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Degrees of irreducible morphisms and finite-representation type

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Abstract

We study the degree of irreducible morphisms in any Auslander–Reiten component of a finite-dimensional algebra over an algebraically closed field. We give a characterization for an irreducible morphism to have finite left (or right) degree in terms of the existence of certain annihilator maps. This is used to prove our main theorem: an algebra is of finite-representation type if and only if for every indecomposable projective the inclusion of the radical in the projective has finite right degree, which is equivalent to requiring that for every indecomposable injective the epimorphism from the injective to its quotient by its socle has finite left degree. As an application of the techniques we develop, we study the behaviour of the composite of paths of irreducible morphisms between indecomposable modules.

Introduction

Let A be an artin algebra over an artin commutative ring k. The representation theory of A deals with the study of the category $\mathsf{mod}\,A$ of (right) finitely generated A-modules, and one of the main problems is to determine when the algebra is of finite-representation type. One of the most powerful tools in this study is the Auslander–Reiten theory, based on irreducible morphisms and almost split sequences (see [1]). Although irreducible morphisms have permitted important advances in representation theory, some of their basic properties still remain mysterious to us. An important example is the composition of two irreducible morphisms: it obviously lies in rad^2 (where rad^l is the lth power of the radical ideal rad of $\mathsf{mod}\,A$) but it may lie in rad^3, rad^∞ or even be the zero morphism. Of course, the situation still makes sense with the composite of arbitrary many irreducible morphisms. A first, but partial, treatment of this situation was given by Igusa and Todorov [10] with the following result. '*If* $X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{n-1} \xrightarrow{f_n} X_n$ *is a sectional path of irreducible morphisms between indecomposable modules, then the composite* $f_n \dots f_1$ *lies in* $\mathsf{rad}^n(X_0, X_n)$ *and not in* $\mathsf{rad}^{n+1}(X_0, X_n)$*, in particular, it is non-zero.*' In [11], Liu introduced the left and right degrees of an irreducible morphism $f : X \to Y$ as follows: the *left degree* $d_l(f)$ of f is the least integer $m \geqslant 1$ such that there exist $Z \in \mathsf{mod}\,A$ and $g \in \mathsf{rad}^m(Z, X)\backslash\mathsf{rad}^{m+1}(Z, X)$ satisfying $fg \in \mathsf{rad}^{m+2}(Z, Y)$. If no such integer m exists, then $d_l(f) = \infty$. The *right degree* is defined dually. This notion was introduced to study the composite of irreducible morphisms. In particular, Liu extended the above study of Igusa and Todorov to presectional paths. Later it was used to determine the possible shapes of the Auslander–Reiten components of A (see [11, 12]). More recently, the composite of irreducible morphisms was studied in [5–7, 9]. The work made in the first three of these papers is based on the notion of degree of irreducible morphisms. The definition of the degree raises the following problem: determine when $d_l(f) = \infty$ or $d_r(f) = \infty$. Consider an irreducible

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morphism $f : X \to Y$ with X indecomposable. Then, the following conditions have been related in the recent literature:

(1) $d_l(f) = n < \infty$;
(2) $\mathsf{Ker}(f)$ lies in the Auslander–Reiten component containing X.

Indeed, these two conditions were proved to be equivalent if the Auslander–Reiten component containing X is convex, generalized standard and with length (see [**8**], actually this equivalence still holds true if one removes the convex hypothesis) and when the algebra is standard (see [**4**]). In this paper, we shall see that such results are key steps to show that the degree of irreducible morphisms is a useful notion to determine the representation type of A. Indeed, we recall the following well-known conjecture appeared first in [**12**] and related to the Brauer–Thrall conjectures. '*If the Auslander–Reiten quiver of A is connected, then A is of finite-representation type.*' This conjecture is related to the degree of irreducible morphisms as follows: in the above situation of assertions (1) and (2), the existence of f such that $d_l(f) = \infty$ is related to the existence of at least two Auslander–Reiten components. Actually, it was proved in [**8**, Theorem 3.11] that if A is of finite-representation type, then every irreducible morphism between indecomposables either has finite right degree or has finite left degree. Conversely, one can wonder if the converse holds true. In this paper, we prove the following main theorem where we assume that k is an algebraically closed field.

THEOREM A. *Let A be a connected finite-dimensional k-algebra over an algebraically closed field. The following conditions are equivalent.*

(a) *The algebra A is of finite-representation type.*
(b) *For every indecomposable projective A-module P, the inclusion $\mathsf{rad}(P) \hookrightarrow P$ has finite right degree.*
(c) *For every indecomposable injective A-module I, the quotient $I \to I/\mathsf{soc}(I)$ has finite left degree.*
(d) *For every irreducible epimorphism $f : X \to Y$ with X or Y indecomposable, the left degree of f is finite.*
(e) *For every irreducible monomorphism $f : X \to Y$ with X or Y indecomposable, the right degree of f is finite.*

Hence, going back to the above conjecture, if one knows that the Auslander–Reiten quiver of A is connected, by (b) and (c), it suffices to study the degree of finitely many irreducible morphisms in order to prove that A is of finite-representation type. Our proof of the above theorem only uses considerations on degrees and their interaction with coverings of translation quivers. In particular, it uses no advanced characterization of finite-representation type (such as the Brauer–Thrall conjectures or multiplicative bases, for example). The theorem shows that the degrees of irreducible morphisms are somehow related to the representation type of A. Note also that our characterization is expressed in terms of the knowledge of the degree of finitely many irreducible morphisms. In order to prove the theorem we investigate the degree of irreducible morphisms and more particularly assertions (1) and (2) above. Assuming that k is an algebraically closed field and given $f : X \to Y$ an irreducible epimorphism with X indecomposable, we prove that the assertion (1) is equivalent to (3) below and implies (2), with no assumption on the Auslander–Reiten component Γ containing X:

(3) there exist $Z \in \Gamma$ and $h \in \mathsf{rad}^n(Z, X)\backslash\mathsf{rad}^{n+1}(Z, X)$ such that $fh = 0$.

Therefore, the existence of an irreducible monomorphism (or epimorphism) with infinite left (or right) degree indicates that there are more than one component in the Auslander–Reiten quiver (at least when Γ is generalized standard). We also prove that (2) implies (1) (and therefore implies (3)) under the additional assumption that Γ is generalized standard. The equivalence between (1) and (3) and the fact that it works for any Auslander–Reiten component are the core facts in the proof of the theorem. For this purpose we use the covering techniques introduced in [**13**]. Indeed, these techniques allow one to reduce the study of the degree of irreducible morphisms in a component to the study of the degree of irreducible morphisms in a suitable covering called the generic covering. Among other things, the generic covering is a translation quiver with length. As was proved in [**8**] such a condition is particularly useful in the study of the degree of an irreducible morphism.

The paper is therefore organized as follows. In Section 1 we recall some needed definitions. In Section 2 we extend to any Auslander–Reiten component the pioneer result [**13**, 2.2, 2.3] on covering techniques which, in its original form, only works for the Auslander–Reiten quiver of representation-finite algebras. The results of this section are used in Section 3 to prove the various implications between assertions (1), (2) and (3) in Theorem C and Proposition 3.4. As explained above, these results have been studied previously and they were proved under additional assumptions. In particular, the corresponding corollaries proved at that time can be generalized accordingly. In Section 4, we prove our main theorem (Theorem A) using the previous results. The proof of our main results is based on the covering techniques developed in the second section. In Section 5, we use these to study when the non-zero composite of n irreducible morphisms lies in the $(n+1)$th power of the radical and we extend the result of Igusa and Todorov [**10**] on the composite of a sectional paths to sums of composites of sectional paths.

1. *Preliminaries*

1.1. *Notation on modules*

Let A be a finite-dimensional k-algebra. We denote by $\mathsf{ind}\,A$ a full subcategory of $\mathsf{mod}\,A$ which contains exactly one representative of each isomorphism class of indecomposable modules. Also, we write rad for the *radical* of $\mathsf{mod}\,A$. Hence, given indecomposable modules X and Y, the space $\mathsf{rad}(X,Y)$ is the subspace of $\mathsf{Hom}_A(X,Y)$ consisting of non-isomorphisms $X \to Y$. For $l \geqslant 1$, we write rad^l for the lth power of the ideal rad, recursively defined by $\mathsf{rad}^1 = \mathsf{rad}$ and $\mathsf{rad}^{l+1} = \mathsf{rad}.\mathsf{rad}^l$ $(= \mathsf{rad}^l.\mathsf{rad})$. For short we shall say that some morphisms $u_1, \ldots, u_r : X \to Y$ are *linearly independent modulo* $\mathsf{rad}^n(X,Y)$ if their respective classes modulo $\mathsf{rad}^n(X,Y)$ are linearly independent in $\mathsf{Hom}_A(X,Y)/\mathsf{rad}^n(X,Y)$. We recall that the *Auslander–Reiten quiver of* A is the translation quiver $\Gamma(\mathsf{mod}\,A)$ with vertices the modules in $\mathsf{ind}\,A$, such that the number of arrows $X \to Y$ equals the dimension of the quotient space $\mathsf{rad}(X,Y)/\mathsf{rad}^2(X,Y)$ for every pair of vertices $X, Y \in \Gamma$ and whose translation is induced by the Auslander–Reiten translation $\tau_A = D\mathsf{Tr}$. Hence, the translation quivers we shall deal with are not valued quivers and may have multiple (parallel) arrows. If Γ is a component of $\Gamma(\mathsf{mod}\,A)$ (or an *Auslander–Reiten component*, for short), we write $\mathsf{ind}\,\Gamma$ for the full subcategory of $\mathsf{ind}\,A$ with objects the modules in Γ. Recall that a *hook* is a path $X \to Y \to Z$ of irreducible morphisms between indecomposable modules such that Z is non-projective and $X = \tau_A Z$. Also, a path $X_0 \to X_1 \to \cdots \to X_{l-1} \to X_l$ of irreducible morphisms is *sectional* if neither of its subpaths of length 2 is a hook.

We refer the reader to [**11**] for properties on the degree of irreducible morphisms.

1.2. *Radical in mesh-categories*

Let Γ be a *translation quiver*, that is, Γ is a quiver with no loops (but possibly with parallel arrows), endowed with two distinguished subsets of vertices, the elements of which are called projectives and injectives, respectively, and endowed with a bijection $\tau : x \mapsto \tau x$

(the *translation*) from the set of non-projectives to the set of non-injectives, such that, for every vertices x, y with x non-projective, there is a bijection $\alpha \mapsto \sigma\alpha$ from the set of arrows $y \to x$ to the set of arrows $\tau x \to y$. All translation quivers are assumed to be *locally finite*, that is, every vertex is the source or the target of at most finitely many arrows (Auslander–Reiten components are always locally finite quivers). The subquiver of Γ formed by the arrows starting from τx and the arrows arriving at x is called the *mesh* ending at x. We write $\mathsf{k}(\Gamma)$ for the *mesh-category* of Γ, that is, the factor category of the path category $\mathsf{k}\Gamma$ by the ideal generated by the morphisms $\sum_{\alpha:\cdot\to x} \alpha\ \sigma\alpha$, where α runs through the arrows arriving at x, for a given non-projective vertex x. If u is a path in Γ, we write $\overline{u}$ for the corresponding morphism in $\mathsf{k}(\Gamma)$. We denote by $\mathfrak{R}\mathsf{k}(\Gamma)$ the ideal of $\mathsf{k}(\Gamma)$ generated by $\{\overline{\alpha} \mid \alpha \text{ an arrow in } \Gamma\}$. Note that, in general, $\mathfrak{R}\mathsf{k}(\Gamma)$ is not a radical of the category $\mathsf{k}(\Gamma)$. The lth power $\mathfrak{R}^l\mathsf{k}(\Gamma)$ is defined recursively by $\mathfrak{R}^1\mathsf{k}(\Gamma) = \mathfrak{R}\mathsf{k}(\Gamma)$ and $\mathfrak{R}^{l+1}\mathsf{k}(\Gamma) = \mathfrak{R}\mathsf{k}(\Gamma).\mathfrak{R}^l\mathsf{k}(\Gamma)$ $(= \mathfrak{R}^l\mathsf{k}(\Gamma).\mathfrak{R}\mathsf{k}(\Gamma))$. If Γ is *with length*, that is, for every $x, y \in \Gamma$ all the paths from x to y have equal length, then the ideal $\mathfrak{R}\mathsf{k}(\Gamma)$ satisfies the following result proved in [**9**, Proposition 2.1]. This result is central in our work. Later we shall use it without further reference.

Proposition 1.1. *Let Γ be a translation quiver with length and $x, y \in \Gamma$. If there is a path of length l from x to y in Γ, then*:

(a) $\mathsf{k}(\Gamma)(x, y) = \mathfrak{R}\mathsf{k}(\Gamma)(x, y) = \mathfrak{R}^2\mathsf{k}(\Gamma)(x, y) = \ldots = \mathfrak{R}^l\mathsf{k}(\Gamma)(x, y)$;
(b) $\mathfrak{R}^i\mathsf{k}(\Gamma)(x, y) = 0$ *if* $i > l$.

In view of the preceding proposition, we call a *length function on* Γ a function l which assigns an integer $l(x) \in \mathbb{Z}$ to every vertex $x \in \Gamma$, in such a way that $l(y) = l(x) + 1$ whenever there is an arrow $x \to y$ in Γ (see [**2**, 1.6]). Clearly, if Γ has a length function, then Γ is with length. Finally, we define hooks and sectional paths in translation quivers as we did for hooks and sectional paths of irreducible morphisms in module categories.

1.3. *Coverings of translation quivers*

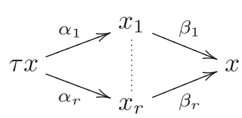
Let Γ be a connected translation quiver. A *covering of translation quivers* [**2**, 1.3] is a morphism $p : \Gamma' \to \Gamma$ of quivers such that:

(a) Γ' is a translation quiver;
(b) a vertex $x \in \Gamma'$ is projective (or injective, respectively) if and only if so is px;
(c) p commutes with the translations in Γ and Γ' (where these are defined);
(d) for every vertex $x \in \Gamma'$, the map $\alpha \mapsto p(\alpha)$ induces a bijection from the set of arrows in Γ' starting from x or ending at x to the set of arrows in Γ starting from $p(x)$ or ending at $p(x)$, respectively.

We shall use a particular covering $\pi : \widetilde{\Gamma} \to \Gamma$ which we call the *generic covering*. Following [**2**, 1.2], we define the equivalence relation $\sim$ on the set of unoriented paths in Γ as generated by the following properties.

(i) If $\alpha : x \to y$ is an arrow in Γ, then $\alpha\alpha^{-1} \sim e_y$ and $\alpha^{-1}\alpha \sim e_x$ (where e_x denotes the stationary path at x, of length 0).

(ii) If x is a non-projective vertex and the mesh in Γ ending at x has the form



then $\beta_i\alpha_i \sim \beta_j\alpha_j$ for every $i, j \in \{1, \ldots, r\}$.

(iii) If α, β are arrows in Γ with the same source and the same target, then $\alpha \sim \beta$.

(iv) If $\gamma_1, \gamma, \gamma', \gamma_2$ are unoriented paths such that $\gamma \sim \gamma'$ and the compositions $\gamma_1\gamma\gamma_2, \gamma_1\gamma'\gamma_2$ are defined, then $\gamma_1\gamma\gamma_2 \sim \gamma_1\gamma'\gamma_2$.

Note that the usual homotopy relation of a translation quiver (see [**2**, 1.2]) is defined using conditions (i), (ii) and (iv) above. Also recall that the universal cover of Γ was defined in [**2**, 1.3] using that homotopy relation. By applying that construction to our equivalence relation $\sim$ instead of to the homotopy relation, we get the covering $\pi : \widetilde{\Gamma} \to \Gamma$ which we call the *generic covering* of Γ. Note that if Γ has no multiple arrows (for example, if $\Gamma = \Gamma(\mathsf{mod}\, A)$ is the Auslander–Reiten quiver of a representation-finite algebra), then the generic covering coincides with the universal covering. The following properties of π are crucial to our work.

Proposition 1.2. *Let Γ be a translation quiver and $\pi : \widetilde{\Gamma} \to \Gamma$ be its generic covering.*

(a) *There is a length function on $\widetilde{\Gamma}$. In particular, $\widetilde{\Gamma}$ is with length.*

(b) *If $\alpha : x \to y$, $\beta : x \to z$ (or $\alpha : y \to x$, $\beta : z \to x$) are arrows in $\widetilde{\Gamma}$ such that $\pi y = \pi z$, then $y = z$.*

(b) *For every pair of vertices $x, y \in \widetilde{\Gamma}$, the covering π induces a bijection from the set of arrows in $\widetilde{\Gamma}$ from x to y to the set of arrows in Γ from πx to πy.*

(c) *Let $x, y \in \widetilde{\Gamma}$ be vertices. If $u : x = x_0 \to \cdots \to x_l = y$ and $v : x = x'_0 \to \cdots \to x'_l = y$ are two paths in $\widetilde{\Gamma}$ from x to y and if u is sectional, then $x_1 = x'_1, \ldots, x_{l-1} = x'_{l-1}$. In particular, all the paths from x to y are sectional.*

(d) *Let $x, y \in \widetilde{\Gamma}$ be vertices, $u_1, \ldots, u_r$ be pairwise distinct sectional paths of length $n \geqslant 0$ in $\widetilde{\Gamma}$ from x to y and $\lambda_1, \ldots, \lambda_r \in \mathsf{k}$ be scalars. Then the following equivalence holds in $\mathsf{k}(\widetilde{\Gamma})$:*

$$\lambda_1\overline{u_1} + \ldots + \lambda_r\overline{u_r} \in \mathfrak{R}^{n+1}\mathsf{k}(\widetilde{\Gamma}) \Leftrightarrow \lambda_1 = \ldots = \lambda_r = 0.$$

Proof. Let Γ' be the translation quiver with no multiple arrows, with the same vertices and the same translation as those of Γ, and such that there is an arrow (and exactly one) $x \to y$ in Γ' if and only if there is (at least) one arrow $x \to y$ in Γ. Define $\widetilde{\Gamma'}$ starting from $\widetilde{\Gamma}$ in a similar way. Then $\widetilde{\Gamma'}$ is the universal cover of Γ' and it is simply connected (in the sense of [**2**, 1.3, 1.6]).

(a) Applying [**2**, 1.6] to $\widetilde{\Gamma'}$ yields a length function on $\widetilde{\Gamma'}$ and, therefore, on $\widetilde{\Gamma}$. The existence of a length function implies that $\widetilde{\Gamma}$ is with length.

Property (b) follows from the construction of the generic covering.

Property (c) follows from (b) and from the fact that $\pi : \widetilde{\Gamma} \to \Gamma$ is a covering of quivers.

(d) The paths u and v define two (unique) paths $x_0 \to \cdots \to x_l$ and $x'_1 \to \cdots \to x'_l$ from x to y in $\widetilde{\Gamma'}$, the first of which is sectional. The conclusion then follows from [**3**, Lemma 1.2].

(e) Assume that $\lambda_1\overline{u_1} + \ldots + \lambda_r\overline{u_r} \in \mathfrak{R}^{n+1}\mathsf{k}(\widetilde{\Gamma})$. It follows from (a) and from Proposition 1.1 that $\lambda_1\overline{u_1} + \ldots + \lambda_r\overline{u_r} = 0$, that is, $\lambda_1 u_1 + \ldots + \lambda_r u_r$ lies in the mesh-ideal. By definition of the mesh-ideal, this implies that u_i contains a hook whenever $\lambda_i \neq 0$. Using (d), we deduce that $\lambda_1 = \ldots = \lambda_r = 0$. The converse is obvious. □

Property (b) in Proposition 1.2 is not satisfied by the universal cover when Γ has multiple arrows. This is the reason for using the generic covering instead.

2. *Well-behaved functors*

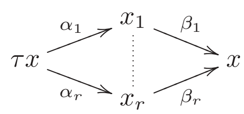
Let A be a finite-dimensional k-algebra, where k is an algebraically closed field, Γ be a component of $\Gamma(\mathsf{mod}\, A)$ and $\pi : \widetilde{\Gamma} \to \Gamma$ be the generic covering. Following [**2**, Example 3.1(b)] (see also [**13**]), a k-linear functor $F : \mathsf{k}(\widetilde{\Gamma}) \to \mathsf{ind}\, \Gamma$ is called *well-behaved* if it satisfies the following conditions for every vertex $x \in \widetilde{\Gamma}$:

(a) $Fx = \pi x$;
(b) if $\alpha_1 : x \to x_1, \dots, \alpha_r : x \to x_r$ are the arrows in $\widetilde{\Gamma}$ starting from x, then $[F(\overline{\alpha}_1), \dots, F(\overline{\alpha}_r)]^t : Fx \to Fx_1 \oplus \dots \oplus Fx_r$ is minimal left almost split in $\mathsf{mod}\, A$;
(c) if $\alpha_1 : x_1 \to x, \dots, \alpha_r : x_r \to x$ are the arrows in $\widetilde{\Gamma}$ ending at x, then $[F(\overline{\alpha}_1), \dots, F(\overline{\alpha}_r)] : Fx_1 \oplus \dots \oplus Fx_r \to Fx$ is minimal right almost split in $\mathsf{mod}\, A$.

Note that these conditions imply that F maps meshes in $\widetilde{\Gamma}$ to almost split sequences in $\mathsf{mod}\, A$.

For convenience, we extend this notion to functors $p : \mathsf{k}\mathcal{X} \to \mathsf{ind}\,\Gamma$ where $\mathcal{X}$ is a subquiver of $\widetilde{\Gamma}$. The functor p is called well-behaved if and only if:

(1) $px = \pi x$ for every vertex $x \in \mathcal{X}$;
(2) given a vertex $x \in \mathcal{X}$, if $x \xrightarrow{\alpha_1} x_1, \dots, x \xrightarrow{\alpha_r} x_r$ are the arrows in $\mathcal{X}$ starting from x, then the morphism $[p(\alpha_1), \dots, p(\alpha_r)]^t : \pi x \to \bigoplus_{i=1}^r \pi x_i$ is irreducible;
(3) given a vertex $x \in \mathcal{X}$, if $x_1 \xrightarrow{\beta_1} x, \dots, x_r \xrightarrow{\beta_r} x$ are the arrows in $\mathcal{X}$ ending at x, then the morphism $[p(\beta_1), \dots, p(\beta_r)] : \bigoplus_{i=1}^r \pi x_i \to \pi x$ is irreducible;
(4) if the vertex x is non-projective and if $\mathcal{X}$ contains the mesh in $\widetilde{\Gamma}$ ending at x



then the sequence $0 \to \tau_A \pi x \xrightarrow{[p(\alpha_1), \dots, p(\alpha_r)]^t} \bigoplus_{i=1}^r \pi x_i \xrightarrow{[p(\beta_1), \dots, p(\beta_r)]} \pi x \to 0$ is exact and almost split.

Recall that if $X \xrightarrow{a_1} X_1, \dots, X \xrightarrow{a_r} X_r$ are all the arrows in Γ starting from some module X and if the morphism $[u_1, \dots, u_r]^t : X \to \bigoplus_{i=1}^r X_i$ is irreducible, then it is minimal left almost split (and dually). Therefore, a well-behaved functor $p : \mathsf{k}\widetilde{\Gamma} \to \mathsf{ind}\,\Gamma$ induces a well-behaved functor $F : \mathsf{k}(\widetilde{\Gamma}) \to \mathsf{ind}\,\Gamma$ by factoring out by the mesh-ideal.

Recall that in the case where A is of finite-representation type, it was proved in [**2**, § 3] that there always exists a well-behaved functor $\mathsf{k}(\widetilde{\Gamma}) \to \mathsf{ind}\,\Gamma$. This result was based on a similar one in [**13**, § 1], where a well-behaved functor $\mathsf{k}(\widetilde{\Gamma}) \to \underline{\mathsf{mod}}\, A$ was constructed when A is self-injective and of finite-representation type and Γ is a stable component of $\Gamma(\mathsf{mod}\, A)$.

In this paper we use the following more general existence result on well-behaved functors. Given a length function l on $\widetilde{\Gamma}$ and an integer $n \in \mathbb{Z}$, we denote by $\widetilde{\Gamma}_{\leqslant n}$ or $\widetilde{\Gamma}_{\geqslant n}$ the full subquiver of $\widetilde{\Gamma}$ with vertices those $x \in \widetilde{\Gamma}$ such that $l(x) \leqslant n$ or $l(x) \geqslant n$, respectively. These are convex subquivers.

PROPOSITION 2.1. *Let $\pi : \widetilde{\Gamma} \to \Gamma$ be the generic covering and let $q : \mathsf{k}\mathcal{Y} \to \mathsf{ind}\,\Gamma$ be a well-behaved functor with $\mathcal{Y}$ a full convex subquiver of $\widetilde{\Gamma}$. Let l be a length function on $\widetilde{\Gamma}$ and assume that at least one of the following conditions is satisfied.*

(a) *There exist integers $m, n \in \mathbb{Z}$ such that $n \leqslant m$ and $\mathcal{Y} \subseteq \widetilde{\Gamma}_{\geqslant n} \cap \widetilde{\Gamma}_{\leqslant m}$.*
(b) *There exists an integer $n \in \mathbb{Z}$ such that $\mathcal{Y} \subseteq \widetilde{\Gamma}_{\geqslant n}$ and $\mathcal{Y}$ is stable under predecessors in $\widetilde{\Gamma}_{\geqslant n}$ (that is, every path in $\widetilde{\Gamma}_{\geqslant n}$ with endpoint lying in $\mathcal{Y}$ lies entirely in $\mathcal{Y}$).*
(c) *There exists an integer $n \in \mathbb{Z}$ such that $\widetilde{\Gamma}_{\geqslant n} \subseteq \mathcal{Y}$ and $\mathcal{Y}$ is stable under successors in $\widetilde{\Gamma}$.*

Then there exists a well-behaved functor $F : \mathsf{k}(\widetilde{\Gamma}) \to \mathsf{ind}\,\Gamma$ such that $F(\overline{\alpha}) = q(\alpha)$ for every arrow $\alpha \in \mathcal{Y}$.

Proof. It is sufficient to prove that there exists a well-behaved functor $p : \mathsf{k}\widetilde{\Gamma} \to \mathsf{ind}\,\Gamma$. For that purpose, we shall prove that:

(i) if $\mathcal{Y}$ satisfies (a), then q extends to a well-behaved functor $p : \mathsf{k}\mathcal{X} \to \mathsf{ind}\,\Gamma$ with $\mathcal{X}$ a full convex subquiver of $\widetilde{\Gamma}$ satisfying (b);

(ii) if $\mathcal{Y}$ satisfies (b), then q extends to a well-behaved functor $p : \mathsf{k}\mathcal{X} \to \mathsf{ind}\,\Gamma$ with $\mathcal{X}$ a full convex subquiver of $\widetilde{\Gamma}$ satisfying (c);

(iii) if $\mathcal{Y}$ satisfies (c), then q extends to a well-behaved functor $p : \mathsf{k}\widetilde{\Gamma} \to \mathsf{ind}\,\Gamma$.

We shall consider pairs $(\mathcal{X}, p)$, where $\mathcal{X}$ is a full convex subquiver of $\widetilde{\Gamma}$ containing $\mathcal{Y}$ and $p : \mathsf{k}\mathcal{X} \to \mathsf{ind}\,\Gamma$ is a well-behaved functor extending q. For any two such pairs $(\mathcal{X}, p)$ and $(\mathcal{X}', p')$, we shall write $(\mathcal{X}, p) \leqslant (\mathcal{X}', p')$ if and only if $\mathcal{X} \subseteq \mathcal{X}'$ and p' extends p. This clearly defines a partial order on the set of such pairs.

Assume that $\mathcal{Y}$ satisfies (a). Consider the set Σ of those pairs $(\mathcal{X}, p)$ where $\mathcal{X}$ is a full convex subquiver of $\widetilde{\Gamma}$ containing $\mathcal{Y}$, contained in $\widetilde{\Gamma}_{\geqslant n} \cap \widetilde{\Gamma}_{\leqslant m}$, and $p : \mathsf{k}\mathcal{X} \to \mathsf{ind}\,\Gamma$ is a well-behaved functor extending q. Then Σ is non-empty since it contains $(\mathcal{X}, p)$. Moreover, $(\Sigma, \leqslant)$ is totally inductive. Therefore, it has a maximal element, say $(\mathcal{X}, p)$. We claim that $\mathcal{X}$ is stable under predecessors in $\widetilde{\Gamma}_{\geqslant n}$. By absurd, assume that this is not the case. Then there exists an arrow $x \to y$ in $\widetilde{\Gamma}_{\geqslant n}$ with $x \notin \mathcal{X}$ and $y \in \mathcal{X}$. We choose such an x with $l(x)$ maximal. This is possible because $\mathcal{X} \subseteq \widetilde{\Gamma}_{\leqslant m}$. Note that there is no arrow $z \to x$ in $\widetilde{\Gamma}$ with $z \in \mathcal{X}$, because, otherwise, the path $z \to x \to y$ would contradict the convexity of $\mathcal{X}$. Therefore, the full subquiver $\mathcal{X}'$ of $\widetilde{\Gamma}$ generated by $\mathcal{X}$ and x has, as arrows, those in $\mathcal{X}$ together with the arrows in $\widetilde{\Gamma}$ starting from x and ending at some vertex in $\mathcal{X}$, say

$$\begin{array}{ccc} & \overset{\alpha_1}{\nearrow} & x_1 = y \\ x & & \vdots \\ & \underset{\alpha_r}{\searrow} & x_r . \end{array}$$

In particular, the convexity of $\mathcal{X}$ and the maximality of $l(x)$ imply that $\mathcal{X}'$ is convex. Assume that x is injective, or else x is non-injective and $\tau^{-1}x \notin \mathcal{X}$. The arrows $\pi(\alpha_1), \ldots, \pi(\alpha_r)$ in Γ are pairwise distinct and start in πx. Therefore, there exists an irreducible morphism $[u_1, \ldots, u_r]^t : \pi x \to \bigoplus_{i=1}^{r} \pi x_i$. We thus extend $p : \mathsf{k}\mathcal{X} \to \mathsf{ind}\,\Gamma$ to a functor $p' : \mathsf{k}\mathcal{X}' \to \mathsf{ind}\,\Gamma$ by setting $p'(\alpha_i) = u_i$, for every i. Note that a mesh in $\widetilde{\Gamma}$ is contained in $\mathcal{X}$ if and only if it is contained in $\mathcal{X}'$, by assumption on x and because there is no arrow in $\widetilde{\Gamma}$ ending at x and starting from some vertex in $\mathcal{X}$. Assume now that x is non-injective and $\tau^{-1}x \in \mathcal{X}$. By maximality of $l(x)$, every arrow in $\widetilde{\Gamma}$ ending at $\tau^{-1}x$ lies in $\mathcal{X}$. Therefore, $\alpha_1, \ldots, \alpha_r$ are all the arrows in $\widetilde{\Gamma}$ starting from x and the mesh in $\widetilde{\Gamma}$ starting from x is as follows:

$$\begin{array}{ccccc} & \overset{\alpha_1}{\nearrow} & x_1 & \overset{\beta_1}{\searrow} & \\ x & & \vdots & & \tau^{-1}x\,. \\ & \underset{\alpha_r}{\searrow} & x_r & \underset{\beta_r}{\nearrow} & \end{array}$$

Since p is well-behaved, the morphism $[p(\beta_1), \ldots, p(\beta_r)] : \bigoplus_{i=1}^{r} \pi x_i \to \tau_A^{-1}\pi x$ is irreducible and, therefore, minimal right almost split, because $\pi(\beta_1), \ldots, \pi(\beta_r)$ are all the arrows in Γ ending at $\tau_A^{-1}\pi x$. Hence, there is an almost split sequence in $\mathsf{mod}\,A$

$$0 \longrightarrow \pi x \xrightarrow{[u_1,\ldots,u_r]^t} \bigoplus_{i=1}^{r} \pi x_i \xrightarrow{[p(\beta_1),\ldots,p(\beta_r)]} \tau_A^{-1}\pi x \longrightarrow 0$$

and we extend $p : \mathsf{k}\mathcal{X} \to \mathsf{ind}\,\Gamma$ to a functor $p' : \mathsf{k}\mathcal{X}' \to \mathsf{ind}\,\Gamma$ by setting $p(\alpha_i) = u_i$, for every i. In any case, p' is well-behaved. Indeed, by construction of p' and because p is well-behaved, we have: p' transforms every mesh in $\widetilde{\Gamma}$ contained in $\mathcal{X}'$ into an almost split sequence in $\mathsf{mod}\,A$; moreover, given vertices $z, t \in \mathcal{X}'$, if $\gamma_1, \ldots, \gamma_s : z \to t$ are all the arrows in $\mathcal{X}'$ from z to t, then the morphism $[p'(\gamma_1), \ldots, p'(\gamma_s)]^t : \pi z \to \bigoplus_{i=1}^{s} \pi t$ is irreducible (or, equivalently, the morphism $[p'(\gamma_1), \ldots, p'(\gamma_s)] : \bigoplus_{i=1}^{s} \pi z \to \pi t$ is irreducible, because k is an algebraically closed field); using Proposition 1.2(b), we deduce that p' is well-behaved. Thus $(\mathcal{X}', p') \in \Sigma$ and $(\mathcal{X}, p) < (\mathcal{X}', p')$, which is a contradiction to the maximality of $(\mathcal{X}, p)$. This proves that $q : \mathsf{k}\mathcal{Y} \to \mathsf{ind}\,\Gamma$ extends to a well-behaved functor $p : \mathsf{k}\mathcal{X} \to \mathsf{ind}\,\Gamma$ with $\mathcal{X}$ a full convex subquiver of $\widetilde{\Gamma}$ containing $\mathcal{Y}$, contained in $\widetilde{\Gamma}_{\geqslant n} \cap \widetilde{\Gamma}_{\leqslant m}$ and stable under predecessors in $\widetilde{\Gamma}_{\geqslant n}$. Therefore, $\mathcal{X}$ satisfies (b).

Now assume that $\mathcal{Y}$ satisfies (b). Let Σ' be the set of those pairs $(\mathcal{X}, p)$ where $\mathcal{X}$ is a full convex subquiver of $\widetilde{\Gamma}$ containing $\mathcal{Y}$, contained in $\widetilde{\Gamma}_{\geqslant n}$ and stable under predecessors in $\widetilde{\Gamma}_{\geqslant n}$, and $p : \mathsf{k}\mathcal{X} \to \mathsf{ind}\,\Gamma$ is a well-behaved functor extending q. Then Σ' is non-empty for it contains $(\mathcal{Y}, q)$, and $(\Sigma', \leqslant)$ is totally inductive. Let $(\mathcal{X}, p)$ be a maximal element in Σ'. We claim that $\mathcal{X} = \widetilde{\Gamma}_{\geqslant n}$. By absurd, assume that this is not the case. Let $x \in \widetilde{\Gamma}_{\geqslant n}$ be a vertex not in $\mathcal{X}$. We may assume that $l(x)$ is minimal for this property. Then x has no successor in $\widetilde{\Gamma}$ lying in $\mathcal{X}$, because $\mathcal{X}$ is stable under predecessors in $\widetilde{\Gamma}_{\geqslant n}$. If there is no arrow $y \to x$ in $\widetilde{\Gamma}$ such that $y \in \mathcal{X}$, then x has no predecessor in $\mathcal{X}$, by minimality of $l(x)$. In such a case the full subquiver $\mathcal{X}'$ of $\widetilde{\Gamma}$ generated by $\mathcal{X}$ and x has the same arrows as those in $\mathcal{X}$ and it is convex. Then p trivially extends to a well-behaved functor $p' : \mathsf{k}\mathcal{X}' \to \mathsf{ind}\,\Gamma$ so that $(\mathcal{X}', p') \in \Sigma'$ and $(\mathcal{X}, p) < (\mathcal{X}', p')$, a contradiction to the maximality of $(\mathcal{X}, p)$. On the other hand, if there is an arrow $y \to x$ in $\widetilde{\Gamma}$ with $y \in \mathcal{X}$, then, using dual arguments to those used on the previous situation (when $\mathcal{Y}$ was supposed to satisfy (a)), we similarly extend p to a well-behaved functor $p' : \mathsf{k}\mathcal{X}' \to \mathsf{ind}\,\Gamma$ where $\mathcal{X}'$ is the (convex) full subquiver of $\widetilde{\Gamma}$ generated by $\mathcal{X}$ and x. As in the previous case, $(\mathcal{X}', p') \in \Sigma'$ and $(\mathcal{X}, p) < (\mathcal{X}', p')$, which contradict the maximality of $(\mathcal{X}, p)$. Therefore, $p : \mathsf{k}\mathcal{X} \to \mathsf{ind}\,\Gamma$ is a well-behaved functor extending q, where $\mathcal{X}$ equals $\widetilde{\Gamma}_{\geqslant n}$ (and therefore satisfies (c)).

Finally, assume that $\mathcal{Y}$ satisfies (c). Let Σ'' be the set of those pairs $(\mathcal{X}, p)$ where $\mathcal{X}$ is a full convex subquiver of $\widetilde{\Gamma}$ containing both $\widetilde{\Gamma}_{\geqslant n}$ and $\mathcal{Y}$, and $\mathcal{X}$ is stable under successors in $\widetilde{\Gamma}$ and $p : \mathsf{k}\mathcal{X} \to \mathsf{ind}\,\Gamma$ is a well-behaved functor extending q. Then Σ'' is non-empty for it contains $(\mathcal{Y}, q)$, and $(\Sigma'', \leqslant)$ is totally inductive (with $\leqslant$ as above). Let $(\mathcal{X}, p)$ be a maximal element in Σ''. We claim that $\mathcal{X} = \widetilde{\Gamma}$. By absurd, assume that this is not the case. Let $x \in \widetilde{\Gamma}$ be a vertex not in $\mathcal{X}$ and with $l(x)$ maximal for this property. This is possible because $\widetilde{\Gamma}_{\geqslant n} \subseteq \mathcal{X}$. Since $\mathcal{X}$ is stable under successors in $\widetilde{\Gamma}$, there is no arrow $z \to x$ with $z \in \mathcal{X}$. If there is no arrow starting from x in $\widetilde{\Gamma}$, then the full subquiver $\mathcal{X}'$ of $\widetilde{\Gamma}$ generated by $\mathcal{X}$ and x has the same arrows as those of $\mathcal{X}$, and p extends trivially to a well-behaved functor $p' : \mathsf{k}\mathcal{X}' \to \mathsf{ind}\,\Gamma$; hence $(\mathcal{X}', p') \in \Sigma$ and $(\mathcal{X}, p) < (\mathcal{X}', p')$ which contradicts the maximality of $(\mathcal{X}, p)$. Thus, there exists an arrow $x \to y$ in $\widetilde{\Gamma}$. The vertex y then lies in $\mathcal{X}$ by maximality of $l(x)$. Let $\mathcal{X}'$ be the full subquiver of $\widetilde{\Gamma}$ generated by $\mathcal{X}$ and x. Therefore, the arrows in $\mathcal{X}'$ are those in $\mathcal{X}$ together with those in $\widetilde{\Gamma}$ starting from x (which, by maximality of $l(x)$ have their endpoint in $\mathcal{X}$) and $\mathcal{X}'$ is convex. Now, using the same arguments as those used in the first situation (when we assumed that $\mathcal{Y}$ satisfied (a)), we extend p to a well-behaved functor $p' : \mathsf{k}\mathcal{X}' \to \mathsf{ind}\,\Gamma$. We thus have $(\mathcal{X}', p') \in \Sigma''$ and $(\mathcal{X}, p) < (\mathcal{X}', p')$, a contradiction to the maximality of $(\mathcal{X}, p)$. This proves that $\mathcal{X} = \widetilde{\Gamma}$ and finishes the proof of the proposition. $\square$

We now present some practical situations where Proposition 2.1 may be applied.

Definition 2.2. Let X be an indecomposable module in Γ and $r \geqslant 1$. A *sectional family of paths* (*starting from* X *and of irreducible morphisms*) is a family

$$
\begin{array}{ccccccccc}
 & & X_{1,1} & \longrightarrow & \cdots & \longrightarrow & X_{1,l_1-1} & \xrightarrow{f_{1,l_1}} & X_{1,l_1} \\
 & \overset{f_{1,1}}{\nearrow} & & & & & & & \\
X & \xrightarrow{f_{2,1}} & X_{2,1} & \longrightarrow & \cdots & \longrightarrow & X_{2,l_2-1} & \xrightarrow{f_{2,l_2}} & X_{2,l_2} \\
 & \underset{f_{r,1}}{\searrow} & \vdots & & & & & & \\
 & & X_{r,1} & \longrightarrow & \cdots & \longrightarrow & X_{r,l_r-1} & \xrightarrow[f_{r,l_r}]{} & X_{r,l_r}
\end{array}
$$

of r paths starting from X and of irreducible morphisms between indecomposables, subject to the following conditions (where $X = X_{i,0}$, for convenience).

(a) For every $M \in \Gamma$ and $l \geqslant 1$, let I be the set of those indices $i \in \{1, \dots, r\}$ such that $l_i \geqslant l$ and $f_{i,l}$ has domain M. Then the morphism $[f_{i,l}\ ;\ i \in I] : M \to \bigoplus_{i \in I} X_{i,l}$ is irreducible.

(b) For every $M \in \Gamma$ and $l \geqslant 1$, let J be the set of those indices $i \in \{1, \dots, r\}$ such that $l_i \geqslant l$ and $f_{i,l}$ has codomain M. Then the morphism $[f_{i,l}\ ;\ i \in J]^t : \bigoplus_{i \in J} X_{i,l-1} \to M$ is irreducible.

(c) There is no hook of the form $\cdot \xrightarrow{f_{i,j}} \cdot \xrightarrow{f_{i',j+1}} \cdot$.

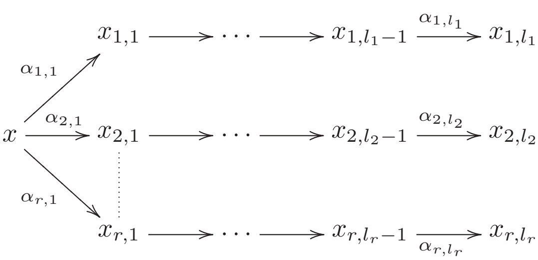
Remark 2.3. (1) The definition implies that each of the paths in the given family is sectional.

(2) If $r = 1$ the definition coincides with that of a sectional path.

(3) If $l_i = 1$ for every i, then the definition is equivalent to say that the morphism $[f_{1,1}, \dots, f_{r,1}]^t : X \to \bigoplus_{i=1}^{r} X_{i,1}$ is irreducible.

(4) Since k is an algebraically closed field, the first two conditions together are equivalent to the following single condition: for every $M, N \in \Gamma$ and $l \geqslant 1$, let K be the set of indices $i \in \{1, \dots, r\}$ such that $l_i \geqslant l$ and $f_{i,l}$ is a morphism from M to N; then the morphisms $f_{i,l} : M \to N$, $i \in K$, are linearly independent modulo $\mathsf{rad}^2(M, N)$.

Proposition 2.4. *Let $\pi : \widetilde{\Gamma} \to \Gamma$ be the generic covering. Let X be in Γ and $x \in \pi^{-1}(X)$. Let $\{X \xrightarrow{f_{i,1}} X_{i,1} \to \cdots \to X_{i,l_i-1} \xrightarrow{f_{i,l_i}} X_{i,l_i}\}_{i=1,\dots,r}$ be a sectional family of paths starting from X and of irreducible morphisms. Then there exist r paths in $\widetilde{\Gamma}$*



starting from x, such that $\pi x_{i,j} = X_{i,j}$, for every i, j, and the arrows $\alpha_{i,j}$ are pairwise distinct. Moreover, for any such data, there exists a well-behaved functor $F : \mathsf{k}(\widetilde{\Gamma}) \to \mathsf{ind}\,\Gamma$ such that $F(\overline{\alpha_{i,j}}) = f_{i,j}$, for every i, j.

Proof. We first construct the vertices $x_{i,j}$ and the arrows $\alpha_{i,j}$. For every $i \in \{1, \dots, r\}$, the path $X \xrightarrow{f_{i,1}} X_{i,1} \to \cdots \to X_{i,l_i-1} \xrightarrow{f_{i,l_i}} X_{i,l_i}$ of irreducible morphisms defines (possibly several) paths $X \to X_{i,1} \to \cdots \to X_{i,l_i-1} \to X_{i,l_i}$ in Γ. Since $\pi : \widetilde{\Gamma} \to \Gamma$ is a covering of quivers, each such path in Γ defines a path $x \to x_{i,1} \to \cdots \to x_{i,l_i-1} \to x_{i,l_i}$ in $\widetilde{\Gamma}$ such that $\pi x_{i,j} = X_{i,j}$. This defines all the vertices $x_{i,j}$. Let $y, z \in \widetilde{\Gamma}$ be vertices and let K be the set of couples (i, j), $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, l_i\}$, such that $y = x_{i,j-1}$ and $z = x_{i,j}$ (with the convention $x_{i,0} = x$). Note that if both (i, j) and (i', j') lie in K, then $j = j'$ because $\widetilde{\Gamma}$ is with length and $x_{i,j}$ or $x_{i',j'}$ is the endpoint of a path in $\widetilde{\Gamma}$ starting from x and of length j or j', respectively. By definition of a sectional family of paths, the irreducible morphisms $f_{i,j} : \pi y \to \pi z$, for $(i, j) \in K$, are linearly independent modulo $\mathsf{rad}^2(\pi y, \pi z)$. Since $\pi : \widetilde{\Gamma} \to \Gamma$ is a covering of quivers, there is an injective map $(i, j) \mapsto \alpha_{i,j}$ from K to the set of arrows from y to z in $\widetilde{\Gamma}$. By proceeding with this construction for every pair of vertices $y, z \in \widetilde{\Gamma}$, one defines all the arrows $\alpha_{i,j}$, for $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, l_i\}$, which are pairwise distinct, by construction.

Given the vertices $x_{i,j}$ and the arrows $\alpha_{i,j}$ as above, we let $\mathcal{Y}$ be the full subquiver of $\widetilde{\Gamma}$ generated by all the $x_{i,j}$. We need some properties on $\mathcal{Y}$. Note that if there exists a path in $\widetilde{\Gamma}$ of the form $x_{i,j} \to y_1 \to \cdots \to y_s \to x_{i',j'}$, for some vertices $y_1, \ldots, y_s \in \widetilde{\Gamma}$, then we have two parallel paths in $\widetilde{\Gamma}$

$$x \xrightarrow{\alpha_{i,1}} x_{i,1} \longrightarrow \cdots \longrightarrow x_{i,j-1} \xrightarrow{\alpha_{i,j}} x_{i,j} \longrightarrow y_1 \longrightarrow \cdots \longrightarrow y_s \longrightarrow x_{i',j'} \quad \text{and}$$
$$x \xrightarrow{\alpha_{i',1}} x_{i',1} \longrightarrow \cdots \longrightarrow x_{i',j'-1} \xrightarrow{\alpha_{i',j'}} x_{i',j'}.$$

Note that the image under π of the second path is a path $X \to X_{i',1} \to \cdots \to X_{i',j'-1} \to X_{i',j'}$ in Γ which is sectional (Remark 2.3(1)). Since $\pi : \widetilde{\Gamma} \to \Gamma$ is a covering of translation quivers, this implies that the path $x \xrightarrow{\alpha_{i',1}} x_{i',1} \to \cdots \to x_{i',j'-1} \xrightarrow{\alpha_{i',j'}} x_{i',j'}$ is sectional. Applying Proposition 1.2 then shows that $j' = j + s + 1$ and the sequence of vertices $(x, x_{i,1}, \ldots, x_{i,j}, y_1, \ldots, y_s, x_{i',j'})$ and $(x, x_{i',1}, \ldots, x_{i',j'})$ coincide. From this, we deduce the following facts.

(1) The quiver $\mathcal{Y}$ is convex in $\widetilde{\Gamma}$.

(2) The quiver $\mathcal{Y}$ contains no path $y \to \cdot \to z$ with z non-projective and $y = \tau z$.

(3) If there is an arrow $y \to z$ in $\mathcal{Y}$, then there exist i, j such that $y = x_{i,j-1}$ and $z = x_{i,j}$. Moreover, given i', j', we have $y = x_{i',j'-1}$ if and only if $z = x_{i',j'}$.

We now define a well-behaved functor $q : \mathsf{k}\mathcal{Y} \to \mathsf{ind}\,\Gamma$ such that $q(\alpha_{i,j}) = f_{i,j}$ for every i, j. Let $y, z \in \mathcal{Y}$ be vertices such that there exists at least one arrow from y to z. Then there is a path in $\widetilde{\Gamma}$ from x to z, say of length n. Since $\widetilde{\Gamma}$ is with length and because of fact (3), we deduce that if $z = x_{i,j}$ for some i, j, then $j = n$ and $y = x_{i,j-1}$. We thus define $I_{y,z}$ to be the set of indices $i \in \{1, \ldots, r\}$ such that $n \geqslant l_i$ and $y = x_{i,n-1}, z = x_{i,n}$. The set of arrows in $\widetilde{\Gamma}$ from y to z is therefore equal to $\{\alpha_{i,n} \mid i \in I_{y,z}\} \cup \{\gamma_1, \ldots, \gamma_s\}$ where $\gamma_1, \ldots, \gamma_s$ are pairwise distinct arrows, none of which is equal to either of the arrows $\alpha_{i',j'}$, $i' \in \{1, \ldots, r\}$ and $j' \in \{1, \ldots, l_{i'}\}$. Recall that the irreducible morphisms $f_{i,n} : \pi y \to \pi z$, for $i \in I_{y,z}$, are linearly independent modulo $\mathsf{rad}^2(\pi y, \pi z)$ (Definition 2.2 and Remark 2.3). Since π induces a bijection from the set of arrows in $\widetilde{\Gamma}$ from y to z to the set of arrows in Γ from πy to πz, we deduce that there exist irreducible morphisms $g_1, \ldots, g_s : \pi y \to \pi z$ such that $g_1, \ldots, g_s$ together with $f_{i,n}$, for $i \in I_{y,z}$, are linearly independent modulo $\mathsf{rad}^2(\pi y, \pi z)$. We then set $q(\alpha_{i,n}) = f_{i,n}$, for every $i \in I_{y,z}$, and $q(\gamma_j) = g_j$, for every $j = 1, \ldots, s$. This defines q on every arrow from y to z, for every pair of vertices $y, z \in \mathcal{Y}$. Hence the functor $q : \mathsf{k}\mathcal{Y} \to \mathsf{ind}\,\Gamma$. The construction of q and the above fact (2) of $\mathcal{Y}$ show that q is well-behaved and $q(\alpha_{i,j}) = f_{i,j}$ for every i, j.

Finally, let l be a length function on $\widetilde{\Gamma}$. Then $\mathcal{Y}$ is a convex full subquiver of $\widetilde{\Gamma}$ such that $l(y) \in \{l(x), l(x)+1, \ldots, l(x) + \mathsf{max}_{i=1,\ldots,r}\, l_i\}$, for every vertex $y \in \mathcal{Y}$. Therefore, Proposition 2.1(a), implies that there exists a well-behaved functor $F : \mathsf{k}(\widetilde{\Gamma}) \to \mathsf{ind}\,\Gamma$ such that $F(\overline{\alpha_{i,j}}) = q(\alpha_{i,j}) = f_{i,j}$ for every i, j. □

Remark 2.5. The proofs we gave for Propositions 2.1 and 2.4 strongly rely on the fact that the generic covering π induces a bijection from the set of arrows in $\widetilde{\Gamma}$ from x to y to the set of arrows in Γ from πx to πy, for every pair of vertices $x, y \in \widetilde{\Gamma}$. In particular, these proofs are not likely to be adapted to the situation where one replaces the generic covering $\widetilde{\Gamma}$ of Γ by the universal covering.

The following result follows from Proposition 2.4. It will be particularly useful to us.

Proposition 2.6. *Let $X, X_1, \ldots, X_r$ lie on Γ and $f = [f_1, \ldots, f_r]^t : X \to X_1 \oplus \ldots \oplus X_r$ be an irreducible morphism in* $\mathsf{mod}\,A$. *Let $x \in \pi^{-1}(X)$ and $x \xrightarrow{\alpha_i} x_i$ be an arrow in $\widetilde{\Gamma}$ such that*

$\pi x_i = X_i$ *for every* $i \in \{1, \dots, r\}$. *Then there exists a well-behaved functor* $F : \mathsf{k}(\widetilde{\Gamma}) \to \mathsf{ind}\,\Gamma$ *such that* $F(\overline{\alpha_i}) = f_i$ *for every* i.

Proof. It follows from Remark 2.3(3), that the family of morphisms $\{f_1, \dots, f_r\}$ is a sectional family of paths starting from X. The conclusion thus follows from Proposition 2.4. □

We now study some properties of well-behaved functors which are essential to our work. We begin with the following basic lemma.

LEMMA 2.7. *Let* $F : \mathsf{k}(\widetilde{\Gamma}) \to \mathsf{ind}\,\Gamma$ *be a well-behaved functor,* x, y *vertices in* $\widetilde{\Gamma}$ *and* $n \geqslant 0$.

(a) *The functor* F *maps a morphism in* $\mathfrak{R}^n\mathsf{k}(\widetilde{\Gamma})(x, y)$ *to a morphism in* $\mathsf{rad}^n(Fx, Fy)$.

(b) *Let* $f \in \mathsf{rad}^{n+1}(Fx, Fy)$ *and* $\alpha_1 : x \to x_1, \dots, \alpha_r : x \to x_r$ *be the arrows in* $\widetilde{\Gamma}$ *starting from* x. *Then there exist* $h_i \in \mathsf{rad}^n(Fx_i, Fy)$, *for every* i, *such that* $f = \sum_i h_i F(\overline{\alpha}_i)$.

Proof. Part (a) follows from the fact that F is well-behaved.

(b) We have a decomposition $f = \sum_j g_j f_j$ where j runs through some index set, $f_j \in \mathsf{rad}(Fx, Y_j)$, $g_j \in \mathsf{rad}^n(Y_j, Fy)$, $Y_j \in \mathsf{ind}\,A$, for every j. The morphism $[F(\overline{\alpha}_1), \dots, F(\overline{\alpha}_r)]^t : Fx \to \bigoplus_{i=1}^r Fx_i$ is minimal left almost split so every f_j factors through it: $f_j = \sum_{i=1}^r f'_{j,i}F(\overline{\alpha}_i)$ with $f'_{i,j} \in \mathsf{Hom}_A(Fx_i, Y_j)$. Setting $h_i = \sum_j g_j f'_{j,i}$ does the trick. □

The following theorem states the main properties of well-behaved functors we shall use. Part (b) of it was first proved in [**13**, § 2] in the case of the stable part of the Auslander–Reiten quiver of a self-injective algebra of finite-representation type (see also [**2**, Example 3.1(b)] for the case of the Auslander–Reiten quiver of an algebra of finite-representation type).

THEOREM B. *Let* $F : \mathsf{k}(\widetilde{\Gamma}) \to \mathsf{ind}\,\Gamma$ *be a well-behaved functor,* x, y *vertices in* $\widetilde{\Gamma}$ *and* $n \geqslant 0$.

(a) *The following two maps induced by* F *are bijective*:

$$\bigoplus_{Fz=Fy} \mathfrak{R}^n\mathsf{k}(\widetilde{\Gamma})(x, z)/\mathfrak{R}^{n+1}\mathsf{k}(\widetilde{\Gamma})(x, z) \longrightarrow \mathsf{rad}^n(Fx, Fy)/\mathsf{rad}^{n+1}(Fx, Fy),$$

$$\bigoplus_{Fz=Fy} \mathfrak{R}^n\mathsf{k}(\widetilde{\Gamma})(z, x)/\mathfrak{R}^{n+1}\mathsf{k}(\widetilde{\Gamma})(z, x) \longrightarrow \mathsf{rad}^n(Fy, Fx)/\mathsf{rad}^{n+1}(Fy, Fx).$$

(b) *The following two maps induced by* F *are injective*:

$$\bigoplus_{Fz=Fy} \mathsf{k}(\widetilde{\Gamma})(x, z) \longrightarrow \mathsf{Hom}_A(Fx, Fy) \quad \text{and} \quad \bigoplus_{Fz=Fy} \mathsf{k}(\widetilde{\Gamma})(z, x) \longrightarrow \mathsf{Hom}_A(Fy, Fx).$$

(c) Γ *is generalized standard if and only if* F *is a covering functor*, *that is, the two maps of* (b) *are bijective* (*see* [**2**, 3.1]).

Proof. We prove the assertions concerning morphisms $Fx \to Fy$. Those concerning $Fy \to Fx$ are proved using similar arguments. Let $\alpha_i : x \to x_i$, $i = 1, \dots, r$, be the arrows in $\widetilde{\Gamma}$ starting from x. So we have a minimal left almost split morphism in $\mathsf{mod}\,A$:

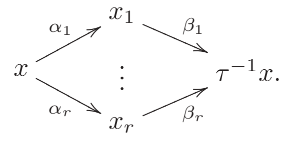
$$Fx \xrightarrow{[F(\overline{\alpha}_1), \dots, F(\overline{\alpha}_r)]^t} \bigoplus_{i=1}^r Fx_i.$$

(a) We denote by F_n the map $\bigoplus_{Fz=Fy} \mathfrak{R}^n \mathsf{k}(\widetilde{\Gamma})(x,z)/\mathfrak{R}^{n+1}\mathsf{k}(\widetilde{\Gamma})(x,z) \to \mathsf{rad}^n(Fx,Fy)/\mathsf{rad}^{n+1}(Fx,Fy)$. We prove that F_n is surjective by induction on $n \geqslant 0$. So, given a morphism $f \in \mathsf{rad}^n(Fx,Fy)$, we prove that there exists $(\phi_z)_z \in \bigoplus_{Fz=Fy} \mathfrak{R}^n\mathsf{k}(\widetilde{\Gamma})(x,z)$ such that $f = \sum_z F(\phi_z) \mod \mathsf{rad}^{n+1}$. We start with $n=0$. Let $f \in \mathsf{Hom}_A(Fx,Fy)$. If $Fx \neq Fy$ then $f \in \mathsf{rad}(Fx,Fy)$. Otherwise, $f = \lambda 1_{Fx} \mod \mathsf{rad}$ with $\lambda \in \mathsf{k}$, that is, $f = F(\lambda 1_x) \mod \mathsf{rad}$ for some $\lambda \in \mathsf{k}$. So F_0 is surjective. Now let $n \geqslant 0$ and assume that F_n is surjective. Let $f \in \mathsf{rad}^{n+1}(Fx,Fy)$. Because of Lemma 2.7(b), there is a decomposition $f = \sum_i h_i F(\overline{\alpha}_i)$ with $h_i \in \mathsf{rad}^n(Fx_i,Fy)$. Moreover, $h_i = \sum_z F(\phi_{i,z}) \mod \mathsf{rad}^{n+1}$ with $(\phi_{i,z})_z \in \bigoplus_{Fz=Fy}\mathfrak{R}^n\mathsf{k}(\widetilde{\Gamma})(x_i,z)$, for every i, because F_n is surjective. Therefore, $f = \sum_z F(\sum_i \phi_{i,z}\overline{\alpha}_i) \mod \mathsf{rad}^{n+2}$ and $\sum_i \phi_{i,z}\overline{\alpha}_i \in \mathfrak{R}^{n+1}\mathsf{k}(\widetilde{\Gamma})(x,z)$, for every $z \in \widetilde{\Gamma}$ such that $Fz = Fy$. So F_{n+1} is surjective. This proves that F_n is surjective for every $n \geqslant 0$.

Now we prove that F_n is injective for every $n \geqslant 0$. Actually, we prove that the following assertion (H_n) holds true. '*Let* $(\phi_z)_z \in \bigoplus_{Fz=Fy}\mathsf{k}(\widetilde{\Gamma})(x,z)$ *be such that* $\sum_z F(\phi_z) \in \mathsf{rad}^n$, *then* $\phi_z \in \mathfrak{R}^n\mathsf{k}(\widetilde{\Gamma})(x,z)$ *for every* z.' Clearly, this will prove the injectivity of all the F_n. We proceed by induction on $n \geqslant 0$. Assume that $n=0$ and that $\sum_z F(\phi_z) \in \mathsf{rad}(Fx,Fy)$. If $Fx \neq Fy$, then $x \neq z$ for every z such that $Fz = Fy$ and, therefore, $\phi_z \in \mathfrak{R}\mathsf{k}(\widetilde{\Gamma})(x,z)$. If $Fx = Fy$, then $\phi_z \in \mathfrak{R}\mathsf{k}(\widetilde{\Gamma})(x,z)$ if $x \neq z$ and there exists $\lambda \in \mathsf{k}$ such that $\phi_x = \lambda 1_x$. So $\lambda 1_{Fx} \in \mathsf{rad}(Fx,Fy)$, that is, $\lambda = 0$. Thus, $\phi_z \in \mathfrak{R}\mathsf{k}(\widetilde{\Gamma})(x,z)$ for every z. This proves that (H_0) holds true. Now let $n \geqslant 0$, assume that (H_n) holds true and let $(\phi_z)_z \in \bigoplus_{Fz=Fy}\mathsf{k}(\widetilde{\Gamma})(x,z)$ be such that $\sum_z F(\phi_z) \in \mathsf{rad}^{n+2}$. So, $\phi_z \in \mathfrak{R}^{n+1}\mathsf{k}(\widetilde{\Gamma})(x,z)$, for every z, because (H_n) holds true. Also, there exists $(\psi_z) \in \bigoplus_{Fz=Fy}\mathfrak{R}^{n+2}\mathsf{k}(\widetilde{\Gamma})(x,z)$ such that $\sum_z F(\phi_z) = \sum_z F(\psi_z) \mod \mathsf{rad}^{n+3}$, because F_{n+2} is surjective and $\sum_z F(\phi_z) \in \mathsf{rad}^{n+2}(Fx,Fy)$. Therefore, there exists $h_i \in \mathsf{rad}^{n+2}(Fx_i,Fy)$, for every i, such that $\sum_z F(\phi_z - \psi_z) = \sum_i h_i F(\overline{\alpha}_i)$, because of Lemma 2.7(b). Since $\phi_z, \psi_z \in \mathfrak{R}\mathsf{k}(\widetilde{\Gamma})(x,z)$, there is a decomposition $\phi_z - \psi_z = \sum_{i=1}^r \theta_{z,i}\overline{\alpha}_i$ with $\theta_{z,i} \in \mathsf{k}(\widetilde{\Gamma})(x_i,z)$ for every i. We deduce that

$$\sum_i \left(\sum_z F(\theta_{z,i}) - h_i\right) F(\overline{\alpha}_i) = 0. \qquad (\star)$$

Now if x is injective, then $\sum_z F(\theta_{z,i}) - h_i = 0$ for every i. Since $h_i \in \mathsf{rad}^{n+2}(Fx_i,Fy)$, we deduce that $\sum_z F(\theta_{z,i}) \in \mathsf{rad}^{n+2}(Fx_i,Fy) \subseteq \mathsf{rad}^{n+1}(Fx_i,F_y)$ for every i. Because (H_n) holds true, we get $\theta_{z,i} \in \mathfrak{R}^{n+1}\mathsf{k}(\widetilde{\Gamma})(x_i,z)$, for every i, and, therefore $\phi_z = \psi_z + \sum_i \theta_{z,i}\overline{\alpha}_i \in \mathfrak{R}^{n+2}\mathsf{k}(\widetilde{\Gamma})$. This proves that (H_n) holds true if x is injective. Now assume that x is not injective. The mesh in $\widetilde{\Gamma}$ starting from x is as follows:



Since F is well-behaved, there is an almost split sequence in $\mathsf{mod}\, A$:

$$0 \longrightarrow Fx \xrightarrow{[F(\overline{\alpha}_1),\ldots,F(\overline{\alpha}_r)]^t} \bigoplus_{i=1}^r Fx_i \xrightarrow{[F(\overline{\beta}_1),\ldots,F(\overline{\beta}_r)]} \tau_A^{-1}Fx \longrightarrow 0.$$

From $(\star)$, we deduce that there exists $h \in \mathsf{Hom}_A(\tau_A^{-1}Fx,Fy)$ such that $\sum_z F(\theta_{z,i}) - h_i = hF(\overline{\beta}_i)$, for every i. Since $F_0,\ldots,F_{n-1}$ are surjective, there exists $(\chi_z)_z \in \bigoplus_{Fz=Fy}\mathsf{k}(\widetilde{\Gamma})(\tau^{-1}x,z)$ such that $h = \sum_z F(\chi_z) \mod \mathsf{rad}^n$. Therefore, the following equality holds true for every i:

$$\sum_z F(\theta_{z,i}) = \sum_z F(\chi_z\overline{\beta}_i) + h_i \mod \mathsf{rad}^{n+1}.$$

Therefore, $\sum_z F(\theta_{z,i} - \chi_z\overline{\beta}_i) \in \mathsf{rad}^{n+1}(Fx_i, Fz)$, for every i, because $h_i \in \mathsf{rad}^{n+2}(Fx_i, Fy)$. Hence, $\theta_{z,i} - \chi_z\overline{\beta}_i \in \mathfrak{R}^{n+1}\mathsf{k}(\widetilde{\Gamma})(x_i, z)$, for every i, z, because (H_n) holds true. This gives, for every z,

$$\phi_z = \psi_z + \sum_i (\theta_{z,i} - \chi_z\overline{\beta}_i)\overline{\alpha}_i \in \mathfrak{R}^{n+2}\mathsf{k}(\widetilde{\Gamma})(x, z).$$

This proves that (H_{n+1}) holds true. Therefore, for every $n \geqslant 0$, the map F_n is injective and, therefore, bijective.

(b) Let $(\phi_z)_z \in \bigoplus_{Fz=Fy} \mathsf{k}(\widetilde{\Gamma})(x, z)$ be such that $\sum_z F(\phi_z) = 0$. In particular, $\sum_z F(\phi_z) \in \mathsf{rad}^n(Fx, Fy)$ for every $n \geqslant 0$. Since F_n is injective for every n, we deduce that $\phi_z \in \mathfrak{R}^n\mathsf{k}(\widetilde{\Gamma})(x, z)$ for every z and n. On the other hand, given z such that $Fz = Fy$, there exists $l \geqslant 0$ such that all the paths from x to z in $\widetilde{\Gamma}$ are of length l, so that $\mathfrak{R}^n\mathsf{k}(\widetilde{\Gamma})(x, z) = 0$ for $n > l$. Therefore $\phi_z = 0$ for every z. This proves the injectivity of the first given map. The second map is dealt with using dual arguments.

(c) Assume that Γ is generalized standard and let $f \in \mathsf{Hom}_A(Fx, Fy)$. So there exists $n \geqslant 0$ such that $\mathsf{rad}^n(Fx, Fy) = 0$. On the other hand, the surjectivity of the maps F_m $(m \geqslant 0)$ shows that $f = \sum_z F(\phi_z) \bmod \mathsf{rad}^n$ for some $(\phi_z)_z \in \bigoplus_{Fz=Fy} \mathsf{k}(\widetilde{\Gamma})(x, z)$. Therefore $f = \sum_z F(\phi_z)$. So the first given map in (c) is surjective and so is the second (using dual arguments). This and (b) prove that F is a covering functor.

Conversely, assume that F is a covering functor and let $x, y \in \widetilde{\Gamma}$ be vertices. Therefore, there are only finitely many vertices $z \in \widetilde{\Gamma}$ such that $Fz = Fy$ and $\mathsf{k}(\widetilde{\Gamma})(x, z) \neq 0$ because $\mathsf{Hom}_A(Fx, Fy)$ is finite-dimensional. This and the fact that $\widetilde{\Gamma}$ is with length imply that there exists $n \geqslant 0$ such that $\mathfrak{R}^n\mathsf{k}(\widetilde{\Gamma})(x, z) = 0$ for every z such that $Fz = Fy$. The injectivity of F_n then implies that $\mathsf{rad}^n(Fx, Fy) = 0$. So Γ is generalized standard. □

REMARK 2.8. It is not difficult to check that the proofs of Proposition 2.6 and Theorem B still work if Γ is an Auslander–Reiten component of $\mathcal{T}$ (instead of $\mathsf{mod}\,A$) where $\mathcal{T}$ is a triangulated Krull–Schmidt category over k with finite-dimensional Hom spaces and Auslander–Reiten triangles.

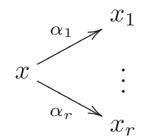
3. *Degrees of irreducible morphisms*

In this section we prove some characterizations for the left (or right) degree of an irreducible morphism to be finite. These shall be used later for the proof of our main result Theorem A. Each statement has its dual counterpart which will be omitted. The following proposition was first proved in [**8**] for generalized standard convex Auslander–Reiten components of an artin algebra. In a weaker form it was also proved in [**4**] for standard Auslander–Reiten components. We thank Shiping Liu for pointing out that the arguments used to prove the first statement can be adapted to prove the second statement. Note that the two statements are not dual to each other.

THEOREM C. *Let $f : X \to Y$ be an irreducible morphism with X indecomposable, Γ be the Auslander–Reiten component of A containing X and $n \in \mathbb{N}$.*

(a) *If $d_l(f) = n$, then there exist $Z \in \Gamma$ and $h \in \mathsf{rad}^n(Z, X)\backslash\mathsf{rad}^{n+1}(Z, X)$ such that $fh = 0$.*
(b) *If $d_r(f) = n$, then there exist $Z \in \Gamma$ and $h \in \mathsf{rad}^n(Y, Z)\backslash\mathsf{rad}^{n+1}(Y, Z)$ such that $hf = 0$.*

Proof. We write $Y = X_1 \oplus \ldots \oplus X_r$ with $X_1, \ldots, X_r \in \Gamma$ and $f = [f_1, \ldots, f_r]^t$ with $f_i : X \to X_i$. Let $\pi : \widetilde{\Gamma} \to \Gamma$ be the generic covering. Because f is irreducible, $\widetilde{\Gamma}$ contains a subquiver of the form



such that $\pi x = X$ and $\pi x_i = X_i$ for every i. Let $F : \mathsf{k}(\widetilde{\Gamma}) \to \mathsf{ind}\,\Gamma$ be a well-behaved functor such that $F(\overline{\alpha}_i) = f_i$ for every i (see (Proposition 2.6)).

(a) If $d_l(f) = n$, then there exist $Z \in \Gamma$ and $g \in \mathsf{rad}^n(Z, X)\backslash\mathsf{rad}^{n+1}(Z, X)$ such that $fg \in \mathsf{rad}^{n+2}(Z, Y)$, that is $f_i g \in \mathsf{rad}^{n+2}(Z, X_i)$ for every i. Because of Theorem B, there exists $(\phi_z)_z \in \bigoplus_{Fz=Z} \mathfrak{R}^n\mathsf{k}(\widetilde{\Gamma})(z, x)$ such that $g = \sum_z F(\phi_z) \mod \mathsf{rad}^{n+1}(Z, X)$ and $\phi_{z_0} \notin \mathfrak{R}^{n+1}\mathsf{k}(\widetilde{\Gamma})(z_0, x)$ for some z_0. Therefore $f_i g = \sum_z F(\overline{\alpha}_i\phi_z) \mod \mathsf{rad}^{n+2}(Z, X_i)$ for every i. Since $f_i g \in \mathsf{rad}^{n+2}(Z, X_i)$ we infer, using Theorem B, that $\overline{\alpha_i}\phi_z \in \mathfrak{R}^{n+2}\mathsf{k}(\widetilde{\Gamma})(z, x_i)$ for every z and every i. On the other hand, $\phi_{z_0} \notin \mathfrak{R}^{n+1}\mathsf{k}(\widetilde{\Gamma})(z_0, x)$ implies that any path in $\widetilde{\Gamma}$ from z_0 to x has length at most n. Hence, any path from z_0 to x_i has length at most $n + 1$ for every i. Thus $\overline{\alpha}_i\phi_{z_0} = 0$ for every i. We then set $h = F(\phi_{z_0})$. Then $fh = \sum_i F(\overline{\alpha}_i\phi_{z_0}) = 0$ and $h \in \mathsf{rad}^n(Z, X)\backslash\mathsf{rad}^{n+1}(Z, X)$, because $\phi_{z_0} \in \mathfrak{R}^n\backslash\mathfrak{R}^{n+1}$ and because of Theorem B.

(b) Now assume that $d_r(f) = n$. There exist $Z \in \Gamma$ and $g \in \mathsf{rad}^n(Y, Z)\backslash\mathsf{rad}^{n+1}(Y, Z)$ such that $gf \in \mathsf{rad}^{n+2}(X, Z)$. We write $g = [g_1, \ldots, g_r]$ with $g_i : X_i \to Z$. Hence, $g_i \in \mathsf{rad}^n(X_i, Z)$; there exists $i_0 \in \{1, \ldots, r\}$ such that $g_{i_0} \notin \mathsf{rad}^{n+1}(X_{i_0}, Z)$; and $\sum_i g_i f_i \in \mathsf{rad}^{n+2}(X, Z)$.

For every i, there exists $(\phi_{i,z})_z \in \bigoplus_{Fz=Z} \mathfrak{R}^n\mathsf{k}(\widetilde{\Gamma})(x_i, z)$ such that $g_i = \sum_z F(\phi_{i,z}) \mod \mathsf{rad}^{n+1}$, and also there exists z_0 such that $Fz_0 = Z$ and $\phi_{i_0,z_0} \notin \mathfrak{R}^{n+1}\mathsf{k}(\widetilde{\Gamma})(x_{i_0}, z_0)$, because of Theorem B and the above properties of the g_i. In particular, the paths in $\widetilde{\Gamma}$ from x_{i_0} to z_0 all have length at most n, and, therefore, the paths from x to z_0 all have length at most $n + 1$.

On the other hand, $\sum_i g_i f_i = \sum_z F(\sum_i \phi_{i,z}\overline{\alpha}_i) \mod \mathsf{rad}^{n+2}$ lies in $\mathsf{rad}^{n+2}(X, Z)$. Hence, $\sum_i \phi_{i,z}\overline{\alpha}_i \in \mathfrak{R}^{n+2}\mathsf{k}(\widetilde{\Gamma})(x, z)$ for every z, because of Theorem B. This and the above property on the length of the paths in $\widetilde{\Gamma}$ from x to z_0 imply that $\sum_i \phi_{i,z_0}\overline{\alpha}_i = 0$, We then set $h_i = F(\phi_{i,z_0}) : X_i \to Z$ and $h = [h_1, \ldots, h_r] : Y \to Z$. Then $hf = F(\sum_i \phi_{i,z_0}\overline{\alpha}_i) = 0$, $h \in \mathsf{rad}^n(Y, Z)$ because $\phi_{i,z_0} \in \mathfrak{R}^n\mathsf{k}(\widetilde{\Gamma})(x_i, z_0)$ for every i, and $h \notin \mathsf{rad}^{n+1}(Y, Z)$ because $\phi_{i_0,z_0} \notin \mathfrak{R}^{n+1}\mathsf{k}(\widetilde{\Gamma})(x_{i_0}, z_0)$ (see Theorem B). □

REMARK 3.1. Keep the notation of Theorem C.

(a) If $d_l(f) = n$, then, by definition, there exist $Z \in \Gamma$ and $g \in \mathsf{rad}^n(Z, X)\backslash\mathsf{rad}^{n+1}(Z, X)$ such that $fg \in \mathsf{rad}^{n+2}(Z, X_1 \oplus \ldots \oplus X_r)$. The proof of Theorem C shows that there exists $h \in \mathsf{rad}^n(Z, X)\backslash\mathsf{rad}^{n+1}(Z, X)$ such that $fh = 0$ (that is, the domain of h is equal to the domain of g). Of course, the same remark holds true if $d_r(f) = n$.

(b) It is still an open question to know whether the morphism h in Theorem C can be chosen to be a composition of irreducible morphisms (instead of a sum of compositions of such). Recall that this is indeed the case if $\alpha(\Gamma) \leqslant 2$ (see [**8**]), where $\alpha(\Gamma)$ is the maximum number of summands in the middle terms of almost split sequences in Γ.

Now we derive some consequences of Theorem C. The following corollary follows directly from Theorem C. We omit its proof. Note that it was proved in [**4**] for irreducible morphisms between indecomposable modules lying in a standard component.

COROLLARY 3.2. *Let $f : X \to Y$ be an irreducible morphism in $\mathsf{mod}\,A$ with X indecomposable. If $d_l(f)$ is finite, then f is not a monomorphism and $d_r(f) = \infty$. In particular, every minimal left almost split morphism in $\mathsf{mod}\,A$ has infinite left degree.*

The following proposition compares $d_l(f)$ and $d_l(g)$ when there is an almost split sequence of the form $0 \to \tau_A Y \xrightarrow{[g,g']^t} X' \oplus X \xrightarrow{[f',f]} Y \to 0$. Recall that it was proved in [11, 1.2] that $d_l(f) < \infty$ implies $d_l(g) \leqslant d_l(f) - 1$ (in the more general setting of artin algebras). Note that the following result was proved in [4] in the case where the indecomposable module Y lies in a standard component.

Proposition 3.3. *Let $f : X \to Y$ be an irreducible morphism with Y indecomposable and non-projective. Assume that the almost split sequence in* $\mathsf{mod}\,A$

$$\begin{array}{ccccccccc} & & & \overset{g}{\nearrow} & X' & \overset{f'}{\searrow} & & & \\ 0 & \longrightarrow & \tau_A Y & & & & Y & \longrightarrow & 0 \\ & & & \underset{g'}{\searrow} & X & \underset{f}{\nearrow} & & & \end{array}$$

is such that $X' \neq 0$. Then $d_l(f) < \infty$ if and only if $d_l(g) < \infty$. In such a case, $d_l(f) = n$ if and only if $d_l(g) = n - 1$.

Proof. It was proved in [11, 1.2] that if $d_l(f) < \infty$, then $d_l(g) \leqslant d_l(f) - 1$. Conversely, assume that $d_l(g) = m < \infty$. Then there exist $Z \in \mathsf{ind}\,A$ and $h \in \mathsf{rad}^m(Z, \tau_A Y)\backslash\mathsf{rad}^{m+1} \times (Z, \tau_A Y)$ such that $gh = 0$, because of Theorem C. Consider the morphism $g'h \in \mathsf{rad}^{m+1}(Z, X)$. The morphism $[g, g']^t$ is minimal left almost split so it has infinite left degree, because of Corollary 3.2. Since $[g, g']^t h = [0, g'h]$ we deduce that $g'h \notin \mathsf{rad}^{m+2}(Z, X)$. On the other hand, $fg'h = (fg' + f'g)h = 0$. This proves that $d_l(f) \leqslant m + 1 = d_l(g) + 1$. $\square$

The following proposition is a key step towards Theorem A.

Proposition 3.4. *Let $f : X \to Y$ be an irreducible morphism in* $\mathsf{mod}\,A$ *with X indecomposable, Γ be the Auslander–Reiten component of A containing X and $n \geqslant 1$ be an integer. The following two conditions are equivalent*:

(a) $d_l(f) = n$;
(b) *f is not a monomorphism and the morphism* $\mathsf{ker}(f) : \mathsf{Ker}(f) \hookrightarrow X$ *lies in* $\mathsf{rad}^n(\mathsf{Ker}(f), X)\backslash\mathsf{rad}^{n+1}(\mathsf{Ker}(f), X)$.

These conditions imply the following one:

(c) *f is not a monomorphism and* $\mathsf{Ker}(f) \in \Gamma$.

If Γ is generalized standard, then the three conditions are equivalent.

Proof. If $d_l(f) = n < \infty$, then Theorem C implies that there exist $n \geqslant 0$, $Z \in \Gamma$ and $h \in \mathsf{rad}^n(Z, X)\backslash\mathsf{rad}^{n+1}(Z, X)$ such that $fh = 0$. In particular, f is not a monomorphism (and, therefore, $\mathsf{Ker}(f)$ is indecomposable, because f is irreducible). Therefore, we have a factorization

$$\begin{array}{ccccc} & \overset{\exists}{\swarrow} & Z & & \\ & & \downarrow h & & \\ \mathsf{Ker}(f) & \hookrightarrow & X & \xrightarrow{f} & Y \end{array}$$

which implies that $\mathsf{ker}(f) \notin \mathsf{rad}^{n+1}(\mathsf{Ker}(f), X)$ and, therefore, $\mathsf{Ker}(f) \in \Gamma$. Let i be such that $\mathsf{ker}(f) \in \mathsf{rad}^i(\mathsf{Ker}(f), X)$. So $i \leqslant n$ and, since $f\mathsf{ker}(f) = 0$, we have $d_l(f) \leqslant i$. Thus, $i = n$ and $\mathsf{ker}(f) \in \mathsf{rad}^n(\mathsf{Ker}(f), X)\backslash\mathsf{rad}^{n+1}(\mathsf{Ker}(f), X)$. This proves that (a) implies (b) and (c).

If f is not a monomorphism and $\mathsf{ker}(f) \in \mathsf{rad}^n(\mathsf{Ker}(f), X)\backslash\mathsf{rad}^{n+1}(\mathsf{Ker}(f), X)$, then $\mathsf{Ker}(f) \in \Gamma$. From the equality $f\mathsf{ker}(f) = 0$ we deduce that $d_l(f) \leqslant n < \infty$. Since (a) implies (b) we deduce that $d_l(f) = n$. This proves that (b) implies (a) and (c).

Finally, if Γ is generalized standard and $\mathsf{Ker}(f) \in \Gamma$, then the inclusion $\mathsf{Ker}(f) \hookrightarrow X$ lies on $\mathsf{rad}^n(\mathsf{Ker}(f), X)\backslash\mathsf{rad}^{n+1}(\mathsf{Ker}(f), X)$ for some $n \geqslant 1$ because $\mathsf{rad}^\infty(\mathsf{Ker}(f), X) = 0$. Thus, (b) and, therefore, (a) hold true. □

Keep the notation of Proposition 3.4 and of its proof and assume that $d_l(f) = n$. Let $g : Z \to \mathsf{Ker}(f)$ be a morphism such that $\mathsf{ker}(f)g = h$. Both morphisms $\mathsf{ker}(f)$ and h lie in $\mathsf{rad}^n\backslash\mathsf{rad}^{n+1}$ so that $g \notin \mathsf{rad}(Z, \mathsf{Ker}(f))$. Since both Z and $\mathsf{Ker}(f)$ are indecomposable, we deduce that $g : Z \to \mathsf{Ker}(f)$ is an isomorphism. In other words, we have the following corollary.

Corollary 3.5. *Let $f : X \to Y$ be an irreducible morphism with X indecomposable. If $d_l(f) = n$ and if there exist $Z \in \mathsf{ind}\, A$ and $h \in \mathsf{rad}^n(Z, X)\backslash\mathsf{rad}^{n+1}(Z, X)$ such that $fh = 0$, then $h = \mathsf{ker}(f)$.*

Proof. This follows from the arguments given before the corollary. □

Using Proposition 3.4 we can prove the following result.

Corollary 3.6. *Let $f, f' : X \to Y$ be irreducible morphisms in $\mathsf{mod}\, A$ with X indecomposable.*

(a) *If f has finite left degree, then $d_l(f) = d_l(f')$ and $\mathsf{Ker}(f) \simeq \mathsf{Ker}(f')$.*
(b) *If f has finite right degree, then $d_r(f) = d_r(f')$.*

Proof. (a) Write $Y = X_1 \oplus \ldots \oplus X_r$ with $X_1, \ldots, X_r$ indecomposable. Let $f_i, f_i' : X \to X_i$ ($i \in \{1, \ldots, r\}$) be the morphisms such that $f = [f_1 \ldots f_r]^t$ and $f' = [f_1' \ldots f_r']^t$. By [**12**, Lemma 1.3], the irreducible morphisms $f_1, \ldots, f_r$ all have finite left degree. By [**11**, Lemma 1.7], we deduce that for every i there exist a scalar $\lambda_i \in \mathsf{k}^*$ and a morphism $r_i \in \mathsf{rad}^2(X, X_i)$ such that $f_i' = \lambda_i f_i + r_i$. This clearly implies that $d_l(f) = d_l(f')$. Let $n = d_l(f)$ and $\iota : \mathsf{Ker}(f) \hookrightarrow X$ be the inclusion. By Proposition 3.4 we know that $\iota \in \mathsf{rad}^n(\mathsf{Ker}(f), X)\backslash\mathsf{rad}^{n+1}(\mathsf{Ker}(f), X)$. On the other hand, we have $f_i'\iota = r_i\iota \in \mathsf{rad}^{n+2}(\mathsf{Ker}(f), X_i)$ for every i, that is, $f'\iota \in \mathsf{rad}^{n+2}(\mathsf{Ker}(f), Y)$. By Theorem C and Remark 3.1, we infer that there exists $h \in \mathsf{rad}^n(\mathsf{Ker}(f), X)\backslash\mathsf{rad}^{n+1}(\mathsf{Ker}(f), X)$ such that $f'h = 0$. Finally, Corollary 3.5 implies that $\mathsf{Ker}(f) \simeq \mathsf{Ker}(f')$.

(b) If X is injective, then there exist $U \in \mathsf{mod}\, A$ and morphisms $u, u' : X \to U$ such that both $[f, u]^t$ and $[f', u']^t$ are minimal right almost split morphisms $X \to Y \oplus U$. The dual version of Corollary 3.2 implies that both $[f, u]^t$ and $[f', u']^t$ have infinite right degree and, therefore, so do f and f'. Therefore, X is not injective and there are minimal almost split sequences in $\mathsf{mod}\, A$

$$\begin{array}{ccccc} & & Y & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ X & & & & \tau_A^{-1}X \\ & \underset{h}{\searrow} & & \underset{i}{\nearrow} & \\ & & Y' & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & Y & & \\ & \overset{f'}{\nearrow} & & \overset{g'}{\searrow} & \\ X & & & & \tau_A^{-1}X. \\ & \underset{h'}{\searrow} & & \underset{i'}{\nearrow} & \\ & & Y' & & \end{array}$$

The dual version of Proposition 3.3 applied to the first sequence yields $d_r(i) = d_r(f) - 1$. Then the dual version of (a) applied to $i, i' : Y' \to \tau_A^{-1}X$ gives $d_r(i) = d_r(i')$. Finally, the dual version of Proposition 3.3 applied to the second sequence above yields $d_r(i') = d_r(f') - 1$. Thus $d_r(f') = d_r(f)$. $\square$

The following example shows that Corollary 3.6 does not necessarily hold true if one drops the finiteness condition on the left degree.

EXAMPLE 3.7. Let A be the path algebra of the following quiver of type $\widetilde{A}_2$:

$$\begin{array}{ccccc} & & 2 & & \\ & \nearrow & & \searrow & \\ 1 & & \longrightarrow & & 3. \end{array}$$

Given a vertex x, we write I_x for the corresponding indecomposable injective A-module. So the irreducible epimorphism $f : I_3 \twoheadrightarrow I_1$ has infinite left degree (see, for example, [**8**, Corollary 4.10]). Then $\mathsf{Ker}(f)$ is as follows:

$$\begin{array}{ccccc} & & \mathsf{k} & & \\ & \overset{\mathsf{Id}}{\nearrow} & & \overset{\mathsf{Id}}{\searrow} & \\ \mathsf{k} & & \underset{0}{\longrightarrow} & & \mathsf{k}. \end{array}$$

On the other hand, let $\mu \in \mathsf{rad}^2(I_3, I_1)$ be the composition $I_3 \twoheadrightarrow I_2 \twoheadrightarrow I_1$ of the two irreducible morphisms. Then $f' = f + \mu : I_3 \twoheadrightarrow I_1$ is also irreducible and its kernel is as follows:

$$\begin{array}{ccccc} & & \mathsf{k} & & \\ & \overset{\mathsf{Id}}{\nearrow} & & \overset{\mathsf{Id}}{\searrow} & \\ \mathsf{k} & & \underset{\mathsf{Id}}{\longrightarrow} & & \mathsf{k}. \end{array}$$

Clearly, $\mathsf{Ker}(f)$ and $\mathsf{Ker}(f')$ lie in distinct homogeneous tubes and are therefore non-isomorphic.

The result below follows from Proposition 3.4. It was first proved for standard algebras in [**4**].

COROLLARY 3.8. *Let A be of finite-representation type and $f : X \to Y$ be an irreducible morphism with X or Y indecomposable. Then the following conditions are equivalent*:

(a) $d_l(f) < \infty$;
(b) $d_r(f) = \infty$;
(c) *f is an epimorphism.*

Proof. Assume first that X is indecomposable. If $d_l(f) < \infty$, then f is not a monomorphism (and, therefore, it is an epimorphism, because it is irreducible) and $d_r(f) = \infty$, because of the dual version of Corollary 3.2. So (a) implies (b) and (c).

If $d_r(f) < \infty$, then f is not an epimorphism, because of Theorem C. Therefore, (c) implies (b). Conversely, if f is not an epimorphism, then $\mathsf{Coker}(f) \in \Gamma(\mathsf{mod}\, A)$ and there exists an integer $n \geqslant 1$ such that $\mathsf{coker(f)} : Y \twoheadrightarrow \mathsf{Coker}(f)$ lies in $\mathsf{rad}^n(Y, \mathsf{Coker}(f)) \backslash \mathsf{rad}^{n+1}(Y, \mathsf{Coker}(f))$ (indeed, $\mathsf{rad}^\infty = 0$ because A is representation-finite). Thus, (b) implies (c) and these two conditions are therefore equivalent.

Finally, assume that f is an epimorphism. Then f is not a monomorphism, $\mathsf{Ker}(f) \in \Gamma(\mathsf{mod}\, A)$ and, as above, $\mathsf{ker}(f) : \mathsf{Ker}(f) \hookrightarrow X$ lies in $\mathsf{rad}^n(\mathsf{Ker}(f), X) \backslash \mathsf{rad}^{n+1}(\mathsf{Ker}(f), X)$. In particular, $d_l(f) < \infty$. Thus (c) implies (a) and, therefore, the three conditions are equivalent if X is indecomposable.

If Y is indecomposable, then, using dual arguments, one proves that the following conditions are equivalent:

(i) $d_l(f) = \infty$;
(ii) $d_r(f) < \infty$;
(iii) f is a monomorphism.

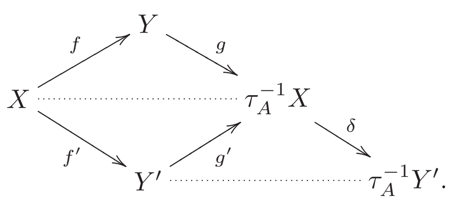
Since an irreducible morphism is either a monomorphism or an epimorphism, this proves that (a)–(c) are equivalent. □

We end this section with another application of Theorem C: the description of the irreducible morphisms with indecomposable domain or indecomposable codomain and with (left or right) degree equal to 2. Again, each statement has its dual counterpart which is omitted. We thus restrict our study to irreducible morphisms with indecomposable domain. We start with a characterization of the equality $d_r(f) = 2$.

COROLLARY 3.9. *Let $f : X \to Y$ be an irreducible morphism with X indecomposable. Then the following conditions are equivalent.*

(a) $d_r(f) = 2$.

(b) *The module X is not injective and there exists an almost split sequence $0 \to X \xrightarrow{[f,f']^t} Y \oplus Y' \xrightarrow{[g,g']} \tau_A^{-1}X \to 0$ with Y' indecomposable non-injective fitting into an almost split sequence $0 \to Y' \xrightarrow{g'} \tau_A^{-1}X \xrightarrow{\delta} \tau_A^{-1}Y' \to 0$. In other words, there is a configuration of almost split sequences in $\mathsf{mod}\, A$*



Proof. Assume that (a) holds true. Then X is not injective (see [**11**, 1.3]) and f is not a minimal left almost split monomorphism (see [**11**, 1.12]). So there is an almost split sequence $0 \to X \xrightarrow{[f,f']^t} Y \oplus Y' \xrightarrow{[g,g']} \tau_A^{-1}X \to 0$ with $Y' \in \mathsf{mod}\, A$ non-zero. On the other hand, there exist $M \in \mathsf{ind}\, A$ and $h \in \mathsf{rad}^2(Y, M)\backslash\mathsf{rad}^3(Y, M)$ such that $hf = 0$, because $d_r(f) = 2$ and because of the dual version of Theorem C. Since $[h, 0][f, f']^t = 0$, there exists $h' : \tau_A^{-1}X \to M$ such that $h'g = h$ and $h'g' = 0$. Clearly, $h' \in \mathsf{rad}(\tau_A^{-1}X, M)\backslash\mathsf{rad}^2(\tau_A^{-1}X, M)$, because g is irreducible and $h = h'g \in \mathsf{rad}^2(Y, M)\backslash\mathsf{rad}^3(Y, M)$. Hence, $d_r(g') = 1$. Using [**11**, 1.12], we deduce that Y' is indecomposable not injective and g' is minimal left almost split. This proves that (b) holds true.

Conversely, assume that (b) holds true. We prove that so does (a). In particular, X is non-injective and f is not minimal left almost split. Using [**11**, 1.12], we infer that $d_r(f) \geqslant 2$. Consider the morphism $\delta g \in \mathsf{rad}^2(Y, \tau_A^{-1}Y')$. Let Z be an indecomposable summand of Y and let $Z \to \tau_A^{-1}X$ be the composition of $g : Y \to \tau_A^{-1}X$ with the section $Z \to Y$. Then $Z \not\simeq Y'$, because the almost split sequence starting from Y' has its middle term indecomposable. Therefore, $Z \to \tau_A^{-1}X \xrightarrow{\delta} \tau_A^{-1}Y'$ is a sectional path of irreducible morphisms so that its composite lies in $\mathsf{rad}^2(Z, \tau_A^{-1}Y')\backslash\mathsf{rad}^3(Z, \tau_A^{-1}Y')$ (see [**10**]). Thus, $\delta g \in \mathsf{rad}^2(Y, \tau_A^{-1}Y')\backslash\mathsf{rad}^3(Y, \tau_A^{-1}Y')$ and $\delta g f = -\delta g' f' = 0$. This proves that $d_r(f) = 2$. So (b) implies (a). □

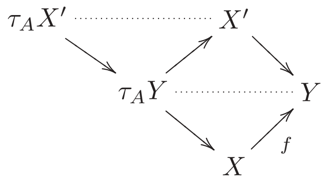
We now give the characterization of the equality $d_l(f) = 2$. The following corollary was first proved in [**4**] for irreducible morphisms in standard components. Using Theorem C, the proof given in [**4**] generalizes to any Auslander–Reiten component. We thus refer the reader to [**4**] for a detailed proof.

Corollary 3.10. *Let $f : X \to Y$ be an irreducible morphism with X indecomposable. Then the following conditions are equivalent.*

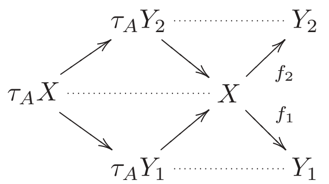
(a) $d_l(f) = 2$.

(b) *The module Y is the direct sum of at most two indecomposables; f is not minimal right almost split; and there exist $Z \in \mathsf{ind}\, A$ and a path of irreducible morphisms with composite h lying on $\mathsf{rad}^2(Z, X)\backslash\mathsf{rad}^3(Z, X)$ and such that $fh = 0$.*

(c) *The translation quiver $\Gamma(\mathsf{mod}\, A)$ contains one of the following two configurations of meshes:*



with $Y, X' \in \mathsf{ind}\, A$, or



with $Y_1, Y_2 \in \mathsf{ind}\, A$, $Y = Y_1 \oplus Y_2$ and $f = [f_1, f_2]^t$.

4. *Algebras of finite-representation type*

In this section we prove Theorem A. First we need two lemmas.

Lemma 4.1. *Let S be a simple A-module, $S \hookrightarrow I$ be its injective hull and $X \in \mathsf{ind}\, A$ be such that S is a direct summand of $\mathsf{soc}(X)$. Assume that $I \twoheadrightarrow I/\mathsf{soc}(I)$ has finite left degree equal to n. Then there is a path in $\Gamma(\mathsf{mod}\, A)$ starting from S, ending at I, of length at most n and going through X. In particular X, S and I lie in the same component of $\Gamma(\mathsf{mod}\, A)$.*

Proof. Let Γ be the component containing I. We denote by π the irreducible epimorphism $I \twoheadrightarrow I/\mathsf{soc}(I)$ and by $\iota : S \hookrightarrow I$ the injective hull. It follows from Proposition 3.4 applied to π that $S \in \Gamma$ and $\iota \in \mathsf{rad}^n(S, I)\backslash\mathsf{rad}^{n+1}(S, I)$. Since S is a direct summand of $\mathsf{soc}(X)$, the injective hull ι factors through X, that is, is equal to some composition $S \xrightarrow{f} X \xrightarrow{g} I$. Therefore there exist $l, m \geqslant 1$ such that $f \in \mathsf{rad}^l(S, X)\backslash\mathsf{rad}^{l+1}(S, X)$, $g \in \mathsf{rad}^m(X, I)\backslash\mathsf{rad}^{m+1}(X, I)$ and $l + m \leqslant n$. Therefore, f or g is a sum of compositions of paths of irreducible morphisms at least one of which has length l or m, respectively. In particular, P, X and S all lie in Γ. □

Of course, Lemma 4.1 has a dual statement which holds true using dual arguments. If S is a simple A-module with projective cover $P \twoheadrightarrow S$ such that $\mathsf{rad}\, P \hookrightarrow P$ has finite right degree equal to n and if S is a direct summand of $\mathsf{top}(X)$ for some $X \in \mathsf{ind}\, A$, then there exists a path in $\Gamma(\mathsf{mod}\, A)$ starting from P, ending at S, going through X and of length at most n. In particular, P, X and S all lie in the same component of $\Gamma(\mathsf{mod}\, A)$.

Lemma 4.2. *Assume A is connected and that for every indecomposable injective I the quotient morphism $I \twoheadrightarrow I/\mathsf{soc}(I)$ has finite left degree. Let n be the supremum of all these left degrees. Then, for every $X \in \mathsf{ind}\, A$, there exists a path in $\Gamma(\mathsf{mod}\, A)$ starting from X, ending at some injective and of length at most n. In particular, $\Gamma(\mathsf{mod}\, A)$ is finite and connected.*

Proof. The first assertion follows directly from Lemma 4.1. In order to prove the second one it suffices to prove that for all I, J indecomposable injectives, I and J lie on the same component of $\Gamma(\mathsf{mod}\, A)$. Since A is connected there exists a sequence $I_0 = I, I_1, \ldots, I_l = Q$ of indecomposable injectives such that, letting $S_i = \mathsf{soc}(I_i)$, we have that S_i is a direct summand of $\mathsf{soc}(I_{i-1}/\mathsf{soc}(I_{i-1}))$ or S_{i-1} is a direct summand of $\mathsf{soc}(I_i/\mathsf{soc}(I_i))$ for every $i = 1, \ldots, l$. Accordingly, Lemma 4.1 implies that there exists a path $I_i \rightsquigarrow I_{i-1}$ or $I_{i-1} \rightsquigarrow I_i$, respectively, in $\Gamma(\mathsf{mod}\, A)$. This proves that I and J lie on the same component of $\Gamma(\mathsf{mod}\, A)$. Since $\Gamma(\mathsf{mod}\, A)$ is locally finite, this proves the lemma. □

Note that the dual statement of Lemma 4.2 holds true using dual arguments and the dual version of Lemma 4.1. In a previous version of this paper, Lemma 4.2 assumed an additional condition, dual to that on the degree of the morphisms $I \twoheadrightarrow I/\mathsf{soc}(I)$ for I injective. The authors thank Juan Cappa for pointing out that this dual statement was unnecessary.

Now we can prove Theorem A.

Proof of Theorem A. If A is of finite-representation type, then $\Gamma(\mathsf{mod}\, A)$ is connected and $\mathsf{rad}^\infty = 0$ (for example, this follows from the Lemma of Harada and Sai) and conditions (b) and (c) follow from Proposition 3.4 and its dual. The implications (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a) follow from Lemma 4.2 and from its dual version, respectively. Thus, conditions (a)–(c) are equivalent. Note that (d) implies (c), and (e) implies (b). On the other hand, Corollary 3.8 and its dual version show that (a) implies both (d) and (e). Therefore, conditions (a)–(e) are equivalent. □

Remark 4.3. Our arguments allow us to recover the following well-known implication using degrees of irreducible morphisms only. *If* $\mathsf{rad}^\infty = 0$, *then A is of finite-representation type and* $\Gamma(\mathsf{mod}\, A)$ *is connected.* Indeed, if $\mathsf{rad}^\infty = 0$ then both (b) and (c) hold true in Theorem A. So A is of finite-representation type and $\Gamma(\mathsf{mod}\, A)$ is connected.

5. *Composition of morphisms*

Let A be a finite-dimensional k-algebra and Γ be a component of $\Gamma(\mathsf{mod}\, A)$. In view of Remark 3.1(b), there seems to be a connection between the degree of an irreducible morphism and the behaviour of the composite of n irreducible morphisms between indecomposable modules (for any n). This motivates the work of the present section, that is, to study when the composite of n irreducible morphisms between indecomposable modules lies in rad^{n+1}. The following result characterizes such a situation when Γ has *trivial valuation* (that is, has no multiple arrows).

Proposition 5.1. *Let $n \geqslant 1$ be an integer and $X_1, \ldots, X_{n+1}$ be modules in Γ. Consider the following assertions.*

(a) *There exist irreducible morphisms $h_i : X_i \to X_{i+1}$ for every i such that $h_n \ldots h_1 \in \mathsf{rad}^{n+1}(X_1, X_{n+1})\backslash\{0\}$.*

(b) *There exist irreducible morphisms $f_i : X_i \to X_{i+1}$ together with morphisms $\varepsilon_i : X_i \to X_{i+1}$ such that $f_n \ldots f_1 = 0$, $\varepsilon_n \ldots \varepsilon_1 \neq 0$ and $\varepsilon_i = f_i$ or $\varepsilon_i \in \mathsf{rad}^2(X_i, X_{i+1})$ for every i.*

Then (b) *implies* (a). *Also, if $h_1, \ldots, h_n$ satisfy* (a) *and represent arrows with trivial valuation, then* (b) *holds true. In particular,* (a) *and* (b) *are equivalent if Γ has trivial valuation.*

Proof. Let $h_i : X_i \to X_{i+1},\ i = 1, \ldots, n$, be irreducible morphisms in $\operatorname{ind} A$ such that $h_n \ldots h_1 \in \operatorname{rad}^{n+1}(X_1, X_{n+1})\backslash\{0\}$ and such that the arrows represented by $h_1, \ldots, h_n$ have trivial valuation. Let $F : \mathsf{k}(\widetilde{\Gamma}) \to \operatorname{ind} \widetilde{\Gamma}$ be a well-behaved functor with respect to the generic covering $\pi : \widetilde{\Gamma} \to \Gamma$ and let $x_1 \in F^{-1}(X_1)$. Since $\pi : \widetilde{\Gamma} \to \Gamma$ is a covering of quivers and the arrow represented by h_1 has trivial valuation, there is exactly one arrow $x_1 \xrightarrow{\alpha_1} x_2$ in $\widetilde{\Gamma}$ starting from x_1 and such that $Fx_2 = X_2$. By repeating the same argument, we deduce that there is exactly one path in $\widetilde{\Gamma}$:

$$x_1 \xrightarrow{\alpha_1} x_2 \longrightarrow \cdots \longrightarrow x_n \xrightarrow{\alpha_n} x_{n+1}$$

starting from x_1 of length n and such that $Fx_i = X_i$ for every i. Let $i \in \{1, \ldots, n\}$; then $F(\overline{\alpha}_i) : X_i \to X_{i+1}$ is irreducible so that $h_i = \lambda_i F(\overline{\alpha}_i) + h'_i$, where $\lambda_i \in \mathsf{k}^*$ and $h'_i \in \operatorname{rad}^2(X_i, X_{i+1})$ because $\pi(\alpha_i)$, represented by h_i, has trivial valuation. Since $h_n \ldots h_1 \neq 0$, we have a non-zero morphism:

$$\lambda F(\overline{\alpha_n \ldots \alpha_1}) + \sum_{t=1}^{n} \sum_{i_1 < \ldots < i_t} F(\overline{\alpha}_n) \ldots F(\overline{\alpha}_{i_t+1}) h'_{i_t} F(\overline{\alpha}_{i_t-1}) \ldots F(\overline{\alpha}_{i_1+1}) h'_{i_1} F(\overline{\alpha}_{i_1-1}) \ldots F(\overline{\alpha}_1), \qquad (\star)$$

where $\lambda = \lambda_1 \ldots \lambda_n \in \mathsf{k}^*$ and the whole sum lies on $\operatorname{rad}^{n+1}(X_1, X_{n+1})$. In particular, $F(\overline{\alpha_n \ldots \alpha_1})$ lies on $\operatorname{rad}^{n+1}(X_1, X_{n+1})$. By Theorem B, we have $\overline{\alpha_n \ldots \alpha_1} \in \mathfrak{R}^{n+1}\mathsf{k}(\widetilde{\Gamma})(x_1, x_{n+1})$. Since $\widetilde{\Gamma}$ is a component with length, we deduce that $\overline{\alpha_n \ldots \alpha_1} = 0$ and therefore $F(\overline{\alpha}_n) \ldots F(\overline{\alpha}_1) = 0$. This and $(\star)$ imply that there exist $t \in \{1, \ldots, n\}$ and $i_1 < \ldots < i_t$ such that

$$F(\overline{\alpha}_n) \ldots F(\overline{\alpha}_{i_t+1}) h'_{i_t} F(\overline{\alpha}_{i_t-1}) \ldots F(\overline{\alpha}_{i_1+1}) h'_{i_1} F(\overline{\alpha}_{i_1-1}) \ldots F(\overline{\alpha}_1) \neq 0\ . \qquad (\star\star)$$

We thus let:

(i) $f_i = F(\overline{\alpha}_i)$ for every $i \in \{1, \ldots, n\}$; so $f_i : X_i \to X_{i+1}$ is irreducible because $F : \mathsf{k}(\widetilde{\Gamma}) \to \operatorname{ind} \Gamma$ is well-behaved;
(ii) $\varepsilon_{i_j} = h'_{i_j}$ for every $j \in \{1, \ldots, t\}$; so $\varepsilon_{i_j} \in \operatorname{rad}^2(X_{i_j}, X_{i_j+1})$;
(iii) $\varepsilon_i = f_i$ for every $i \in \{1, \ldots, n\}\backslash\{i_1, \ldots, i_t\}$.

In particular, $\varepsilon_n \ldots \varepsilon_1 \neq 0$ because of $(\star\star)$. The morphisms f_i and ε_i $(i \in \{1, \ldots, n\})$ satisfy the conclusion of (b). This proves (b) when $h_1, \ldots, h_n$ satisfy (a) and represent arrows with trivial valuation.

For the implication (b) implies (a), we refer the reader to the proof of [9, Theorem 2.7] (where the standard hypothesis made therein is not used for that implication).

The equivalence between (a) and (b) when Γ has trivial valuation follows from the above considerations. □

Recall [6] that the path $h_1, \ldots, h_n$ is called *almost sectional* if either the path $X_1 \xrightarrow{h_1} X_2 \to \cdots \to X_{n-1} \xrightarrow{h_{n-1}} X_n$ is sectional and $X_{n-1} = \tau_A X_{n+1}$ or else the path $X_2 \xrightarrow{h_2} X_3 \to \cdots \to X_n \xrightarrow{h_n} X_{n+1}$ is sectional and $X_1 = \tau_A X_3$.

Remark 5.2. (a) Let $h_1 : X_1 \to X_2, \ldots, h_n : X_n \to X_{n+1},\ i = 1, \ldots, n$, be irreducible morphisms such that $h_n \ldots h_1 \in \operatorname{rad}^{n+1}(X_1, X_{n+1})\backslash\{0\}$. Under additional assumption such as, $\alpha(\Gamma) \leqslant 2$ (see [8]), or $n = 2$ (see [5]), or $n = 3$ (see [7]) or the path $h_1, \ldots, h_n$ is almost sectional (see [6]), it was shown that the arrows in Γ represented by $h_1, \ldots, h_n$ all have trivial valuation.

(b) It is still an open question to know whether $h_n \ldots h_1 \in \operatorname{rad}^{n+1}(X_1, X_{n+1})\backslash\{0\}$ implies that the arrow $X_i \to X_{i+1}$ has trivial valuation for $i = 1, \ldots, n$.

Our last result concerns sums of composites of paths in a sectional family (Definition 2.2). Note that this result extends the well-known result of Igusa and Todorov [10] and which asserts that if $\cdot \xrightarrow{f_1} \cdot \to \cdots \to \cdot \xrightarrow{f_l} \cdot$ is a sectional path of irreducible morphisms between indecomposables, then the composite $f_n \dots f_1$ does not lie in rad^{n+1} and, therefore, is non-zero. Recall that a sectional path of irreducible morphisms between indecomposables is a particular case of a sectional family of paths (Remark 2.3).

PROPOSITION 5.3. *Let X and Y be indecomposable modules in Γ. Let $\{X \xrightarrow{f_{i,1}} X_{i,1} \to \cdots \to X_{i,l_i-1} \xrightarrow{f_{i,l_i}} Y\}_{i=1,\dots,r}$ be a sectional family of paths starting from X, ending at Y and of irreducible morphisms between indecomposables. Let $n = \mathsf{min}_{i=1,\dots,r}\, l_i$. Then $\sum_{i=1}^{r} f_{i,l_i} \dots f_{i,1}$ lies in $\mathsf{rad}^n(X,Y)$ and does not lie in $\mathsf{rad}^{n+1}(X,Y)$. In particular it is non-zero.*

Proof. Let $\pi : \widetilde{\Gamma} \to \Gamma$ be the generic covering and let $x \in \pi^{-1}(X)$. We apply Proposition 2.4 from which we adopt the notation $(x_{i,j}, \alpha_{i,j})$. In particular, there exists a well-behaved functor $F : \mathsf{k}(\widetilde{\Gamma}) \to \mathsf{ind}\,\Gamma$ such that $F(\overline{\alpha_{i,j}}) = f_{i,j}$ for every i, j.

For every $y \in \pi^{-1}(Y)$ let I_y be the set of indices such that $x_{i,l_i} = y$. For each i, let u_i be the path $x \xrightarrow{\alpha_{i,1}} x_{i,1} \to \cdots \to x_{i,l_i-1} \xrightarrow{\alpha_{i,l_i}} x_{i,l_i}$. Therefore, there exists some $y_0 \in \pi^{-1}(Y)$ such that I_{y_0} is non-empty and all the paths u_i, $i \in I_{y_0}$, have length n. Moreover, each path u_i, for $i \in \{1, \dots, r\}$, is sectional, because $\pi : \widetilde{\Gamma} \to \Gamma$ is a covering of translation quivers and $\pi(u_i)$ is a sectional path $X \to X_{i,1} \to \cdots \to X_{i,l_i-1} \to X_{i,l_i}$ (Remark 2.3(1)).

Clearly, the sum $\sum_{i=1}^{r} f_{i,l_i} \dots f_{i,1}$ equals $\sum_{i=1}^{r} F(\overline{u_i})$ and lies in $\mathsf{rad}^n(X,Y)$. By absurd, assume that it lies in $\mathsf{rad}^{n+1}(X,Y)$. Using Theorem B, we deduce that $\sum_{i \in I_y} \overline{u_i} \in \mathfrak{R}^{n+1}\mathsf{k}(\widetilde{\Gamma})(x,y)$ for every $y \in \pi^{-1}(Y)$. In particular, $\sum_{i \in I_{y_0}} \overline{u_i} \in \mathfrak{R}^{n+1}\mathsf{k}(\widetilde{\Gamma})(x,y_0)$. This contradicts Proposition 1.2(e), because the paths u_i, for $i \in I_{y_0}$, are sectional and of length n. □

We end this paper with a final remark.

REMARK 5.4. It is a natural question to ask whether the results in this paper still hold true if k is an artin ring and A is an artin k-algebra. In a forthcoming paper, we shall explain that this is the case when k is a perfect field. In particular, in this more general context, the equivalence in Proposition 5.1 will be proved without assumption on valuations.

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