A NON-CONSTRUCTIVE PROOF OF THE EXISTENCE OF STABLE MATCHINGS IN THE MARRIAGE MODEL

Juan Carlos Cesco

Abstract: In this note we present a non-constructive proof of the existence of stable matchings in the marriage model which uses a game theoretic approach. To this end, we develop a theory of hedonic partitioning games. Our approach differs from that used by Sotomayor (1996) based upon fixed point theory.

Keywords: *Marriage model, stable matchings, hedonic games*
2000 AMS Subject Classification: 91A12

1 INTRODUCTION

The seminal paper of Gale and Shapley [2] was the starting point of modern matching theory. There, the marriage model is developed and the central notion of stable matching is introduced. A proof of its existence for any marriage model is carried out by designing a computational procedure, the deferred acceptance algorithm, which is proven to converge to a stable matching. It was until 1996 when Sotomayor [6] presented the first non-constructive proof about the existence of stable matchings in the marriage model. She used a fixed point approach to get a very simple proof. In this note, from another point of view, which takes advantages of the relationships existing between two models of coalition formation, namely, matching models and hedonic games, we present another non-constructive proof of the existence of stable matchings for the marriage model. Toward this end, we study properties of a subclass of hedonic games with a restricted family of coalitions which captures, in a hedonic framework, the relevant characteristics of the partitioning games studied by Kaneko and Wooders [4] in the context of games with and without transferable utility.

2 PARTITIONING HEDONIC GAMES

We start with a finite set $N = \{1, ..., n\}$ whose elements are going to be called the players, while a subset of them will be a coalition. Given any family $\mathcal{B}$ of coalitions, and a player $i \in N$, let us denote by $\mathcal{B}(i)$ the subfamily of those coalitions in $\mathcal{B}$ containing player i. A family of coalitions $\mathcal{A}$ such that $\{i\} \in \mathcal{A}$ for all $i \in N$ is called a family of basic coalitions (Kaneko and Wooders [4]). A hedonic game with $\mathcal{B}$ as its family of basic coalitions is a 3-tuple $(N, \succeq; \mathcal{A})$, where N is the set of players and $\succeq = (\succeq_i)_{i\in N}$ is a preference profile with $\succeq_i$ being a reflexive, complete and transitive binary relation on $\mathcal{A}(i)$ for each $i \in N$. An individual preference is strict if $S \succeq_i T$ and $S \neq T$ implies that not $T \succeq_i S$). For each $i \in N, \succeq_i$ will stand for the strict preference relation related to $\succeq_i$ ($S \succ_i T$ iff $S \succeq_i T$ but not $T \succeq_i S$). $\mathcal{P}^{\mathcal{A}}(N)$ will denote the family of partitions of N having all its elements in $\mathcal{A}$. Given $\pi = \{\pi_1, ..., \pi_p\} \in \mathcal{P}^{\mathcal{A}}(N)$ and $i \in N, \pi(i)$ will denote the unique set in π containing player i.

Given a hedonic game $(N, \succeq; \mathcal{A})$ and $\pi \in \mathcal{P}^{\mathcal{A}}(N)$, we say that $T \in \mathcal{A}$ blocks π if for each $i \in T, T \succ_i \pi(i)$. The core $C(N, \succeq; \mathcal{A})$ of $(N, \succeq; \mathcal{A})$ is the set of partitions in $\mathcal{P}^{\mathcal{A}}(N)$ blocked by no coalition $T \in \mathcal{A}$.

A family of non-empty coalitions $\mathcal{B} \subseteq \mathcal{N}$ is called balanced if there exists a set of positive real numbers (the balancing weights) $(\lambda_S)_{S\in\mathcal{B}}$ satisfying $\sum_{S\in\mathcal{B}(i)} \lambda_S = 1$, for all $i \in N$. $\mathcal{B}$ is minimal balanced if there is no proper balanced subfamily of it. In this case, the set of balanced weights is unique. Let us call a family of basic coalitions $\mathcal{A}$ partitionable (Kaneko and Wooders [4]) if the only minimal subfamilies that it contains are partitions.

A family $\mathcal{I} = (\mathcal{I}(A))_{A\in\mathcal{A}}$ is called an $\mathcal{A}$-distribution, or simply a distribution (Iehlé [i]) if, for each non-empty coalition $A \in \mathcal{A}$, $\phi \neq \mathcal{I}(A) \subseteq A$ Given a distribution $\mathcal{I}$, a family $\mathcal{B} \subseteq \mathcal{A}$ is $\mathcal{I}$-balanced if the family $(\mathcal{I}(B))_{B\in\mathcal{B}}$ is balanced.

Definition 1 $(N, \succeq; \mathcal{A})$ *is ordinally balanced if for each balanced family $\mathcal{B} \subseteq \mathcal{A}$ there exists a partition π, whose elements belong to $\mathcal{A}$, such that, for each $i \in N, \pi(i) \succeq_i B$ for some $B \in \mathcal{B}(i)$.*

Definition 2 *$(N, \succeq; \mathcal{A})$ is pivotally balanced with respect to an $\mathcal{A}$-distribution $\mathcal{I}$, if for each $\mathcal{I}$-balanced family $\mathcal{B}$, there exists a partition π whose elements belong to $\mathcal{A}$ such that, for each $i \in N, \pi(i) \succeq_i B$ for some $B \in \mathcal{B}(i)$. The game is pivotally balanced if it is pivotally balanced with respect to some distribution $\mathcal{I}$.*

Note 1 *Ordinal balancedness was first introduced by Bogomolnaia and Jackson [1], while the notion of pivotal balancedness was first employed by Iehlé [3], both notions stated for the case in which $\mathcal{A}$ is the whole family of non-empty coalitions.*

The first part of the following theorem is a sufficient condition for the existence of core-partitions for hedonic games with coalitional restrictions which parallels the first part of Theorem 1 in Bogomolnaia and Jackson [1], while the second part parallels the characterization given in Theorem 3 of Iehlé [3], and whose proofs are carried out in a similar way.

Theorem 1 *Let $(N, \succeq; \mathcal{A})$ be a hedonic game with $\mathcal{A}$ as its family of admissible coalitions. $a)$ If the game is ordinally balanced, and has strict individual preferences, then $C(N, \succeq; \mathcal{A})$ is non-empty. $b)$ $C(N, \succeq; \mathcal{A})$ is non-empty if and only if the game is pivotally balanced.*

Note 2 *Ordinal balancedness implies pivotal balancedness with respect to the distribution $\mathcal{I} = (\chi_A)_{A \in \mathcal{A}}$, being χ_A the indicator vector of the coalition A.*

Definition 3 *A basic partitioning hedonic game is a hedonic game $(N, \succeq; \mathcal{A})$ where the family of admissible coalitions is partitionable.*

Theorem 2 *Let a partitionable essential family of coalitions $\mathcal{A}$ be given. Then, every basic partitioning hedonic game $(N, \succeq; \mathcal{A})$ has non-empty core.*

Proof. The proof follows from the fact that the basic partitioning hedonic game $(N, \succeq; \mathcal{A})$ is ordinally balanced. To see this, let $\mathcal{B}$ be a balanced family of coalitions and because $\mathcal{A}$ is partitionable, $\mathcal{B}$ contains a partition π. Then, since for each $i \in N$ it holds that $\pi(i) \succeq_i \pi(i)$, we conclude that the game is ordinally balanced. Thus, by part $b)$ of Theorem 1 we conclude that its core is non-empty. □

Note 3 *We point out that being the individual preferences in the game not necessarily strict, the non-emptiness of the core is guaranteed by part $b)$ rather than by part $a)$ of Theorem 1.*

3 Existence of stable matchings

As a simple but important consequence of Theorem 2, we derive a new proof about the existence of stable matchings in the marriage model of Gale and Shapley [2]. To do this, we first associate a basic partitioning hedonic game to each matching problem. Then, we will show that the core of the game is related to the set of stable matchings. Finally, we will use the fact that the family of admissible coalitions in the game is partitionable to get our result.

The marriage model consists of two finite sets of agents, the sets M of 'men' and the set W of 'women'. It is assumed that each man $m \in M$ is endowed with a preference $\succeq^m$over the set $W \cup \{m\}$, and that each woman w has a preference $\succeq^w$on the set $M \cup \{w\}$. Individual preferences are assumed to be reflexive, complete and transitive on their corresponding domains. Let us denote by $(M, W, \succeq^M, \succeq^W)$ a marriage problem, where $\succeq^M = (\succeq^m)_{m \in M}$ and $\succeq^W = (\succeq^w)_{w \in W}$ are the preference profile corresponding to the men and women.

A matching is a function $\mu : M \cup W \to M \cup W$ satisfying:

$a)$ For each $m \in M$,if $\mu(m) \neq m$, then $\mu(m) \in W$.

$b)$ For each $w \in W$,if $\mu(w) \neq w$, then $\mu(w) \in M$.

$c)$ $\mu(\mu(k)) = k$ for all $k \in M \cup W$.

A matching μ is stable if $\mu(k) \succeq^k k$ for all $k \in M \cup W$ (individual stability) and if there is no pair $m \in M, w \in W$ such that $\mu(m) \neq w, \mu(w) \neq m$, and $w \succ^m \mu(m)$ and $m \succ^w \mu(w)$ (pairwise stability).

The pair (m, w) is called a blocking pair. Given a matching problem $(M, W, \succeq^M, \succeq^W)$, let us consider the family $\mathcal{A} = \{S \subseteq M \cup W : |S \cap M| = 1 \text{ or } |S \cap W| = 1\}$. Clearly $\mathcal{A}$ is basic. Moreover, it is a partitionable family as follows from the result of Kaneko and Wooders [4] about the existence of core-points for every assignment game (Shapley and Shubik [5]).

With each matching problem $(M, W, \succeq^M, \succeq^W)$ we associate a basic partitioning hedonic game $(N, \hat{\succeq}; \mathcal{A})$ where $N = M \cup W$, and for each $i \in N$, $\hat{\succeq}_i$ is defined on $\mathcal{A}(i)$ as follows. If $i = m$ for some $m \in M$, $S \hat{\succeq}_i T$ if and only if

$S \cap W \succeq^m T \cap W$ if $|S| = |T| = 2$,
$S \cap W \succeq^m m$ if $|S| = 2$ and $T = 1$,
$m \succeq^m T \cap W$ if $|S| = 1$ and $T = 2$.
If $i = w$ for some $w \in W$, $S \hat{\succeq}_i T$ if and only if
$S \cap M \succeq^m T \cap M$ if $|S| = |T| = 2$,
$S \cap M \succeq^m w$ if $|S| = 2$ and $T = 1$,
$w \succeq^m T \cap M$ if $|S| = 1$ and $T = 2$.
For any $i \in N$, we also declare that $S \succeq_i T$ when $|S| = |T| = 1$.

With each partition π in the game $(N, \hat{\succeq}; \mathcal{A})$, we associate the matching μ^π, where, for each $m \in M$, $\mu^\pi(m) = \pi(m) \cap W$ if $|\pi(w)| = 2$ and $\mu^\pi(m) = m$ if $|\pi(m)| = 1$. Similarly, for each $w \in W, \mu^\pi(w) = \pi(w) \cap W$ if $|\pi(w)| = 2$ and $\mu^\pi(w) = w$ if $|\pi(w)| = 1$.

Now, we are ready to state the following result.

Theorem 3 *Let $(M, W, \succeq^M, \succeq^W)$ be a marriage problem. Then, its set of stable matchings is non-empty.*

Proof. From Theorem 1 we get that $C(N, \hat{\succeq}; \mathcal{A}) \neq \phi$. We claim μ^π is a stable matching for each core-partition π. To see this, let $m \in M$. If $\mu^\pi(m) \neq m$, then $|\pi(m)| = 2$ and since π is a core partition, m can not be strictly preferred to $\pi(m)$. Thus, $\pi(m) \hat{\succeq}_m m$, and according to the definition $\hat{\succeq}_m$this implies that $\pi(m) \cap W = \mu^\pi(m) \succeq^m m$. A similar argument shows that, for each $w \in W, \mu^\pi(w) \succeq^w w$ for any $w \in W$ such that $\mu^\pi(w) \neq w$. Then, μ^π is individually stable.

On the other hand, let us assume that there is a blocking pair (m, w) to μ^π. We claim that the coalition $S = \{m, w\}$ objects the partition π, leading to a contradiction. Indeed, from $w \succ^m \mu^\pi(m)$ we get that $S \cap W \succ^m \mu^\pi(m) \cap W$ or, equivalently, that $S \hat{\succ}_m \pi(m)$ when $\mu^\pi(m) \neq m$ ($|\pi(m)| = 2$), and we also get that $S \hat{\succ}^m \pi(m)$ when $\pi(m) = m$ ($|\pi(m)| = 1$). In a similar way we obtain that $m \succ^w \mu^\pi(w)$ implies that $S \hat{\succ}_w \pi(w)$ showing that S blocks π. □

REFERENCES

[1] BOGOMOLNAIA AND A., JACKSON, M. *The stability of hedonic coalition structures.* Games and Economic Behavior 38 (2002), 201-230.

[2] D. GALE AND L. SHAPLEY. *College admissions and the stability of marriage.* American Mathematical Monthly 69 (1962), 9-15.

[3] V. IEHLÉ. emphThe core-partition of a hedonic game. Mathematical Social Sciences 54 (2007), 176-185.

[4] M. KANEKO AND M. WOODERS. *Cores of partitioning games.* Mathematical Social Sciences 3 (1982), 313-327.

[5] L. SHAPLEY AND M. SHUBIK. *The assignment game I: The core.* International Journal of Game Theory 1 (1972), 111-130.

[6] M. SOTOMAYOR. *A non-constructive elementary proof of the existence of stable marriages.* Games and Economic Behavior 13 (1996), 135–137.