shows that they are not pairwise conjugate.

Let G be a regular subgroup with $|\pi_2(G)| = p^2 q$. According to [1, Lemma 2.1], the p -Sylow subgroup of the multiplicative group of the skew brace associated to G is cyclic, and then so is the p -Sylow subgroup of $\pi_2(G)$. Up to conjugation we can assume that $\pi_2(G)$ is $\langle \varphi_{1,1}, \varphi_{0,t} \rangle \cong A$, i.e.

$$G = \langle \sigma^a \tau^b \varphi_{1,1}, \sigma^c \tau^d \varphi_{0,t} \rangle.$$

From the conditions (R) we have

$$(7) \quad (\sigma^a \tau^b \varphi_{1,1})^{p^2} = 1,$$

$$(8) \quad (\sigma^c \tau^d \varphi_{0,t})^q = 1,$$

$$(9) \quad (\sigma^c \tau^d \varphi_{0,t}) \sigma^a \tau^b \varphi_{1,1} (\sigma^c \tau^d \varphi_{0,t})^{-1} = (\sigma^a \tau^b \varphi_{1,1})^t.$$

The equality in (7) implies that the same relation is true modulo \mathfrak{H}_1 and so $b = 0$. If $d = 0$, then G is not regular by Lemma 2.4(1). Accordingly $d \neq 0$ and from (9) we have $a = \frac{1}{t-1}$ and so the standard presentation of G is

$$G = \langle \sigma^{\frac{1}{t-1}} \varphi_{1,1}, \sigma^c \tau^d \varphi_{0,t} \rangle.$$

If $d = -1$ then from (8) it follows that $c = 0$. Otherwise, G is conjugate to G_d by $\varphi_{1,1}^n$ where $n = \frac{c(t-1)}{1-t^{d+1}}$. \square

We summarize the content of this subsection in the following table:

$ \ker \lambda $	$\mathbb{Z}_{p^2 q}$	$\mathbb{Z}_{p^2} \rtimes_t \mathbb{Z}_q$
1	-	$q - 1$
p	1	-
q	2	-
p^2	1	$q - 2$
$p^2 q$	-	1

TABLE 3. Enumeration of skew braces of $\mathbb{Z}_{p^2} \rtimes_t \mathbb{Z}_q$ -type for $p \equiv 1 \pmod{q}$.

3.2. Skew braces of \mathcal{G}_k -type for $k \neq 0, \pm 1$. In this section we assume that $k \in \mathfrak{B} \setminus \{0, 1, -1\}$ and accordingly $q > 3$. Recall that a presentation of the group \mathcal{G}_k is the following

$$(10) \quad \mathcal{G}_k = \langle \sigma, \tau, \epsilon \mid \sigma^p = \tau^p = \epsilon^q = [\sigma, \tau] = 1, \epsilon \sigma \epsilon^{-1} = \sigma^g, \epsilon \tau \epsilon^{-1} = \tau^{g^k} \rangle$$

where g is a fixed element of order q in \mathbb{Z}_p^\times as before. An automorphism of \mathcal{G}_k is determined by its image on the generators, i.e. by its restriction to $\langle \sigma, \tau \rangle$ given by a matrix and by the image on ϵ . According to [9, Subsections 4.1, 4.3], the mapping

$$\begin{aligned} \phi : \mathbb{Z}_p^2 \rtimes_\rho (\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times) &\longrightarrow \text{Aut}(\mathcal{G}_k), \\ [(n, m), (a, b)] &\mapsto h = \begin{cases} h|_{\langle \sigma, \tau \rangle} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \\ \epsilon \mapsto \sigma^n \tau^m \epsilon, \end{cases} \end{aligned}$$

is a group isomorphism (the action ρ is defined by setting $\rho(a, b)(n, m) = (an, bm)$). In particular, $|\text{Aut}(\mathcal{G}_k)| = p^2(p-1)^2$ and the unique p -Sylow subgroup of $\text{Aut}(\mathcal{G}_k)$ is generated by $\alpha_1 = [(1, 0), (1, 1)]$ and $\alpha_2 = [(0, 1), (1, 1)]$.

Let G be a regular subgroup of $\text{Hol}(\mathcal{G}_k)$. Since p^2q divides $|\text{Aut}(\mathcal{G}_k)|$ we need to discuss all possible cases for the size of $\pi_2(G)$.

In the group $\text{Hol}(\mathcal{G}_k)$ we have that

$$(11) \quad (\sigma^a \tau^b \alpha_1) \epsilon (\sigma^a \tau^b \alpha_1)^{-1} = \sigma^{1+(1-g)a} \tau^{(1-g^k)b} \epsilon,$$

$$(12) \quad (\sigma^c \tau^d \alpha_2) \epsilon (\sigma^c \tau^d \alpha_2)^{-1} = \sigma^{(1-g)c} \tau^{1+(1-g^k)d} \epsilon,$$

hold for every $0 \leq a, b, c, d \leq p-1$.

Lemma 3.6. *A set of representative regular subgroups G of $\text{Hol}(\mathcal{G}_k)$ with $|\pi_2(G)| = p$ is*

$$H_i = \langle \epsilon, \tau, \sigma^{\frac{1}{g-1}} \alpha_1 \alpha_2^i \rangle \cong \mathcal{G}_0, \quad K_i = \langle \epsilon, \sigma, \tau^{\frac{1}{g^k-1}} \alpha_1^i \alpha_2 \rangle \cong \mathcal{G}_0,$$

for $i = 0, 1$.

Proof. Up to conjugation, the subgroups of order p of $\text{Aut}(\mathcal{G}_k)$ are $\langle \alpha_1 \rangle, \langle \alpha_2 \rangle$ and $\langle \alpha_1 \alpha_2 \rangle$. The groups in the statement are not conjugate since either their images or their kernels with respect to π_2 are not. Each of them has the form

$$G = \langle \epsilon, u, v\theta \rangle$$

with $\theta(v) = v$ and $\mathcal{G}_k = \langle \epsilon, u, v \rangle$. According to Lemma 2.4(2), $\pi_1(G) = \mathcal{G}_k$, i.e. G is regular.

Let G be a regular subgroup of $\text{Hol}(\mathcal{G}_k)$ with $|\pi_2(G)| = p$. The kernel of π_2 has size pq and therefore it has an element of order q of the form $u\epsilon$ where $u \in \langle \sigma, \tau \rangle$. The subgroup $L = \langle \alpha_1, \alpha_2 \rangle$ normalizes $\pi_2(G)$, so we can conjugate G by a suitable element of L and according to (11) and (12) we can assume that $u = 0$. Then the kernel has the form $\langle \epsilon, v \rangle$ for $v \in \langle \sigma, \tau \rangle$. The p -Sylow subgroup of $\ker \pi_2$ is normal and therefore $v = \sigma$ or τ . Therefore the group G has the following form

$$G = \langle \epsilon, v, w\theta \rangle$$

where $\theta \in \{\alpha_1, \alpha_2, \alpha_1 \alpha_2\}$ and, either $v = \sigma$ and $1 \neq w \in \langle \tau \rangle$ or $v = \tau$ and $1 \neq w \in \langle \sigma \rangle$. By condition (K) we need to check that

$$w\theta\epsilon\theta^{-1}w^{-1} = w\theta(\epsilon)w^{-1} \in \ker \pi_2|_G.$$

Using (11) and (12) it follows that G is either H_i or K_i for $i = 0, 1$. □

Lemma 3.7. *The unique skew brace of \mathcal{G}_k -type with $|\ker \lambda| = q$ is $(B, +, \circ)$ where $(B, +) = \mathbb{Z}_p^2 \rtimes_{\mathcal{D}_{1,k}} \mathbb{Z}_q$ and*

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \circ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 g^{y_3} + y_1 g^{x_3} \\ x_2 g^{ky_3} + y_2 g^{kx_3} \\ x_3 + y_3 \end{pmatrix}.$$

for every $0 \leq x_1, x_2, y_1, y_2 \leq p-1$ and $0 \leq x_3, y_3 \leq q-1$. In particular, $(B, \circ) \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q$.

Proof. According to (11) and (12), the subgroup

$$H = \langle \epsilon, \sigma^{\frac{1}{g-1}} \alpha_1, \tau^{\frac{1}{g^k-1}} \alpha_2 \rangle$$

is isomorphic to $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ and $|\pi_2(G)| = p^2$. According to Lemma 2.4(2), $\pi_1(H) = \langle \epsilon, \sigma, \tau \rangle$ and so H is regular.

Let G be a regular subgroup with $|\pi_2(G)| = p^2$. Then the image of π_2 is the unique p -Sylow subgroup of $\text{Aut}(\mathcal{G}_k)$. The kernel is a subgroup of order q of G , and, according to (11) and (12), we can assume that it is generated by ϵ . The group G has a standard presentation as

$$G = \langle \epsilon, \sigma^a \tau^b \alpha_1, \sigma^c \tau^d \alpha_2 \rangle$$

and it is abelian, since it has a normal q -Sylow subgroup. Using (11) and (12) again, we have that $b = c = 0$, $a = \frac{1}{g-1}$ and $d = \frac{1}{g^k-1}$, i.e. $G = H$.

Let B be the skew brace associated to H . Since $\epsilon \in \ker \lambda$ and $\sigma, \tau \in \text{Fix}(B)$, by [1, Lemma 2.1] we can compute the formula for the \circ operation of B as in the statement. \square

The subgroup

$$(13) \quad \mathfrak{H}_2 = \langle \sigma, \tau, \alpha_1, \alpha_2 \rangle \cong \mathbb{Z}_p^4$$

is normal in $\text{Hol}(\mathcal{G}_k)$. Let μ be a generator of \mathbb{Z}_p^\times , then $\text{Hol}(\mathcal{G}_k)/\mathfrak{H}_2 = \langle \epsilon, [(0, 0), (1, \mu)], [(0, 0), (\mu, 1)] \rangle \cong \mathbb{Z}_q \times \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$. In particular, it is abelian.

The formula

$$(F_k) \quad \alpha_1^n \alpha_2^m u \epsilon^d \theta \alpha_2^{-m} \alpha_1^{-n} = u \sigma^{xn \frac{g^d-1}{g-1}} \alpha_1^{(1-x)n} \tau^{ym \frac{g^{kd}-1}{g^k-1}} \alpha_2^{(1-y)m} \epsilon^d \theta,$$

where $u \in \langle \sigma, \tau \rangle$ and $\theta = [(0, 0), (x, y)]$ holds in $\text{Hol}(\mathcal{G}_k)$.

Let us denote $\beta_s = [(0, 0), (g, g^s)]$ and $\tilde{\beta} = [(0, 0), (1, g)]$ for $0 \leq s \leq q-1$.

Proposition 3.8. *The skew braces of \mathcal{G}_k -type with $|\ker \lambda| = p^2$ are $(B_{a,b}, +, \circ)$ where $(B_{a,b}, +) = \mathbb{Z}_p^2 \rtimes_{\mathcal{D}_{1,k}} \mathbb{Z}_q$ and*

$$(B_{a,b}, \circ) = \mathbb{Z}_p^2 \rtimes_{\mathcal{D}_{a+1, b+k}} \mathbb{Z}_q \cong \begin{cases} \mathbb{Z}_p^2 \times \mathbb{Z}_q, & \text{if } a = q-1, b = -k \pmod{q}, \\ \mathcal{G}_0, & \text{if } a = q-1, b \neq -k \pmod{q}, \\ \mathcal{G}_{\frac{b+k}{a+1}}, & \text{otherwise,} \end{cases}$$

for $0 \leq a, b \leq q-1$ and $(a, b) \neq (0, 0)$. In particular, there are $q^2 - 1$ such skew braces and they are bi-skew braces.

Proof. The subgroups

$$G_{a,b} = \langle \sigma, \tau, \epsilon \beta_0^a \tilde{\beta}^b \rangle$$

are regular. If $G_{a,b}$ and $G_{c,d}$ are conjugate, then so are their images in the quotient $\text{Hol}(\mathcal{G}_k)/\mathfrak{H}_2$, which is abelian. Then $\langle \epsilon \beta_0^a \tilde{\beta}^b \rangle = \langle \epsilon \beta_0^c \tilde{\beta}^d \rangle$ and so it follows that $(a, b) = (c, d)$.

Let G be a regular subgroup of $\text{Hol}(\mathcal{G}_k)$ such that $|\pi_2(G)| = q$. Then the kernel of π_2 is the unique p -Sylow subgroup of \mathcal{G}_k . The elements of order q of $\text{Aut}(\mathcal{G}_k)$ are of the form $\alpha_1^s \alpha_2^t \beta_0^a \tilde{\beta}^b$ where $s = 0$ (resp. $t = 0$) whenever $a = 0$ (resp. $b = 0$). According to formula (F_k) for $u = 1$ and $d = 0$, up to conjugation by an element in $\alpha_1 \alpha_2$, we can assume that $\pi_2(G)$ is generated by $\beta_0^a \tilde{\beta}^b$ where $(a, b) \neq (0, 0)$. Therefore

$$G = \langle \sigma, \tau, \epsilon^n \beta_0^a \tilde{\beta}^b \rangle$$

and so $n \neq 0$ since G is regular. Using that $(\epsilon^n \beta_0^a \tilde{\beta}^b)^{n-1} = \epsilon \beta_0^{an-1} \tilde{\beta}^{bn-1}$ we can conclude that $G = G_{an-1, bn-1}$.

Let $B_{a,b}$ be the skew brace associated to the regular subgroup $G_{a,b}$. Note that $\epsilon \in \text{Fix}(B_{a,b})$ and $\sigma, \tau \in \ker \lambda$. Therefore we can apply [1, Lemma 2.2] and it follows that

$$(14) \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \circ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + g^{(a+1)x_3} y_1 \\ x_2 + g^{(b+k)x_3} y_2 \\ x_3 + y_3 \end{pmatrix}$$

for every $0 \leq x_1, x_2, y_1, y_2 \leq p-1$, $0 \leq x_3, y_3 \leq q-1$. So $(B_{a,b}, +) = \mathbb{Z}_p^2 \rtimes_{\mathcal{D}_{1,k}} \mathbb{Z}_q$, $(B_{a,b}, \circ) = \mathbb{Z}_p^2 \rtimes_{\mathcal{D}_{a+1, b+k}} \mathbb{Z}_q$ and the images of the two actions commute. According to [2, Proposition 1.1], $B_{a,b}$ is a bi-skew brace. \square

To compute the regular subgroups of $\text{Hol}(\mathcal{G}_k)$ with image under π_2 of size pq and p^2q we are computing the conjugacy classes of subgroups of size pq and p^2q of $\text{Aut}(\mathcal{G}_k)$.

Lemma 3.9. *A set of representatives of conjugacy classes of subgroups of $\text{Aut}(\mathcal{G}_k)$ of order pq and order p^2q are*

Size	G	Parameters	Class
pq	$\mathcal{H}_{1,s} = \langle \alpha_1, \beta_s \rangle$	$0 \leq s \leq q-1$	$\mathbb{Z}_p \rtimes_t \mathbb{Z}_q$
	$\mathcal{H}_{2,s} = \langle \alpha_2, \beta_s \rangle$	$0 \leq s \leq q-1$	\mathbb{Z}_{pq} , if $s = 0$, $\mathbb{Z}_p \rtimes_t \mathbb{Z}_q$, otherwise.
	$\mathcal{K}_i = \langle \alpha_i, \tilde{\beta} \rangle$	$i = 1, 2$	\mathbb{Z}_{pq} , if $i = 1$, $\mathbb{Z}_p \rtimes_t \mathbb{Z}_q$, if $i = 2$.
	$\mathcal{W} = \langle \alpha_1 \alpha_2, \beta_1 \rangle$	-	$\mathbb{Z}_p \rtimes_t \mathbb{Z}_q$
p^2q	$\mathcal{T}_s = \langle \alpha_1, \alpha_2, \beta_s \rangle$	$0 \leq s \leq q-1$	\mathcal{G}_s
	$\mathcal{U} = \langle \alpha_1, \alpha_2, \tilde{\beta} \rangle$	-	\mathcal{G}_0

Proof. Let G be a subgroup of size pq of $\text{Aut}(\mathcal{G}_k)$. Then G is generated by an element of order p and an element of order q . The elements of order p belong to the subgroup generated by α_1 and α_2 and up to conjugation there are three elements of order p , namely $\alpha_1, \alpha_2, \alpha_1 \alpha_2$. The elements of order q can be chosen of the form $\alpha_1^n \alpha_2^m \beta_s$ for $s \neq 0$, $\alpha_1^n \beta_0$ or $\alpha_2^m \tilde{\beta}$.

If the p -Sylow subgroup of G is generated by α_i , $i = 1, 2$, using that

$$(15) \quad \begin{aligned} [(a, b), (x, y)] \alpha_1^n \alpha_2^m \beta_s [(a, b), (x, y)]^{-1} &= \alpha_1^{xn+(1-g)a} \alpha_2^{ym+(1-g^s)b} \beta_s, \\ [(a, b), (x, y)] \alpha_2^m \tilde{\beta} [(a, b), (x, y)]^{-1} &= \alpha_2^{ym+(1-g)b} \tilde{\beta}, \end{aligned}$$

we have that G is conjugate to $\mathcal{H}_{i,s}$ or to \mathcal{K}_i . If the p -Sylow subgroup of G is generated by $\alpha_1 \alpha_2$ then necessarily the element of order q has form $\alpha_1^n \alpha_2^m \beta_1$, since the p -Sylow subgroup has to be normal. In this case, then $G = \langle \alpha_1 \alpha_2, \alpha_1^n \alpha_2^m \beta_1 \rangle = \langle \alpha_1 \alpha_2, \alpha_1^{n-m} \beta_1 \rangle$ and according to (15), G is conjugate to \mathcal{W} by a suitable power of α_1 .

If G is a subgroup of order p^2q , then G is generated by α_1, α_2 and an element of order q , which can be chosen to be β_s for $0 \leq s \leq q-1$ or $\tilde{\beta}$. Such groups are not conjugate, since their restrictions to $\langle \sigma, \tau \rangle$ are not. \square

In the following we are employing the same notation as in Lemma 3.9 and the following formula:

$$(P_k) \quad (\sigma^a \tau^b \epsilon^c \theta)^n = \sigma^a \sum_{i=0}^{n-1} g^{i(c+x)} \tau^a \sum_{i=0}^{n-1} g^{i(ck+y)} \epsilon^{nc} \theta^n$$

where $\theta = [(0, 0), (g^x, g^y)]$.

Lemma 3.10. *A set of representatives of conjugacy classes of subgroups G of $\text{Hol}(\mathcal{G}_k)$ with $|\pi_2(G)| = pq$ is*

$\pi_2(G)$	Subgroups	Parameters	Isomorphism class	#
$\mathcal{H}_{1,s}$	$\tilde{H}_{1,s,d} = \langle \tau, \sigma^{\frac{1}{g-1}} \alpha_1, \epsilon^d \beta_s \rangle$	$0 \leq s \leq q-1,$ $1 \leq d \leq q-1$	\mathcal{G}_{dk+s}	$q(q-1)$
	$\hat{H}_{1,s} = \langle \sigma, \tau \alpha_1, \epsilon^{\frac{1-s}{k}} \beta_s \rangle$	$2 \leq s \leq q$	$\mathcal{G}_{\frac{k-s+1}{k}}$	$q-1$
	$\bar{H}_{1,s} = \langle \sigma\tau, \sigma^{\frac{1}{g-1}} \alpha_1, \epsilon^{\frac{s-1}{1-k}} \beta_s \rangle$	$2 \leq s \leq q$	$\mathcal{G}_{\frac{s-k}{1-k}}$	$q-1$
$\mathcal{H}_{2,s}$	$\tilde{H}_{2,s,d} = \langle \sigma, \tau^{\frac{1}{g^{k-1}}} \alpha_2, \epsilon^d \beta_s \rangle$	$0 \leq s \leq q-1,$ $1 \leq d \leq q-1$	$\mathbb{Z}_p^2 \times \mathbb{Z}_q$, if $s=0, d=-1$, \mathcal{G}_0 , if $s \neq 0, d=-1$, $\mathcal{G}_{\frac{s}{d+1}}$, otherwise.	$q(q-1)$
	$\hat{H}_{2,s} = \langle \tau, \sigma \alpha_2, \epsilon^{s-1} \beta_s \rangle$	$2 \leq s \leq q$	\mathcal{G}_0 , if $s=0$, $\mathcal{G}_{\frac{s(k+1)-k}{s}}$, otherwise.	$q-1$
	$\bar{H}_{2,s} = \langle \sigma\tau, \tau^{\frac{1}{g^{k-1}}} \alpha_2, \epsilon^{\frac{s-1}{1-k}} \beta_s \rangle$	$2 \leq s \leq q$	\mathcal{G}_0 if $s=0$, $\mathcal{G}_{\frac{s-k}{s(1-k)}}$, otherwise.	$q-1$
\mathcal{K}_1	$\tilde{K}_{1,d} = \langle \tau, \sigma^{\frac{1}{g-1}} \alpha_1, \epsilon^d \tilde{\beta} \rangle$	$1 \leq d \leq q-1$	$\mathbb{Z}_p^2 \times \mathbb{Z}_q$ if $d = -k^{-1}$, \mathcal{G}_0 , otherwise.	$q-1$
	$\hat{K}_1 = \langle \sigma, \tau \alpha_1, \epsilon^{-k^{-1}} \tilde{\beta} \rangle$	-	\mathcal{G}_0	1
	$\bar{K}_1 = \langle \sigma\tau, \sigma^{\frac{1}{g-1}} \alpha_1, \epsilon^{\frac{1}{1-k}} \tilde{\beta} \rangle$	-	\mathcal{G}_0	1
\mathcal{K}_2	$\tilde{K}_{2,d} = \langle \sigma, \tau^{\frac{1}{g^{k-1}}} \alpha_2, \epsilon^d \tilde{\beta} \rangle$	$1 \leq d \leq q-1$	\mathcal{G}_d	$q-1$
	$\hat{K}_2 = \langle \tau, \sigma \alpha_2, \epsilon \tilde{\beta} \rangle$	-	\mathcal{G}_{k+1}	1
	$\bar{K}_2 = \langle \sigma\tau, \tau^{\frac{1}{g^{k-1}}} \alpha_2, \epsilon^{\frac{1}{1-k}} \tilde{\beta} \rangle$	-	$\mathcal{G}_{\frac{1}{1-k}}$	1
\mathcal{W}	$\tilde{W}_d = \langle \sigma, \tau^{\frac{1}{g^{k-1}}} \alpha_1 \alpha_2, \epsilon^d \beta_1 \rangle$	$1 \leq d \leq q-1$	\mathcal{G}_{d+1}	$q-1$
	$\hat{W}_d = \langle \tau, \sigma^{\frac{1}{g-1}} \alpha_1 \alpha_2, \epsilon^d \beta_1 \rangle$	$1 \leq d \leq q-1$	\mathcal{G}_{dk+1}	$q-1$

Proof. Any of the subgroups G in the table have the form

$$(16) \quad G = \langle u, w\delta, \epsilon^c h \rangle,$$

for $\langle u, w \rangle = \langle \sigma, \tau \rangle$, $\delta \in \langle \alpha_1, \alpha_2 \rangle$, $c \neq 0$, $\delta(u) = u$, $\delta(w) = w$, $h(\epsilon) = \epsilon$. According to Lemma 2.4(2), we have $\langle \sigma, \tau \rangle \subseteq \pi_1(G)$. Since $\epsilon^c \in \pi_1(G)$, then $|\pi_1(G)| > p^2$ and so $\pi_1(G) = \mathcal{G}_k$.

A set of representatives of the orbits of the elements of $\langle \sigma, \tau \rangle$ under the action of $\text{Aut}(\mathcal{G}_k)$ is $\{\sigma, \tau, \sigma\tau\}$. Then the subgroups with the same image under π_2 and belonging to different rows of the tables are not conjugate, since their kernels with respect to π_2 are not.

Assume that two groups in the same row are conjugate. Then their images in the abelian quotient $\text{Hol}(\mathcal{G}_k)/\mathfrak{H}_2$ coincide. Then according to (16), their images are generated respectively by $\epsilon^c h$ and $\epsilon^d h$ and so it follows that $c = d$.

We focus on the case when the image under π_2 is $\mathcal{H}_{1,s}$, for the other cases we can apply the same ideas, so we omit the computation.

Let G be a regular subgroup of $\text{Hol}(\mathcal{G}_k)$ with $\pi_2(G) = \mathcal{H}_{1,s}$. The kernel of π_2 is a subgroup of order p of \mathcal{G}_k and we can choose it up to the action of the normalizer of $\mathcal{H}_{1,s}$ on the subgroups of $\langle \sigma, \tau \rangle$. Since $\{[(0,0), (a,b)] : 1 \leq a, b \leq p-1\} \leq N_{\text{Aut}(\mathcal{G}_k)}(\mathcal{H}_{1,s})$, we can assume that the kernel is generated by σ , by τ or by $\sigma\tau$. The group G has the form

$$G = \langle u, v\epsilon^b \alpha_1, w\epsilon^d \beta_s \rangle$$

where $u \in \{\sigma, \tau, \sigma\tau\}$ and $v, w \in \langle \sigma, \tau \rangle$. From the (R) condition $(v\epsilon^b \alpha_1)^p \in \ker \pi_2 = \langle u \rangle$ we have that $(v\epsilon^b \alpha_1)^p \mathfrak{H}_2 = \epsilon^{bp} \mathfrak{H}_2 = \mathfrak{H}_2$ and so $b = 0$. If $d = 0$ then according to Lemma 2.4(1), $\pi_1(G) \subseteq \langle \sigma, \tau \rangle$ and so G is not regular. Thus $d \neq 0$. From the (R) condition $\beta_s \alpha_1 \beta_s^{-1} = \alpha_1^g$ we have that

$$(17) \quad (w\epsilon^d \beta_s) v \alpha_1 (w\epsilon^d \beta_s)^{-1} = (v \alpha_1)^g = v^g \alpha_1^g \pmod{\langle u \rangle}.$$

(i) Assume that $\ker \pi_2|_G = \langle \sigma \rangle$. Then G has a standard presentation as

$$G = \langle \sigma, \tau^a \alpha_1, \tau^c \epsilon^d \beta_s \rangle$$

for $a, d \neq 0$ and so up to conjugation by $[(0, 0), (1, a^{-1})]$ we can assume that $a = 1$. Hence G has the form

$$G = \langle \sigma, \tau\alpha_1, \tau^c \epsilon^d \beta_s \rangle.$$

By (17) we have

$$(18) \quad (\tau^c \epsilon^d \beta_s) \tau \alpha_1 (\tau^c \epsilon^d \beta_s)^{-1} = \sigma^g \frac{g^c - 1}{g - 1} \tau^{g^{dk+s}} \alpha_1^g = \tau^g \alpha_1^g \pmod{\langle \sigma \rangle}$$

and it follows that $d = \frac{1-s}{k}$ and so $s \neq 1$. If there exists $h \in \text{Aut}(\mathcal{G}_k)$ that normalizes $\langle \sigma, \tau\alpha_1 \rangle$ and that $h\tau^c \epsilon^{\frac{1-s}{k}} \beta_s h^{-1} \in \widehat{H}_{1,s}$, the group G is conjugate to $\widehat{H}_{1,s}$. Hence, according to (F_k) we can choose h to be a suitable power of α_1 .

(ii) Assume that $\ker \pi_2|_G = \langle \tau \rangle$. Then

$$G = \langle \tau, \sigma^a \alpha_1, \sigma^c \epsilon^d \beta_s \rangle$$

where $a, d \neq 0$. Similarly to the previous case, equation (17) implies that $a = \frac{1}{g-1}$. The relation $(\beta_s)^q = 1$ holds and so $(\sigma^c \epsilon^{-1} \beta_s)^q \in \langle \tau \rangle$. According to (P_k) if $d = -1$ then $(\sigma^c \epsilon^d \beta_s)^q = \sigma^{qc}$, and so $c = 0$. Otherwise, according to (F_k) , the group G is conjugate to $\widehat{H}_{1,s,d}$ by a suitable power of α_1 .

(iii) Assume that $\ker \pi_2|_G = \langle \sigma\tau \rangle$. Then

$$G = \langle \sigma\tau, \sigma^a \alpha_1, \sigma^c \epsilon^d \beta_s \rangle.$$

Arguing as in (ii) we have $a = \frac{1}{g-1}$, $d = \frac{s-1}{1-k} \neq 0$, $s \neq 1$ and if $d = -1$ then $c = 0$. If $d \neq -1$, according to (F_k) we have that G is conjugate to $\overline{H}_{1,s}$ by a suitable power of α_1 . \square

Lemma 3.11. *A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(\mathcal{G}_k)$ with $|\pi_2(G)| = p^2q$ is*

$\pi_2(G)$	Subgroups	Parameters	Isomorphism class	#
\mathcal{T}_s	$\widetilde{T}_{c,s} = \langle \sigma^{\frac{1}{g-1}} \alpha_1, \tau^{\frac{1}{g^{k-1}}} \alpha_2, \epsilon^c \beta_s \rangle$	$0 \leq s \leq q-1$ $1 \leq c \leq q-1$	\mathcal{G}_s	$q(q-1)$
	$\widehat{T}_s = \langle \sigma^{\frac{1}{g-1}} \alpha_1, \sigma \tau^{\frac{1}{g^{k-1}}} \alpha_2, \epsilon^{s-1} \beta_s \rangle$	$2 \leq s \leq q$	\mathcal{G}_s	$q-1$
	$\overline{T}_s = \langle \tau \sigma^{\frac{1}{g-1}} \alpha_1, \tau^{\frac{1}{g^{k-1}}} \alpha_2, \epsilon^{\frac{1-s}{k}} \beta_s \rangle$	$2 \leq s \leq q$	\mathcal{G}_s	$q-1$
\mathcal{U}	$\widetilde{U}_c = \langle \sigma^{\frac{1}{g-1}} \alpha_1, \tau^{\frac{1}{g^{k-1}}} \alpha_2, \epsilon^c \widetilde{\beta} \rangle$	$1 \leq c \leq q-1$	\mathcal{G}_0	$q-1$
	$\widehat{U} = \langle \tau \sigma^{\frac{1}{g-1}} \alpha_1, \tau^{\frac{1}{g^{k-1}}} \alpha_2, \epsilon^{-k-1} \widetilde{\beta} \rangle$	-	\mathcal{G}_0	1
	$\overline{U} = \langle \sigma^{\frac{1}{g-1}} \alpha_1, \tau^{\frac{1}{g^{k-1}}} \sigma \alpha_2, \widetilde{\beta} \rangle$	-	\mathcal{G}_0	1

Proof. The groups in different rows of the tables are not conjugate since their p -Sylow subgroups are not. The same argument used in Lemma 3.10 shows that the groups in the table are regular and that the groups in the same row and with the same image under π_2 are not conjugate (e.g. if $\widetilde{T}_{c,s}$ and $\widetilde{T}_{d,s}$ are conjugate then $c = d$).

Let G be a regular subgroup of $\text{Hol}(\mathcal{G}_k)$ with $\pi_2(G) = \mathcal{T}_s$. Then

$$G = \langle u\epsilon^a \alpha_1, v\epsilon^b \alpha_2, w\epsilon^c \beta_s \rangle$$

for $u, v, w \in \langle \sigma, \tau \rangle$ and $0 \leq a, b, c \leq q-1$. We need to check the (R) conditions for the generators of G . Since

$$(u\epsilon^a \alpha_1)^p = (v\epsilon^b \alpha_2)^p \stackrel{(R)}{=} 1,$$

then we have that $(u\epsilon^a \alpha_1)^p \mathfrak{H}_2 = \epsilon^a \mathfrak{H}_2 = (v\epsilon^b \alpha_2)^p \mathfrak{H}_2 = \epsilon^b \mathfrak{H}_2$ and so $a = b = 0$. If $c = 0$, according to Lemma 2.4(1), $\pi_1(G) \subseteq \langle \sigma, \tau \rangle$ and so G is not regular. Hence $c \neq 0$ and

$$(19) \quad (w\epsilon^c \beta_s) u \alpha_1 (w\epsilon^c \beta_s)^{-1} \stackrel{(R)}{=} (u \alpha_1)^g = u^g \alpha_1^g,$$

$$(20) \quad (w\epsilon^c \beta_s) v \alpha_2 (w\epsilon^c \beta_s)^{-1} \stackrel{(R)}{=} (v \alpha_2)^{g^s} = v^{g^s} \alpha_2^{g^s}.$$

Equation (19) implies that $u = \sigma^{\frac{1}{g-1}}\tau^n$ where either $n = 0$ or $c = k^{-1}(1-s)$ and equation (20) that $v = \sigma^m\tau^{\frac{1}{g^k-1}}$ where either $m = 0$ or $c = s-1$. The equality $c = k^{-1}(1-s) = s-1$ implies that $k = -1$ or $s = 1$ and so $c = 0$, contradiction. Thus, we have to discuss the following cases:

- (i) $n = m = 0$.
 - (ii) $m = 0, c = k^{-1}(1-s), n \neq 0$ and $s \neq 1$.
 - (iii) $n = 0, c = 1-s, n \neq 0$ and $s \neq 1$.
- (i) If $n = m = 0$, then we have that

$$G = \langle \sigma^{\frac{1}{g-1}}\alpha_1, \tau^{\frac{1}{g^k-1}}\alpha_2, \sigma^a\tau^b\epsilon^c\beta_s \rangle$$

for some a, b . Let $h = \alpha_1^r\alpha_2^t$. To see that $hGh^{-1} = \widetilde{T}_{c,s}$ for some r, t , it is sufficient to show that $h\sigma^a\tau^b\epsilon^c\beta_s h^{-1} \in \widetilde{T}_{c,s}$. Using (F_k) , this is equivalent to have that (r, t) is a solution of the following system of linear equations:

$$\begin{cases} r(g^{c+1} - 1) = a(1-g) \\ t(1 - g^{ck+s}) = b(1-g^k). \end{cases}$$

Using (P_k) and the (R) condition $(\sigma^a\tau^b\epsilon^c\beta_s)^q = 1$, if $c+1 = 0$ then $a = 0$ and if $ck+s = 0$ then $b = 0$. So the system admits a solution for every a, b .

(ii) Let $m = 0$ and $c = 1-s$. Up to conjugation by the automorphism $h = [(0, 0), (n^{-1}, 1)]$ we can assume that $n = 1$. Accordingly G has the following form

$$G = \langle \tau\sigma^{\frac{1}{g-1}}\alpha_1, \tau^{\frac{1}{g^k-1}}\alpha_2, \sigma^a\tau^b\epsilon^{\frac{1-s}{k}}\beta_s \rangle$$

for some a, b . Let (r, t) be a solution of the following system of linear equations:

$$\begin{cases} r(g^{\frac{1-s+k}{k}} - 1) = a(1-g) \\ t(1-g) = b(1-g^k) + r(1-g^k)(1-g). \end{cases}$$

We have $(\sigma^a\tau^b\epsilon^{\frac{1-s}{k}}\beta_s)^q = 1$. According to (P_k) , if $\frac{1-s}{k} + 1 = \frac{1-s+k}{k} = 0$ then $a = 0$. Hence the system admits a solution for every a, b and $\alpha_1^r\alpha_2^t G(\alpha_1^r\alpha_2^t)^{-1} = \widehat{T}_s$.

(iii) Let $n = 0$ and $c = s-1$. Up to conjugation by $[(0, 0), (1, m^{-1})]$ we can assume that $m = 1$, i.e.

$$G = \langle \sigma^{\frac{1}{g-1}}\alpha_1, \sigma\tau^{\frac{1}{g^k-1}}\alpha_2, \sigma^a\tau^b\epsilon^{s-1}\beta_s \rangle$$

for some a, b . Let (r, t) be a solution of the following system of linear equations:

$$\begin{cases} r(g^s - 1) = (1-g^s)t + (1-g)a \\ t(g^{sk-k+s} - 1) = (1-g^k)b. \end{cases}$$

From the condition $(\sigma^a\tau^b\epsilon^{s-1}\beta_s)^q = 1$ and (P_k) we have that if $sk-k+s = 0$ then $b = 0$ and if $s = 0$ then $a = 0$. Accordingly the system has solution and $\alpha_1^r\alpha_2^t G(\alpha_1^r\alpha_2^t)^{-1} = \overline{T}_s$.

We can use the same ideas when $\pi_2(G) = \mathcal{U}$. In such case, G is conjugate to $\widetilde{U}_c, \widehat{U}$ or \overline{U} . □

We summarize the content of this subsection in the following table:

$ \ker \lambda $	$\mathbb{Z}_p^2 \times \mathbb{Z}_q$	\mathcal{G}_k	$\mathcal{G}_s, s \neq 0, \pm 1, k$	\mathcal{G}_0	\mathcal{G}_{-1}	\mathcal{G}_1
1	-	$2(q+1)$	$2(q+1)$	$2(q+1)$	$q+1$	$q-1$
p	2	$4(q+2)$	$4(q+2)$	$4(q+1)$	$2(q+2)$	$2(q-1)$
q	1	-	-	-	-	-
pq	-	-	-	4	-	-
p^2	1	$2q-3$	$2(q-1)$	$2(q-1)$	$q-1$	$q-1$
p^2q	-	1	-	-	-	-

TABLE 4. Enumeration of skew braces of \mathcal{G}_k -type for $p = 1 \pmod{q}$.

3.3. Skew braces of \mathcal{G}_0 -type. Let us consider the group \mathcal{G}_0 with presentation

$$\mathcal{G}_0 = \langle \sigma, \tau, \epsilon \mid \sigma^p = \tau^p = \epsilon^q = [\sigma, \tau] = [\tau, \epsilon] = 1, \epsilon\sigma\epsilon^{-1} = \sigma^g \rangle.$$

According to the description of the automorphisms of \mathcal{G}_0 provided in [9, Theorem 3.1, 3.4], the map

$$\phi : \mathbb{Z}_p \rtimes_{\rho} (\mathbb{Z}_p^{\times} \times \mathbb{Z}_p^{\times}) \longrightarrow \text{Aut}(\mathcal{G}_0), \quad [n, (a, b)] \mapsto h = \begin{cases} h|_{\langle \sigma, \tau \rangle} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, & \epsilon \mapsto \sigma^n \epsilon \end{cases}$$

where $\rho(a, b)(n) = an$, is a group isomorphism. In particular, $|\text{Aut}(\mathcal{G}_0)| = p(p-1)^2$ and its p -Sylow subgroup is generated by $\alpha = [1, (1, 1)]$.

If G is a regular subgroup of $\text{Hol}(\mathcal{G}_0)$, then $|\pi_2(G)|$ divides p^2q and $|\text{Aut}(\mathcal{G}_0)|$, so it divides pq .

In this section we can employ the same computations we did in Section 3.2, taking into account that in this case the p -Sylow subgroup of $\text{Aut}(\mathcal{G}_0)$ is cyclic and generated by α .

Lemma 3.12. *A set of representative of regular subgroups G of $\text{Hol}(\mathcal{G}_0)$ with $|\pi_2(G)| = p$ is*

$$H = \langle \epsilon, \tau, \sigma^{\frac{1}{g-1}} \alpha \rangle \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q, \quad K = \langle \epsilon, \sigma, \tau \alpha \rangle \cong \mathcal{G}_0.$$

Proof. We can argue as in Lemma 3.6 taking into account that the subgroups of order q of \mathcal{G}_0 are generated by elements of the form $\sigma^n \epsilon$. \square

The subgroup

$$(21) \quad \mathfrak{H}_3 = \langle \sigma, \tau, \alpha \rangle$$

is a normal subgroup of $\text{Hol}(\mathcal{G}_0)$ with $\text{Hol}(\mathcal{G}_0)/\mathfrak{H}_3 \cong \mathbb{Z}_q \times \mathbb{Z}_p^{\times} \times \mathbb{Z}_p^{\times}$, formula (P_k) holds in $\text{Hol}(\mathcal{G}_0)$ setting $k = 0$ and the formula

$$(F_0) \quad \alpha^n u \epsilon^d \theta \alpha^{-n} = u \sigma^{xn \frac{q^d-1}{q-1}} \alpha^{(1-x)n} \epsilon^d \theta,$$

where $u \in \langle \sigma, \tau \rangle$ and $\theta = [(0, 0), (x, y)]$ holds in \mathcal{G}_0 . Then, using the same argument of Proposition 3.8, we have the following.

Proposition 3.13. *The skew braces of \mathcal{G}_0 -type with $|\ker \lambda| = p^2$ are $(B_{a,b}, +, \circ)$, where $(B_{a,b}, +) = \mathbb{Z}_p^2 \rtimes_{\mathcal{D}_{1,0}} \mathbb{Z}_q$ and*

$$(B_{a,b}, \circ) = \mathbb{Z}_p^2 \rtimes_{\mathcal{D}_{a+1,b}} \mathbb{Z}_q \cong \begin{cases} \mathbb{Z}_p^2 \times \mathbb{Z}_q, & \text{if } a = q-1, b = 0, \\ \mathcal{G}_0, & \text{if } a = q-1, b \neq 0, \\ \mathcal{G}_{\frac{b}{a+1}}, & \text{otherwise,} \end{cases}$$

for $0 \leq a, b \leq q-1$ and $(a, b) \neq (0, 0)$. In particular, there are $q^2 - 1$ such skew braces and they are bi-skew braces.

Similarly to Lemma 3.9 we obtain that a set of representative of conjugacy classes of subgroups of order pq of $\text{Aut}(\mathcal{G}_0)$ is

$$\mathcal{H}_s = \langle \alpha, \beta_s \rangle, \quad \mathcal{K} = \langle \alpha, \tilde{\beta} \rangle$$

where $\beta_s = [0, (g, g^s)]$ and $0 \leq s \leq q-1$ and $\tilde{\beta} = [0, (1, g)]$.

Lemma 3.14. *A set of representatives of regular subgroups G of $\text{Hol}(\mathcal{G}_0)$ with $|\pi_2(G)| = pq$ is*

$\pi_2(G)$	Subgroups	Isomorphism class	Parameters	#
\mathcal{H}_s	$\tilde{H}_{s,c} = \langle \tau, \sigma^{\frac{1}{g-1}} \alpha, \epsilon^c \beta_s \rangle$	\mathcal{G}_s	$0 \leq s \leq q-1,$ $1 \leq c \leq q-1$	$q(q-1)$
	$\hat{H}_s = \langle \sigma\tau, \sigma^{\frac{1}{g-1}} \alpha, \epsilon^{s-1} \beta_s \rangle$	\mathcal{G}_s	$2 \leq s \leq q$	$q-1$
\mathcal{H}_1	$\bar{H}_{1,c} = \langle \sigma, \tau \alpha, \epsilon^c \beta_1 \rangle$	\mathcal{G}_{c+1}	$1 \leq c \leq q-1$	$q-1$
\mathcal{K}	$\tilde{K}_c = \langle \tau, \sigma^{\frac{1}{g-1}} \alpha, \epsilon^c \tilde{\beta} \rangle$	\mathcal{G}_0	$1 \leq c \leq q-1$	$q-1$
	$\hat{K} = \langle \sigma\tau, \sigma^{\frac{1}{g-1}} \alpha, \epsilon \tilde{\beta} \rangle$	\mathcal{G}_0	-	1

Proof. By the same argument used in Lemma 3.10 we can show that the groups in the table are regular and they are not conjugate (in this case we are going to use the group \mathfrak{H}_3 as in (21)). In order to prove that the groups in the table form a set of representatives, we can employ the very same computations we performed in Lemma 3.10 for the groups with image under π_2 equal to $\langle \alpha_1, \beta_s \rangle$ and $\langle \alpha_1, \tilde{\beta} \rangle$.

If we assume that $\pi_2(G) = \mathcal{H}_s$ and $\ker \pi_2|_G = \langle \sigma \rangle$ the (R) condition (18) implies that $s = 1$. For the other cases we can just set $k = 0$ and the same computations will do.

If we assume that $\pi_2(G) = \mathcal{K}$ and $\ker \pi_2|_G = \langle \sigma \rangle$ then

$$G = \langle \sigma, \tau^a \alpha, \tau^b \epsilon^c \tilde{\beta} \rangle$$

and the same condition implies that $a = 0$, i.e. G is not regular. The other cases are basically unchanged.

Moreover we can use (F_0) instead of (F_k) to prove that any regular subgroup G is conjugate to one in the table by a suitable power of α . \square

Remark 3.15. A skew brace B is a direct product if and only if there exists I, J ideals of B such that $I + J = B$ and $I \cap J = 0$. The direct product of a skew brace of size p^2 and a skew brace of size q is of abelian type. The following are the skew braces of \mathcal{G}_0 -type which are direct products of the trivial skew brace of size p and a skew brace of size pq . Since \mathcal{G}_0 is the direct product of \mathbb{Z}_p and $\mathbb{Z}_p \rtimes_g \mathbb{Z}_q$ we need to consider just the skew braces of size pq with non-abelian additive group. According to the classification provided in [2], there are $2q - 1$ non trivial such skew braces, then we need to find $2q - 1$ skew braces that are a direct product:

- (i) The skew braces $B_{a,0}$ for $a \neq 0$ as defined in Proposition 3.13 are direct product of the trivial skew brace of size p and a skew brace of size pq with $|\ker \lambda| = p$.
- (ii) The skew brace associated to the group H of Lemma 3.12 is the direct product of the trivial skew brace of size p and the unique skew brace of size pq with $|\ker \lambda| = q$.
- (iii) The skew brace associated to $\tilde{T}_{0,c}$ for $c \neq 0$ as defined in Lemma 3.14 is the direct product of the trivial skew brace of size p and a skew brace of size pq with $|\ker \lambda| = 1$.

We summarize the content of this subsection in the following table:

$ \ker \lambda $	$\mathbb{Z}_p^2 \times \mathbb{Z}_q$	$\mathcal{G}_k, k \in \mathfrak{B} \setminus \{0, \pm 1\}$	\mathcal{G}_0	\mathcal{G}_{-1}	\mathcal{G}_1
p	-	$2(q+1)$	$2q+1$	$q+1$	$q-1$
pq	1	-	1	-	-
p^2	1	$2(q-1)$	$2q-3$	$q-1$	$q-1$
p^2q	-	-	1	-	-

TABLE 5. Enumeration of skew braces of \mathcal{G}_0 -type for $p = 1 \pmod{q}$.

3.4. Skew braces of \mathcal{G}_{-1} -type. Let us consider the group \mathcal{G}_{-1} with presentation

$$\mathcal{G}_{-1} = \langle \sigma, \tau, \epsilon \mid \sigma^p = \tau^p = \epsilon^q = [\sigma, \tau] = 1, \epsilon \sigma \epsilon^{-1} = \sigma^g, \epsilon \tau \epsilon^{-1} = \tau^{g^{-1}} \rangle.$$

Let

$$\mathfrak{G} = (\mathbb{Z}_p^2 \rtimes_{\rho} (\mathbb{Z}_p^{\times} \times \mathbb{Z}_p^{\times})) \rtimes_{\rho'} \mathbb{Z}_2$$

where $\rho(a, b)(n, m) = (an, bm)$, $\rho'(1)[(n, m), (a, b)] = [(-gm, -g^{-1}n), (b, a)]$ for every $0 \leq n, m \leq p-1$, $1 \leq a, b \leq p-1$. According to [9, Subsections 4.1, 4.3], the mapping

$$\phi : \mathfrak{G} \longrightarrow \text{Aut}(\mathcal{G}_{-1}), \quad [(n, m), (a, b), i] \mapsto h = \begin{cases} h|_{\langle \sigma, \tau \rangle} = H_i(a, b), \\ \epsilon \mapsto \sigma^n \tau^m \epsilon^{(-1)^i}, \end{cases}$$

where

$$H_0(a, b) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad H_1(a, b) = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix},$$

is an isomorphism. The map

$$\nu : \text{Aut}(\mathcal{G}_{-1}) \longrightarrow \mathbb{Z}_2, \quad [(n, m), (a, b), i] \mapsto i$$

is a group homomorphism and the group $\text{Aut}(\mathcal{G}_{-1})$ is generated by $\text{Aut}(\mathcal{G}_{-1})^+ = \ker \nu$ and by the involution $\psi = [(0, 0), (1, 1), 1]$, defined by

$$(22) \quad \psi = \left\{ \begin{array}{l} \sigma \mapsto \tau, \quad \tau \mapsto \sigma, \quad \epsilon \mapsto \epsilon^{-1}. \end{array} \right.$$

Note that $\text{Aut}(\mathcal{G}_{-1})^+ \cong \text{Aut}(\mathcal{G}_k)$ for $k \neq 0, \pm 1$ and it contains the p -Sylow subgroup of $\text{Aut}(\mathcal{G}_{-1})$, generated by $\alpha_1 = [(1, 0), (1, 1), 0]$ and $\alpha_2 = [(0, 1), (1, 1), 0]$, and the elements of odd order of $\text{Aut}(\mathcal{G}_{-1})$. Since $p > 2$, then $\pi_2(G) \leq \text{Aut}(\mathcal{G}_{-1})^+$ and $G \leq \mathcal{G}_{-1} \rtimes \text{Aut}(\mathcal{G}_{-1})^+ \leq \text{Hol}(\mathcal{G}_{-1})$ for every regular subgroup $G \leq \text{Hol}(\mathcal{G}_{-1})$. Nevertheless the conjugacy classes of regular subgroups might be different, since we need to take into account the conjugation by ψ .

In some cases, the computations of the conjugacy classes can be obtained from those in Subsection 3.2 just by setting $k = -1$, as for the following that follows directly from Lemma 3.7.

Lemma 3.16. *The unique skew brace of \mathcal{G}_{-1} -type with $|\ker \lambda| = q$ is $(B, +, \circ)$ where $(B, +) = \mathbb{Z}_p^2 \rtimes_{\mathcal{D}_{1,-1}} \mathbb{Z}_q$ and*

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \circ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 g^{y_3} + y_1 g^{x_3} \\ x_2 g^{-y_3} + y_2 g^{-x_3} \\ x_3 + y_3 \end{pmatrix}$$

for every $0 \leq x_1, x_2, y_1, y_2 \leq p-1$ and $0 \leq x_3, y_3 \leq q-1$. In particular, $(B, \circ) \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q$.

We claim that the computations in Lemma 3.6, Proposition 3.8, Lemma 3.9 and Lemma 3.10 depend just on the condition $k \neq 0, 1$ and so we can employ them in the proofs of the next results. Note also that the subgroup

$$\mathfrak{H}_4 = \langle \sigma, \tau, \alpha_1, \alpha_2 \rangle$$

is a normal subgroup of $\text{Hol}(\mathcal{G}_{-1})$ and setting $k = -1$ in (F_k) we get a formula that holds in $\text{Hol}(\mathcal{G}_{-1})$.

Lemma 3.17. *A set of representatives of regular subgroups G of $\text{Hol}(\mathcal{G}_{-1})$ with $|\pi_2(G)| = p$ is*

$$H_i = \langle \epsilon, \tau, \sigma^{\frac{1}{g-1}} \alpha_1 \alpha_2^i \rangle \cong \mathcal{G}_0$$

for $i = 0, 1$.

Proof. The groups H_0 and H_1 are not conjugate since their images under π_2 are not. Arguing as in Lemma 3.6 we can show that every regular subgroup of $\text{Hol}(\mathcal{G}_{-1})$ with $|\pi_2(G)| = p$ is conjugate to the groups H_i, K_i for $i = 0, 1$ defined by setting $k = -1$ in the groups in Lemma 3.6. It is easy to see that H_0 is conjugate to K_0 by the automorphism ψ and that H_1 is conjugate to K_1 by the automorphism $[(0, 0), (g^2, 1), 0]$. \square

Let us denote $\beta_s = [(0, 0), (g, g^s), 0]$ and $\tilde{\beta} = [(0, 0), (1, g), 0]$ for $0 \leq s \leq q-1$.

Proposition 3.18. *The skew braces of \mathcal{G}_{-1} -type with $|\ker \lambda| = p^2$ are $(B_{a,b}, +, \circ)$ where $(B_{a,b}, +) = \mathbb{Z}_p^2 \rtimes_{\mathcal{D}_{1,-1}} \mathbb{Z}_q$ and*

$$(B_{a,b}, \circ) = \mathbb{Z}_p^2 \rtimes_{\mathcal{D}_{a+1, b-1}} \mathbb{Z}_q \cong \begin{cases} \mathbb{Z}_p^2 \times \mathbb{Z}_q, & \text{if } a = q-1, b = 1, \\ \mathcal{G}_0, & \text{if } a = q-1, b \neq 1, \\ \mathcal{G}_{\frac{b-1}{a+1}}, & \text{otherwise,} \end{cases}$$

for $0 \leq a, b \leq q-1$ and $(a, b) \neq (0, 0)$. In particular, they are bi-skew braces and $B_{a,b} \cong B_{c,d}$ if and only if $(c, d) = (-b, -a)$ and so there are $\frac{(q-1)(q+2)}{2}$ such skew braces.

Proof. Let G be a regular subgroup of $\text{Hol}(\mathcal{G}_{-1})$ such that $|\pi_2(G)| = q$. Arguing as in Proposition 3.8 we can show that every such group is conjugate to a subgroup

$$G_{a,b} = \langle \sigma, \tau, \epsilon \beta_0^a \tilde{\beta}^b \rangle$$

by an element of $\text{Aut}(\mathcal{G}_{-1})^+$. Since $\psi G_{a,b} \psi = G_{-b, -a}$, the groups of the form $G_{a, -a}$ are normalized by ψ and the other orbits have length 2. Therefore there are

$$q-1 + \frac{q(q-1)}{2} = \frac{(q-1)(q+2)}{2}$$

orbits under the action of ψ . The statement about the structure of the associated skew braces follows by the same argument of Proposition 3.8. \square

Lemma 3.19. *A set of representatives of conjugacy classes of subgroups of $\text{Aut}(\mathcal{G}_{-1})$ of order pq and p^2q is*

Size	G	Parameters	Isomorphism class
pq	$\mathcal{H}_{1,s} = \langle \alpha_1, \beta_s \rangle$	$0 \leq s \leq q-1$	$\mathbb{Z}_p \rtimes_g \mathbb{Z}_q$
	$\mathcal{H}_{2,0} = \langle \alpha_2, \beta_0 \rangle$	-	\mathbb{Z}_{pq}
	$\mathcal{W} = \langle \alpha_1 \alpha_2, \beta_1 \rangle$	-	$\mathbb{Z}_p \rtimes_g \mathbb{Z}_q$
p^2q	$\mathcal{T}_s = \langle \alpha_1, \alpha_2, \beta_s \rangle \cong \mathcal{G}_s$	$s \in \mathfrak{B}$	\mathcal{G}_s

where \mathfrak{B} is as in Remark 2.5.

Proof. The subgroups of size pq of $\text{Aut}(\mathcal{G}_{-1})^+$ up to conjugation by elements of $\text{Aut}(\mathcal{G}_{-1})^+$ are collected in Lemma 3.9. We need to compute the orbits of such groups under the action by conjugation of ψ . It is easy to see that

- $\psi \mathcal{H}_{1,s} \psi = \mathcal{H}_{2,s-1}$ and $\psi \mathcal{T}_s \psi = \mathcal{T}_{s-1}$ for $s \neq 0$;
- $\psi \mathcal{H}_{1,0} \psi = \mathcal{K}_2$, $\psi \mathcal{H}_{2,0} \psi = \mathcal{K}_1$ and $\psi \mathcal{W} \psi = \mathcal{T}_0$;
- $\psi \mathcal{W} \psi = \langle \alpha_1^{-g} \alpha_2^{-g-1}, \beta_1 \rangle$ is not conjugate to any other group in the table, since their p -Sylow subgroups are not. \square

Lemma 3.20. *A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(\mathcal{G}_{-1})$ with $|\pi_2(G)| = pq$ is the following:*

$\pi_2(G)$	Subgroups	Parameters	Isomorphism class	#
$\mathcal{H}_{1,s}$	$\tilde{H}_{1,s,d} = \langle \tau, \sigma^{\frac{1}{g-1}} \alpha_1, \epsilon^d \beta_s \rangle$	$0 \leq s \leq q-1,$ $1 \leq d \leq q-1$	\mathcal{G}_{s-d}	$q(q-1)$
	$\hat{H}_s = \langle \sigma, \tau \alpha_1, \epsilon^{s-1} \beta_s \rangle$	$2 \leq s \leq q$	\mathcal{G}_s	$q-1$
	$\bar{H}_s = \langle \sigma \tau, \sigma^{\frac{1}{g-1}} \alpha_1, \epsilon^{\frac{s-1}{2}} \beta_s \rangle$	$2 \leq s \leq q$	$\mathcal{G}_{\frac{s+1}{2}}$	$q-1$
$\mathcal{H}_{2,0}$	$\tilde{H}_{2,0,d} = \langle \sigma, \tau^{\frac{1}{g-1-1}} \alpha_2, \epsilon^d \beta_0 \rangle$	$1 \leq d \leq q-1$	$\mathbb{Z}_p^2 \times \mathbb{Z}_q$, if $d = -1$ \mathcal{G}_0 , otherwise	$q-1$
	$\hat{H}_{2,0} = \langle \tau, \sigma \alpha_2, \epsilon^{-1} \beta_0 \rangle$	-	\mathcal{G}_0	1
	$\bar{H}_{2,0} = \langle \sigma \tau, \tau^{\frac{1}{g-1-1}} \alpha_2, \epsilon^{-\frac{1}{2}} \beta_0 \rangle$	-	\mathcal{G}_0	1
\mathcal{W}	$\tilde{W}_d = \langle \sigma, \tau^{\frac{1}{g-1-1}} \alpha_1 \alpha_2, \epsilon^d \beta_1 \rangle$	$1 \leq d \leq q-1$	\mathcal{G}_{d+1}	$q-1$

Proof. Let G be a regular subgroup of $\text{Hol}(\mathcal{G}_{-1})$ and assume that $\pi_2(G) = \mathcal{H}_{i,s}$ for s and i as in Lemma 3.19. Using the same computations as in Lemma 3.10 we can show that G is conjugate to one of the groups in the table with the same image under π_2 by some element of $\text{Aut}(\mathcal{G}_{-1})^+$. Such groups are not conjugate. Indeed, if $\pi_2(G) \neq \mathcal{W}$, then its normalizer is contained in $\text{Aut}(\mathcal{G}_{-1})^+ \cong \text{Aut}(\mathcal{G}_k)$ for $k \neq 0, \pm 1$ and so we can prove that the corresponding groups in the table are not conjugate as we did in Lemma 3.10.

If $\pi_2(G) = \mathcal{W}$, the groups with $\pi_2(G) = \mathcal{W}$ are conjugate to \tilde{W}_d for some $d \neq 0$ by some element in $\text{Aut}(\mathcal{G}_{-1})^+$. If \tilde{W}_c and \tilde{W}_d are conjugate in $\text{Aut}(\mathcal{G}_{-1})$ by $h\psi$ for $h \in \text{Aut}(\mathcal{G}_{-1})^+$ so are their images modulo the subgroup \mathfrak{H}_4 . Hence we have

$$h\psi \epsilon^c \beta_1 \psi h^{-1} \mathfrak{H}_4 = \epsilon^{-c} \beta_1 \mathfrak{H}_4 \in \langle \epsilon^d \beta_1 \mathfrak{H}_4 \rangle,$$

and so $c = -d$. It is easy to see that the groups \tilde{W}_d and \tilde{W}_{-d} are conjugate by $[(0, 0), (1, g^2), 0]$.

Therefore the table collects a set of representatives of regular subgroups with the desired properties. \square

Lemma 3.21. *A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(\mathcal{G}_{-1})$ with $|\pi_2(G)| = p^2q$ is*

$\pi_2(G)$	Subgroups	Class	Parameters	#
\mathcal{T}_s	$\tilde{T}_{d,s} = \langle \sigma^{\frac{1}{g-1}} \alpha_1, \tau^{\frac{1}{g-1-1}} \alpha_2, \epsilon^d \beta_s \rangle$	\mathcal{G}_s	$s \in \mathfrak{B} \setminus \{1\},$ $1 \leq d \leq q-1$	$\frac{q^2-1}{2}$
	$\hat{T}_{m,s} = \langle \tau \sigma^{\frac{1}{g-1}} \alpha_1, \sigma^m \tau^{\frac{1}{g-1-1}} \alpha_2, \epsilon^{s-1} \beta_s \rangle$	\mathcal{G}_s	$s \in \mathfrak{B} \setminus \{1, -1\},$ $0 \leq m \leq p-1, m \neq -\frac{g}{(g-1)^2} \pmod{p}$	$\frac{(p-1)(q-1)}{2}$
	$\bar{T}_s = \langle \sigma^{\frac{1}{g-1}} \alpha_1, \sigma \tau^{\frac{1}{g-1-1}} \alpha_2, \epsilon^{s-1} \beta_s \rangle$	\mathcal{G}_s	$s \in \mathfrak{B} \setminus \{1, -1\}$	$\frac{q-1}{2}$
\mathcal{T}_1	$\tilde{T}_{d,1} = \langle \sigma^{\frac{1}{g-1}} \alpha_1, \tau^{\frac{1}{g-1-1}} \alpha_2, \epsilon^d \beta_1 \rangle$	\mathcal{G}_1	$d \in \mathfrak{A}$	$\frac{q-1}{2}$
\mathcal{T}_{-1}	$\hat{T}_{c,-1} = \langle \tau \sigma^{\frac{1}{g-1}} \alpha_1, \sigma^n \tau^{\frac{1}{g-1-1}} \alpha_2, \epsilon^{-2} \beta_{-1} \rangle$	\mathcal{G}_{-1}	$0 \leq n \leq p-1, n \neq -\frac{g}{(g-1)^2} \pmod{p}$	$p-1$

where $\mathfrak{A} \subseteq \mathbb{Z}_q^\times$ containing one out of d and $-d$ for each $d \in \mathbb{Z}_q^\times$.

Proof. Using the same argument as in Lemma 3.10 we can show that the groups in the list are regular. Let us show that the groups with the same image under π_2 in the table are not conjugate. Let $\delta = h\psi$ for $h \in \text{Aut}(\mathcal{G}_{-1})^+$.

- if $\pi_2(G) = \mathcal{T}_s$ and $s \neq \pm 1$ then $N_{\text{Aut}(\mathcal{G}_1)}(\mathcal{T}_s) = \text{Aut}(\mathcal{G}_{-1})^+$. So the groups in different rows and the groups in the family $\hat{T}_{m,s}$ are not pairwise conjugate, since their p -Sylow subgroups are not conjugate by elements in $\text{Aut}(\mathcal{G}_{-1})^+$.
- If $s = 1$, then \mathcal{T}_1 is normal in $\text{Aut}(\mathcal{G}_{-1})$ and

$$\delta \epsilon^c \beta_1 \delta^{-1} \mathfrak{H}_4 = \psi \epsilon^c \beta_1 \psi \mathfrak{H}_4 = \epsilon^{-c} \beta_1 \mathfrak{H}_4.$$

So, if $\tilde{T}_{d,1}$ and $\tilde{T}_{c,1}$ are conjugate by δ then $c = -d$.

- If $s = -1$, then \mathcal{T}_{-1} is normal in $\text{Aut}(\mathcal{G}_{-1})$ and

$$\delta \epsilon^c \beta_{-1} \delta^{-1} \mathfrak{H}_4 = \psi \epsilon^{-c} \beta_{-1} \psi \mathfrak{H}_4 = \epsilon^{-c} \beta_{-1}^{-1} \mathfrak{H}_4.$$

If $\tilde{T}_{c,-1}$ and $\tilde{T}_{d,-1}$ are conjugate by δ , then $c = d$. The groups $\tilde{T}_{c,-1}$ and $\hat{T}_{n,-1}$ are not conjugate since their p -Sylow subgroups are not.

Let G be a regular subgroup of $\text{Hol}(\mathcal{G}_{-1})$. According to Lemma 3.19 we can assume that $\pi_2(G) = \mathcal{T}_s$ for $s \in \mathfrak{B}$. From the same computations as in the beginning of Lemma 3.11, in which we set $k = -1$, we have that

$$G = \langle \sigma^{\frac{1}{g-1}} \tau^n \alpha_1, \sigma^m \tau^{\frac{1}{g-1-1}} \alpha_2, w \epsilon^c \beta_s \rangle$$

for $c \neq 0$ and some $w \in \langle \sigma, \tau \rangle$. So we need to discuss two cases:

- $n = m = 0$;
- $(n, m) \neq (0, 0)$, $c = s - 1$ and so $s \neq 1$.

In the second case, if $n = 0$ we can set $m = 1$ up to conjugation by $[(0, 0), (m^{-1}, m^{-1}), 0]$, otherwise we can set $n = 1$ conjugating by $[(0, 0), (n^{-1}, n^{-1}), 0]$. Then we reduce the cases to $(n, m) = (0, 1)$ or $n = 1$ and $0 \leq m \leq p-1$. If $n = 1$ the elements $\tau \sigma^{\frac{1}{g-1}}$ and $\sigma^m \tau^{\frac{1}{g-1-1}}$ need to be linearly independent, i.e.

$$\det \begin{bmatrix} \frac{1}{g-1} & m \\ 1 & \frac{1}{g-1-1} \end{bmatrix} = -\frac{g}{(g-1)^2} - m \neq 0 \pmod{p}.$$

In both cases, up to conjugation by an element in $\langle \alpha_1, \alpha_2 \rangle$ (using (F_k) with $k = -1$) we can assume that $w = 1$ and so the regular groups are identified by the parameters (s, c, n, m) .

The groups identified by $(1, c, 0, 0)$ and $(1, -c, 0, 0)$ are conjugate by ψ . The groups identified by $(-1, -2, 0, 1)$ and $(-1, -2, 1, 0)$ are conjugate by $[(0, 0), (1, -g), 0]\psi$. Accordingly, G is conjugate to one of the groups in the table. \square

We summarize the content of this subsection in the following table:

$ \ker \lambda $	$\mathbb{Z}_p^2 \times \mathbb{Z}_q$	\mathcal{G}_k	\mathcal{G}_0	\mathcal{G}_{-1}	\mathcal{G}_1
1	-	$p+q-1$	$p+q-1$	$p+q-2$	$\frac{q-1}{2}$
p	1	$2(q+2)$	$2(q+1)$	$q+2$	$q-1$
q	1	-	-	-	-
pq	-	-	2	-	-
p^2	1	$q-1$	$q-1$	$q-2$	$\frac{q-1}{2}$
p^2q	-	-	-	1	-

TABLE 6. Enumeration of skew braces of \mathcal{G}_{-1} -type for $p = 1 \pmod{q}$.

3.5. **Skew braces of \mathcal{G}_1 -type.** A presentation of the group \mathcal{G}_1 is the following

$$\mathcal{G}_1 = \langle \sigma, \tau, \epsilon \mid \sigma^p = \tau^p = \epsilon^q = [\sigma, \tau] = 1, \epsilon\sigma\epsilon^{-1} = \sigma^g, \epsilon\tau\epsilon^{-1} = \tau^g \rangle.$$

According to [9, Subsections 4.1, 4.2], the mapping

$$\phi : \mathbb{Z}_p^2 \rtimes GL_2(p) \longrightarrow \text{Aut}(\mathcal{G}_1), \quad [(n, m), H] \mapsto h = \begin{cases} h|_{\langle \sigma, \tau \rangle} = H, \\ \epsilon \mapsto \sigma^n \tau^m \epsilon \end{cases}$$

is a group isomorphism. In particular, $|\text{Aut}(\mathcal{G}_1)| = p^3(p-1)^2(p+1)$, a p -Sylow subgroup of $\text{Aut}(\mathcal{G}_1)$ is generated by $\alpha_1 = [(1, 0), Id]$, $\alpha_2 = [(0, 1), Id]$, $\gamma = [(0, 0), C]$, where C is defined as in Remark 2.5 and α_1 and α_2 generate a normal subgroup in $\text{Aut}(\mathcal{G}_1)$.

Note that for $k \neq 0, -1$ the map

$$\iota : \text{Aut}(\mathcal{G}_k) \longrightarrow \text{Aut}(\mathcal{G}_1), \quad [(n, m), (a, b)] \mapsto \left[(n, m), \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right]$$

is an injective group homomorphism. Moreover, the formula (F_k) with $k = 1$ holds in $\text{Hol}(\mathcal{G}_1)$.

The procedure to check the regularity of a subgroup has already been established in the previous subsections, so we are not showing that part of the classification strategy explicitly in this subsection.

In the following we are using the formulas

$$(23) \quad \alpha_2 \gamma \alpha_1^{-1} \alpha_2^{-1} = \alpha_1^{-1} \gamma,$$

$$(24) \quad (\tau^a \alpha_2^b \gamma)^n = \sigma^{\frac{an(n-1)}{2}} \tau^{an} \alpha_1^{\frac{bn(n-1)}{2}} \alpha_2^{bn} \gamma^n$$

for $a, b, n \in \mathbb{Z}$.

Lemma 3.22. *A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(\mathcal{G}_1)$ with $|\pi_2(G)| = p$ is*

$$H = \langle \epsilon, \tau, \sigma^{\frac{1}{g-1}} \alpha_1 \rangle \cong \mathcal{G}_0, \quad W = \langle \epsilon, \sigma, \tau^{\frac{1}{g-1}} \alpha_2 \gamma \rangle \cong \mathcal{G}_0.$$

Proof. Up to conjugation the subgroups of order p of $\text{Aut}(\mathcal{G}_1)$ are $\langle \alpha_1 \rangle$, $\langle \alpha_2^i \gamma \rangle$ for $i = 1, 2$. So H and W are not conjugate.

If $\pi_2(G) = \langle \alpha_1 \rangle$, arguing as in Lemma 3.6 we can show that G is conjugate to H .

Let G be a regular subgroup of $\text{Hol}(\mathcal{G}_1)$ with $\pi_2(G) = \langle \alpha_2^i \gamma \rangle$ for $i = 0, 1$. Then $K = \ker \pi_2|_G$ contains an element of order q and so, up to conjugation by a power of α_1 we can assume that K is generated by an element of the form $\tau^n \epsilon$ and by some $v \in \langle \sigma, \tau \rangle$. So G has the following standard presentation

$$G = \langle \tau^n \epsilon, v, u \alpha_2^i \gamma \rangle$$

where $u, v \in \langle \sigma, \tau \rangle$. The p -Sylow subgroup of K is characteristic in K and K is normal in G then $\langle v \rangle$ is normal in G . Thus

$$w \alpha_2^i \gamma (u) w^{-1} = \gamma(u) \in \langle u \rangle.$$

Therefore we can assume that $u = \sigma$ and so $u = \tau^c$ for $c \neq 0$. According to condition (K), we have

$$\tau^c \alpha_2^i \gamma \tau^n \epsilon \gamma^{-1} \alpha_2^{-i} \tau^{-c} = \sigma^n \tau^{(1-g)c+i+n} \epsilon \in K.$$

Hence, $(1-g)c+i=0$. If $i=0$ then $c=0$ and so G is not regular. Therefore $i=1$ and $c = \frac{1}{g-1}$. Finally, using (23), we have that G is conjugate to W by $\alpha_2^{-n} \gamma^{-n}$. \square

Lemma 3.23. *A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(\mathcal{G}_1)$ with $|\pi_2(G)| = p^2$ is*

$$H_i = \langle \epsilon, \sigma^{\frac{1}{g-1}} \alpha_1, \tau^{\frac{1}{g-1}} \alpha_2 \gamma^i \rangle \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q$$

for $i = 0, 1$.

Proof. The subgroups of $\text{Aut}(\mathcal{G}_1)$ of size p^2 are $\langle \alpha_1, \alpha_2 \rangle$, $\langle \alpha_1, \gamma \rangle$ and $\langle \alpha_1, \alpha_2 \gamma \rangle$, up to conjugation. So, the groups in the statement are not conjugate since their image under π_2 are not.

Let G be a regular subgroup of $\text{Hol}(\mathcal{G}_1)$ such that $|\pi_2(G)| = p^2$. The kernel of π_2 is the normal q -Sylow subgroup of G and so G is abelian. Moreover, up to conjugation by the normalizer of $\pi_2(G)$ we can assume that it is generated by ϵ .

If $\pi_2(G) = \langle \alpha_1, \alpha_2 \rangle$ then G is conjugate to H_0 by an element of $\iota(\text{Aut}(\mathcal{G}_k)) \leq \text{Aut}(\mathcal{G}_1)$ (see Lemma 3.7). Otherwise we have that

$$G = \langle \epsilon, u\alpha_1, v\alpha_2^i \gamma \rangle$$

for some $u, v \in \langle \sigma, \tau \rangle$. From abelianness of G it follows that $u = \sigma^{\frac{1}{g-1}}$ and $v = \tau^{\frac{i}{g-1}}$ by (23). Since $v \neq 1$ then $i = 1$ and so $G = H_1$. \square

Employing the notation in Remark 2.5 let us define $\beta_s = [(0, 0), \mathcal{D}_{1,s}]$ and $\tilde{\beta} = [(0, 0), \mathcal{D}_{0,1}]$ for $1 \leq s \leq q-1$.

Proposition 3.24. *The skew braces of \mathcal{G}_1 -type with $|\ker \lambda| = p^2$ are $(B_{a,b}, +, \circ)$ where $(B_{a,b}, +) = \mathbb{Z}_p^2 \rtimes_{\mathcal{D}_{1,1}} \mathbb{Z}_q$ and*

$$(B_{a,b}, \circ) = \mathbb{Z}_p^2 \rtimes_{\mathcal{D}_{a+1, b+1}} \mathbb{Z}_q \cong \begin{cases} \mathbb{Z}_p^2 \times \mathbb{Z}_q, & \text{if } a = b = q-1, \\ \mathcal{G}_0, & \text{if } a = q-1, b \neq q-1, \\ \mathcal{G}_{\frac{b+1}{a+1}}, & \text{otherwise,} \end{cases}$$

for $0 \leq a, b \leq q-1$ and $(a, b) \neq (0, 0)$. Moreover, they are bi-skew braces and $B_{a,b} \cong B_{c,d}$ if and only if $(c, d) = (b, a)$ and so there are $\frac{(q-1)(q+2)}{2}$ such skew braces.

Proof. Let G be a regular subgroup of $\text{Hol}(\mathcal{G}_1)$ such that $|\pi_2(G)| = q$. By Remark 2.5 we have that, up to conjugation, an element of order q of $GL_2(p)$ is a diagonal matrix with entries a power of g . Arguing as in Proposition 3.8, also in this case we can show that every such group is conjugate to a subgroup of the form

$$G_{a,b} = \langle \sigma, \tau, \epsilon \beta_0^a \tilde{\beta}^b \rangle.$$

Hence $G_{a,b}$ and $G_{c,d}$ are conjugate by $h \in \text{Aut}(\mathcal{G}_1)$ if and only if

$$h(\epsilon)h\beta_0^a \tilde{\beta}^b h^{-1} = \epsilon h\beta_0^c \tilde{\beta}^d h^{-1} = \epsilon \beta_0^c \tilde{\beta}^d \pmod{\langle \sigma, \tau \rangle},$$

i.e. $\beta_0^a \tilde{\beta}^b|_{\langle \sigma, \tau \rangle} = \mathcal{D}_{a,b}$ and $\beta_0^c \tilde{\beta}^d|_{\langle \sigma, \tau \rangle} = \mathcal{D}_{c,d}$ are conjugate. And so either $(a, b) = (c, d)$ or $(a, b) = (d, c)$. In particular there are $\frac{(q-1)(q+2)}{2}$ such classes. The other claims follow as in Proposition 3.8. \square

The following lemma collects the conjugacy classes of subgroups of size pq and p^2q of $\text{Aut}(\mathcal{G}_1)$.

Lemma 3.25. *A set of representatives of conjugacy classes of subgroups of size pq and p^2q of $\text{Aut}(\mathcal{G}_1)$ is*

Size	G	Parameters	Class
pq	$\mathcal{H}_s = \langle \alpha_1, \beta_s \rangle$	$0 \leq s \leq q-1$	$\mathbb{Z}_p \rtimes_g \mathbb{Z}_q$
	$\mathcal{U} = \langle \alpha_2, \beta_0 \rangle$	-	\mathbb{Z}_{pq}
	$\mathcal{K}_s = \langle \gamma, \beta_s \rangle$	$0 \leq s \leq q-1$	\mathbb{Z}_{pq} , if $s = 1$, $\mathbb{Z}_p \rtimes_g \mathbb{Z}_q$, otherwise
	$\mathcal{M} = \langle \alpha_1 \alpha_2^2 \gamma, \beta_{2-1} \rangle$	-	$\mathbb{Z}_p \rtimes_g \mathbb{Z}_q$
	$\mathcal{V} = \langle \gamma, \tilde{\beta} \rangle$	-	$\mathbb{Z}_p \rtimes_g \mathbb{Z}_q$
p^2q	$\mathcal{T}_s = \langle \alpha_1, \alpha_2, \beta_s \rangle$	$s \in \mathfrak{B}$	\mathcal{G}_s
	$\mathcal{R}_s = \langle \alpha_1, \gamma, \beta_s \rangle$	$0 \leq s \leq q-1$	\mathcal{G}_{1-s}
	$\mathcal{N} = \langle \alpha_1, \gamma, \tilde{\beta} \rangle$	-	\mathcal{G}_0
	$\mathcal{L} = \langle \alpha_1, \alpha_2 \gamma, \beta_{2-1} \rangle$	-	\mathcal{G}_2

where \mathfrak{B} is as in Remark 2.5.

Proof. The groups in the table are not pairwise conjugate. Indeed if $h = [(n, m), M]$ conjugate a pair of such groups then their p -Sylow subgroups are conjugate by h and their restrictions to $\langle \sigma, \tau \rangle$ are conjugate by M . A case-by-case discussion shows that these two conditions can not be satisfied by any pair of groups in the table.

Let H be a subgroup of order pq or p^2q of $\text{Aut}(\mathcal{G}_1)$ and let $H|_{\langle \sigma, \tau \rangle}$ denotes the restriction of the action of H to $\langle \sigma, \tau \rangle$. If $H|_{\langle \sigma, \tau \rangle}$ has size q then, according to Remark 2.5 we have that $H|_{\langle \sigma, \tau \rangle}$ is generated by $\mathcal{D}_{1,s}$ for $s \in \mathcal{B}$. Hence, up to conjugation, $H \leq \iota(\text{Aut}(\mathcal{G}_k))$. Using the same notation as in Lemma 3.9, we have that H is conjugate to one of the groups $\{\mathcal{H}_{1,s}, \mathcal{H}_{2,s}, \mathcal{W}, \mathcal{T}_s : s \in \mathcal{B}\}$ by an element of $\iota(\text{Aut}(\mathcal{G}_k))$. Moreover:

- if $s \neq 0, \pm 1$, $\psi \mathcal{H}_{1,s} \psi = \mathcal{H}_{2,s^{-1}}$, where ψ is the automorphism swapping σ and τ .
- The group \mathcal{W} is conjugate to $\mathcal{H}_{1,1}$ by a suitable automorphism of the form $[(0, 0), M]$, since $\mathcal{D}_{1,1}$ is central in $GL_2(p)$.

Hence H is conjugate to one of the groups $\{\mathcal{H}_s, \mathcal{U} : 0 \leq s \leq q-1\}$ if $|H| = pq$ or to one of $\{\mathcal{T}_s : 0 \leq s \leq q-1\}$ if $|H| = p^2q$ as in the statement.

If $H|_{\langle \sigma, \tau \rangle}$ has size pq then, according to Remark 2.5, up to conjugation we have that $H|_{\langle \sigma, \tau \rangle} = \langle C, \mathcal{D}_{1,s} \rangle$ or $H|_{\langle \sigma, \tau \rangle} = \langle C, \mathcal{D}_{0,1} \rangle$ for $0 \leq s \leq q-1$. Therefore

$$H = \langle \alpha_1^n \alpha_2^m \gamma, \alpha_1^r \alpha_2^t \theta \rangle$$

for $\theta \in \{\beta_s, \tilde{\beta} : 0 \leq s \leq q-1\}$. If $\theta = \beta_0$ then $t = 0$ and if $\theta = \tilde{\beta}$ then $r = 0$. Using (F_k) for $k = 1$, up to conjugation we can assume that $r = t = 0$. The p -Sylow subgroup of H has to be normal, i.e. if $\theta = [(0, 0), \mathcal{D}_{t,s}]$, using (24) we have

$$\theta \alpha_1^n \alpha_2^m \gamma \theta^{-1} = \alpha_1^{ng^t} \alpha_2^{mg^s} \gamma^{g^{t-s}} = (\alpha_1^n \alpha_2^m \gamma)^r = \alpha_1^{nr+mr\frac{r-1}{2}} \alpha_2^{mr} \gamma^r$$

for some $r \in \mathbb{Z}$. Thus, $r = g^{s-t}$ and accordingly n, m solve the following system of linear equations:

$$\begin{cases} 2(g^{t-s} - g^t)n + g^{t-s}(g^{t-s} - 1)m = 0 \\ (g^{t-s} - g^s)m = 0. \end{cases}$$

Hence:

- if $\theta = \beta_s, \tilde{\beta}$ with $s \neq 0, 2^{-1}$ then $n = m = 0$, i.e. $H = \mathcal{K}_s$ or $H = \mathcal{V}$.
- If $\theta = \beta_{2^{-1}}$ then $m = 2n$ and so, either $n = m = 0$ or up to conjugation by $[(0, 0), n^{-1}Id]$, $n = 1, m = 2$, i.e. $H = \mathcal{K}_{2^{-1}}$ or H is conjugate to \mathcal{M} .
- If $\theta = \beta_0$ then either $n = m = 0$ or, up to conjugation by $[(0, 0), n^{-1}Id]$, $m = 0$ and $n = 1$. Since $\alpha_2 \langle \alpha_1 \gamma, \beta_0 \rangle \alpha_2^{-1} = \mathcal{K}_0$ then H is conjugate to \mathcal{K}_0 .

Let H be a subgroup of $\text{Aut}(\mathcal{G}_1)$ of size p^2q and let $H|_{\langle \sigma, \tau \rangle}$ has size pq . Then, up to conjugation,

$$H = \langle \alpha_1^n \alpha_2^m, \alpha_1^r \alpha_2^t \gamma, \theta \rangle$$

where θ is either β_s or $\tilde{\beta}$. Using (23) to check the abelianness of the p -Sylow subgroup of H it follows that $m = 0$ and therefore we can assume that $n = 1$ and $r = 0$. From the normality of the p -Sylow subgroup it follows that:

- if $\theta = \beta_s$ for $s \neq 2^{-1}$ then $t(g^s - g^{1-s}) = 0$ and so $t = 0$, i.e. $H = \mathcal{R}_s$.
- If $\theta = \tilde{\beta}$ then $(1-g)t = 0$ and so $t = 0$, i.e. $H = \mathcal{N}$.
- If $\theta = \beta_{2^{-1}}$ then up to conjugation by an element of the form $[(0, 0), xId]$ we have $t \in \{0, 1\}$, i.e. H is conjugate either to $\mathcal{R}_{2^{-1}}$ or to \mathcal{L} . \square

The group

$$(25) \quad \mathfrak{H}_5 = \langle \sigma, \tau, \alpha_1, \alpha_2 \rangle$$

is a normal subgroup of $\text{Hol}(\mathcal{G}_1)$ and $\text{Hol}(\mathcal{G}_1)/\mathfrak{H}_5 \cong \mathbb{Z}_q \times GL_2(p)$.

Lemma 3.26. *A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(\mathcal{G}_1)$ with $|\pi_2(G)| = pq$ is*

$\pi_2(G)$	Subgroups	Parameters	Class	#
\mathcal{H}_s	$\tilde{H}_{s,c} = \langle \tau, \sigma^{\frac{1}{q-1}} \alpha_1, \epsilon^c \beta_s \rangle$	$0 \leq s \leq q-1,$ $1 \leq c \leq q-1$	\mathcal{G}_{c+s}	$q(q-1)$
	$\hat{H}_s = \langle \sigma, \tau \alpha_1, \epsilon^{1-s} \beta_s \rangle$	$2 \leq s \leq q$	\mathcal{G}_{2-s}	$q-1$
\mathcal{K}_s	$\tilde{K}_s = \langle \sigma, \tau \gamma, \epsilon^{1-2s} \beta_s \rangle$	$0 \leq s \leq q-1,$ $s \neq 2^{-1}$	$\mathbb{Z}_p^2 \times \mathbb{Z}_q,$ if $s = 1,$ $\mathcal{G}_2,$ otherwise.	$q-1$
\mathcal{U}	$\tilde{U}_c = \langle \sigma, \tau^{\frac{1}{q-1}} \alpha_2, \epsilon^c \beta_0 \rangle$	$1 \leq c \leq q-1$	$\mathbb{Z}_p^2 \times \mathbb{Z}_q,$ if $c = -1,$ $\mathcal{G}_0,$ otherwise.	$q-1$
	$\tilde{U} = \langle \tau, \sigma \alpha_2, \epsilon^{-1} \beta_0 \rangle$	-	\mathcal{G}_0	1
\mathcal{M}	$\tilde{M}_c = \langle \sigma, \tau^{\frac{2}{q-1}} \alpha_1 \alpha_2^2 \gamma, \epsilon^c \beta_{2-1} \rangle$	$1 \leq c \leq q-1$	$\mathcal{G}_{2(c+1)}$	$q-1$
\mathcal{V}	$\tilde{V} = \langle \sigma, \tau \gamma, \epsilon^{-2} \tilde{\beta} \rangle$	-	\mathcal{G}_2	1

Proof. The groups with the same image under π_2 but in different rows of the table are not pairwise conjugate because their p -Sylow subgroups are not conjugate by the normalizer of their image under π_2 . The same argument used in Lemma 3.10 shows that the groups in the same row are not pairwise conjugate (employing the group \mathfrak{H}_5).

Let G be a regular subgroup of $\text{Hol}(\mathcal{G}_1)$ with $\pi_2(G) \in \{\mathcal{H}_s, \mathcal{U} : 0 \leq s \leq q-1\}$, i.e. with $\pi_2(G) \leq \iota(\text{Aut}(\mathcal{G}_k))$. Then $G \leq \mathcal{G}_1 \rtimes \iota(\text{Aut}(\mathcal{G}_k))$ and we can assume that

$$G = \langle u, v \alpha_i, w \epsilon^c \beta_s \rangle$$

for $c \neq 0$ and $u, v, w \in \langle \sigma, \tau \rangle$. Up to conjugation by an element in the normalizer of $\pi_2(G)$ we can assume that $u \in \{\sigma, \tau, \sigma\tau\}$.

If $\ker \pi_2 = \langle \sigma\tau \rangle$, the condition (K) forces $s = 1$ and $i = 1$. In such case, up to conjugation by $h \in N_{\text{Aut}(\mathcal{G}_1)}(\mathcal{H}_1)$, we can assume that the kernel is generated by σ .

Therefore, setting $k = 1$ in the proof of Lemma 3.10, it follows that G is conjugate to one of the groups in the table corresponding to the same image under π_2 by an element of $\iota(\text{Aut}(\mathcal{G}_k))$ (we are using that (F_k) and (P_k) holds for $k = 1$).

In all the other cases we can assume that G has the form

$$G = \langle u, v \alpha_1^n \alpha_2^m \gamma, w \epsilon^c \theta \rangle$$

for $c \neq 0$ and $u, v, w \in \langle \sigma, \tau \rangle$. From condition (K) we have that $\ker \pi_2|_G = \langle \sigma \rangle$ and so $v, w \in \langle \tau \rangle$.

Let $\pi_2(G) = \mathcal{K}_s$. Then

$$G = \langle \sigma, \tau^a \gamma, \tau^b \epsilon^c \beta_s \rangle$$

for $a \neq 0$ and so up to conjugation by $h = [(0, 0), a^{-1} Id]$ we can assume that $a = 1$. In \mathcal{K}_s the relation $\beta_s \gamma \beta_s^{-1} = \gamma^{g^{1-s}}$ holds. Then the condition (R) and formula (24) imply that

$$(\tau^b \epsilon^c \beta_s) \tau \gamma (\tau^b \epsilon^c \beta_s)^{-1} = \tau^{g^{s+c}} \gamma^{g^{1-s}} \stackrel{(R)}{=} (\tau \gamma)^{g^{1-s}} = \tau^{g^{1-s}} \gamma^{g^{1-s}} \pmod{\langle \sigma \rangle}$$

and so $c = 1 - 2s$ and then $s \neq 2^{-1}$. Since $(\tau^b \epsilon^{1-2s} \beta_s)^q \in \ker \pi_2$, if $s = 1$ then $b = 0$ (see (P_k)). Otherwise, G is conjugate to \tilde{K}_s by γ^n where $n = \frac{b}{1-g^{1-s}}$.

The other cases are similar and so we omit the computations. \square

Lemma 3.27. *A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(\mathcal{G}_1)$ with $|\pi_2(G)| = p^2 q$ is*

$\pi_2(G)$	Subgroups	Parameters	Class	#
\mathcal{T}_s	$\tilde{T}_{s,d} = \langle \sigma^{\frac{1}{g-1}} \alpha_1, \tau^{\frac{1}{g-1}} \alpha_2, \epsilon^d \beta_s \rangle$	$s \in \mathfrak{B} \setminus \{-1\},$ $1 \leq d \leq q-1$	\mathcal{G}_s	$\frac{(q-1)(q+1)}{2}$
	$\tilde{T}_{-1,d} = \langle \sigma^{\frac{1}{g-1}} \alpha_1, \tau^{\frac{1}{g-1}} \alpha_2, \epsilon^d \beta_{-1} \rangle$	$d \in \mathfrak{A}$	\mathcal{G}_{-1}	$\frac{q-1}{2}$
	$\hat{T}_s = \langle \sigma^{\frac{1}{g-1}} \alpha_1, \sigma \tau^{\frac{1}{g-1}} \alpha_2, \epsilon^{s-1} \beta_s \rangle$	$s \in \mathfrak{B} \setminus \{1\}$	\mathcal{G}_s	$\frac{q+1}{2}$
	$\bar{T}_s = \langle \tau \sigma^{\frac{1}{g-1}} \alpha_1, \tau^{\frac{1}{g-1}} \alpha_2, \epsilon^{1-s} \beta_s \rangle$	$s \in \mathfrak{B} \setminus \{1, -1\}$	\mathcal{G}_s	$\frac{q-1}{2}$
\mathcal{R}_s	$\tilde{R}_s = \langle \sigma^{\frac{1}{g-1}} \alpha_1, \tau \gamma, \tau^{\frac{1-g^{1-s}}{2}} \epsilon^{1-2s} \beta_s \rangle$	$0 \leq s \leq q-1, s \neq 2^{-1}$	\mathcal{G}_{1-s}	$q-1$
\mathcal{N}	$\tilde{N} = \langle \sigma^{\frac{1}{g-1}} \alpha_1, \tau \gamma, \tau^{\frac{1-g^{-1}}{2}} \epsilon^{-2} \tilde{\beta} \rangle$	-	\mathcal{G}_0	1
\mathcal{L}	$\tilde{L}_d = \langle \sigma^{\frac{1}{g-1}} \alpha_1, \tau^{\frac{1}{g-1}} \alpha_2 \gamma, \epsilon^d \beta_{2^{-1}} \rangle$	$1 \leq d \leq q-1$	\mathcal{G}_2	$q-1$

where \mathfrak{A} is as in Lemma 3.21.

Proof. Let us show that the groups in the table are not pairwise conjugate:

- If $s \neq -1$, if $\tilde{T}_{s,d}$ (resp. \tilde{L}_d) and $\tilde{T}_{s,c}$ (resp. \tilde{L}_c) are conjugate by an element in the normalizer of their image under π_2 then, by looking at their images in $\text{Hol}(\mathcal{G}_1)/\mathfrak{H}_5$ it follows that $c = d$ (and the same argument shows that the groups \hat{T}_s and \bar{T}_s for $s \neq -1$ and the groups \tilde{L}_d are not pairwise conjugate). If $s = -1$ the same argument shows that $c = -d$.
- If $\pi_2(G) = \mathcal{T}_s$ for $s \neq \pm 1$ then the action of $h \in N_{\text{Aut}(\mathcal{G}_1)}(\pi_2(G))$ restricted to $\langle \sigma, \tau \rangle$ is a diagonal matrix. Therefore the groups $\tilde{T}_{s,d}$, \hat{T}_s and \bar{T}_s are not conjugate, since their p -Sylow subgroups are not.
- If $s \neq -1$ and $\tilde{T}_{s,d}$ and $\tilde{T}_{s,c}$ are conjugate then, by looking at their images in $\text{Hol}(\mathcal{G}_1)/\mathfrak{H}_5$ it follows that $c = d$ (and the same argument shows that the groups \hat{T}_s and \bar{T}_s for $s \neq -1$ and the groups \tilde{L}_d are not pairwise conjugate).
- If $\pi_2(G) = \mathcal{T}_{-1}$ then the action of $h \in N_{\text{Aut}(\mathcal{G}_1)}(\pi_2(G))$ restricted to $\langle \sigma, \tau \rangle$ is either a diagonal matrix or it has the form

$$h|_{\langle \sigma, \tau \rangle} = \begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix}.$$

The groups \hat{T}_{-1} and \bar{T}_{-1} and the groups $\tilde{T}_{-1,c}$ and $\tilde{T}_{-1,-c}$ are conjugate by the automorphism ψ swapping σ and τ . The other groups are not conjugate, since their p -Sylow subgroups are not.

Let G be a regular subgroup and let $\pi_2(G) = \mathcal{T}_s \leq \iota(\text{Aut}(\mathcal{G}_k))$ for $s \in \mathfrak{B}$. Then we can argue as in Lemma 3.11 to show that G is conjugate to one of the groups in the table (formulas (F_k) and (P_k) hold for $k = 1$).

Otherwise, as in Lemma 3.26 we have that G has the form

$$G = \langle \sigma^{\frac{1}{g-1}} \alpha_1, u \alpha_2^i \gamma, w \epsilon^c \theta \rangle$$

for $c \neq 0$, $i = 0, 1$ and $\theta \in \{\beta_s, \tilde{\beta} : 0 \leq s \leq q-1\}$.

- If $\pi_2(G) = \mathcal{R}_s$, up to conjugation by a power of α_1 we can assume that $w \in \langle \tau \rangle$ and that $u \in \langle \tau \rangle$ up to conjugation by a power of γ . Conjugating by an automorphism of the form $[(0, 0), xId]$ we can assume that $u = \tau$. Since

$$\beta_s \gamma \beta_s^{-1} = (\gamma)^{g^{1-s}}$$

the condition (R) implies that the last two generators of G satisfy the same relation. Accordingly $c = 1 - 2s$ and $w = \tau^{\frac{1-g^{1-s}}{2}}$. Therefore G is conjugate to \tilde{R}_s . If $\pi_2(G) = \mathcal{N}$ we can similarly conclude that G is conjugate to \tilde{N} .

- If $\pi_2(G) = \mathcal{L}$, up to conjugation by an element of the form

$$h = \left[(0, 0), \begin{bmatrix} x^2 & y \\ 0 & x \end{bmatrix} \right]$$

we can assume that $u \in \langle \tau \rangle$. Using (24) we have that

$$\beta_{2^{-1}} \alpha_2 \gamma (\beta_{2^{-1}})^{-1} = \alpha_1^{-g^{2^{-1}-1}} (\alpha_2 \gamma)^{g^{2^{-1}}},$$

so, as in the previous case, condition (R) implies that $w \in \langle \tau \rangle$ and $u = \tau^{\frac{1}{q-1}}$. Finally G is conjugate to \tilde{L}_c by a suitable power of α_1 . \square

The following tables summarize the content of this subsection.

ker λ	$\mathbb{Z}_p^2 \times \mathbb{Z}_q$	$\mathcal{G}_k, k \neq 0, \pm 1, 2$	\mathcal{G}_2	\mathcal{G}_0	\mathcal{G}_{-1}	\mathcal{G}_1	ker λ	$\mathbb{Z}_p^2 \times \mathbb{Z}_3$	\mathcal{G}_0	\mathcal{G}_{-1}	\mathcal{G}_1
1	-	$q+3$	$2q+1$	$q+3$	$\frac{q+3}{2}$	q	1	-	6	4	3
p	2	$2(q+1)$	$3q$	$2q$	$q+1$	q	p	2	6	5	3
q	2	-	-	-	-	-	3	2	-	-	-
pq	-	-	-	2	-	-	$3p$	-	2	-	-
p^2	1	$q-1$	$q-1$	$q-1$	$\frac{q-1}{2}$	$q-2$	p^2	1	2	1	1
p^2q	-	-	-	-	-	1	$3p^2$	-	-	-	1

TABLE 7. Enumeration of skew braces of \mathcal{G}_1 -type for $p = 1 \pmod{q}$: the first table assumes $q > 3$ and the second table assumes $q = 3$.

4. SKEW BRACES OF SIZE p^2q WITH $p = -1 \pmod{q}$

In this section we assume that p and q are odd primes such that $p = -1 \pmod{q}$ (recall that we are omitting the case $q = 2$ and $p^2q = 12$) and that $H(x) = x^2 + \xi x + 1$ is an irreducible polynomial over \mathbb{Z}_p such that its companion matrix

$$(26) \quad F = \begin{bmatrix} 0 & -1 \\ 1 & -\xi \end{bmatrix}$$

has order q .

According to [7, Proposition 21.17], the unique non-abelian group of size p^2q is:

$$\mathcal{G}_F = \langle \sigma, \tau, \epsilon \mid \sigma^p = \tau^p = \epsilon^q = 1, \epsilon\sigma\epsilon^{-1} = \tau, \epsilon\tau\epsilon^{-1} = \sigma^{-1}\tau^{-\xi} \rangle \cong \mathbb{Z}_p^2 \rtimes_F \mathbb{Z}_q.$$

An automorphism of \mathcal{G}_F is determined by its image on the generators, i.e. by its restriction to $\langle \sigma, \tau \rangle$ given by a matrix and by the image on ϵ . According to [9, subsections 4.1, 4.4], the map

$$\phi : \mathbb{Z}_p^2 \rtimes_{\rho} (N_{GL_2(p)}(F)) \longrightarrow \text{Aut}(\mathcal{G}_F), \quad [(n, m), M] \mapsto h_{\pm} = \begin{cases} h|_{\langle \sigma, \tau \rangle} = M_{\pm} \\ \epsilon \mapsto \sigma^n \tau^m \epsilon^{\pm 1} \end{cases}$$

where $\phi([(n, m), M]) = h_+$ if $M \in C_{GL_2(p)}(F)$ and h_- otherwise, is a group isomorphism. The form of $M_{\pm} = h_{\pm}|_{\langle \sigma, \tau \rangle}$ is the following:

$$(27) \quad M_+ = \begin{bmatrix} x & -y \\ y & x - \xi y \end{bmatrix}, \quad M_- = \begin{bmatrix} x & y - \xi x \\ y & -x \end{bmatrix}$$

where $x, y \in \mathbb{Z}_p$ and $x^2 + y^2 - \xi xy \neq 0$. The p -Sylow subgroup of \mathcal{G}_F is characteristic, so the map

$$\nu : \text{Aut}(\mathcal{G}_F) \longrightarrow \text{Aut}(\mathcal{G}_F / \langle \sigma, \tau \rangle), \quad h_{\pm} \mapsto \pm 1$$

is a group homomorphism. The kernel of ν is $\text{Aut}(\mathcal{G}_F)^+ = \mathbb{Z}_p^2 \rtimes C_{GL_2(p)}(F)$ and it contains the p -Sylow subgroup of $\text{Aut}(\mathcal{G}_F)$, generated by $\alpha_1 = [(1, 0), Id]$ and $\alpha_2 = [(0, 1), Id]$, and the elements of odd order of $\text{Aut}(\mathcal{G}_F)$.

Proposition 4.1. *Let G be a regular subgroup of $\text{Hol}(\mathcal{G}_F)$. Then $|\pi_2(G)| \neq p, pq$.*

Proof. Let G be a regular subgroup of $\text{Hol}(\mathcal{G}_F)$. The group \mathcal{G}_F has no subgroups of order pq and then $|\pi_2(G)| \neq p$. The group \mathcal{G}_F is the unique non-abelian group of size p^2q and has no normal subgroup of order p . If $\pi_2(G) = pq$ then G has a normal subgroup of order p and so it has to be abelian. Accordingly, $\pi_2(G)$ is an abelian subgroup of order pq of $\text{Aut}(\mathcal{G}_F)$. On the other hand, $\text{Aut}(\mathcal{G}_F)$ has no such subgroup, contradiction. \square

The subgroup

$$(28) \quad \mathfrak{H}_F = \langle \sigma, \tau, \alpha_1, \alpha_2 \rangle$$

is normal in $\text{Hol}(\mathcal{G}_F)$.

Lemma 4.2. *The skew braces of \mathcal{G}_F -type with $|\ker \lambda| = p^2$ are $(B_a, +, \circ)$ where $(B_a, +) = \mathbb{Z}_p^2 \rtimes_F \mathbb{Z}_q$ and*

$$(B_a, \circ) = \mathbb{Z}_p^2 \rtimes_{F^{\frac{a+1}{a}}} \mathbb{Z}_q \cong \begin{cases} \mathbb{Z}_p^2 \times \mathbb{Z}_q, & \text{if } a = q - 1 \\ \mathcal{G}_F, & \text{otherwise} \end{cases}$$

for $1 \leq a \leq q - 1$. In particular they are bi-skew braces.

Proof. The groups

$$G_a = \langle \sigma, \tau, \epsilon^a f \rangle$$

where $f = [(0, 0), F]$ have size $p^2 q$ and they are regular. Let \mathfrak{H}_F be as in (28) and assume that G_a and G_b are conjugate by h , then

$$h(\epsilon)^a h f h^{-1} \mathfrak{H}_F = \epsilon^{\pm a} f^{\pm 1} \mathfrak{H}_F = (\epsilon^b f)^n \mathfrak{H}_F$$

for some n . So $n = \pm 1$ and $a = b$.

Let G be a regular subgroup of $\text{Hol}(\mathcal{G}_F)$ such that $|\pi_2(G)| = q$. Up to conjugation, we can assume that $\pi_2(G)$ is generated by f . The kernel of π_2 is the p -Sylow subgroup of \mathcal{G}_F and then we can assume that

$$G = \langle \sigma, \tau, \epsilon^a f \rangle$$

where $a \neq 0$. Arguing as in Proposition 3.8 we can provide the structure of the skew braces associated to the group G_a . \square

Lemma 4.3. *There exists a unique conjugacy class of regular subgroups G of $\text{Hol}(\mathcal{G}_F)$ with $|\pi_2(G)| = p^2$. A representative is*

$$H = \langle \epsilon, u\alpha_1, w\alpha_2 \rangle \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q$$

where $u = (F - 1)^{-1}(\sigma)$ and $w = (F - 1)^{-1}(\tau)$.

Proof. The group H has the desired properties. Assume that G is a regular subgroup of $\text{Hol}(\mathcal{G}_F)$ with $|\pi_2(G)| = p^2$. Then the image of π_2 is the normal p -Sylow subgroup of $\text{Aut}(\mathcal{G}_F)^+$ generated by α_1 and α_2 . Up to conjugation, we can assume that the kernel is generated by ϵ and so we have $G = \langle \epsilon, u\alpha_1, w\alpha_2 \rangle$. The kernel of π_2 is a normal subgroup of size q of G , and so G is abelian. Thus, by abelianness, it follows that $u = (F - 1)^{-1}(\sigma)$ and $w = (F - 1)^{-1}(\tau)$. \square

Remark 4.4. Let us collect some basic properties of the linear mapping defined by the matrix (26).

(i) Let W be a polynomial. Then

$$(29) \quad \ker(W(F)) \neq 0 \text{ if and only if } W(F) = 0.$$

Indeed $\ker(W(F))$ is an F -invariant subgroup and therefore it is either 0 or \mathbb{Z}_p^2 . Therefore, the polynomial algebra generated by F is a field and moreover, if W_1 and W_2 are polynomials, then $W_1(F) = W_2(F)$ if and only if $W_1(F)(x) = W_2(F)(x)$ for some $x \neq 0$.

(ii) Let $n \in \mathbb{N}$. Since

$$\begin{aligned} H(F^n) - H(F) &= F^{2n} - \xi F^n + 1 - (F^2 + \xi F + 1) \\ &= (F^n - F)(F^n + \underbrace{F + \xi}_{=-F^{-1}}) = (F^n - F)(F^n - F^{-1}) \end{aligned}$$

and the matrix F satisfies the equation $H(F) = F^2 + \xi F + 1 = 0$, we have that $H(F^n) = 0$ if and only if $n = \pm 1 \pmod{q}$.

(iii) The automorphism F acts irreducibly on \mathbb{Z}_p^2 and so

$$(30) \quad \langle x, F(x) \rangle = \mathbb{Z}_p^2$$

for every $x \neq 0$.

Let us define the map Ψ by setting

$$\Psi : \mathbb{Z}_p^2 \longrightarrow \mathbb{Z}_p, \quad (x, y) \mapsto x^2 + y^2 - x + y - \xi xy.$$

It is easy to check that Ψ is surjective.

Lemma 4.5. *Let $v \in \langle \sigma, \tau \rangle$, $\tilde{v} = F(v) - (1 + F)\sigma$ and $S_v = \langle \tilde{v}\alpha_1, v\alpha_2 \rangle \leq \text{Hol}(\mathcal{G}_F)$. Then*

$$|N_{\text{Hol}(\mathcal{G}_F)}(S_v) \cap \text{Aut}(\mathcal{G}_F)| = \begin{cases} 2p^2(p^2 - 1), & \text{if } v = (\frac{1}{\xi+2}, -\frac{1}{\xi+2}), \\ 2p^2(p - 1), & \text{otherwise.} \end{cases}$$

Moreover, if S_v and S_w are conjugate by an element of $\text{Aut}(\mathcal{G}_F)$ then $\Psi(v) = \Psi(w)$.

Proof. Let $h = [(n, m), M_+] \in \text{Aut}(\mathcal{G}_F)$ as defined in (27), i.e. $M_+ \in C_{GL_2(p)}(F)$. Then

$$hS_vh^{-1} = \langle M(\tilde{v})\alpha_1^x\alpha_2^y, M(v)\alpha_1^{-y}\alpha_2^{x-\xi y} \rangle.$$

The equality $hS_vh^{-1} = S_w$ is equivalent to a system of linear equations, namely:

$$(31) \quad \begin{cases} M(\tilde{v}) = x\tilde{w} + yw \\ M(v) = -y\tilde{w} + (x - \xi y)w. \end{cases}$$

Given $v = \sigma^n\tau^m$ and $w = \sigma^s\tau^t$, we can translate (31) into a system of linear equations for x and y . The initial system of 4 equations in x and y is equivalent to the following homogeneous system:

$$(32) \quad \begin{bmatrix} m - t & -\xi m + s + n - 1 \\ n - s & -m + \xi s - t - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = L \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

If the system admits non-trivial solutions, i.e. S_v and S_w are conjugate, then $\det(L) = \psi(n, m) - \psi(s, t) = 0$.

Setting $v = w$, it is easy to deduce that if $v = (\frac{1}{\xi+2}, -\frac{1}{\xi+2})$, every $(x, y) \in \mathbb{Z}_p^2$ is a solution of (32), i.e. $\text{Aut}(\mathcal{G}_1)$ normalizes S_v . Otherwise, the solutions to (32) are given by $y = 0$.

A similar argument leads to the same condition if we consider M_- as in (27), with the property $M_-F = F^{-1}M_-$. In such case we get that $S_{(\frac{1}{\xi+2}, -\frac{1}{\xi+2})}$ is normalized by any $h = [(n, m), M_-]$, otherwise the coefficients of M_- satisfy $x = 0$ and $y \neq 0$. \square

Lemma 4.6. *A set of representatives of regular subgroups G of $\text{Hol}(\mathcal{G}_F)$ such that $|\pi_2(G)| = p^2q$ is*

Groups	Parameters	Class	#
$G_c = \langle u_c\alpha_1, f(u_c)\alpha_2, \epsilon^c f \rangle$	$1 \leq c \leq q - 1, c \neq -2$	\mathcal{G}_F	$q - 2$
$H_a = \langle \tilde{v}_a\alpha_1, v_a\alpha_2, \epsilon^{-2} f \rangle$	$1 \leq a \leq p - 1$	\mathcal{G}_F	$p - 1$

where $f = [(0, 0), F]$, $u_c = H(F^{c+1})^{-1}(F - 1)^{-1}(F^c - 1)(F^{c+2} - 1)(\sigma)$, and $\{v_a \in \mathbb{Z}_p^2 : a \in \mathbb{Z}_p\}$ is a set of representatives of the level sets of Ψ with $\Psi(v_a) = a$ and $\tilde{v}_a = F(v_a) - (1 + F)(\sigma)$ (as defined in Lemma 4.5).

Proof. The subgroups G_c in the statement are regular. Indeed $\epsilon^c \in \pi_1(G_c)$ and according to Lemma 2.4(2) we have $\langle u_c, f(u_c) \rangle \subseteq \pi_1(G_c)$ and by Remark 4.4(iii), $\{u_c, F(u_c)\}$ is a basis of \mathbb{Z}_p^2 . So $|\pi_1(G_c)| > p^2$ and then $\pi_1(G_c) = \mathcal{G}_F$ (and similarly for H_a).

If G_c is conjugate to G_d or to H_a , then arguing as in Lemma 4.2 it follows that $c = d$ or $c = -2$. If H_a and H_b are conjugate then their p -Sylow subgroups are conjugate. Thus, by Lemma 4.5 we have $a = \Psi(v_a) = \Psi(v_b) = b$.

The unique subgroup of order p^2q of $\text{Aut}(\mathcal{G}_F)$ is generated by α_1, α_2 and $f = [(0, 0), F]$ and it is isomorphic to \mathcal{G}_F . Therefore, the standard presentation of the regular subgroup G of $\text{Hol}(\mathcal{G}_F)$ with $|\pi_2(G)| = p^2q$ is

$$G = \langle u\epsilon^a\alpha_1, v\epsilon^b\alpha_2, w\epsilon^c f \rangle$$

where $u, v, w \in \langle \sigma, \tau \rangle$. The (R) conditions are:

$$(33) \quad (u\epsilon^a\alpha_1)^p = (v\epsilon^b\alpha_2)^p = (w\epsilon^c f)^q = 1$$

$$(34) \quad (w\epsilon^c f)u\epsilon^a\alpha_1(w\epsilon^c f)^{-1} = v\epsilon^b\alpha_2$$

$$(35) \quad (w\epsilon^c f)v\epsilon^b\alpha_2(w\epsilon^c f)^{-1} = (u\epsilon^a\alpha_1)^{-1}(v\epsilon^b\alpha_2)^{-\xi}.$$

From (33) we have $a = b = 0$. If $c = 0$, then by Lemma 2.4(1), $\pi_1(G) \subseteq \langle \sigma, \tau \rangle$ and then G is not regular. Thus $c \neq 0$ and (34) and (35) are equivalent to

$$(36) \quad \begin{cases} v = F^{c+1}(u) - (F - 1)^{-1}(F^c - 1)F(\sigma) \\ h(F^{c+1})(u) = (F - 1)^{-1}(F^c - 1)(F^{c+2} - 1)(\sigma). \end{cases}$$

According to Remark 4.4(ii) and (i), if $c \neq -2$ then $H(F^{c+1})$ is an automorphism of \mathbb{Z}_p^2 . If $c \neq -2$ then, substituting the second equation in the first one in (36) and using that $F^2 = -\xi F - 1$ we have that (36) is equivalent to

$$(37) \quad \begin{cases} v = F(u) \\ u = h(F^{c+1})^{-1}(F-1)^{-1}(F^c-1)(F^{c+2}-1)(\sigma). \end{cases}$$

Therefore $u = u_c$ is uniquely determined by c and

$$G = \langle u_c \alpha_1, f(u_c) \alpha_2, w \epsilon^c f \rangle$$

where $c \neq -2$. If $c+1=0$, the last condition in (33) implies that $w=0$ and so $G = G_{-1}$.

Otherwise, we will prove that G is conjugate to G_c by an element of the form $h = \alpha_1^n \alpha_2^m$ for some n, m . Indeed, h centralizes $\langle u \alpha_1, f(u) \alpha_2 \rangle$ and so $h G h^{-1} = G_c$ if and only if the conjugate of the last generator is in G_c , i.e.

$$(38) \quad h w \epsilon^c f h^{-1} = w' \alpha_1^{n+m} \alpha_2^{-n+(1+\xi)m} \epsilon^c f = (u_c \alpha_1)^{n+m} (f(u_c) \alpha_2)^{-n+(1+\xi)m} \epsilon^c f \in G_c.$$

Equation (38) is equivalent to have that $w = nx + mF(x)$, where

$$x = \frac{(F^c-1)(F^{c+1}-1)(F^{c+2}-1)}{(F-1)H(F^{c+1})} \neq 0.$$

According to Remark (4.4)(iii), $\{x, F(x)\}$ is a basis of \mathbb{Z}_p^2 and therefore $w \in \langle x, F(x) \rangle$.

Let $c = -2$. In such case, the second condition in (36) is trivial according to (29) and the first one is equivalent to

$$(39) \quad u = F(v) - (F+1)(\sigma) = \tilde{v}.$$

If $v = (x, y)$ then $\tilde{v} = (-y-1, x-\xi y-1)$. Since G is regular, then u and v are linearly independent, i.e.

$$(40) \quad \det [v \quad \tilde{v}] = \det \begin{bmatrix} x & -y-1 \\ y & x-\xi y-1 \end{bmatrix} = \Psi(x, y) \neq 0.$$

The number of solutions of the equation $\Psi(x, y) = 0$ is $p+1$, so there are $p^2 - p - 1$ choices for v .

The p -Sylow subgroup of G is the group $S_v = \langle \tilde{v} \alpha_1, v \alpha_2 \rangle$. According to Lemma 4.5, if $v = (\frac{1}{\xi+2}, -\frac{1}{\xi+2})$ then S_v is normalized by $\text{Aut}(\mathcal{G}_F)$, otherwise its normalizer has index $p+1$ in $\text{Aut}(\mathcal{G}_1)$. Accordingly, there are

$$\frac{p^2 - p - 2}{p+1} + 1 = \frac{(p-2)(p+1)}{p+1} + 1 = p-1$$

choices for v up to conjugation and the representatives can be chosen to be $\{v_a : 0 \neq a \in \mathbb{Z}_p\}$, since the value $\Psi(v)$ is an invariant. Thus, we can assume that G is conjugate to a group of the form

$$G = \langle \tilde{v}_a \alpha_1, v_a \alpha_2, w \epsilon^{-2} f \rangle$$

for $1 \leq a \leq p-1$. We show that G is conjugate to some H_a by $h \in \langle \alpha_1, \alpha_2 \rangle$. Indeed, this is equivalent to have that

$$h w \epsilon^{-2} f h^{-1} = w \sigma^m \frac{F^{-2}-1}{F-1} \tau^{(\xi m-n) \frac{F^{-2}-1}{F-1}} \alpha_1^{n+m} \alpha_2^{m(1+\xi)-n} = (\tilde{v}_a \alpha_1)^{n+m} (v_a \alpha_2)^{(1+\xi)m-n}.$$

Working out this condition, it turns out to be equivalent that $w \in \langle x, y \rangle$, where

$$x = \tilde{v} - v + z, \quad y = \tilde{v} + (1+\xi)v + F(z)$$

with $z = (F+\xi-1)(\sigma)$. It is easy to check that

$$\det [x \quad y] = -(\xi+2)\Psi(v_a) \neq 0,$$

i.e. x, y is a basis and so G is conjugate to H_a . □

We summarize the content of this subsection in the following table:

$ \ker \lambda $	$\mathbb{Z}_p^2 \times \mathbb{Z}_q$	\mathcal{G}_F
1	-	$p + q - 3$
q	1	-
p^2	1	$q - 2$
p^2q	-	1

TABLE 8. Enumeration of skew braces of \mathcal{G}_F -type for $p = -1 \pmod{q}$.

5. SKEW BRACES OF SIZE p^2q WITH $q = 1 \pmod{p}$ AND $q \neq 1 \pmod{p^2}$

In this section we assume that p and q are primes such that $q = 1 \pmod{p}$ and $q \neq 1 \pmod{p^2}$ (including the case $p = 2$ unless stated otherwise). Let r be a fixed element of order p in \mathbb{Z}_q^\times . Since $p^2q \neq 12$, the relevant non-abelian groups for this section are the following:

- (i) $\mathbb{Z}_q \rtimes_r \mathbb{Z}_{p^2} = \langle \sigma, \tau \mid \sigma^{p^2} = \tau^q = 1, \sigma\tau\sigma^{-1} = \tau^r \rangle$,
- (ii) $\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_r \mathbb{Z}_p) = \langle \sigma, \tau, \epsilon \mid \sigma^p = \tau^p = \epsilon^q = 1, [\epsilon, \tau] = [\tau, \sigma] = 1, \sigma\epsilon\sigma^{-1} = \epsilon^r \rangle$.

Tables 9 and 10 summarize the enumeration of skew braces according to the isomorphism class of their additive and multiplicative groups.

$+\backslash\circ$	\mathbb{Z}_{p^2q}	$\mathbb{Z}_q \rtimes_r \mathbb{Z}_{p^2}$	$\mathbb{Z}_p^2 \times \mathbb{Z}_q$	$\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_r \mathbb{Z}_p)$
$\mathbb{Z}_q \rtimes_r \mathbb{Z}_{p^2}$	$2p$	$2p(p-1)$	-	-
$\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_r \mathbb{Z}_p)$	-	-	4	$6p - 4$

TABLE 9. Enumeration of skew braces of size p^2q with $q = 1 \pmod{p}$, $q \neq 1 \pmod{p^2}$ and $p > 2$.

$+\backslash\circ$	\mathbb{Z}_{4q}	$\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_4$	$\mathbb{Z}_2^2 \times \mathbb{Z}_q$	$\mathbb{Z}_2 \times (\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_2)$
$\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_4$	2	2	2	4
$\mathbb{Z}_2 \times (\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_2)$	2	2	2	4

TABLE 10. Enumeration of skew braces of order $4q$ with $q = 1 \pmod{2}$ and $q \neq 1 \pmod{4}$. Notice that in this case we can assume $r = -1$.

5.1. Skew braces of $\mathbb{Z}_q \rtimes_r \mathbb{Z}_{p^2}$ -type. In this section we denote by A the group $\mathbb{Z}_q \rtimes_r \mathbb{Z}_{p^2}$. Such group has the following presentation:

$$A = \langle \sigma, \tau \mid \sigma^{p^2} = \tau^q = 1, \sigma\tau\sigma^{-1} = \tau^r \rangle.$$

According to [9, Subsection 4.5], the map

$$\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes \mathbb{Z}_q^\times) \longrightarrow \text{Aut}(A), \quad (k, j, i) \mapsto \varphi_{i,j}^k = \begin{cases} \tau \mapsto \tau^i \\ \sigma \mapsto \tau^j \sigma^{kp+1} \end{cases}$$

is an isomorphism. In particular, p^2q divides $|\text{Aut}(A)| = pq(q-1)$ and so we need to check all the possible values for the size of the image of regular subgroups under π_2 . Note that $Z(A) = \langle \sigma^p \rangle$ and so $\langle \tau, \sigma^p \rangle$ is an abelian characteristic subgroup of A of order pq .

The conjugacy classes of subgroups of $\text{Aut}(A)$ are given in Table 11.

Size	Group	Parameters	Class
p	$\langle \varphi_{r,0}^k \rangle$	$0 \leq k \leq p-1$	\mathbb{Z}_p
	$\langle \varphi_{1,0}^1 \rangle$	-	
q	$\langle \varphi_{1,1}^0 \rangle$	-	\mathbb{Z}_q
p^2	$\langle \varphi_{1,0}^1, \varphi_{r,0}^0 \rangle$	-	\mathbb{Z}_p^2
pq	$\langle \varphi_{r,0}^k, \varphi_{1,1}^0 \rangle$	$0 \leq k \leq p-1$	$\mathbb{Z}_q \rtimes_r \mathbb{Z}_p$
	$\langle \varphi_{1,0}^1, \varphi_{1,1}^0 \rangle$	-	\mathbb{Z}_{pq}
p^2q	$\langle \varphi_{1,0}^1, \varphi_{r,0}^0, \varphi_{1,1}^0 \rangle$	-	$\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_r \mathbb{Z}_p)$

TABLE 11. Conjugacy classes of subgroups of $\text{Aut}(A)$.

Lemma 5.1. *The unique skew brace of A-type with $|\ker \lambda| = p^2$ is $(B, +, \circ)$ where*

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + r^{y_1} x_2 \\ y_1 + y_2 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \circ \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} r^{y_2} x_1 + r^{y_1} x_2 \\ y_1 + y_2 \end{pmatrix}$$

for every $0 \leq x_1, x_2 \leq q-1$ and every $0 \leq y_1, y_2 \leq p^2-1$. In particular, $(B, \circ) \cong \mathbb{Z}_{p^2q}$.

Proof. Let us consider the group

$$G = \langle \sigma, \tau^{\frac{1}{r-1}} \varphi_{1,1}^0 \rangle \cong \mathbb{Z}_{p^2q}.$$

The subset $\pi_1(G)$ contains $\langle \sigma \rangle$ and $\tau^{\frac{1}{r-1}}$ and then $|\pi_1(G)| > p^2$ and it divides p^2q . So $\pi_1(G) = A$ and according to Lemma 2.3 we have that G is regular.

Let G be a regular subgroup of $\text{Hol}(A)$ with $|\pi_2(G)| = q$. According to Table 11, we have that $\pi_2(G) = \langle \varphi_{1,1}^0 \rangle$, that is a normal subgroup of $\text{Aut}(A)$. The subgroups of size p^2 of A are all conjugate to $\langle \sigma \rangle$, so we have that G has the standard presentation:

$$G = \langle \sigma, \tau^a \sigma^b \varphi_{1,1}^0 \rangle = \langle \sigma, \tau^a \varphi_{1,1}^0 \rangle.$$

By condition (K), we have that $a = \frac{1}{r-1}$. The formula follows by the same argument of Lemma 3.4. \square

In the following we consider the group $\mathfrak{G}_1 = \langle \sigma^p, \tau, \varphi_{1,1}^0 \rangle \trianglelefteq \text{Hol}(A)$ with $\text{Hol}(A)/\mathfrak{G}_1 \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q^\times$.

Lemma 5.2. *A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(A)$ with $|\pi_2(G)| = p$ is*

$\pi_2(G)$	Groups	Parameters	Class for $p > 2$	Class for $p = 2$	#
$\langle \varphi_{1,0}^1 \rangle$	$G_a = \langle \tau, \sigma^p, \sigma^a \varphi_{1,0}^1 \rangle$	$1 \leq a \leq p-1$,	A	$\mathbb{Z}_2 \times (\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_2)$	$p-1$
$\langle \varphi_{r,0}^k \rangle$	$H_{a,k} = \langle \tau, \sigma^p, \sigma^a \varphi_{r,0}^k \rangle$	$1 \leq a \leq p-1$,	\mathbb{Z}_{p^2q} , if $a = p-1$,	\mathbb{Z}_{4p} , if $k = 0$,	$p(p-1)$
		$0 \leq k \leq p-1$	A , otherwise	$\mathbb{Z}_2^2 \times \mathbb{Z}_p$, if $k = 1$	

Proof. The groups G of order p^2q in the statement have the general form:

$$\langle \tau, \sigma^p, \sigma^a \theta \rangle$$

for $1 \leq a \leq p-1$ and $\theta \in \{\varphi_{1,0}^1, \varphi_{r,0}^k\}$. So $\langle \tau, \sigma^p \rangle \subseteq \pi_1(G)$ and $\sigma^a \in \pi_1(G)$. Thus $|\pi_1(G)| > pq$ and it divides p^2q . So, $\pi_1(G) = A$ and according to Lemma 2.3 we have that G is regular.

If G_a and G_b are conjugate, then their images in $\text{Hol}(A)/\mathfrak{G}_1$ coincide, i.e. $\langle \sigma^a \varphi_{1,0}^1 \rangle = \langle \sigma^b \varphi_{1,0}^1 \rangle$ and so $a = b$ (the same idea applies to the groups $H_{a,k}$). Also, G_a and $H_{b,k}$ are not conjugate since their images under π_2 are not.

The only subgroup of size pq in A is $\langle \tau, \sigma^p \rangle$. If G is a regular subgroup of $\text{Hol}(A)$, we have two cases to consider according to Table 11. If $\pi_2(G) = \langle \varphi_{1,0}^1 \rangle$, then G has the following standard presentation:

$$G = \langle \tau, \sigma^p, \tau^b \sigma^a \varphi_{1,0}^1 \rangle = \langle \tau, \sigma^p, \sigma^a \varphi_{1,0}^1 \rangle$$

where $1 \leq a \leq p-1$, i.e. $G = G_a$.

If $\pi_2(G) = \langle \varphi_{r,0}^k \rangle$ for $0 \leq k \leq p-1$, a standard presentation for G is

$$G = \langle \tau, \sigma^p, \sigma^a \varphi_{r,0}^k \rangle$$

with $1 \leq a \leq p-1$, i.e. $G = H_{a,k}$. \square

Lemma 5.3. *A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(A)$ with $|\pi_2(G)| = pq$ is*

$\pi_2(G)$	Groups	Parameters	Class for $p > 2$	Class for $p = 2$	#
$\langle \varphi_{1,0}^1, \varphi_{1,1}^0 \rangle$	$G_a = \langle \sigma^p, \sigma^a \varphi_{1,0}^1, \tau^{\frac{1}{r-1}} \varphi_{1,1}^0 \rangle$	$1 \leq a \leq p-1$	\mathbb{Z}_{p^2q}	$\mathbb{Z}_2 \times (\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_2)$	$p-1$
$\langle \varphi_{r,0}^k, \varphi_{1,1}^0 \rangle$	$H_{a,k} = \langle \sigma^p, \sigma^a \varphi_{r,0}^k, \tau^{\frac{1}{r-1}} \varphi_{1,1}^0 \rangle$	$1 \leq a \leq p-1,$ $0 \leq k \leq p-1$	A	\mathbb{Z}_{4q} , if $k = 0$, $\mathbb{Z}_2^2 \times \mathbb{Z}_q$, if $k = 1$	$p(p-1)$

Proof. The same argument of Lemma 5.2 shows that the groups in the statement are regular and that they are not pairwise conjugate.

Let G be a regular subgroup of $\text{Hol}(A)$ such that $|\pi_2(G)| = pq$. According to Table 11 we have two cases, and in both cases the image of π_2 is normal. Up to conjugation, the unique subgroup of size p of A is $\langle \sigma^p \rangle$ and so $\ker \pi_2 = \langle \sigma^p \rangle$. If $\pi_2(G) = \langle \varphi_{1,0}^1, \varphi_{1,1}^0 \rangle$, then

$$G = \langle \sigma^p, \tau^b \sigma^a \varphi_{1,0}^1, \tau^c \sigma^d \varphi_{1,1}^0 \rangle.$$

From (R) conditions we have:

- the third generator has order q modulo $\langle \sigma^p \rangle$ and so it follows that $d = 0 \pmod{p}$.
- the last two generators commute modulo $\langle \sigma^p \rangle$ and so $c = \frac{1}{r-1}$.

If $a = 0 \pmod{p}$ then by Lemma 2.4(1) we have $\pi_1(G) \subseteq \langle \sigma^p, \tau \rangle$, contradiction by regularity. Thus we can assume that $1 \leq a \leq p-1$ and so

$$G = \langle \sigma^p, \tau^b \sigma^a \varphi_{1,0}^1, \tau^{\frac{1}{r-1}} \varphi_{1,1}^0 \rangle.$$

Now, the group G is conjugate to G_a by $h = \varphi_{1,n}^0$ where $n = b \frac{1-r}{r^a-1}$.

If $\pi_2(G) = \langle \varphi_{r,0}^k, \varphi_{1,1}^0 \rangle$ then, by the same computations of the previous case the standard presentation of G is

$$G = \langle \sigma^p, \tau^b \sigma^a \varphi_{r,0}^k, \tau^{\frac{1}{r-1}} \varphi_{1,1}^0 \rangle.$$

By the same argument, we can also assume that $1 \leq a \leq p-1$. If $a = p-1$, from $(\tau^b \sigma^{p-1} \varphi_{r,0}^k)^p = \tau^{bp} (\sigma^{p-1} \varphi_{r,0}^k)^p \in \langle \sigma^p \rangle$ it follows that $b = 0$. Otherwise, G is conjugate to $H_{a,k}$ by a power of $\varphi_{1,1}^0$. To do this, notice that $\varphi_{1,n}^0 = (\varphi_{1,1}^0)^n$ centralizes the first and the third generators of G and that

$$(\varphi_{1,n}^0)(\tau^b \sigma^a \varphi_{r,0}^k)(\varphi_{1,n}^0)^{-1} = \tau^{b+n \frac{r^a-1}{r-1}} \sigma^a (\varphi_{1,1}^0)^{(1-r)n} \varphi_{r,0}^k = \tau^{b+n \frac{r^a+1-1}{r-1}} (\tau^{\frac{1}{r-1}} \varphi_{1,1}^0)^{(1-r)n} \sigma^a \varphi_{r,0}^k.$$

holds. Thus, $n = b \frac{1-r}{r^a+1-1}$ will do. \square

Proposition 5.4. *Let $p > 2$ and let G be a regular subgroup of $\text{Hol}(A)$. Then $|\pi_2(G)| \notin \{p^2, p^2q\}$.*

Proof. If $p > 2$ then

$$(41) \quad (\sigma^a \varphi_{1,0}^1)^n = \sigma^{a(p \frac{n(n-1)}{2} + n)} (\varphi_{1,0}^1)^n$$

for every $n \in \mathbb{N}$.

Let G be a subgroup of $\text{Hol}(A)$ of size p^2q . If $|\pi_2(G)| = p^2$, then according to Table 11 and the fact that the kernel of π_2 is the normal q -Sylow subgroup of A , we have the following standard presentation for G :

$$G = \langle \tau, \tau^b \sigma^a \varphi_{1,0}^1, \tau^d \sigma^c \varphi_{r,0}^0 \rangle = \langle \tau, \sigma^a \varphi_{1,0}^1, \sigma^c \varphi_{r,0}^0 \rangle.$$

By condition (R), $(\sigma^a \varphi_{1,0}^1)^p = (\sigma^c \varphi_{r,0}^0)^p \in \langle \tau \rangle$. Using (41), we have $a = c = 0 \pmod{p}$.

Assume that $|\pi_2(G)| = p^2q$. According to Table 11, G has the standard presentation:

$$G = \langle \tau^a \sigma^b \varphi_{1,0}^1, \tau^c \sigma^d \varphi_{r,0}^0, \tau^e \sigma^f \varphi_{1,1}^0 \rangle$$

where the first two generators have order p and the third one has order q . So, $f = 0$ and $b = d = 0 \pmod{p}$ where we are using formula (41) again. The group $\langle \sigma^p, \tau \rangle$ is characteristic in A . Hence in both cases, according to Lemma 2.4(1) we have $\pi_1(G) \subseteq \langle \sigma^p, \tau \rangle$, and so G is not regular. \square

The last result is quite different when we consider the case $p = 2$.

Lemma 5.5. *Let $p = 2$. There exists a unique conjugacy class of regular subgroups of $\text{Hol}(A)$ with $|\pi_2(G)| = 4$. A representative is given by*

$$H = \langle \tau, \sigma \varphi_{1,0}^1, \sigma^2 \varphi_{-1,0}^0 \rangle \cong \mathbb{Z}_2 \times (\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_2).$$

Proof. For the group H , we have that $\{\tau^n, \tau^n \sigma : 0 \leq n \leq q-1\} \subseteq \pi_1(H)$ and $\sigma^2 \in \pi_1(H)$. So $|\pi_1(H)| > 2q$ and it divides $4q$. Thus, H is regular by Lemma 2.3.

Let G be a regular subgroup with $|\pi_2(G)| = 4$. According to Table 11, we have that G has the following standard presentation:

$$G = \langle \tau, \sigma^a \varphi_{1,0}^1, \sigma^b \varphi_{-1,0}^0 \rangle$$

where $1 \leq a, b \leq 3$. By the condition (R), $(\sigma^b \varphi_{-1,0}^0)^2 = \sigma^{2b} \in \langle \tau \rangle$, thus we have $b = 2$. So $a = 1$ or $a = 3$. If $a = 1$, we have $G = H$ and if $a = 3$, the group G is conjugate to H by $\varphi_{1,0}^1$. \square

Lemma 5.6. *Let $p = 2$. There exists a unique conjugacy class of regular subgroups with $|\pi_2(G)| = 4q$. A representative is given by*

$$H = \langle \sigma \varphi_{1,0}^1, \sigma^2 \varphi_{-1,0}^0, \tau^{-\frac{1}{2}} \varphi_{1,1}^0 \rangle \cong \mathbb{Z}_2 \times (\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_2).$$

Proof. Analogously to Lemma 5.5, it is straightforward to check that the group in the statement is a regular subgroup of $\text{Hol}(A)$. A regular subgroup with $|\pi_2(G)| = 4q$ has the following standard presentation:

$$G = \langle \tau^a \sigma^b \varphi_{1,0}^1, \tau^c \sigma^d \varphi_{-1,0}^0, \tau^e \sigma^f \varphi_{1,1}^0 \rangle.$$

By condition (R) we have $(\tau^a \sigma^b \varphi_{1,0}^1)^2 = (\tau^c \sigma^d \varphi_{-1,0}^0)^2 = (\tau^e \sigma^f \varphi_{1,1}^0)^q = 1$, and so $b \in \{1, 3\}$, $d \in \{0, 2\}$ and $f = 0$. Since $\varphi_{1,0}^1$ is a central element in $\pi_2(G)$, by condition (R) again, we have that $\tau^a \sigma^b \varphi_{1,0}^1$ is central in G , hence $a = c$ and $e = -\frac{1}{2}$ and so

$$G = \langle \tau^a \sigma^b \varphi_{1,0}^1, \tau^c \sigma^d \varphi_{-1,0}^0, \tau^{-\frac{1}{2}} \varphi_{1,1}^0 \rangle$$

for some $1 \leq a \leq q-1$ and the constraints on b and d from above. We can assume $b = 1$, otherwise we conjugate by $\varphi_{1,0}^1$. If we conjugate by $\varphi_{1,-a}^0$, we get

$$G = \langle \sigma \varphi_{1,0}^1, \sigma^d \varphi_{-1,0}^0, \tau^{-\frac{1}{2}} \varphi_{1,1}^0 \rangle.$$

Finally, from regularity, it follows that $d \neq 0$, so $d = 2$ and we have the group in the statement. \square

We summarize the content of this subsection in the following tables:

$ \ker \lambda $	\mathbb{Z}_{p^2q}	$\mathbb{Z}_q \rtimes_r \mathbb{Z}_{p^2}$	$ \ker \lambda $	\mathbb{Z}_{4q}	$\mathbb{Z}_2^2 \times \mathbb{Z}_q$	$\mathbb{Z}_2 \times (\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_2)$	$\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_4$
p	$p-1$	$p(p-1)$	1	-	-	1	-
pq	p	$p^2 - p - 1$	2	-	1	1	1
p^2	1	-	4	1	-	-	-
p^2q	-	1	q	-	-	1	-
			$2q$	1	1	1	-
			$4q$	-	-	-	1

TABLE 12. Enumeration of skew braces of A -type for $q = 1 \pmod{p}$, $q \neq 1 \pmod{p^2}$ and $p > 2$ and $p = 2$ respectively.

5.2. Skew braces of $\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_r \mathbb{Z}_p)$ -type. In this section we denote by A the group $\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_r \mathbb{Z}_p)$. A presentation of this group is

$$A = \langle \sigma, \tau, \epsilon \mid \sigma^p = \tau^p = \epsilon^q = 1, [\epsilon, \tau] = [\tau, \sigma] = 1, \sigma \epsilon \sigma^{-1} = \epsilon^r \rangle.$$

According to [9, Subsection 4.6] the mapping

$$\phi : (\mathbb{Z}_p \rtimes \mathbb{Z}_p^\times) \times (\mathbb{Z}_q \rtimes \mathbb{Z}_q^\times) \longrightarrow \text{Aut}(A), \quad [(l, i), (s, j)] \mapsto \alpha_{l,i} \beta_{s,j}$$

where

$$\alpha_{l,i} = \begin{cases} \epsilon \mapsto \epsilon \\ \tau \mapsto \tau^i \\ \sigma \mapsto \tau^l \sigma \end{cases} \quad \text{and} \quad \beta_{s,j} = \begin{cases} \epsilon \mapsto \epsilon^j \\ \tau \mapsto \tau \\ \sigma \mapsto \epsilon^s \sigma \end{cases}$$

is an isomorphism of groups. In particular, note that $\alpha_{l,i}$ and $\beta_{s,j}$ commute. In particular, p^2q divides $|\text{Aut}(A)| = pq(p-1)(q-1)$ and so we need to discuss all the possible values for the size of the image of regular subgroups under π_2 . The conjugacy classes of subgroups of $\text{Aut}(A)$ are collected in Table 13.

Size	Group	Class
p	$\langle \alpha_{1,1} \rangle$ $\langle \beta_{0,r} \rangle$ $\langle \alpha_{1,1}\beta_{0,r} \rangle$	\mathbb{Z}_p
q	$\langle \beta_{1,1} \rangle$	\mathbb{Z}_q
pq	$\langle \alpha_{1,1}, \beta_{1,1} \rangle$ $\langle \beta_{0,r}, \beta_{1,1} \rangle$ $\langle \alpha_{1,1}\beta_{0,r}, \beta_{1,1} \rangle$	\mathbb{Z}_{pq} $\mathbb{Z}_q \rtimes_r \mathbb{Z}_p$ $\mathbb{Z}_q \rtimes_r \mathbb{Z}_p$
p^2	$\langle \alpha_{1,1}, \beta_{0,r} \rangle$	\mathbb{Z}_p^2
p^2q	$\langle \alpha_{1,1}, \beta_{1,1}, \beta_{0,r} \rangle$	A

TABLE 13. Conjugacy classes of subgroups of $\text{Aut}(A)$.

The procedure to check the regularity of a subgroup has already been established in the previous sections, so we are not showing that part of the classification strategy explicitly in what remains of Section 5.

Proposition 5.7. *The unique skew brace of A -type with $|\ker \lambda| = p^2$ is $B = B_1 \times B_2$, where B_1 is the trivial skew brace of size p and B_2 is the unique skew brace of non-abelian type of size pq with $|\ker \lambda_{B_2}| = p$. In particular, $(B, \circ) \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q$.*

Proof. The unique subgroup of size q of $\text{Aut}(A)$ is generated by $\beta_{1,1}$ and the kernel is a p -Sylow subgroup of A which can be taken as $\langle \sigma, \tau \rangle$ up to conjugation. Therefore $I = \langle \tau \rangle_+ \leq \ker \lambda \cap \text{Fix}(B) \cap Z(B, +)$ and so I is an ideal of B contained in $Z(B, \circ)$. The subgroup $J = \langle \epsilon, \sigma \rangle_+ \trianglelefteq (B, +)$ is a left ideal and since $\tau \in Z(B, \circ)$ then J is an ideal of B . Therefore $B = I + J$ and $I \cap J = 0$ and so B is a direct product of the trivial skew brace of size p and a skew brace B_2 of size pq with $|\ker \lambda_{B_2}| = p$. According to [2, Theorem 3.6], there exists a unique such skew brace and $(B_2, \circ) \cong \mathbb{Z}_{pq}$. \square

In the following we consider the subgroup $\mathfrak{G}_2 = \langle \epsilon, \tau, \alpha_{1,1}, \beta_{1,1} \rangle \trianglelefteq \text{Hol}(A)$. In particular, $\text{Hol}(A)/\mathfrak{G}_2 \cong \mathbb{Z}_p \times \mathbb{Z}_p^\times \times \mathbb{Z}_q^\times$.

Lemma 5.8. *A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(A)$ with $|\pi_2(G)| = p$ is*

$\pi_2(G)$	Groups	Parameters	Class for $p > 2$	Class for $p = 2$	#
$\langle \alpha_{1,1} \rangle$	$K = \langle \epsilon, \tau, \sigma \alpha_{1,1} \rangle$	-	A	$\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_4$	1
$\langle \beta_{0,r} \rangle$	$L = \langle \epsilon, \sigma, \tau \beta_{0,r} \rangle$	-	A	A	1
	$G_c = \langle \epsilon, \tau, \sigma^c \beta_{0,r} \rangle$	$1 \leq c \leq p-1$	$\mathbb{Z}_p^2 \times \mathbb{Z}_q$, if $c = -1$, A , otherwise	$\mathbb{Z}_2^2 \times \mathbb{Z}_q$	$p-1$
$\langle \alpha_{1,1}\beta_{0,r} \rangle$	$H_c = \langle \epsilon, \tau, \sigma^c \alpha_{1,1}\beta_{0,r} \rangle$	$1 \leq c \leq p-1$	$\mathbb{Z}_p^2 \times \mathbb{Z}_q$, if $c = -1$, A , otherwise	\mathbb{Z}_{4q}	$p-1$

Proof. The groups K, L, G_c and H_c are not conjugate since their images under π_2 or their kernels are not. If G_c and G_d (resp. H_c and H_d) are conjugate, then their images in $\text{Hol}(A)/\mathfrak{G}_2$ coincide, i.e. $\langle \sigma^c \beta_{0,r} \mathfrak{G}_2 \rangle = \langle \sigma^d \beta_{0,r} \mathfrak{G}_2 \rangle$. Thus, $c = d$.

Let G be a regular subgroup of $\text{Hol}(A)$ such that $|\pi_2(G)| = p$. According to Table 13 we need to discuss three cases. Assume that $\pi_2(G) = \langle \alpha_{1,1} \rangle$. Then the kernel has order pq and so G has the form

$$G = \langle \epsilon, \sigma^n \tau^m, \sigma^a \tau^b \alpha_{1,1} \rangle.$$

By condition (K) we have $n = 0$. So we can assume $b = 0$ and $G = \langle \epsilon, \tau, \sigma^a \alpha_{1,1} \rangle$. By regularity, $a \neq 0$ and so G is conjugate to K by $\alpha_{0,a}$.

Assume that $\pi_2(G) = \langle \beta_{0,r} \rangle$. Then G has the following standard presentation:

$$G = \langle \epsilon, \sigma^n \tau^m, \sigma^a \tau^b \beta_{0,r} \rangle.$$

If $n = 0$ then $G = G_a$. If $n \neq 0$, we can assume $m = 0$, otherwise, we conjugate by $\alpha_{1,1}^{-\frac{m}{n}}$. Thus we can assume $a = 0$ and $n = 1$. By regularity, $b \neq 0$ and so G is conjugate to L by $\alpha_{0,b^{-1}}$.

Finally, assume that $\pi_2(G) = \langle \alpha_{1,1}\beta_{0,r} \rangle$. Then

$$G = \langle \epsilon, \sigma^n \tau^m, \sigma^a \tau^b \alpha_{1,1} \beta_{0,r} \rangle.$$

The condition (K) implies that $n = 0$ and so $G = \langle \epsilon, \tau, \sigma^a \alpha_{1,1} \beta_{0,r} \rangle = H_a$.

Notice that for the case $p = 2$, the groups K and H_1 (which is the only H_c) have elements of order 4 since $(\sigma \alpha_{1,1})^2 = (\sigma \alpha_{1,1} \beta_{0,r})^2 = \tau$. For $p > 2$, there are no elements of order p^2 and then we have a different isomorphism class for the corresponding groups in the table of the statement. \square

Lemma 5.9. *A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(A)$ with $|\pi_2(G)| = pq$ is*

$\pi_2(G)$	Groups	Parameters	Class for $p > 2$	Class for $p = 2$	#
$\langle \alpha_{1,1}, \beta_{1,1} \rangle$	$K_1 = \langle \tau, \sigma \alpha_{1,1}, \epsilon^{\frac{1}{r-1}} \beta_{1,1} \rangle$	-	$\mathbb{Z}_p^2 \times \mathbb{Z}_q$	\mathbb{Z}_{4q}	1
$\langle \beta_{1,1}, \beta_{0,r} \rangle$	$K_2 = \langle \sigma, \tau \beta_{0,r}, \epsilon^{\frac{1}{r-1}} \beta_{1,1} \rangle$	-	A	A	1
	$G_a = \langle \tau, \sigma^a \beta_{0,r}, \epsilon^{\frac{1}{r-1}} \beta_{1,1} \rangle$	$1 \leq a \leq p-1$	A	A	$p-1$
$\langle \beta_{1,1}, \alpha_{1,1} \beta_{0,r} \rangle$	$H_a = \langle \tau, \sigma^a \alpha_{1,1} \beta_{0,r}, \epsilon^{\frac{1}{r-1}} \beta_{1,1} \rangle$	$1 \leq a \leq p-1$	A	$\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_4$	$p-1$

Proof. The same argument as in Lemma 5.8 shows that the groups in the statement are not conjugate. Let G be a regular subgroup of $\text{Hol}(A)$ with $|\pi_2(G)| = pq$. According to Table 13 we have to check three cases. In all such cases, up to the action of the normalizer of $\pi_2(G)$, we can assume that $\ker \pi_2|_G$ is generated by τ or σ .

Let $\pi_2(G) = \langle \alpha_{1,1}, \beta_{1,1} \rangle$. By the condition (K) we have that $\ker \pi_2|_G = \langle \tau \rangle$. So,

$$G = \langle \tau, \epsilon^a \sigma^b \alpha_{1,1}, \epsilon^c \sigma^d \beta_{1,1} \rangle.$$

The condition (R) implies that $d = 0$ and $c = \frac{1}{r-1}$. If $b = 0$ then, by Lemma 2.4(1), $\pi_1(G) \subseteq \langle \epsilon, \tau \rangle$, a contradiction since G is regular. So $b \neq 0$ and therefore G is conjugate to K_1 by $\alpha_{0,b} \beta_{1,1}^n$ where $n = a \frac{1-r}{r-1}$.

Let $\pi_2(G) = \langle \beta_{0,r}, \beta_{1,1} \rangle$. If $\ker \pi_2|_G = \langle \sigma \rangle$, then

$$G = \langle \sigma, \epsilon^a \tau^b \beta_{0,r}, \epsilon^c \tau^d \beta_{1,1} \rangle.$$

According to the (K) condition we have $c = \frac{1}{r-1}$ and $a = 0$ (so, $b \neq 0$ by regularity) and by (R) we have $d = 0$. Thus G is conjugate to K_2 by $\alpha_{0,b-1}$.

If $\ker \pi_2|_G$ is generated by τ then

$$G = \langle \tau, \epsilon^a \sigma^b \beta_{0,r}, \epsilon^c \sigma^d \beta_{1,1} \rangle.$$

From the (R) conditions we have that $d = 0$ and $c = \frac{1}{r-1}$. So, $b \neq 0$ otherwise $\pi_1(G) \subseteq \langle \tau, \epsilon \rangle$. If $b+1 = 0$, then ϵ commutes with $\sigma^b \beta_{0,r}$ and since $(\epsilon^a \sigma^b \beta_{0,r})^p \in \langle \tau \rangle$ we have $a = 0$, and so $G = G_{-1}$. Otherwise G is conjugate to G_b by $\beta_{1,1}^n$ where $n = a \frac{1-r}{r^{b+1}-1}$.

Let $\pi_2(G) = \langle \alpha_{1,1} \beta_{0,r}, \beta_{1,1} \rangle$. The condition (K) implies that $\ker \pi_2|_G = \langle \tau \rangle$. Then

$$G = \langle \tau, \epsilon^a \sigma^b \alpha_{1,1} \beta_{0,r}, \epsilon^c \sigma^d \beta_{1,1} \rangle.$$

The (R) conditions imply that $d = 0$, $c = \frac{1}{r-1}$. We also have $b \neq 0$ as above. In the same fashion, if $b+1 = 0$, then $a = 0$ and thus $G = H_{-1}$. Otherwise, G is conjugate to H_b by $\beta_{1,1}^n$ where $n = a \frac{1-r}{r^{b+1}-1}$.

Similar to Lemma 5.8, note that for the case $p = 2$, the groups K_1 and H_1 (which is the only H_c) have elements of order 4 because $(\sigma \alpha_{1,1})^2 = (\sigma \alpha_{1,1} \beta_{0,r})^2 = \tau$. So, the isomorphism classes of these groups are different to the case $p > 2$ in the table of the statement. \square

As in the previous subsection, for the following results we have different consequences when considering $p = 2$ and $p > 2$. Recall that if $p = 2$ then we can assume $r = -1$.

Proposition 5.10. *Let $p = 2$. If G is a regular subgroup of $\text{Hol}(A)$, then $|\pi_2(G)| \notin \{4, 4q\}$.*

Proof. Let us assume that G is a subgroup of size $p^2 q$ of $\text{Hol}(A)$ such that $|\pi_2(G)| = 4$. By Table 13 and the fact that A has a unique q -Sylow subgroup, we have that G has the following standard presentation:

$$G = \langle \epsilon, \sigma^a \tau^b \alpha_{1,1}, \sigma^c \tau^d \beta_{0,-1} \rangle$$

for some $0 \leq a, b, c, d \leq 1$. By condition (R), since $(\sigma^a \tau^b \alpha_{1,1})^2 \in \langle \epsilon \rangle$ we have $a = 0$. We also have that the second and the third generators must commute modulo the kernel, so $c = 0$.

Now, assume that $|\pi_2(G)| = 4q$, so G has the following standard presentation:

$$G = \langle \epsilon^a u \alpha_{1,1}, \epsilon^b v \beta_{0,-1}, \epsilon^c w \beta_{1,1} \rangle$$

where $u, v, w \in \langle \sigma, \tau \rangle$. By condition (K), the first generator has order 2 and we have $u \in \langle \tau \rangle$. Since $\alpha_{1,1} \beta_{0,-1} = \beta_{0,-1} \alpha_{1,1}$, condition (R) leads to $v \in \langle \tau \rangle$. For the same reason, $\alpha_{1,1} \beta_{1,1} = \beta_{1,1} \alpha_{1,1}$ implies $w \in \langle \tau \rangle$.

Then, in both cases $\pi_1(G) \subseteq \langle \tau, \epsilon \rangle$ by Lemma 2.4(1) and so G is not regular. \square

The following two lemmas hold just for odd primes. In the proofs we are using that if $p > 2$ then

$$(42) \quad (\sigma^a \alpha_{1,1})^n = \tau^{a \frac{n(n-1)}{2}} \sigma^{na} \alpha_{1,1}^n$$

for every $n \in \mathbb{N}$.

Lemma 5.11. *Let $p > 2$. A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2$ is*

$$G_a = \langle \epsilon, \sigma^a \alpha_{1,1}, \tau \beta_{0,r} \rangle \cong A$$

for $1 \leq a \leq p-1$.

Proof. If G_a is conjugate to G_b by $h \in \text{Aut}(A)$ then $h \in N(\pi_2(G))$ and so $h = \alpha_{i,j} \beta_{0,j}$. Then

$$\begin{aligned} h \tau \beta_{0,r} h^{-1} &= \tau^i \beta_{0,r} \in G_b \\ h \sigma^a \alpha_{1,1} h^{-1} &= \tau^{la} \sigma^a \alpha_{1,1}^i \in G_b. \end{aligned}$$

Then $i = 1$ and $l = 0$ and so we have that $a = b$.

Let G be a regular subgroup with $|\pi_2(G)| = p^2$. Then up to conjugation $\pi_2(G) = \langle \alpha_{1,1}, \beta_{0,r} \rangle$ and the kernel of $\pi_2|_G$ is the subgroup generated by ϵ . Hence, we can assume that

$$G = \langle \epsilon, \sigma^a \tau^b \alpha_{1,1}, \sigma^c \tau^d \beta_{0,r} \rangle.$$

From the (R) condition, the second and the third generators must commute modulo the kernel, so we have $c = 0$ and then $d \neq 0$ by regularity. If $a = 0$ then according to Lemma 2.4(1) we have $\pi_1(G) \subseteq \langle \tau, \epsilon \rangle$, a contradiction. Therefore $a \neq 0$ and G is conjugate to G_a by the automorphism $\alpha_{0,d-1} \alpha_{1,1}^{-\frac{b}{a}}$. \square

Lemma 5.12. *Let $p > 2$. A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2 q$ is*

$$G_a = \langle \sigma \alpha_{1,1}, \tau^a \beta_{0,r}, \epsilon^{\frac{1}{r-1}} \beta_{1,1} \rangle \cong A$$

for $1 \leq a \leq p-1$.

Proof. If G_a and G_b are conjugate by h , then h normalizes $\langle \sigma \alpha_{1,1} \rangle$, the center of G_a and G_b , and so $h = \beta_{0,j}$ for some j . Then $h \tau^a \beta_{0,r} h^{-1} = \tau^a \beta_{0,r} \in G_b$. Therefore $a = b$.

Let G be a regular subgroup of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2 q$. The unique subgroup of order $p^2 q$ up to conjugation is $\langle \alpha_{1,1}, \beta_{0,g}, \beta_{1,1} \rangle \cong A$. Hence a standard presentation of G is the following:

$$G = \langle \epsilon^a u \alpha_{1,1}, \epsilon^b v \beta_{0,r}, \epsilon^c w \beta_{1,1} \rangle$$

for some $u, v, w \in \langle \tau, \sigma \rangle$. Let $u = \sigma^x \tau^y$. Since $\alpha_{1,1}$ is central in $\pi_2(G)$, by condition (R), we have that $\epsilon^a u \alpha_{1,1} \in Z(G)$ and so $v, w \in \langle \tau \rangle$ and

$$a = b \frac{r^x - 1}{r - 1} \quad \text{and} \quad c(r^x - 1) = \frac{r^x - 1}{r - 1}.$$

Now, lifting the relation $\beta_{0,r} \beta_{1,1} = \beta_{1,1}^r \beta_{0,r}$ we have $w = 1$. If $x = 0$, by Lemma 2.4(1) we have $\pi_1(G) \subseteq \langle \epsilon, \tau \rangle$, a contradiction. So, we assume $x \neq 0$ and then $c = \frac{1}{r-1}$. Thus,

$$G = \langle \epsilon^{b \frac{r^x - 1}{r - 1}} \sigma^x \tau^y \alpha_{1,1}, \epsilon^b \tau^t \beta_{0,r}, \epsilon^{\frac{1}{r-1}} \beta_{1,1} \rangle$$

for some $0 \leq t \leq p-1$. Hence, using (42) we have that G is conjugate to G_{xt} by $h = \alpha_{n,x} \beta_{-b,1}$ where $n = \frac{x-1}{2} - y$. \square

The following remark is analogous to Remark 3.15.

Remark 5.13. The skew braces of A -type which decompose as direct products are the following:

- (i) the skew brace in Proposition 5.7.
- (ii) the skew brace associated to the group G_c for $1 \leq c \leq p-1$ as defined in Lemma 5.8 is the direct product of the trivial skew brace of size p and a skew brace of size pq with $|\ker \lambda| = q$, see [2, Theorem 3.9].
- (iii) The skew brace associated to G_a for $1 \leq a \leq p-1$ as defined in Lemma 5.9 is the direct product of the trivial skew brace of size p and a skew brace of size pq with $|\ker \lambda| = 1$, see [2, Theorem 3.12].

According to the enumeration of skew braces of size pq collected in [2, Theorems 3.6, 3.9 and 3.12], a complete list of skew braces of non-abelian type of size p^2q that decompose as a direct product is given by the skew braces above together with those in Remark 3.15.

We summarize the content of this subsection in the following tables.

$ \ker \lambda $	$\mathbb{Z}_p^2 \times \mathbb{Z}_q$	$\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_g \mathbb{Z}_p)$	$ \ker \lambda $	\mathbb{Z}_{4q}	$\mathbb{Z}_2^2 \times \mathbb{Z}_q$	$\mathbb{Z}_2 \times (\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_2)$	$\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_4$
1	-	$p-1$	2	1	-	2	1
p	1	$2p-1$	$2q$	1	1	1	1
q	-	$p-1$	4	-	1	-	-
pq	2	$2(p-1)$	$4q$	-	-	1	-
p^2	1	-					
p^2q	-	1					

TABLE 14. Enumeration of skew braces of A -type for $q = 1 \pmod{p}$, $q \neq 1 \pmod{p^2}$: the left table assumes $p > 2$ and the right table assumes $p = 2$.

6. SKEW BRACES OF SIZE p^2q WITH $q = 1 \pmod{p^2}$

In this section we assume that $q = 1 \pmod{p^2}$ and we will denote by h a fixed element of order p^2 in \mathbb{Z}_q^\times . Accordingly, we have the following non-abelian groups of size p^2q :

- (i) $\mathbb{Z}_q \rtimes_{h^{p^2}} \mathbb{Z}_{p^2} = \langle \sigma, \tau \mid \tau^q = \sigma^{p^2} = 1, \sigma\tau\sigma^{-1} = \tau^{h^{p^2}} \rangle$,
- (ii) $\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_p) = \langle \sigma, \tau, \epsilon \mid \sigma^p = \tau^p = \epsilon^q = 1, [\epsilon, \tau] = [\tau, \sigma] = 1, \sigma\epsilon\sigma^{-1} = \epsilon^{h^p} \rangle$,
- (iii) $\mathbb{Z}_q \rtimes_h \mathbb{Z}_{p^2} = \langle \sigma, \tau \mid \tau^q = \sigma^{p^2} = 1, \sigma\tau\sigma^{-1} = \tau^h \rangle$.

In Tables 15 and 16 we have the enumeration of skew braces according to their additive and multiplicative groups.

$+\backslash\circ$	\mathbb{Z}_{p^2q}	$\mathbb{Z}_q \rtimes_{h^{p^2}} \mathbb{Z}_{p^2}$	$\mathbb{Z}_q \rtimes_h \mathbb{Z}_{p^2}$	$\mathbb{Z}_p^2 \times \mathbb{Z}_q$	$\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_p)$
$\mathbb{Z}_q \rtimes_{h^{p^2}} \mathbb{Z}_{p^2}$	$2p$	$2p(p-1)$	$2p(p-1)$	-	-
$\mathbb{Z}_q \rtimes_h \mathbb{Z}_{p^2}$	2	$2(p-1)$	$2p(p-1)$	-	-
$\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_p)$	-	-	-	4	$6p-4$

TABLE 15. Enumeration of skew braces of size p^2q with $q = 1 \pmod{p^2}$ for $p > 2$.

$+\backslash\circ$	\mathbb{Z}_{4q}	$\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_4$	$\mathbb{Z}_q \rtimes_h \mathbb{Z}_4$	$\mathbb{Z}_2^2 \times \mathbb{Z}_q$	$\mathbb{Z}_2 \times (\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_2)$
$\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_4$	2	2	2	2	4
$\mathbb{Z}_q \rtimes_h \mathbb{Z}_4$	2	2	4	-	-
$\mathbb{Z}_2 \times (\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_2)$	2	2	2	2	4

TABLE 16. Enumeration of skew braces of order $4q$ with $q = 1 \pmod{4}$.

As the strategy to check regularity has been widely discussed in the previous sections, we are not referring to it from now on.

6.1. Skew braces of $\mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_{p^2}$ -type. In this section we denote by A the group $\mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_{p^2}$. The automorphism group of A can be described as in Subsection 5.1 and we employ the same notation. In particular, the subgroups of order p, q and pq of $\text{Aut}(A)$ coincide with the subgroups in Table 11. Therefore, if G is a regular subgroup with $|\pi_2(G)| \in \{q, p, pq\}$ we can apply Lemma 5.1, 5.2 and 5.3, respectively.

Up to conjugation, the elements of order dividing p^2 are contained in the subgroup of $\text{Aut}(A)$ given by $\langle \varphi_{1,0}^1, \varphi_{h,0}^0 \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p^2$. So, according to [26, Theorem 3.3], it has $p+1$ subgroups of order p^2 , namely

$$(43) \quad \langle \varphi_{h,0}^k \rangle \cong \mathbb{Z}_{p^2} \quad \text{and} \quad \langle \varphi_{1,0}^1, \varphi_{h^p,0}^0 \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$$

for $0 \leq k \leq p-1$.

On the other hand, since $\langle \varphi_{1,1}^0 \rangle$ is the unique q -Sylow subgroup of $\text{Aut}(A)$, we have that the subgroups of size p^2q in $\text{Aut}(A)$ up to conjugation are the following $p+1$ subgroups:

$$(44) \quad \langle \varphi_{h,0}^k, \varphi_{1,1}^0 \rangle \quad \text{and} \quad \langle \varphi_{1,0}^1, \varphi_{h^p,0}^0, \varphi_{1,1}^0 \rangle$$

for $0 \leq k \leq p-1$.

The following lemmas enumerate the regular subgroups of $\text{Hol}(A)$ up to conjugation with $|\pi_2(G)| \in \{p^2, p^2q\}$. In order to do this, we separate $p=2$ and $p>2$.

Lemma 6.1. *Let $p > 2$. A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2$ is*

$$G_{b,k} = \langle \tau, \sigma^b \varphi_{h,0}^k \rangle \cong \mathbb{Z}_q \rtimes_h \mathbb{Z}_{p^2}$$

for $0 \leq k \leq p-1$ and $1 \leq b \leq p-1$.

Proof. Arguing as in Lemma 5.2 we can show that the groups in the statement are not conjugate. Let G be a regular subgroup of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2$. The unique subgroup of order q of A is $\langle \tau \rangle$. According to (43) we have two cases to consider.

(i) If $\pi_2(G) = \langle \varphi_{h,0}^k \rangle$, then a standard presentation is

$$G = \langle \tau, \tau^a \sigma^b \varphi_{h,0}^k \rangle = \langle \tau, \sigma^b \varphi_{h,0}^k \rangle$$

where $b \neq 0 \pmod{p}$. Up to conjugation by the normalizer of $\langle \varphi_{h,0}^k \rangle$ in $\text{Aut}(A)$, we can assume $1 \leq b \leq p-1$, so $G = G_{b,k}$.

(ii) If $\pi_2(G) = \langle \varphi_{1,0}^1, \varphi_{h^p,0}^0 \rangle$, then we can argue as in Proposition 5.4, and prove that there are no such regular subgroups. \square

Lemma 6.2. *Let $p > 2$. A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2q$ is*

$$G_{b,k} = \langle \sigma^b \varphi_{h,0}^k, \tau^{\frac{1}{p-1}} \varphi_{1,1}^0 \rangle \cong \mathbb{Z}_q \rtimes_h \mathbb{Z}_{p^2}$$

for $1 \leq b \leq p-1$ and $0 \leq k \leq p-1$.

Proof. The groups in the statement are not conjugate by the same argument of Lemma 5.2. Let G be a regular subgroup of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2q$. According to (44) we have two cases to consider. In the first one, G has the standard presentation:

$$G = \langle \tau^a \sigma^b \varphi_{h,0}^k, \tau^c \sigma^d \varphi_{1,1}^0 \rangle.$$

From the (R) condition $(\tau^a \sigma^b \varphi_{h,0}^k)^q = 1$, we have $d = 0$ and so $b \neq 0 \pmod{p}$ (otherwise by Lemma 2.4(1) it follows that $\pi_1(G) \subseteq \langle \sigma^p, \tau \rangle$). Moreover, we can also assume that $1 \leq b \leq p-1$, up to conjugation by a power of $\varphi_{1,0}^1$. From the (R) condition, lifting the relation $\varphi_{h,0}^k \varphi_{1,1}^0 = (\varphi_{1,1}^0)^h \varphi_{h,0}^k$ we have that $c = \frac{1}{h^p-1}$.

Now, if $a = 0$, we have one of the representatives in the statement. If not, we conjugate by $\varphi_{n,1}^0$ where $n = -\frac{h^{pb+1}-1}{a(h^p-1)}$.

In the second case we have no regular subgroups by the same argument of Proposition 5.4. \square

Now, we deal with the case $p=2$. Recall that since h is an element of order 4 in \mathbb{Z}_q^\times , we can assume that $h^2 = -1$.

Lemma 6.3. *Let $p = 2$. A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(A)$ with $|\pi_2(G)| = 4$ is given by*

$$G_1 = \langle \tau, \sigma\varphi_{1,0}^1, \sigma^2\varphi_{-1,0}^0 \rangle \cong \mathbb{Z}_2 \times (\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_2), \quad G_2 = \langle \tau, \sigma\varphi_{h,0}^0 \rangle \cong \mathbb{Z}_q \rtimes_h \mathbb{Z}_4.$$

Proof. The groups G_1 and G_2 are not conjugate since their images under π_2 are not. Let G be a regular subgroup of $\text{Hol}(A)$. If $\pi_2(G) = \langle \varphi_{1,0}^1, \varphi_{-1,0}^0 \rangle$ we can argue as in Lemma 5.5 to get G_1 . If $\pi_2(G) = \langle \varphi_{h,0}^k \rangle$ for $k = 0, 1$, arguing as in Lemma 6.1, we have

$$G = \langle \tau, \sigma\varphi_{h,0}^k \rangle.$$

If $k = 1$ then $(\sigma\varphi_{h,0}^1)^2 = (\varphi_{h,0}^1)^2 = \varphi_{-1,1}^0 \in G$ and so G is not regular. Hence $k = 0$ and so $G = G_2$. \square

Lemma 6.4. *Let $p = 2$. There exists two regular subgroups G of $\text{Hol}(A)$ with $|\pi_2(G)| = 4q$ up to conjugation. A set of representatives is given by*

$$G_1 = \langle \sigma\varphi_{1,0}^1, \sigma^2\varphi_{-1,0}^0, \tau^{-\frac{1}{2}}\varphi_{1,1}^0 \rangle \cong \mathbb{Z}_2 \times (\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_2), \quad G_2 = \langle \sigma\varphi_{h,0}^0, \tau^{-\frac{1}{2}}\varphi_{1,1}^0 \rangle \cong \mathbb{Z}_q \rtimes_h \mathbb{Z}_4.$$

Proof. The groups G_1 and G_2 are not conjugate since their images under π_2 are not. Let G be a regular subgroup of $\text{Hol}(A)$. If $\pi_2(G) = \langle \varphi_{1,0}^1, \varphi_{-1,0}^0, \varphi_{1,1}^0 \rangle$, arguing as in Lemma 5.6 we get G_1 . If $\pi_2(G) = \langle \varphi_{h,0}^k, \varphi_{1,1}^0 \rangle$, by the same argument as in Lemma 6.2, we have that G has the following standard presentation:

$$G = \langle \sigma\varphi_{h,0}^k, \tau^{-\frac{1}{2}}\varphi_{1,1}^0 \rangle.$$

If $k = 1$ then $(\sigma\varphi_{h,0}^1)^2 = (\varphi_{h,0}^1)^2 = \varphi_{-1,0}^0 \in G$ and so G is not regular. So $k = 0$ and we get G_2 . \square

The results of this subsection are summarized in the following tables:

$ \ker \lambda $	\mathbb{Z}_{p^2q}	$\mathbb{Z}_q \rtimes_{hp} \mathbb{Z}_{p^2}$	$\mathbb{Z}_q \rtimes_h \mathbb{Z}_{p^2}$	$ \ker \lambda $	\mathbb{Z}_{4q}	$\mathbb{Z}_2^2 \times \mathbb{Z}_q$	$\mathbb{Z}_2 \times (\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_2)$	$\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_4$	$\mathbb{Z}_q \rtimes_h \mathbb{Z}_4$
1	-	-	$p(p-1)$	1	-	-	1	-	1
p	$p-1$	$p(p-1)$	-	2	-	1	1	1	-
q	-	-	$p(p-1)$	q	-	-	1	-	1
p^2	1	-	-	4	1	-	-	-	-
pq	p	$p^2 - p - 1$	-	$2q$	1	1	1	-	-
p^2q	-	1	-	$4q$	-	-	-	1	-

TABLE 17. Enumeration of skew braces of $\mathbb{Z}_q \rtimes_{hp} \mathbb{Z}_{p^2}$ -type for $q = 1 \pmod{p^2}$: the first table assumes $p > 2$ and the second table assumes $p = 2$.

6.2. Skew braces of $\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_{hp} \mathbb{Z}_p)$ -type. In this section we denote by A the group $\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_{hp} \mathbb{Z}_p)$. The regular subgroups of $\text{Hol}(A)$ with size of the image under π_2 equal to q, p, pq are the same as the one described in Proposition 5.7, Lemma 5.8 and Lemma 5.9, respectively, since the subgroups of the automorphism group of A of order p, q and pq coincide with the ones in the case $q = 1 \pmod{p}$ and $q \neq 1 \pmod{p^2}$. Following the notation in Subsection 5.2, the subgroups of order p^2 in $\text{Aut}(A)$ are

$$(45) \quad \langle \alpha_{1,1}, \beta_{0,h^p} \rangle, \quad \langle \alpha_{1,1}^k \beta_{0,h} \rangle$$

for $0 \leq k \leq p-1$, up to conjugation. As we did before, we consider the cases $p = 2$ and $p > 2$ separately. First, we start with the case $p > 2$.

Lemma 6.5. *Let $p > 2$. A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2$ is*

$$G_a = \langle \epsilon, \sigma^a \alpha_{1,1}, \tau \beta_{0,h^p} \rangle \cong A$$

for $1 \leq a \leq p-1$.

Proof. If $\pi_2(G) = \langle \alpha_{1,1}, \beta_{0,h^p} \rangle$ we can argue as in Lemma 5.11 to get G_a as in the statement. We show that if $\pi_2(G) = \langle \alpha_{1,1}^k \beta_{0,h} \rangle$ for $0 \leq k \leq p-1$ then G is not regular. In such case we have

$$G = \langle \epsilon, \sigma^a \tau^b \alpha_{1,1}^k \beta_{0,h} \rangle.$$

If $k = 0$ then $(\sigma^a \tau^b \beta_{0,h})^p = \beta_{0,h}^p \in G$. Otherwise, since $p > 2$, we have

$$(\sigma^a \tau^b \alpha_{1,1}^k \beta_{0,h})^p = \tau^{bp+k \frac{p(p-1)a}{2}} \sigma^{ap} \alpha_{1,1}^{pk} \beta_{0,h}^p = \beta_{0,h}^p \in G.$$

In both cases, G is not regular according to Lemma 2.3. \square

Now, we deal with the case $p = 2$. Note that in this case we have $h^2 = -1$.

Lemma 6.6. *Let $p = 2$. There exists a unique conjugacy class of regular subgroups G of $\text{Hol}(A)$ with $|\pi_2(G)| = 4$. A representative is given by*

$$H = \langle \epsilon, \sigma \alpha_{1,1} \beta_{0,h} \rangle \cong \mathbb{Z}_q \rtimes_h \mathbb{Z}_4.$$

Proof. If $\pi_2(G)$ is generated by $\alpha_{1,1}$ and $\beta_{0,-1}$, then G is not regular by the same computations we performed in Proposition 5.10. Assume that $\pi_2(G) = \langle \alpha_{1,1}^k \beta_{0,h} \rangle$ for $k = 0, 1$, then G has the following standard presentation

$$G = \langle \epsilon, \sigma^a \tau^b \alpha_{1,1}^k \beta_{0,h} \rangle$$

for some $0 \leq a, b \leq 1$. If $k = 0$, then $(\sigma^a \tau^b \beta_{0,h})^2 = \beta_{0,h}^2 = \beta_{0,-1} \in G$ and so G is not regular. Then $k = 1$ and $a = 1$, otherwise $\pi_1(G) \subseteq \langle \tau, \epsilon \rangle$. Finally, we can assume that $b = 0$, up to conjugation by $\alpha_{1,1}$. \square

The subgroups of order $p^2 q$ in $\text{Aut}(A)$ are

$$\langle \alpha_{1,1}, \beta_{0,h^p}, \beta_{1,1} \rangle, \quad \langle \beta_{1,1}, \alpha_{1,1}^k \beta_{0,h} \rangle$$

for $0 \leq k \leq p-1$, up to conjugation. We also consider the cases $p > 2$ and $p = 2$ separately. We start with the case $p > 2$.

Lemma 6.7. *Let $p > 2$. A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2 q$ is*

$$G_a = \langle \sigma \alpha_{1,1}, \tau^a \beta_{0,h^p}, \epsilon^{\frac{1}{h^{p-1}}} \beta_{1,1} \rangle \cong A$$

for $1 \leq a \leq p-1$.

Proof. If the p -Sylow subgroup of the image of $\pi_2(G)$ is not cyclic we can conclude as in Lemma 5.12. We show that if $\pi_2(G) = \langle \beta_{1,1}, \alpha_{1,1}^k \beta_{0,h} \rangle$, then G is not regular. Indeed if

$$G = \langle \epsilon^a u \beta_{1,1}, \epsilon^b v \alpha_{1,1}^k \beta_{0,h} \rangle$$

for some $u, v \in \langle \sigma, \tau \rangle$ then, from $(\epsilon^a u \beta_{1,1})^q = 1$ by condition (R), we have $u = 1$, i.e. $G = \langle \epsilon^a \beta_{1,1}, \epsilon^b v \alpha_{1,1}^k \beta_{0,h} \rangle$ where $v = \sigma^c \tau^d$. Moreover, we can assume $a \neq 0$ by regularity. Since $p > 2$, we have

$$(\epsilon^b \sigma^c \tau^d \alpha_{1,1}^k \beta_{0,h})^p = \epsilon^s \tau^{dp+ck \frac{p(p-1)}{2}} \sigma^{cp} \alpha_{1,1}^{kp} \beta_{0,h}^p = \epsilon^s \beta_{0,h}^p \in G$$

for some s . So,

$$(\epsilon^a \beta_{1,1})^{-\frac{s}{a}} (\epsilon^s \beta_{0,h}^p) = \beta_{1,1}^{-\frac{s}{a}} \beta_{0,h}^p \in G$$

and so G is not regular. \square

Lemma 6.8. *Let $p = 2$. There exists a unique conjugacy class of regular subgroups G of $\text{Hol}(A)$ with $|\pi_2(G)| = 4q$. A representative is given by*

$$H = \langle \sigma \alpha_{1,1} \beta_{0,h}, \epsilon^{-\frac{1}{2}} \beta_{1,1} \rangle \cong \mathbb{Z}_q \rtimes_h \mathbb{Z}_4.$$

Proof. If $\pi_2(G) = \langle \alpha_{1,1}, \beta_{0,-1}, \beta_{1,1} \rangle$, then G is not regular using the same computations we performed in Proposition 5.10. Assume that $\pi_2(G) = \langle \alpha_{1,1}^k \beta_{0,h}, \beta_{1,1} \rangle$ for $k = 0, 1$, so G has the following standard presentation

$$\langle \tau^a \epsilon^b \sigma^c \alpha_{1,1}^k \beta_{0,h}, \tau^d \epsilon^e \sigma^f \beta_{1,1} \rangle$$

for $0 \leq a, c, d, f \leq 1$ and $0 \leq b, e \leq q-1$. By condition (R), since $(\tau^d \epsilon^e \sigma^f \beta_{1,1})^q = 1$ we have $f = d = 0$. By regularity, $e \neq 0 \pmod{q}$. If $c = 0$, then $\pi_1(G) \subseteq \langle \tau, \epsilon \rangle$. Thus, $c = 1$. By condition (R) and the fact that $\beta_{0,h} \beta_{1,1} = \beta_{1,1}^h \beta_{0,h}$, we have $e = -\frac{1}{2}$. Up to conjugation by $\alpha_{1,1}$ we can assume $a = 0$.

If $k = 0$ then $(\epsilon^{-\frac{1}{2}} \beta_{1,1})^{-2b(h-1)} (\epsilon^b \sigma \beta_{0,h})^2 = \beta_{1,1}^{-2b(h-1)} \beta_{0,h}^2 \in G$ and then G is not regular. Then $k = 1$ and G is conjugate to H by $\beta_{1,1}^n$ where $n = -\frac{2b}{h+1}$. \square

According to Lemma 6.5 and Lemma 6.7, the enumeration of the skew braces of $\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_p)$ -type with $q = 1 \pmod{p^2}$ and $p > 2$ is as in Table 14. Table 18 collects the enumeration for the case $p = 2$.

ker λ	\mathbb{Z}_{4q}	$\mathbb{Z}_2^2 \times \mathbb{Z}_q$	$\mathbb{Z}_2 \times (\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_2)$	$\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_4$	$\mathbb{Z}_q \rtimes_h \mathbb{Z}_4$
1	-	-	-	-	1
2	1	-	2	1	-
4	-	1	-	-	-
q	-	-	-	-	1
$2q$	1	1	1	1	-
$4q$	-	-	1	-	-

TABLE 18. Enumeration of skew braces of $\mathbb{Z}_2 \times (\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_2)$ -type with $q = 1 \pmod{4}$.

6.3. Skew braces of $\mathbb{Z}_q \rtimes_h \mathbb{Z}_{p^2}$ -type. In this section, we denote by A the group $\mathbb{Z}_q \rtimes_h \mathbb{Z}_{p^2}$. A presentation of such group is

$$G = \langle \sigma, \tau \mid \sigma^{p^2} = \tau^q = 1, \sigma\tau\sigma^{-1} = \tau^h \rangle.$$

According to [9, Theorem 3.4], the map

$$\phi : \mathbb{Z}_q \rtimes \mathbb{Z}_q^\times \longrightarrow \text{Aut}(A), \quad (i, j) \mapsto \varphi_{i,j} = \begin{cases} \tau \mapsto \tau^j \\ \sigma \mapsto \tau^i \sigma \end{cases}$$

is a group isomorphism. Since $q = 1 \pmod{p^2}$, then p^2q divides $|\text{Aut}(G)| = q(q-1)$ and so we need to discuss all the possible values of $|\pi_2(G)|$.

A set of representatives of the conjugacy classes of subgroups of $\text{Aut}(A)$ is displayed in Table 19.

Size	Group	Class
p	$\langle \varphi_{0,h^p} \rangle$	\mathbb{Z}_p
q	$\langle \varphi_{1,1} \rangle$	\mathbb{Z}_q
pq	$\langle \varphi_{0,h^p}, \varphi_{1,1} \rangle$	$\mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_p$
p^2	$\langle \varphi_{0,h} \rangle$	\mathbb{Z}_{p^2}
p^2q	$\langle \varphi_{1,1}, \varphi_{0,h} \rangle$	A

TABLE 19. Conjugacy classes of groups of $\text{Aut}(A)$.

Lemma 6.9. *The unique skew brace of A -type with $|\ker \lambda| = p^2$ is $(B, +, \circ)$ where*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + h^{x_2} y_1 \\ x_2 + y_2 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \circ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h^{y_2} x_1 + h^{x_2} y_1 \\ x_2 + y_2 \end{pmatrix}$$

for every $0 \leq x_1, y_1 \leq q-1$ and $0 \leq x_2, y_2 \leq p^2-1$. In particular, $(B, \circ) \cong \mathbb{Z}_{p^2q}$.

Proof. Let G be a regular subgroup of $\text{Hol}(A)$ with $|\pi_2(G)| = q$. Then we can assume that $\pi_2(G) = \langle \varphi_{1,1} \rangle$ which is normal in $\text{Aut}(A)$. Since all p -Sylow subgroups of A are conjugate to each other and have order p^2 , we can assume that G has the standard presentation

$$G = \langle \sigma, \tau^a \sigma^b \varphi_{1,1} \rangle = \langle \sigma, \tau^a \varphi_{1,1} \rangle,$$

for some $a \neq 0$. The condition (K) is fulfilled if and only if $a = \frac{1}{h-1}$.

The structure of the skew brace associated to G can be obtained as in Lemma 5.1. □

The group $\mathfrak{G}_3 = \langle \tau, \varphi_{1,1} \rangle$ is normal in $\text{Hol}(A)$ and $\text{Hol}(A)/\mathfrak{G}_3 \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_q^\times$.

Lemma 6.10. *The skew braces of A -type with $|\ker \lambda| = pq$ are $(B_c, +, \circ)$ where $(B_c, +) = \mathbb{Z}_q \rtimes_h \mathbb{Z}_{p^2}$ and $(B_c, \circ) = \mathbb{Z}_q \rtimes_{h^{\frac{p}{c+1}}} \mathbb{Z}_{p^2} \cong A$ for $1 \leq c \leq p-1$. In particular, B_c is a bi-skew brace.*

Proof. The subgroup

$$G_c = \langle \tau, \sigma^p, \sigma^c \varphi_{0,h^p} \rangle \cong A$$

is regular. If G_c and G_d are conjugate, then their images in $\text{Hol}(A)/\mathfrak{G}_3$ coincide and so $\langle \sigma^p, \sigma^c \varphi_{0,h^p} \rangle = \langle \sigma^p, \sigma^d \varphi_{0,h^p} \rangle$. Then $c = d$.

The unique subgroup of A of order pq is $\langle \tau, \sigma^p \rangle$. Hence if G is a regular subgroup of $\text{Hol}(A)$, we can assume that $G = G_c$ for some $1 \leq c \leq p-1$.

Using the same argument as in Lemma 3.4 we can describe the structure of the skew brace B_c associated to G_c and according to [2, Corollary 1.2], the associated skew brace is a bi-skew brace. \square

Lemma 6.11. *A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(A)$ such that $|\pi_2(G)| = pq$ is*

$$G_c = \langle \sigma^p, \sigma^c \varphi_{0,h^p}, \tau^{\frac{1}{h-1}} \varphi_{1,1} \rangle \cong A$$

for $1 \leq c \leq p-1$.

Proof. Arguing as in Lemma 6.10 we can show that the groups G_c are not pairwise conjugate. Let G be a regular subgroup of $\text{Hol}(A)$ with $|\pi_2(G)| = pq$. According to Table 19, we can assume that $\pi_2(G) = \langle \varphi_{0,h^p}, \varphi_{1,1} \rangle$. The subgroups of order p of A are $\langle \tau^a \sigma^p \rangle$ for some a . Since $\varphi_{1,1}$ is in the normalizer of $\pi_2(G)$, up to conjugation by a power of $\varphi_{1,1}$ we can assume that $a = 0$. So, G has the standard presentation

$$G = \langle \sigma^p, \tau^b \sigma^c \varphi_{0,h^p}, \tau^d \sigma^e \varphi_{1,1} \rangle$$

for some $0 \leq c, e \leq p-1$ and $0 \leq b, d \leq q-1$. From the condition (R) and the fact that $\varphi_{0,h^p} \varphi_{1,1} = \varphi_{1,1}^{h^p} \varphi_{0,h^p}$ it follows that $e(1-h^p) = 0 \pmod{p^2}$, so $e = 0 \pmod{p}$, and $d(h^c - 1) = \frac{h^c - 1}{h-1}$. If $c = 0$, then $\pi_1(G) \subseteq \langle \tau, \sigma^p \rangle$ by Lemma 2.4(1), so we have $c \neq 0$ and then $d = \frac{1}{h-1}$. From $(\tau^b \sigma^c \varphi_{0,h^p})^p \in \langle \sigma^p \rangle$ we get $b = 0$. Thus $G = G_c$. \square

Lemma 6.12. *A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(A)$ such that $|\pi_2(G)| = p^2$ is*

$$G_b = \langle \tau, \sigma^b \varphi_{0,h} \rangle \cong \begin{cases} \mathbb{Z}_{p^2q}, & \text{if } b = -1 \pmod{p^2}, \\ \mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_{p^2}, & \text{if } b = -1 \pmod{p} \text{ and } b \neq -1 \pmod{p^2}, \\ \mathbb{Z}_q \rtimes_h \mathbb{Z}_{p^2}, & \text{otherwise,} \end{cases}$$

where $1 \leq b \leq p^2 - 1$ and $b \neq 0 \pmod{p}$.

Proof. By the same argument as in Lemma 6.10, the groups in the statement are not pairwise conjugate. Let G be a regular subgroup of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2$. Since A has a unique subgroup of order q and according to Table 19, we have that every regular subgroup G has the standard presentation

$$G_b = \langle \tau, \sigma^b \varphi_{0,h} \rangle$$

for $b \neq 0 \pmod{p}$ (otherwise, $\pi_1(G) \subseteq \langle \tau, \sigma^p \rangle$), i.e. $G = G_b$. \square

Lemma 6.13. *A set of representatives of conjugacy classes of regular subgroups G of $\text{Hol}(A)$ with $|\pi_2(G)| = p^2q$ is*

$$G_d = \langle \tau^{\frac{1}{h-1}} \varphi_{1,1}, \sigma^d \varphi_{0,h} \rangle \cong A$$

for $1 \leq d \leq p^2 - 1$ and $d \neq 0 \pmod{p}$.

Proof. Let G be a regular subgroup with $|\pi_2(G)| = p^2q$. According to Table 19, we can assume that a standard presentation is

$$G = \langle \tau^a \sigma^b \varphi_{1,1}, \tau^c \sigma^d \varphi_{0,h} \rangle.$$

According to the (R) conditions, from $(\tau^a \sigma^b \varphi_{1,1})^q = 1$ we have $b = 0$. Lifting the relation $\varphi_{0,h} \varphi_{1,1} = \varphi_{1,1}^h \varphi_{0,h}$, we have the equation $a(h^d - 1) = \frac{h^d - 1}{h-1}$. If $d = 0 \pmod{p}$, then $\pi_1(G) \subseteq \langle \tau, \sigma^p \rangle$. So $d \neq 0 \pmod{p}$ and $a = \frac{1}{h-1}$. Hence,

$$G = \langle \tau^{\frac{1}{h-1}} \varphi_{1,1}, \tau^c \sigma^d \varphi_{0,h} \rangle$$

for $d \neq 0 \pmod{p}$. If $d = -1$, then $c = 0$ since $\tau^c \sigma^d \varphi_{0,h}$ has order p^2 . Otherwise, G is conjugate to G_d by $\varphi_{-c \frac{h-1}{hd+1-1}, 1}$. \square

We summarize the content of this subsection in the following table:

$ \ker \lambda $	\mathbb{Z}_{p^2q}	$\mathbb{Z}_q \rtimes_{h^p} \mathbb{Z}_{p^2}$	$\mathbb{Z}_q \rtimes_h \mathbb{Z}_{p^2}$
1	-	-	$p(p-1)$
p	-	$p-1$	-
q	1	$p-1$	$p(p-2)$
p^2	1	-	-
pq	-	-	$p-1$
p^2q	-	-	1

TABLE 20. Enumeration of skew braces of $\mathbb{Z}_q \rtimes_h \mathbb{Z}_{p^2}$ -type for $q \equiv 1 \pmod{p^2}$.

7. A PROOF TO A CONJECTURE ON SKEW BRACES OF SIZE p^2q

In this section, we give a proof to Conjecture 4.1 of [6]. To do this, we need the tables we obtained in the present work and also in [1] for the skew braces of abelian type (also known as *classical* braces).

For an integer n , denote by $A(n)$ the number of non-isomorphic skew braces of abelian type of size n (i.e. classical braces) and by $B(n)$ the number of non-isomorphic skew braces of non-abelian type of size n . So the total number of non-isomorphic skew braces $s(n)$ is given by:

$$(46) \quad s(n) = A(n) + B(n).$$

From [12], it is known the value of $A(4q)$ for a prime $q \geq 5$ and also the value of $A(p^2q)$ for primes p, q such that $q > p + 1 > 3$. As a particular case, we reprove these results.

Theorem 7.1. [6, Conjecture 4.1] *Let p and q be prime integers. If $q \geq 5$, then*

$$s(4q) = \begin{cases} 29, & \text{if } q \equiv 3 \pmod{4} \\ 43, & \text{if } q \equiv 1 \pmod{4} \end{cases}$$

and if $q > p + 1 > 3$, then

$$s(p^2q) = \begin{cases} 4, & \text{if } q \not\equiv 1 \pmod{p} \\ 2p^2 + 7p + 8, & \text{if } q \equiv 1 \pmod{p} \text{ and } q \not\equiv 1 \pmod{p^2} \\ 6p^2 + 6p + 8, & \text{if } q \equiv 1 \pmod{p^2}. \end{cases}$$

Proof. For the first part of the statement, according to [1, Table 1] we have that

$$A(4q) = \begin{cases} 9, & \text{if } q \equiv 3 \pmod{4} \\ 11, & \text{if } q \equiv 1 \pmod{4} \end{cases}$$

and adding up all the entries on Tables 10 and 16, we have:

$$B(4q) = \begin{cases} 20, & \text{if } q \equiv 3 \pmod{4} \\ 32, & \text{if } q \equiv 1 \pmod{4}. \end{cases}$$

So, by (46) we have $s(4q)$ as desired.

For the second part, according to [7, p. 237], we have that both conditions $q > p + 1 > 3$ and $q \not\equiv 1 \pmod{p}$ can only be fulfilled if p and q are arithmetically independent (that is $p \not\equiv \pm 1 \pmod{q}$ and $q \not\equiv 1 \pmod{p}$). So, by [1, Table 1] we have:

$$A(p^2q) = \begin{cases} 4, & \text{if } q \not\equiv 1 \pmod{p} \\ p + 8, & \text{if } q \equiv 1 \pmod{p} \text{ and } q \not\equiv 1 \pmod{p^2} \\ 2p + 8, & \text{if } q \equiv 1 \pmod{p^2}. \end{cases}$$

For $B(n)$, we have that the case $q \equiv 1 \pmod{p}$ and $q \not\equiv 1 \pmod{p^2}$ comes from Table 9 and the condition $q \equiv 1 \pmod{p^2}$ comes from Table 15. Finally, since every group of order p^2q with p and q arithmetically

independent must be abelian, we only have braces of abelian type. Summarizing:

$$B(p^2q) = \begin{cases} 0, & \text{if } q \not\equiv 1 \pmod{p} \\ 2p^2 + 6p, & \text{if } q \equiv 1 \pmod{p} \text{ and } q \not\equiv 1 \pmod{p^2} \\ 6p^2 + 4p, & \text{if } q \equiv 1 \pmod{p^2} \end{cases}$$

and then the second part of the conjecture follows. Notice that the numbers in the last equation were obtained adding up all the entries in Table 9 and 15. \square

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