BALANCED TU-GAMES WITH A SINGLE POINT CORE

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Abstract: In this note we present a characterization of the sub-class of TU-games having non-empty core with only one point. It generalizes the adding up (AU) property stated in Brandenburger and Stuart [2] in the study of biform games, a new class of models recently introduced by those authors.

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1 INTRODUCTION

The core of a game with transferable utilities (TU-game) is the most appealing and widely studied solution concept for this class of games. However, sometimes it can be the empty set, but in general, when this is not the case, it contains more than one point. The classical theorem of Bondareva [1] and Shapley [5] gives a necessary and sufficient condition for the non-emptiness of the core. Recently Brandenburger and Stuart [2] introduce a sufficient condition guaranteeing that the core has only one element, and it is used extensively to study a rather new class of games, namely, biform games. In this note we generalize that condition and show that it is also necessary to assure that the core of a game contains only one imputation. The note is organized as follows. In the next section we present some basic facts related to the theory of TU-games and the adding up property due to Brandenburger and Stuart [2]. In Section 2 we prove our main result.

2 PRELIMINARIES

A TU-game is an ordered pair (N, v) where $N = \{1, 2, ..., n\}$ is a finite non-empty set, the set of players, v is the *c*characteristic function, which is a real valued function defined on the family of subsets of $N, \mathcal{P}(N)$ satisfying $v(\Phi) = 0$. The elements in $\mathcal{P}(N)$ are called coalitions.

The set of pre-imputations is $E = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i\in N} x_i = v(N)\}$, and the set of imputations is $A = \{x \in E : x_i \geq 0 \text{ for all } i \in N\}$.

Given a pre-imputation x and a coalition S in a game (N, v), the excess of the coalition S with respect to x is $e(S, x) = v(S) - x(S)$, where $x(S) = \sum_{i\in S} x_i$ if $S{\neq}\Phi$ and 0 otherwise. The core of (N, v) is the set $C = \{x \in E : e(S, x) \leq 0 \text{ for all } S \in \mathcal{P}(N)\}$.

The core of a game may be the empty set. The Shapley-Bondareva theorem [1], [5] characterizes the sub-class of TU-games with non-empty core, where the notion of balanced subfamily of coalitions plays a central role. A non-empty family $\mathcal{B}$ is balanced if there exists a set of positive real numbers $\lambda_{\mathcal{B}} = (\lambda_S)_{S\in\mathcal{B}}$, the balancing weights, such that $\sum\limits_{S\in\mathcal{B}(i)} \lambda_S = 1$ for all $i \in N$. Here $\mathcal{B}(i) = \{S \in B : i \in S\}$. A game (N, v) is balanced if $\sum_{S\in\mathcal{B}} \lambda_S v(S) \leq v(N)$ for all balanced family $\mathcal{B}$ with balancing weights $\lambda_{\mathcal{B}}$. Balancedness can also be defined as follows. Let $\chi_S \in \mathbb{R}^n$ denotes the vector defined by $(\chi_S)_i = 1$ if $i \in S$ and 0 otherwise (the indicator vector of S). Then, a family $\mathcal{B}$ is balanced if there exist positive balancing weights $(\lambda_S)_{S\in\mathcal{B}}$ such that $\sum_{S\in\mathcal{B}} \lambda_S \chi_S = \chi_N$.

Shapley- Bondareva's theorem states that the core of a $TU-$game is non-empty if and only if the game is balanced.

A minimal balanced family is one including no other proper balanced subfamily, and it has a unique set of balancing weights [5]. The worth of a minimal balanced family $\mathcal{B}$ with respect to a set of balancing weights $\lambda_{\mathcal{B}} = (\lambda_S)_{S\in\mathcal{B}}$ in a game (N, v) is $w(\mathcal{B}, \lambda_{\mathcal{B}}) = \sum\limits_{S\in\mathcal{B}} \lambda_S v(S)$.

Usually, when the core is non-empty, it contains more than one point, but in some context is convenient to restrict oneself to work in the subclass of balanced games with core having a single point. This happens, for instances, in the framework of biform games, recently studied in Brandenburger and Stuart [2]. There, they propose the adding up condition which determines a subclass of games having core with only one imputation. A TU-game (N, v) possesses the adding up condition (AU) if

$$\sum_{i \in N} (v(N) - v(N \backslash \{i\})) = v(N).$$

This is only a sufficient condition guaranteeing the uniqueness property of the core. In the next section we extend the AU property and show that it is also necessary.

3 MAIN RESULT

We note that $\mathcal{B} = \{N \backslash \{i\}\}_{i \in N}$ is a minimal balanced family of coalitions with cardinality n whose indicator vectors $\chi_S, S \in \mathcal{B}$, are linearly independent. Furthermore, when the adding up property is present, it satisfies that $w(\mathcal{B}, \lambda_{\mathcal{B}}) = v(N)$, since the unique collection of balancing weights for $\mathcal{B}$ is $\lambda_{\mathcal{B}} = (\frac{1}{n-1})_{S \in \mathcal{B}}$. Given a family of coalitions $\mathcal{B}$ with m members, $M_{\mathcal{B}}$ will stand for a $m \times n$ matrix whose rows are the indicator vectors χ_S of the coalitions S belonging to $\mathcal{B}$. We say that a balanced family of coalitions $\mathcal{B}$ is determining in a game (N, v) if it satisfies the following two conditions:

$i)$ $rank(M_{\mathcal{B}}) = n$,

$ii)$ There is a collection of balancing weights $\lambda_{\mathcal{B}}$ for $\mathcal{B}$ such that $w(\mathcal{B}, \lambda_{\mathcal{B}}) \geq w(\mathcal{B}', \lambda_{\mathcal{B}'})$ for all balanced family $\mathcal{B}'$ with balancing weights $\lambda_{\mathcal{B}'}$.

When (N, v) is balanced, any determining family $\mathcal{B}$ satisfies that $w(\mathcal{B}, \lambda_{\mathcal{B}}) = v(N)$.

Theorem 1 *Let (N, v) be a TU-game. Then $|C| = 1$ if and only if there is a determining family $\mathcal{B}$ with $w(\mathcal{B}, \lambda_{\mathcal{B}}) = v(N)$ for some collection $\lambda_{\mathcal{B}}$ of balancing weights.*

Proof. Let us assume first that there exists a determining family $\mathcal{B}$, and a collection of balancing weights $\lambda_{\mathcal{B}}$ such that $w(\mathcal{B}, \lambda_{\mathcal{B}}) = v(N)$. Then the Shapley-Bondareva theorem guarantees the non-emptiness of the core. Moreover, if $x \in C$, then $x(S) = v(S)$ for all $S \in \mathcal{B}$. But since $\mathcal{B}$ is determining, the linear system

$$M_{\mathcal{B}} y = v_{\mathcal{B}}$$

has a unique solution. Thus, $|C| = 1$. Here $v_{\mathcal{B}}$ is the vector $(v(S))_{S \in \mathcal{B}}$.

On the other hand, if $|C| = 1$, the game is balanced and $w(\mathcal{B}, \lambda_{\mathcal{B}}) \leq v(N)$ for any balanced family of coalitions $\mathcal{B}$ with balancing weights given by $\lambda_{\mathcal{B}}$. Moreover, the unique point in C is the nucleolus $\hat{x}$ of (N, v) ([sc]). The nucleolus satisfies that $\max_S e(S, \hat{x}) = 0$, and $\hat{\mathcal{B}} = \{S : e(S, \hat{x}) = 0\}$ is a balanced family of coalitions ([3]). For any coalition S outside of $\hat{\mathcal{B}}, e(S, \hat{x}) < 0$. We claim that $\hat{\mathcal{B}}$ is a determining subfamily. If rank of $M_{\hat{\mathcal{B}}} < n$, there is $0 \neq y \in Kernel(M_{\hat{\mathcal{B}}})$. Let $x^{\varepsilon} = \hat{x} + \varepsilon y$. Then $M_{\hat{\mathcal{B}}} x^{\varepsilon} = v_{\hat{\mathcal{B}}}$, so $e(S, x^{\varepsilon}) = 0$ for all $S \in \hat{\mathcal{B}}$. Furthermore, if ε is small enough, $e(S, x^{\varepsilon}) < 0$ for all $S \notin \hat{\mathcal{B}}$. Finally, since for any collection $\lambda_{\hat{\mathcal{B}}}$ of balancing weights for $\hat{\mathcal{B}}$, it holds that

$$\lambda_{\hat{\mathcal{B}}} M_{\hat{\mathcal{B}}} y = \chi_N y = 0,$$

it turns to be that x^{ε} is a pre-imputation, different from $\hat{x}$, belonging to C, in contradiction with the assumed cardinality for C. Then, $rank(\hat{\mathcal{B}}) = n$. Finally, since

$$\begin{aligned} w(\hat{\mathcal{B}}, \lambda_{\hat{\mathcal{B}}}) &= \sum_{S \in \hat{\mathcal{B}}} \lambda_S v(S) \\ &= \sum_{S \in \hat{\mathcal{B}}} \lambda_S \hat{x}(S) = v(N), \end{aligned}$$

we conclude that $\hat{\mathcal{B}}$ is a determining family of coalitions. □

Corollary 1 *A sufficient condition for the core of a game* (N, v) *have only one imputation is that there exist a minimal balanced family* $\mathcal{B}$ *with* $|\mathcal{B}| = n$ *and* $w(\mathcal{B}, \lambda_{\mathcal{B}}) = v(N)$.

The latter condition is not however, a necessary condition as the following example shows.

Example 1 *Let* (N, v) *a game with* $N = \{1, 2, 3\}$ *and* $v(N) = 1, v(\{1, 3\}) = v(\{2, 3\} = 1, v(\{1, 2\}) = -1$, *and* $v(S) = 0$ *otherwise. The only core imputation* x *in this game is* $x = (0, 0, 1)$, *and the only determining families are:*

$\mathcal{B}_1 = \{\{1\}, \{2\}, \{1, 3\}, \{2, 3\}\}$ *and* $\mathcal{B}_2 = \{\{1\}, \{2\}, \{1, 3\}, \{2, 3\}, N\}$, *none of them being a minimal balanced family of coalitions.*

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