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A local monotonicity formula for an inhomogeneous singular perturbation problem and applications: Part II

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Abstract In this paper we continue with our work in Lederman and Wolanski (Ann Math Pura Appl 187(2):197–220, 2008) where we developed a local monotonicity formula for solutions to an inhomogeneous singular perturbation problem of interest in combustion theory. There we proved local monotonicity formulae for solutions u^ε to the singular perturbation problem and for $u = \lim u^\varepsilon$, assuming that both u^ε and u were defined in an arbitrary domain $\mathcal{D}$ in $\mathbb{R}^{N+1}$. In the present work we obtain global monotonicity formulae for limit functions u that are globally defined, while u^ε are not. We derive such global formulae from a local one that we prove here. In particular, we obtain a global monotonicity formula for blow up limits u_0 of limit functions u that are not globally defined. As a consequence of this formula, we characterize blow up limits u_0 in terms of the value of a density at the blow up point. We also present applications of the results in this paper to the study of the regularity of $\partial\{u > 0\}$ (the flame front in combustion models). The fact that our results hold for the inhomogeneous singular perturbation problem allows a very wide applicability, for instance to problems with nonlocal diffusion and/or transport.

Keywords Singular perturbation problems · Monotonicity formula · Inhomogeneous problems · Combustion

Mathematics Subject Classification (2000) 35R35 · 35J65 · 80A25 · 35B65

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1 Introduction

In this paper we continue with our work in [7] where we developed a local monotonicity formula for solutions to an inhomogeneous singular perturbation problem of interest in combustion theory. That formula was inspired on a global monotonicity formula that G. S. Weiss developed for solutions of the global homogeneous problem (see [10]). As in [7], the problem under consideration here is the following: for $\varepsilon > 0$ we let u^ε be a solution to

$$\Delta u^\varepsilon - u^\varepsilon_t = \beta_\varepsilon(u^\varepsilon) + f_\varepsilon \quad \text{in } \mathcal{D}, \qquad (P_\varepsilon(f_\varepsilon))$$

where $\varepsilon > 0$, $\mathcal{D}$ is a domain in $\mathbb{R}^{N+1}$, $f_\varepsilon \in L^\infty(\mathcal{D})$, $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$ with β a Lipschitz continuous function, $\beta(s) > 0$ for $0 < s < 1$ and $\beta(s) = 0$ otherwise. This type of reaction term appears in the study of the propagation of deflagration flames. In that context ε represents the inverse of the activation energy (see, for instance, [1,2,9] and the references therein).

We are looking at the inhomogeneous equation—this is, we allow $f_\varepsilon \not\equiv 0$—which makes the applicability of our results much wider. In particular, our results apply to more general equations that include nonlocal diffusion and/or transport (see [6–8] for a discussion and applications).

In [10], Weiss obtained a *global* monotonicity formula for solutions u^ε of the global homogeneous version of problem $P_\varepsilon(f_\varepsilon)$ (i.e., with u^ε defined in $\mathbb{R}^N \times (0, T)$ and $f_\varepsilon \equiv 0$), as well as an analogous formula for $u = \lim u^\varepsilon (\varepsilon \to 0)$.

In [7], we proved *local* monotonicity formulae for solutions u^ε to the inhomogeneous problem and for $u = \lim u^\varepsilon$, assuming that both u^ε and u were defined in an arbitrary domain $\mathcal{D}$ in $\mathbb{R}^{N+1}$.

In the present work we deal with the inhomogeneous problem and we obtain *global* monotonicity formulae for limit functions u that are globally defined (i.e., defined in the whole region $\mathbb{R}^N \times (0, T)$) while u^ε are not. Such formulae cannot be derived from the ones in the previous papers [7,10] that were described above.

In fact, we obtain the first of our *global* monotonicity formulae in Theorem 2.2. We derive such *global* formula from a *local* one that we prove in Theorem 2.1.

We also obtain a *global* monotonicity formula for blow up limits u_0 of limit functions u that are not globally defined (here $u_0 = \lim_{\lambda_n \to 0} \frac{1}{\lambda_n} u(x_0+\lambda_n x, t_0+\lambda_n^2 t)$ with $(x_0, t_0) \in \partial\{u > 0\}$ and $u = \lim u^\varepsilon$) (see Corollary 2.1).

As a consequence of this last formula, we are able to characterize blow up limits u_0 in terms of the value of a density at the blow up point (x_0, t_0). Namely, in terms of

$$\delta(x_0, t_0) = \lim_{r\to 0} \frac{1}{r^2} \int_{t_0-4r^2}^{t_0-r^2} \int_{\mathbb{R}^N} \left(|\nabla(u\psi)|^2 + 2\psi^2\chi + \frac{1}{2}\frac{(u\psi)^2}{t-t_0} \right) G(x-x_0, t_0-t)\mathrm{d}x\mathrm{d}t, \tag{1.1}$$

where $G(x,t) = \frac{1}{(4\pi t)^{N/2}} \exp(-\frac{|x|^2}{4t})$, $\chi = \lim B_\varepsilon(u^\varepsilon)$ and $\psi = \psi(x) \in C_0^\infty$, $0 \le \psi \le 1$, $\psi \equiv 1$ in a neighborhood of x_0 (this limit exists and it is finite and independent of the cut off function ψ, by the local monotonicity formula we proved in [7]).

More precisely, in the stationary case, we prove that if $\delta(x_0, t_0) = 3M$ and u_0 is a blow up limit at (x_0, t_0), then $u_0 = \alpha x_1^+ - \gamma x_1^-$ for some $\alpha > 0$ and $\gamma \ge 0$, in some coordinate system (Theorem 3.2). In addition, we show that if $\delta(x_0, t_0) = 6M$, then $u_0 = \alpha|x_1|$ for some $\alpha \ge 0$ (Theorem 3.1). Here $M = \int_0^1 \beta(s)\mathrm{d}s$. See also Remark 3.2.

Furthermore, we prove that $3M \leq \delta(x_0, t_0) \leq 6M$ and that $\delta(x_0, t_0)$ is $3M$ or $6M$ almost everywhere on the free boundary. Moreover, in dimension 2, $\delta(x_0, t_0)$ is $3M$ or $6M$ everywhere on the free boundary (Propositions 3.1–3.3).

In Sect. 4 we present applications of Theorems 3.1 and 3.2 to the study of the regularity of the boundary of $\{u > 0\}$ for $u = \lim u^\varepsilon$ (the flame front in combustion models). We proved these regularity results in [8].

Let us remark that our global monotonicity formula proven in Theorem 2.2 allows us to show, in the particular case that $\|f_\varepsilon\|_{L^\infty} \to 0$, that limit functions u that are globally defined satisfy inequality (2.12), which is the same one proven in [10] for limit functions of the global homogeneous problem—even though in our case u^ε are not globally defined and $f_\varepsilon \not\equiv 0$.

In particular, we obtain this same inequality for blow up limits u_0 of limit functions u that are not globally defined (see inequality (2.14) in Corollary 2.1).

On the other hand, let us mention that the results on characterization of blow up limits described above are similar to the ones obtained in [10] for the global homogeneous problem but for a different density. The density in [10]—unlike the one given by (1.1)—is defined only for global functions.

We also remark that the applications given in [10] to these results (namely, classification of points in $\partial\{u > 0\}$ and rectifiability of the singular set) are different from the applications we are presenting in Sect. 4.

We finally want to point out that all the results in this paper are new when $f_\varepsilon \not\equiv 0$, even in the case that u^ε are globally defined. Moreover, our results are also new when $f_\varepsilon \equiv 0$, in case u^ε are not globally defined.

An outline of the paper is as follows. In Sect. 2 we prove the monotonicity formulae. In Sect. 3 we use these results to characterize blow up limits in terms of the density at the blow up point and Sect. 4 contains applications to regularity results of $\partial\{u > 0\}$ for $u = \lim u^\varepsilon$.

Notation

Given a point $(\bar{x}, \bar{t}) \in \mathbb{R}^{N+1}$ and $R, R_0 > 0$, we will denote

$$Q_{R,R_0}(\bar{x}, \bar{t}) := B_R(\bar{x}) \times (\bar{t} - 4R_0^2, \bar{t}].$$

We will be considering rescalings of functions in a neighborhood of $(\bar{x}, \bar{t})$ and we will denote

$$v_r(x,t) = \frac{1}{r} v(\bar{x} + rx, \bar{t} + r^2 t) \quad \text{and} \quad v^r(x,t) = v(\bar{x} + rx, \bar{t} + r^2 t).$$

We will say that a function v is in the class $Lip\,(1, 1/2)$ in a domain $\mathcal{D} \subset \mathbb{R}^{N+1}$, if v is bounded and there exists a constant $L = L(\mathcal{D})$ such that

$$|v(x,t) - v(y,\tau)| \leq L(|x - y| + |t - \tau|^{\frac{1}{2}})$$

for every (x, t), (y, τ) in $\mathcal{D}$. The norm in $Lip\,(1, 1/2)$ in $\mathcal{D}$ is

$$\|v\|_{Lip\,(1,1/2)} = \|v\|_{L^\infty(\mathcal{D})} + \sup_{(x,t),(y,\tau)\in\mathcal{D}} \frac{|v(x,t) - v(y,\tau)|}{|x - y| + |t - \tau|^{1/2}}.$$

We will denote by

$$|v|_{Lip\,(1,1/2)} = \sup_{(x,t),(y,\tau)\in\mathcal{D}} \frac{|v(x,t) - v(y,\tau)|}{|x - y| + |t - \tau|^{1/2}}$$

the $Lip\,(1, 1/2)$ seminorm in $\mathcal{D}$.

Finally, we will denote

$$B_\varepsilon(r) = \int_0^r \beta_\varepsilon(s)\mathrm{d}s, \quad M = \int_0^1 \beta(s)\mathrm{d}s$$

and $G(x,t) = \frac{1}{(4\pi t)^{N/2}} \exp(-\frac{|x|^2}{4t})$.

2 Monotonicity formulae

In this section we prove monotonicity formulae for solutions u^ε of problem $P_\varepsilon(f_\varepsilon)$ and for $u = \lim u^\varepsilon$ ($\varepsilon \to 0$).

In fact, in Theorem 2.1 we prove a *local* monotonicity formula for solutions u^ε to problem $P_\varepsilon(f_\varepsilon)$ that are defined in bounded domains of $\mathbb{R}^{N+1}$. This formula is an improvement of the one we obtained in Theorem 2.1 in [7].

As a consequence we obtain, in Theorem 2.2, a *global* monotonicity formula for limit functions u that are globally defined (i.e., defined in the whole region $\mathbb{R}^N \times (0,T)$) while u^ε are not.

In particular, in Corollary 2.1 we obtain a *global* monotonicity formula for blow up limits u_0 of limit functions u that are not globally defined (here $u_0 = \lim_{\lambda_n \to 0} \frac{1}{\lambda_n} u(x_0 + \lambda_n x, t_0 + \lambda_n^2 t)$ with $(x_0, t_0) \in \partial\{u > 0\}$ and $u = \lim u^\varepsilon$).

We first prove

Theorem 2.1 (ε-Local monotonicity formula) *Let u^ε be a solution to $P_\varepsilon(f_\varepsilon)$ in $Q_{R,R_0}(\bar{x},\bar{t})$ where $R_0 \le R$ and $R > 1$. Let $\psi = \psi(x) \in C_0^\infty(B_R(\bar{x}))$, $0 \le \psi \le 1$ and $\psi \equiv 1$ in $B_{R/2}(\bar{x})$. Assume that $f_\varepsilon \in L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))$ and*

$$|u^\varepsilon(x,t)| \le A_1\left(1 + |x-\bar{x}| + |t-\bar{t}|^{1/2}\right) \text{ in } Q_{R,R_0}(\bar{x},\bar{t}), \quad |u^\varepsilon|_{Lip_{(1,1/2)}(Q_{R,R_0}(\bar{x},\bar{t}))} \le A_2$$
$$||\nabla\psi||_{L^\infty(B_R(\bar{x}))} \le \frac{A_4}{R}, \quad ||D^2\psi||_{L^\infty(B_R(\bar{x}))} \le A_4. \tag{2.1}$$

For $0 < r \le R_0$, let

$$\mathcal{W}^\varepsilon_{(\bar{x},\bar{t})}(r) = \mathcal{W}^\varepsilon_{(\bar{x},\bar{t})}(r, u^\varepsilon, \psi) = \frac{1}{r^2}\int_{\bar{t}-4r^2}^{\bar{t}-r^2}\int_{\mathbb{R}^N}\left(|\nabla(u^\varepsilon\psi)|^2 + 2\psi^2 B_\varepsilon(u^\varepsilon) + \frac{1}{2}\frac{(u^\varepsilon\psi)^2}{t-\bar{t}}\right)G(x-\bar{x},\bar{t}-t)\mathrm{d}x\mathrm{d}t. \tag{2.2}$$

Then,

$$\begin{aligned}\frac{\partial\mathcal{W}^\varepsilon_{(\bar{x},\bar{t})}}{\partial r}(r) \ge &\int_{-4}^{-1}\int_{\mathbb{R}^N}\left(\partial_r w_r^\varepsilon\right)^2 \frac{rG(x,-t)}{-t}\mathrm{d}x\mathrm{d}t \\ &- C||f_\varepsilon||_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}\left(1 + r\|f_\varepsilon\|_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}\right) \\ &- C||f_\varepsilon||_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}\frac{|u^\varepsilon(\bar{x},\bar{t})|}{r} - C\left(1 + ||f_\varepsilon||_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}\right)R^2 \mathrm{e}^{-C'\frac{R^2}{r^2}}.\end{aligned} \tag{2.3}$$

Here $w^\varepsilon(x,t) = \psi(x)u^\varepsilon(x,t)$ and $w_r^\varepsilon(x,t) = \frac{1}{r}w^\varepsilon(\bar{x}+rx, \bar{t}+r^2t)$.

The constant C in (2.3) depends only on A_i, the dimension N, $M_1 = \max_{0\le s\le 1} s\beta(s)$ and $M = \int_0^1 \beta(s)\mathrm{d}s$. The constant C' is universal.

Proof By rescaling we get, for $0 < r \le R_0$,

$$\mathcal{W}^{\varepsilon}_{(\bar x,\bar t)}(r) = \int_{-4}^{-1}\int_{\mathbb{R}^N} \left(|\nabla w^{\varepsilon}_r|^2 + 2(\psi^r)^2 B_\varepsilon(ru^{\varepsilon}_r) + \frac{1}{2}\frac{(w^{\varepsilon}_r)^2}{t}\right) G(x,-t)\mathrm{d}x\mathrm{d}t. \tag{2.4}$$

Proceeding as in Theorem 2.1 in [7], we get

$$\begin{aligned}
\frac{\partial \mathcal{W}^{\varepsilon}_{(\bar x,\bar t)}}{\partial r}(r) \ge & \int_{-4}^{-1}\int_{\mathbb{R}^N} (\partial_r w^{\varepsilon}_r)^2 \frac{rG(x,-t)}{-t}\mathrm{d}x\mathrm{d}t \\
& + \int_{-4}^{-1}\int_{\mathbb{R}^N} (\partial_r w^{\varepsilon}_r)(-2\psi^r r f_\varepsilon{}^r) G(x,-t)\mathrm{d}x\mathrm{d}t \\
& + \int_{-4}^{-1}\int_{\mathbb{R}^N} (\partial_r w^{\varepsilon}_r)(-2u^{\varepsilon}_r \Delta\psi^r - 4\nabla\psi^r\nabla u^{\varepsilon}_r) G(x,-t)\mathrm{d}x\mathrm{d}t \\
& + \int_{-4}^{-1}\int_{\mathbb{R}^N} \left(-2\psi^r \beta_\varepsilon(ru^{\varepsilon}_r)\, r u^{\varepsilon}_r\, \partial_r\psi^r + 4\psi^r\, \partial_r\psi^r\, B_\varepsilon(ru^{\varepsilon}_r)\right) G(x,-t)\mathrm{d}x\mathrm{d}t \\
= & \int_{-4}^{-1}\int_{\mathbb{R}^N} (\partial_r w^{\varepsilon}_r)^2 \frac{rG(x,-t)}{-t}\mathrm{d}x\mathrm{d}t + I + II + III.
\end{aligned} \tag{2.5}$$

Now,

$$\partial_r w^{\varepsilon}_r(x,t) = -\frac{w^{\varepsilon}(\bar x + rx, \bar t + r^2 t)}{r^2} + \frac{\nabla w^{\varepsilon}(\bar x + rx, \bar t + r^2 t)}{r}\cdot x + 2t w^{\varepsilon}_t(\bar x + rx, \bar t + r^2 t).$$

There holds,

$$|w^{\varepsilon}(\bar x + rx, \bar t + r^2 t)| \le |w^{\varepsilon}(\bar x, \bar t)| + (A_1A_4 + A_2)\,(|x| + |t|^{1/2})\, r,$$

and

$$\begin{aligned}
&\left|\left(-\frac{w^{\varepsilon}(\bar x + rx, \bar t + r^2 t)}{r^2} + \frac{\nabla w^{\varepsilon}(\bar x + rx, \bar t + r^2 t)}{r}\cdot x\right)\left(-2\psi^r r f_\varepsilon{}^r\right) G(x,-t)\right| \\
&\quad \le \left(2\frac{|u^{\varepsilon}(\bar x,\bar t)|}{r} + 2(5A_1A_4 + 2A_2)\,(|x| + |t|^{1/2})\right) \|f_\varepsilon\|_{L^\infty}\, G(x,-t).
\end{aligned}$$

So that,

$$\begin{aligned}
&\int_{-4}^{-1}\int_{\mathbb{R}^N} \left(-\frac{w^{\varepsilon}(\bar x + rx, \bar t + r^2 t)}{r^2} + \frac{\nabla w^{\varepsilon}(\bar x + rx, \bar t + r^2 t)}{r}\cdot x)(-2\psi^r r f_\varepsilon{}^r)\right) G(x,-t)\mathrm{d}x\mathrm{d}t \\
&\quad \ge -C_2\|f_\varepsilon\|_{L^\infty(Q_{R,R_0}(\bar x,\bar t))}\frac{|u^{\varepsilon}(\bar x,\bar t)|}{r} - \bar C_1\|f_\varepsilon\|_{L^\infty(Q_{R,R_0}(\bar x,\bar t))}.
\end{aligned}$$

Now, as in [7],

$$\left|\int_{-4}^{-1}\int_{\mathbb{R}^N}\left(2tw_t^{\varepsilon}(\bar{x}+rx,\bar{t}+r^2t)(-2\psi^r rf_{\varepsilon}{}^r)\right)G(x,-t)\mathrm{d}x\mathrm{d}t\right|$$
$$\leq 32\|f_{\varepsilon}\|_{L^{\infty}(Q_{R,R_0}(\bar{x},\bar{t}))}\left(\int_{-4}^{-1}\int_{\mathbb{R}^N}|\partial_t(u_r^{\varepsilon})|^2(\psi^r)^2G(x,-t)\mathrm{d}x\mathrm{d}t\right)^{1/2}.$$

Let $v=u_r^{\varepsilon}$. Then v is a solution to $\Delta v-v_t=\beta_{\varepsilon/r}(v)+rf_{\varepsilon}{}^r$. Thus,

$$\begin{aligned}\int_{-4}^{-1}\int_{\mathbb{R}^N}v_t^2(\psi^r)^2G(x,-t)\mathrm{d}x\mathrm{d}t &= \int_{-4}^{-1}\int_{\mathbb{R}^N}v_t\Delta v(\psi^r)^2G(x,-t)\mathrm{d}x\mathrm{d}t\\ &\quad-\int_{-4}^{-1}\int_{\mathbb{R}^N}\beta_{\varepsilon/r}(v)\,v_t\,(\psi^r)^2G(x,-t)\mathrm{d}x\mathrm{d}t\\ &\quad-\int_{-4}^{-1}\int_{\mathbb{R}^N}rf_{\varepsilon}{}^r v_t(\psi^r)^2G(x,-t)\mathrm{d}x\mathrm{d}t\\ &=\text{(i)}+\text{(ii)}+\text{(iii)}.\end{aligned}$$

There holds,

$$\begin{aligned}\text{(i)}=&-\int_{-4}^{-1}\int_{\mathbb{R}^N}\nabla v_t\nabla v\,(\psi^r)^2G(x,-t)\mathrm{d}x\mathrm{d}t-2\int_{-4}^{-1}\int_{\mathbb{R}^N}v_t\nabla v\,\psi^r\nabla\psi^r G(x,-t)\mathrm{d}x\mathrm{d}t\\ &-\int_{-4}^{-1}\int_{\mathbb{R}^N}v_t\nabla v\,(\psi^r)^2\nabla G\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Arguing as in [7], but using that in the present case $r\leq R_0\leq R$ and $|\nabla\psi|\leq\frac{A_4}{R}$, we get, for $0<\eta<1$,

$$\text{(i)}\leq\eta\int_{-4}^{-1}\int_{\mathbb{R}^N}v_t^2(\psi^r)^2G(x,-t)\mathrm{d}x\mathrm{d}t+C_{\eta}(A_2,A_4).$$

Proceeding in a similar way,

$$\text{(iii)}\leq\eta\int_{-4}^{-1}\int_{\mathbb{R}^N}v_t^2(\psi^r)^2G(x,-t)\mathrm{d}x\mathrm{d}t+C_{\eta}r^2\|f_{\varepsilon}\|^2_{L^{\infty}(Q_{R,R_0}(\bar{x},\bar{t}))}.$$

Using that $\beta_{\varepsilon/r}(v)\,v_t=\partial_tB_{\varepsilon/r}(v)$ and $0\leq B_{\varepsilon/r}(s)\leq M$ we get for (ii)

$$\text{(ii)}\leq C(M).$$

Thus,

$$\left|\int_{-4}^{-1}\int_{\mathbb{R}^N}(\partial_t u_r^\varepsilon)^2(\psi^r)^2 G(x,-t)\mathrm{d}x\mathrm{d}t\right| \le C\left(1+r^2\|f_\varepsilon\|^2_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}\right). \tag{2.6}$$

Summing up,

$$I \ge -C\|f_\varepsilon\|_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}\frac{|u^\varepsilon(\bar{x},\bar{t})|}{r} - C||f_\varepsilon||_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}\left(1+r\|f_\varepsilon\|_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}\right).$$

On the other hand, since

$$\begin{aligned}\partial_r w_r^\varepsilon(x,t) = &-\frac{\psi(\bar{x}+rx)u^\varepsilon(\bar{x}+rx,\bar{t}+r^2t)}{r^2}+\frac{\psi(\bar{x}+rx)\nabla u^\varepsilon(\bar{x}+rx,\bar{t}+r^2t)}{r}\cdot x\\ &+\frac{u^\varepsilon(\bar{x}+rx,\bar{t}+r^2t)\nabla\psi(\bar{x}+rx)}{r}\cdot x+2t\psi(\bar{x}+rx)\partial_t u^\varepsilon(\bar{x}+rx,\bar{t}+r^2t),\end{aligned} \tag{2.7}$$

then, for x in the support of ψ^r and $-4\le t\le -1$, there holds

$$|\partial_r w_r^\varepsilon| \le \frac{4A_1R}{r^2}+\frac{A_2}{r}\frac{R}{r}++\frac{4A_1R}{r}\frac{A_4}{R}\frac{R}{r}+\frac{8}{r}|\psi^r\partial_t u_r^\varepsilon| = C\frac{R}{r^2}+\frac{C}{r}|\psi^r\partial_t u_r^\varepsilon|$$

and

$$|-2u_r^\varepsilon\Delta\psi^r - 4\nabla\psi^r\nabla u_r^\varepsilon| \le \frac{8A_1R}{r}r^2A_4+4r\frac{A_4}{R}A_2 \le CR^2$$

(we have used that $r\le R_0\le R$ and $R>1$). Since $\psi\equiv 1$ in $B_{R/2}(\bar{x})$, it follows that

$$\begin{aligned}|II| \le &\frac{CR^3}{r^2}\int_{-4}^{-1}\int_{\frac{R}{2r}\le|x|\le\frac{R}{r}} G(x,-t)\mathrm{d}x\mathrm{d}t\\ &+\frac{CR^2}{r}\left(\int_{-4}^{-1}\int_{\mathbb{R}^N}(\partial_t u_r^\varepsilon)^2(\psi^r)^2G(x,-t)\mathrm{d}x\mathrm{d}t\right)^{1/2}\left(\int_{-4}^{-1}\int_{\frac{R}{2r}\le|x|\le\frac{R}{r}} G(x,-t)\mathrm{d}x\mathrm{d}t\right)^{1/2}.\end{aligned}$$

Now, observing that

$$\int_{-4}^{-1}\int_{\frac{R}{2r}\le|x|\le\frac{R}{r}} G(x,-t)\mathrm{d}x\mathrm{d}t \le Ce^{-C'\frac{R^2}{r^2}} \tag{2.8}$$

and recalling (2.6), we obtain

$$II \ge -C\left(1+||f_\varepsilon||_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}\right)R^2\mathrm{e}^{-C''\frac{R^2}{r^2}}.$$

Since $0\le s\,\beta_\varepsilon(s)\le M_1$ and $0\le B_\varepsilon(s)\le M$, we have

$$|-2\psi^r\beta_\varepsilon(ru_r^\varepsilon)\,ru_r^\varepsilon\,\partial_r\psi^r+4\psi^r\,\partial_r\psi^r\,B_\varepsilon(ru_r^\varepsilon)| \le 2M_1\frac{A_4}{R}|x|+4\frac{A_4}{R}|x|M \le \frac{CR}{r},$$

for x in the support of ψ^r and $-4 \le t \le -1$. Then, using again (2.8) we conclude that

$$II + III \ge -\bar{C}\left(1 + ||f_\varepsilon||_{L^\infty(Q_{R,R_0}(\bar{x},\bar{t}))}\right) R^2 \mathrm{e}^{-C''' \frac{R^2}{r^2}}.$$

The theorem is proved. □

As a consequence of Theorem 2.1, we obtain

Theorem 2.2 (Global monotonicity formula) *Let u^{ε_j} be a family of solutions to $P_{\varepsilon_j}(f_{\varepsilon_j})$ in $Q_{R_j,R_0}(\bar{x},\bar{t})$ with $R_j \to \infty$. Let $\psi_j = \psi_j(x) \in C_0^\infty(B_{R_j}(\bar{x}))$, $0 \le \psi_j \le 1$, $\psi_j \equiv 1$ in $B_{R_j/2}(\bar{x})$. Assume that*

$$|u^{\varepsilon_j}(x,t)| \le A_1\big(1 + |x-\bar{x}| + |t-\bar{t}|^{1/2}\big) \text{ in } Q_{R_j,R_0}(\bar{x},\bar{t}),\ |u^{\varepsilon_j}|_{Lip_{(1,1/2)}(Q_{R_j,R_0}(\bar{x},\bar{t}))} \le A_2$$
$$||\nabla\psi_j||_{L^\infty(B_{R_j}(\bar{x}))} \le \frac{A_4}{R_j}, \quad ||D^2\psi_j||_{L^\infty(B_{R_j}(\bar{x}))} \le A_4. \tag{2.9}$$

Let $u = \lim u^{\varepsilon_j}$ uniformly on compact sets of $\mathbb{R}^N \times (\bar{t} - 4R_0^2, \bar{t}]$, $\chi = \lim B_{\varepsilon_j}(u^{\varepsilon_j})$∗-weakly in $L^\infty_{\mathrm{loc}}(\mathbb{R}^N \times (\bar{t} - 4R_0^2, \bar{t}])$, $A \ge \|f_{\varepsilon_j}\|_{L^\infty(Q_{R_j,R_0}(\bar{x},\bar{t}))}$ and $\varepsilon_j \to 0$.

For $0 < r < R_0$, let

$$\begin{aligned}\mathcal{W}_{(\bar{x},\bar{t})}(r) &= \mathcal{W}_{(\bar{x},\bar{t})}(r,u,\chi)\\ &= \frac{1}{r^2}\int_{\bar{t}-4r^2}^{\bar{t}-r^2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 + 2\chi + \frac{1}{2}\frac{u^2}{t-\bar{t}}\right)G(x-\bar{x},\bar{t}-t)\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{2.10}$$

Then, for $R_0 > \rho_1 > \rho_2 > 0$,

$$\begin{aligned}\mathcal{W}_{(\bar{x},\bar{t})}(\rho_1) - \mathcal{W}_{(\bar{x},\bar{t})}(\rho_2) \ge& \int_{\rho_2}^{\rho_1}\int_{-4}^{-1}\int_{\mathbb{R}^N}(\partial_r u_r)^2\ \frac{rG(x,-t)}{-t}\mathrm{d}x\mathrm{d}t\mathrm{d}r\\ &- C\,A\,(\rho_1-\rho_2)\,(1 + (\rho_1+\rho_2)A) - C\,A\,|u(\bar{x},\bar{t})|\log\left(\frac{\rho_1}{\rho_2}\right)\end{aligned} \tag{2.11}$$

where C is as in Theorem 2.1 and $u_r(x,t) = \frac{1}{r}u(\bar{x} + rx, \bar{t} + r^2 t)$.

In addition, if $\|f_{\varepsilon_j}\|_{L^\infty(Q_{R_j,R_0}(\bar{x},\bar{t}))} \to 0$ as $j \to \infty$ there holds that, for $R_0 > \rho_1 > \rho_2 > 0$,

$$\mathcal{W}_{(\bar{x},\bar{t})}(\rho_1) - \mathcal{W}_{(\bar{x},\bar{t})}(\rho_2) \ge \int_{\rho_2}^{\rho_1}\int_{-4}^{-1}\int_{\mathbb{R}^N}(\partial_r u_r)^2\ \frac{rG(x,-t)}{-t}\mathrm{d}x\mathrm{d}t\mathrm{d}r. \tag{2.12}$$

Proof First of all, it is clear that the second assertion follows immediately from the first one since, when $\|f_{\varepsilon_j}\|_{L^\infty(Q_{R_j,R_0}(\bar{x},\bar{t}))} \to 0$ as $j \to \infty$, we may take as A an arbitrarily small constant.

In order to obtain (2.11), we will apply Theorem 2.1 to our family and pass to the limit.

In fact, integrating Eq. (2.3) for j fixed and bounding $||f_{\varepsilon_j}||_{L^\infty(Q_{R_j,R_0}(\bar{x},\bar{t}))}$ by A, we get

$$\mathcal{W}^{\varepsilon_j}_{(\bar{x},\bar{t})}(\rho_1) - \mathcal{W}^{\varepsilon_j}_{(\bar{x},\bar{t})}(\rho_2) \geq \int_{\rho_2}^{\rho_1}\int_{-4}^{-1}\int_{\mathbb{R}^N} \left(\partial_r w_r^{\varepsilon_j}\right)^2 \frac{rG(x,-t)}{-t}\mathrm{d}x\mathrm{d}t\mathrm{d}r - CA\int_{\rho_2}^{\rho_1}(1+rA)\,\mathrm{d}r$$
$$-CA|u^{\varepsilon_j}(\bar{x},\bar{t})|\log\left(\frac{\rho_1}{\rho_2}\right) - C\,(1+A)\,R_j^2\int_{\rho_2}^{\rho_1} \mathrm{e}^{-C'\frac{R_j^2}{r^2}}\,\mathrm{d}r = I - II - III - IV. \quad (2.13)$$

It is easy to see that, as $j \to \infty$,

$$II = C\,A(\rho_1-\rho_2)\left(1 + A\,\frac{\rho_1+\rho_2}{2}\right),$$
$$III \to C\,A\,|u(\bar{x},\bar{t})|\log\left(\frac{\rho_1}{\rho_2}\right),$$

and

$$0 \leq IV \leq C\,(1+A)\,R_j^2\,\mathrm{e}^{-C'\frac{R_j^2}{\rho_1^2}}(\rho_1-\rho_2) \to 0.$$

On the other hand, we recall that, by (2.7), $\partial_r w_r^{\varepsilon_j}(x,t)$ is the sum of four terms,

$$\partial_r w_r^{\varepsilon_j}(x,t) = \text{(i)} + \text{(ii)} + \text{(iii)} + \text{(iv)}.$$

Now, observe that $\psi_j \to 1$ and $\nabla\psi_j \to 0$ uniformly on compact sets of $\mathbb{R}^N$. Moreover, we know that $\nabla u^{\varepsilon_j} \to \nabla u$ strongly in L^2_{loc} (this convergence was proved in [6] for nonnegative functions, but the same proof holds in the present case). Thus, by taking a subsequence, we obtain in $\mathbb{R}^N \times (-4,-1)$,

$$\text{(i)} \to -\frac{u(\bar{x}+rx,\bar{t}+r^2t)}{r^2} \quad \text{a.e.} \quad \text{and}$$
$$|\text{(i)}| \leq \frac{A_1\left(1+r(|x|+|t|^{1/2})\right)}{r^2},$$
$$\text{(ii)} \to \frac{\nabla u(\bar{x}+rx,\bar{t}+r^2t)}{r}\cdot x \quad \text{a.e.} \quad \text{and}$$
$$|\text{(ii)}| \leq \frac{A_2|x|}{r},$$
$$\text{(iii)} \to 0 \quad \text{a.e.} \quad \text{and}$$
$$|\text{(iii)}| \leq \frac{2A_1A_4|x|}{r}.$$

So that,

$$\int_{\rho_2}^{\rho_1}\int_{-4}^{-1}\int_{\mathbb{R}^N}\left(-\frac{\psi_j(\bar{x}+rx)u^{\varepsilon_j}(\bar{x}+rx,\bar{t}+r^2t)}{r^2} + \frac{\psi_j(\bar{x}+rx)\nabla u^{\varepsilon_j}(\bar{x}+rx,\bar{t}+r^2t)}{r}\cdot x\right.$$
$$\left. + \frac{u^{\varepsilon_j}(\bar{x}+rx,\bar{t}+r^2t)\nabla\psi_j(\bar{x}+rx)}{r}\cdot x\right)^2 \frac{rG(x,-t)}{-t}\mathrm{d}x\mathrm{d}t\mathrm{d}r$$
$$\to \int_{\rho_2}^{\rho_1}\int_{-4}^{-1}\int_{\mathbb{R}^N}\left(-\frac{u(\bar{x}+rx,\bar{t}+r^2t)}{r^2} + \frac{\nabla u(\bar{x}+rx,\bar{t}+r^2t)}{r}\cdot x\right)^2 \frac{rG(x,-t)}{-t}\mathrm{d}x\mathrm{d}t\mathrm{d}r.$$

Next, we use the convergence of u^{ε_j}, estimate (2.6) and the fact that $\psi_j u_t^{\varepsilon_j} = \partial_t(\psi_j u^{\varepsilon_j})$ to deduce that

$$\psi_j(\bar{x}+rx)\partial_t u^{\varepsilon_j}(\bar{x}+rx,\bar{t}+r^2t) \rightharpoonup u_t(\bar{x}+rx,\bar{t}+r^2t)$$

weakly in $L^2\big(\mathbb{R}^N\times(-4,-1),(-t)G(x,-t)\mathrm{d}x\mathrm{d}t\big)$. Then,

$$\liminf_{j\to\infty}\int_{\rho_2}^{\rho_1}\int_{-4}^{-1}\int_{\mathbb{R}^N}\big(2t\psi_j(\bar{x}+rx)\partial_t u^{\varepsilon_j}(\bar{x}+rx,\bar{t}+r^2t)\big)^2\ \frac{rG(x,-t)}{-t}\mathrm{d}x\mathrm{d}t\mathrm{d}r$$
$$\geq\int_{\rho_2}^{\rho_1}\int_{-4}^{-1}\int_{\mathbb{R}^N}\big(2tu_t(\bar{x}+rx,\bar{t}+r^2t)\big)^2\ \frac{rG(x,-t)}{-t}\mathrm{d}x\mathrm{d}t\mathrm{d}r$$

and

$$\int_{\rho_2}^{\rho_1}\int_{-4}^{-1}\int_{\mathbb{R}^N}\big(2t\psi_j(\bar{x}+rx)\partial_t u^{\varepsilon_j}(\bar{x}+rx,\bar{t}+r^2t)\big)$$
$$\times\Bigg(-\frac{\psi_j(\bar{x}+rx)u^{\varepsilon_j}(\bar{x}+rx,\bar{t}+r^2t)}{r^2}+\frac{\psi_j(\bar{x}+rx)\nabla u^{\varepsilon_j}(\bar{x}+rx,\bar{t}+r^2t)}{r}\cdot x$$
$$+\frac{u^{\varepsilon_j}(\bar{x}+rx,\bar{t}+r^2t)\nabla\psi_j(\bar{x}+rx)}{r}\cdot x\Bigg)\frac{rG(x,-t)}{-t}\mathrm{d}x\mathrm{d}t\mathrm{d}r$$
$$\to\int_{\rho_2}^{\rho_1}\int_{-4}^{-1}\int_{\mathbb{R}^N}\big(2tu_t(\bar{x}+rx,\bar{t}+r^2t)\big)$$
$$\times\Bigg(-\frac{u(\bar{x}+rx,\bar{t}+r^2t)}{r^2}+\frac{\nabla u(\bar{x}+rx,\bar{t}+r^2t)}{r}\cdot x\Bigg)\frac{rG(x,-t)}{-t}\mathrm{d}x\mathrm{d}t\mathrm{d}r.$$

Returning to (2.13) we conclude that

$$\liminf_{j\to\infty} I\geq\int_{\rho_2}^{\rho_1}\int_{-4}^{-1}\int_{\mathbb{R}^N}(\partial_r u_r)^2\ \frac{rG(x,-t)}{-t}\mathrm{d}x\mathrm{d}t\mathrm{d}r.$$

Similarly, we can prove that

$$\frac{1}{r^2}\int_{\bar{t}-4r^2}^{\bar{t}-r^2}\int_{\mathbb{R}^N}\left(\left|\nabla(\psi_j u^{\varepsilon_j})\right|^2+\frac{1}{2}\frac{(\psi_j u^{\varepsilon_j})^2}{t-\bar{t}}\right)G(x-\bar{x},\bar{t}-t)\mathrm{d}x\mathrm{d}t$$
$$\to\frac{1}{r^2}\int_{\bar{t}-4r^2}^{\bar{t}-r^2}\int_{\mathbb{R}^N}\left(|\nabla u|^2+\frac{1}{2}\frac{u^2}{t-\bar{t}}\right)G(x-\bar{x},\bar{t}-t)\mathrm{d}x\mathrm{d}t.$$

Finally, given $\sigma>0$, let $R>0$ be such that

$$2M\int_{\bar{t}-4r^2}^{\bar{t}-r^2}\int_{|x-\bar{x}|>R}G(x-\bar{x},\bar{t}-t)\mathrm{d}x\mathrm{d}t<\frac{\sigma}{2}.$$

Then, for j large,

$$
\begin{aligned}
&\left| \int_{\bar{t}-4r^2}^{\bar{t}-r^2} \int_{\mathbb{R}^N} \left\{ \psi_j^2 B_{\varepsilon_j}(u^{\varepsilon_j}) - \chi \right\} G(x-\bar{x}, \bar{t}-t) \mathrm{d}x\mathrm{d}t \right| \\
&\quad \le \left| \int_{\bar{t}-4r^2}^{\bar{t}-r^2} \int_{|x-\bar{x}|<R} \left\{ B_{\varepsilon_j}(u^{\varepsilon_j}) - \chi \right\} G(x-\bar{x}, \bar{t}-t) \mathrm{d}x\mathrm{d}t \right| \\
&\qquad + 2M \left| \int_{\bar{t}-4r^2}^{\bar{t}-r^2} \int_{|x-\bar{x}|>R} G(x-\bar{x}, \bar{t}-t) \mathrm{d}x\mathrm{d}t \right| \\
&\quad < \left| \int_{\bar{t}-4r^2}^{\bar{t}-r^2} \int_{|x-\bar{x}|<R} \left\{ B_{\varepsilon_j}(u^{\varepsilon_j}) - \chi \right\} G(x-\bar{x}, \bar{t}-t) \mathrm{d}x\mathrm{d}t \right| + \frac{\sigma}{2} \\
&\quad < \sigma \qquad \text{if } j \ge j_0(R).
\end{aligned}
$$

Therefore, $\mathcal{W}^{\varepsilon_j}_{(\bar{x},\bar{t})}(r) \to \mathcal{W}_{(\bar{x},\bar{t})}(r)$ as $j \to \infty$ and the result follows. □

We apply Theorem 2.2 to derive

Corollary 2.1 (Global monotonicity formula for blow up limits) *Let u^{ε_j} be a family of solutions to $P_{\varepsilon_j}(f_{\varepsilon_j})$ in a domain $\mathcal{D} \subset \mathbb{R}^{N+1}$, uniformly bounded in $Lip\,(1, 1/2)$ norm with f_{ε_j} uniformly bounded in L^∞ norm in $\mathcal{D}$. Assume $u^{\varepsilon_j} \to u$ uniformly on compact subsets of $\mathcal{D}$ and $B_{\varepsilon_j}(u^{\varepsilon_j}) \to \chi$ $*$-weakly in $L^\infty(\mathcal{D})$ with $\varepsilon_j \to 0$. Let $(x_0, t_0) \in \mathcal{D} \cap \partial\{u > 0\}$, $u_{\lambda_n}(x,t) = \frac{1}{\lambda_n} u(x_0 + \lambda_n x, t_0 + \lambda_n^2 t)$, $\chi^{\lambda_n}(x,t) = \chi(x_0 + \lambda_n x, t_0 + \lambda_n^2 t)$ and $\lambda_n \to 0$. Assume $u_{\lambda_n} \to u_0$ uniformly on compact sets of $\mathbb{R}^{N+1}$, $\chi^{\lambda_n} \to \chi_0$ $*$-weakly in $L^\infty_{\mathrm{loc}}(\mathbb{R}^{N+1})$. Let $(\bar{x}, \bar{t}) \in \mathbb{R}^{N+1}$. Then, for $\mathcal{W}_{(\bar{x},\bar{t})}(r) = \mathcal{W}_{(\bar{x},\bar{t})}(r, u_0, \chi_0)$ and $\rho_1 > \rho_2 > 0$ there holds that*

$$
\mathcal{W}_{(\bar{x},\bar{t})}(\rho_1) - \mathcal{W}_{(\bar{x},\bar{t})}(\rho_2) \ge \int_{\rho_2}^{\rho_1} \int_{-4}^{-1} \int_{\mathbb{R}^N} (\partial_r (u_0)_r)^2 \, \frac{rG(x,-t)}{-t} \mathrm{d}x\mathrm{d}t\mathrm{d}r, \tag{2.14}
$$

where $(u_0)_r(x,t) = \frac{1}{r} u_0(\bar{x} + rx, \bar{t} + r^2 t)$.

Proof We will apply Theorem 2.2 to the functions $u^{\delta_n}(x,t) := \frac{1}{\lambda_n} u^{\varepsilon_{j_n}}(x_0 + \lambda_n x, t_0 + \lambda_n^2 t)$ where j_n is chosen such that (see for instance [7], proof of Theorem 3.1),

- $\delta_n := \frac{\varepsilon_{j_n}}{\lambda_n} \to 0$.
- $u^{\delta_n} \to u_0$ uniformly on compact sets of $\mathbb{R}^{N+1}$.
- $B_{\delta_n}(u^{\delta_n}) \to \chi_0$ $*$-weakly in $L^\infty_{\mathrm{loc}}(\mathbb{R}^{N+1})$.
- $\left|\frac{u(x_0,t_0) - u^{\varepsilon_{j_n}}(x_0,t_0)}{\lambda_n}\right| \le 1$.

Now, let $\sigma > 0$ such that $B_{2\sigma}(x_0) \times [t_0 - 4\sigma^2, t_0 + 4\sigma^2] \subset \mathcal{D}$. Let $R_n = \sigma/\lambda_n$ and let $R_0 > 0$ be fixed. Then, for $(x,t) \in Q_{R_n, R_0}(\bar{x}, \bar{t})$ we have that $|\lambda_n x| \le \lambda_n |\bar{x}| + \lambda_n |x - \bar{x}| \le \lambda_n |\bar{x}| + \sigma < 2\sigma$ and $-4\sigma^2 \le \lambda_n^2(\bar{t} - 4R_0^2) < \lambda_n^2 t < \lambda_n^2 \bar{t} \le 4\sigma^2$ if n is large enough. Thus, u^{δ_n} is defined in $Q_{R_n, R_0}(\bar{x}, \bar{t})$ for n large.

Let $\psi \in C_0^\infty(B_\sigma(0))$, $0 \le \psi \le 1$, $\psi \equiv 1$ in $B_{\sigma/2}(0)$, and $\psi_n(x) = \psi(\lambda_n(x-\bar{x}))$. Then, $|\nabla\psi_n| \le A_4\frac{\lambda_n}{\sigma}$, $|D^2\psi_n| \le A_4$ in $B_{R_n}(\bar{x})$ for a certain constant A_4, $\psi_n \in C_0^\infty\big(B_{R_n}(\bar{x})\big)$, $\psi_n \equiv 1$ in $B_{R_n/2}(\bar{x})$.

Observe that u^{δ_n} are solutions to $P_{\delta_n}(f_{\delta_n})$ with $f_{\delta_n}(x,t) = \lambda_n f_{\varepsilon_{j_n}}(x_0+\lambda_n x, t_0+\lambda_n^2 t)$. Moreover, $\|f_{\delta_n}\|_{L^\infty(Q_{R_n,R_0}(\bar{x},\bar{t}))} \to 0$ as $n\to\infty$.

In order to apply Theorem 2.2, let $\rho_1 > \rho_2 > 0$ arbitrary and then, take $R_0 > \rho_1$. We only need to show that the hypotheses (2.9) are satisfied. In fact, the bounds of ψ_n and its derivatives follow immediately by construction as observed above. On the other hand, taking $L>0$ such that $|u^{\varepsilon_{j_n}}|_{Lip_{(1,1/2)}(\mathcal{D})} \le L$ we get, for $(x,t)\in Q_{R_n,R_0}(\bar{x},\bar{t})$,

$$
\begin{aligned}
|u^{\delta_n}(x,t)| &= \frac{1}{\lambda_n}|u^{\varepsilon_{j_n}}(x_0+\lambda_n x, t_0+\lambda_n^2 t)| \le \left|\frac{u^{\varepsilon_{j_n}}(x_0,t_0)}{\lambda_n}\right| + L(|x|+|t|^{1/2})\\
&\le \left|\frac{u(x_0,t_0)-u^{\varepsilon_{j_n}}(x_0,t_0)}{\lambda_n}\right| + L(|\bar{x}|+|\bar{t}|^{1/2}) + L(|x-\bar{x}|+|t-\bar{t}|^{1/2})\\
&\le A_1(1+|x-\bar{x}|+|t-\bar{t}|^{1/2})
\end{aligned}
$$

for a certain constant A_1 depending on $\bar{x}$ and $\bar{t}$ but independent of n. Moreover,

$$
|u^{\delta_n}|_{Lip_{(1,1/2)}(Q_{R_n,R_0}(\bar{x},\bar{t}))} \le L.
$$

Thus, (2.14) follows and the corollary is proved. □

3 Characterization of blow up limits in terms of the density at the blow up point

In this section we apply the results of Sect. 2 to characterize blow up limits u_0 in terms of the value of a density at the blow up point, in the stationary case.

In fact, we consider a family u^{ε_j} of stationary solutions to $P_{\varepsilon_j}(f_{\varepsilon_j})$ in a domain $\Omega\subset\mathbb{R}^N$ with $\|f_{\varepsilon_j}\|_{L^\infty(\Omega)} \le C$, $\|u^{\varepsilon_j}\|_{L^\infty(\Omega)} \le C'$ and $\varepsilon_j \to 0$. By the results in [8], it follows that u^{ε_j} are locally uniformly bounded in Lip norm in Ω, so that the results of Sect. 2 apply to this family.

Let $u = \lim u^{\varepsilon_j}$ uniformly on compact subsets of Ω and $\chi = \lim B_{\varepsilon_j}(u^{\varepsilon_j})$ $*$-weakly in $L^\infty(\Omega)$. For $x_0\in\Omega\cap\partial\{u>0\}$, we consider

$$
\delta(x_0) := \delta(x_0,0) = \lim_{r\to 0}\frac{1}{r^2}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^N}\left(|\nabla(\psi u)|^2 + 2\psi^2\chi + \frac{(\psi u)^2}{2t}\right)G(x-x_0,-t)\,\mathrm{d}x\mathrm{d}t,
\tag{3.1}
$$

where $\psi\in C_0^\infty(B_\sigma(x_0))$, $0\le\psi\le 1$, $\psi\equiv 1$ in $B_{\sigma/2}(x_0)$ and $B_\sigma(x_0)\subset\subset\Omega$ (this limit exists and it is finite and independent of the cut off function ψ, by Theorem 2.2 and Corollary 2.1 in [7]).

First, we prove Theorems 3.1 and 3.2, where we characterize blow up limits u_0 at free boundary points x_0 when $\delta(x_0)\in\{3M,6M\}$ (here $u_0 = \lim_{\lambda_n\to 0}\frac{1}{\lambda_n}u(x_0+\lambda_n x)$, with $x_0\in\Omega\cap\partial\{u>0\}$ and $M=\int_0^1\beta(s)\mathrm{d}s$). See Remark 3.2 for the reciprocal of these results.

Then, we show that $\delta(x_0)\in[3M,6M]$. Moreover, $\delta(x_0)\in\{3M,6M\}$ for $\mathcal{H}^{N-1}$-almost every $x_0\in\Omega\cap\partial\{u>0\}$ and also for every $x_0\in\Omega\cap\partial\{u>0\}$, if $N=2$ (Propositions 3.1–3.3).

We start with

Theorem 3.1 *Let u^{ε_j} be stationary solutions to $P_{\varepsilon_j}(f_{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ with $\|f_{\varepsilon_j}\|_{L^\infty(\Omega)} \leq C$, $\|u^{\varepsilon_j}\|_{L^\infty(\Omega)} \leq C'$ and $\varepsilon_j \to 0$. Let $u = \lim u^{\varepsilon_j}$ uniformly on compact subsets of Ω and $\chi = \lim B_{\varepsilon_j}(u^{\varepsilon_j})$ $*$-weakly in $L^\infty(\Omega)$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ and $\delta(x_0)$ as in (3.1).*

Assume $\delta(x_0) = 6M$. Let $\lambda_n \to 0$ be such that there exists $u_0(x) = \lim_{n\to\infty} \frac{1}{\lambda_n} u(x_0 + \lambda_n x)$ uniformly on compact sets of $\mathbb{R}^N$. Then, there exists $\alpha \geq 0$ such that, in a certain coordinate system,

$$u_0(x) = \alpha|x_1|.$$

Proof By taking a subsequence, we may assume that $\chi^{\lambda_n}(x) = \chi(x_0 + \lambda_n x) \to \chi_0(x)$ $*$-weakly in $L^\infty_{\text{loc}}(\mathbb{R}^N)$. By the results of [7] (Corollaries 2.1 and 2.2) we know that, for $r > 0$,

$$\delta(x_0) = \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(|\nabla u_0|^2 + 2\chi_0 + \frac{u_0^2}{2t} \right) G(x, -t)\,\mathrm{d}x\mathrm{d}t = \int_{-4}^{-1} \int_{\mathbb{R}^N} 2\chi_0(x) G(x, -t)\,\mathrm{d}x\mathrm{d}t. \tag{3.2}$$

Since $0 \leq \chi_0 \leq M$ there holds that $0 \leq \delta(x_0) \leq 6M$. Thus,

$$\delta(x_0) = 6M \quad \Rightarrow \quad \chi_0 \equiv M \quad \text{a.e. in} \quad \mathbb{R}^N.$$

On the other hand, we know that u_0 is homogeneous (see Corollary 2.1 in [7]). This is,

$$u_0(rx) = r u_0(x) \quad \text{for every } r > 0, \quad x \in \mathbb{R}^N.$$

Since $u_0 = \lim u^{\delta_n}$ and $\chi_0 = \lim B_{\delta_n}(u^{\delta_n})$ with δ_n and u^{δ_n} as in Corollary 2.1, we can apply the results in Lemma 3.1 in [8] to deduce that $\chi_0(x) \in \{0, M\}$ for almost every $x \in \mathbb{R}^N$, $\chi_0 = 0$ in $\{u_0 < 0\}$, $\chi_0 = M$ in $\{u_0 > 0\}$ and χ_0 is constant on every connected component of $\{u_0 \leq 0\}^\circ$. In particular, since $\chi_0 \equiv M$, we have that $u_0 \geq 0$.

If $u_0 \equiv 0$ in $\mathbb{R}^N$, then $u_0(x) = \alpha|x_1|$ with $\alpha = 0$ and the theorem is proved in this case.

Thus, we may assume that $u_0 > 0$ somewhere.

Now, let us show that the theorem holds when u_0 depends only on 1 variable. In fact, if $u_0 = u_0(x_1)$ depends on 1 variable, the only possible components of $\{u_0 > 0\}$ are $\{x_1 > 0\}$ and $\{x_1 < 0\}$.

If $u_0 > 0$ in $\{x_1 > 0\}$ there holds that $u_0(x_1) = \alpha x_1$ in $\{x_1 > 0\}$ with $\alpha > 0$ since it is harmonic in this set and $u_0(0) = 0$.

Assume $u_0 = 0$ in $\{x_1 < 0\}$. Using that $u_0 = \lim u^{\delta_n}$, $\chi_0 = \lim B_{\delta_n}(u^{\delta_n})$ and that u^{δ_n} are solutions to $P_{\delta_n}(f_{\delta_n})$ with $f_{\delta_n} \to 0$ uniformly on compact subsets of $\mathbb{R}^N$, we proceed as in Proposition 3.2 in [8] (see also Proposition 5.1 in [6] and Proposition 5.2 in [4]) and deduce that

$$\frac{\alpha^2}{2} = M - M = 0,$$

which is a contradiction.

By similar arguments, if $u_0 > 0$ in $\{x_1 < 0\}$ there exists $\gamma > 0$ such that $u_0(x_1) = -\gamma x_1$ in $\{x_1 < 0\}$.

This time we proceed as in Proposition 3.3 in [8] (see also Proposition 5.3 in [4]), and deduce that

$$\frac{\alpha^2}{2} - \frac{\gamma^2}{2} = M - M = 0.$$

So that, $\gamma = \alpha > 0$ and $u_0(x_1) = \alpha|x_1|$.

Therefore, the theorem is proved when u_0 depends on 1 variable.

We will devote the rest of the proof to showing that this is necessarily the case. The proof will follow from a dimension reduction argument.

Let us first observe that if u_0 depends on k variables, then χ_0 depends on the same k variables. In fact, assume that u_0 does not depend on a direction $\bar\nu$. We will show that χ_0 does not depend on $\bar\nu$. For simplicity, we assume that $\bar\nu = e_1$.

Multiplying equation $P_{\delta_n}(f_{\delta_n})$ by $u^{\delta_n}_{x_1}\varphi$, where $\varphi \in C_0^\infty(\mathbb{R}^N)$, and integrating by parts, we get

$$-\frac{1}{2}\int |\nabla u^{\delta_n}|^2 \varphi_{x_1} + \int \nabla u^{\delta_n}\,\nabla\varphi\, u^{\delta_n}_{x_1} = \int B_{\delta_n}(u^{\delta_n})\varphi_{x_1} - \int f_{\delta_n}\, u^{\delta_n}_{x_1}\,\varphi.$$

Passing to the limit, we obtain

$$-\frac{1}{2}\int |\nabla u_0|^2\,\varphi_{x_1} = \int \chi_0\varphi_{x_1}. \tag{3.3}$$

We observe the left-hand side of (3.3) vanishes because u_0 does not depend on x_1 and thus, this equality implies that χ_0 does not depend on x_1, as we claimed.

Next, in order to develop the dimension reduction argument, let us assume that u_0 depends on k variables. Thus, we will assume that $u_0 = u_0(x_1, \ldots, x_k)$ and correspondingly, $\chi_0 = \chi_0(x_1, \ldots, x_k)$.

Since the rest of the proof relies on the definition of the functional $\mathcal{W}_{(\bar x,0)}(r) = \mathcal{W}_{(\bar x,0)}(r, u_0, \chi_0)$ (more precisely, on (2.14) and (3.2)), we see that we can assume that we are working in $\mathbb{R}^k$ instead of $\mathbb{R}^N$.

We will show that there is a direction ν in $\mathbb{R}^k$ such that u_0 does not depend on this direction. This is, we will show that $\nabla u_0 \cdot \nu = 0$ in $\mathbb{R}^k$, thus deducing that u_0 actually depends on $k-1$ variables. Iterating this argument we finally get that u_0 only depends on 1 variable.

Let us assume that $k \geq 2$.

In order to find ν we first observe that there exists $\tilde x \in \mathbb{R}^k\setminus\{0\}$ such that $u_0(\tilde x) = 0$. In fact, if this is not the case we have

$$\begin{cases} \Delta u_0 = 0 & \text{in } \mathbb{R}^k\setminus\{0\} \\ |u_0(x) - u_0(y)| \leq L|x-y| & \end{cases}$$

for some $L > 0$, and we deduce that $\Delta u_0 = 0$ in $\mathbb{R}^k$, $u_0 \geq 0$, $u_0(0) = 0$ which is a contradiction.

Therefore, there exists $\tilde x \neq 0$ such that $u_0(\tilde x) = 0$. Since u_0 is homogeneous there holds that $u_0(\lambda\tilde x) = 0$ for every $\lambda > 0$. By rotating this line we find in $\mathbb{R}^k$ a point $0 \neq \bar x \in \partial\{u_0 > 0\}$. If not, we would have $u_0 \equiv 0$ and this is not the case.

Let us apply the monotonicity formula (2.14) at $(\bar x, \bar t)$ with $\bar t = 0$. We have, for $\rho_1 > \rho_2 > 0$,

$$0 \leq \int_{\rho_2}^{\rho_1}\int_{-4}^{-1}\int_{\mathbb{R}^k} (\partial_r(u_0)_r)^2\ \frac{rG(x,-t)}{-t}\mathrm{d}x\mathrm{d}t\mathrm{d}r \leq \mathcal{W}_{(\bar x,0)}(\rho_1) - \mathcal{W}_{(\bar x,0)}(\rho_2),$$

where $(u_0)_r(x) = \frac{1}{r}u_0(\bar x + rx)$ and $\mathcal{W}_{(\bar x,0)}(r) = \mathcal{W}_{(\bar x,0)}(r, u_0, \chi_0)$.

Now let $R > 0$ be fixed. Letting $\rho_1 \to R^-$ and $\rho_2 \to 0^+$ (the limits exist by the monotonicity of $\mathcal{W}_{(\bar{x},0)}(r)$ shown in Corollary 2.1) we get

$$0 \le \int_0^R \int_{-4}^{-1} \int_{\mathbb{R}^k} (\partial_r(u_0)_r)^2 \, \frac{rG(x,-t)}{-t} \mathrm{d}x\mathrm{d}t\mathrm{d}r \le \mathcal{W}_{(\bar{x},0)}(R) - \mathcal{W}_{(\bar{x},0)}(0^+).$$

Now, it is easy to see from the rescaling invariance of $\mathcal{W}_{(\bar{x},0)}(r)$, by arguments similar to those used in Corollary 2.2 in [7], that

$$\mathcal{W}_{(\bar{x},0)}(0^+) = \int_{-4}^{-1} \int_{\mathbb{R}^k} 2\chi_{00}\, G(x,-t)\mathrm{d}x\mathrm{d}t,$$

where $\chi_{00}(x) = \lim_{j\to\infty} \chi_0(\bar{x} + \mu_j x)$ for a certain sequence $\mu_j \to 0$.

Since $\chi_0 \equiv M$, the same is true for χ_{00}. Thus, $\mathcal{W}_{(\bar{x},0)}(0^+) = 6M$.

On the other hand, we know that $\mathcal{W}_{(0,0)}(r)$ is constant (recall (3.2)). Therefore,

$$\mathcal{W}_{(\bar{x},0)}(0^+) = 6M = \delta(x_0) = \mathcal{W}_{(0,0)}(R).$$

Thus,

$$0 \le \int_0^R \int_{-4}^{-1} \int_{\mathbb{R}^k} (\partial_r(u_0)_r)^2 \, \frac{rG(x,-t)}{-t} \mathrm{d}x\mathrm{d}t\mathrm{d}r \le \mathcal{W}_{(\bar{x},0)}(R) - \mathcal{W}_{(0,0)}(R). \tag{3.4}$$

Let us see that the right-hand side converges to 0 as $R \to \infty$. In fact, since u_0 is homogeneous,

$$\begin{aligned}
\mathcal{W}_{(\bar{x},0)}(R) - \mathcal{W}_{(0,0)}(R) &= \frac{1}{R^2} \int_{-4R^2}^{-R^2} \int_{\mathbb{R}^k} \left(|\nabla u_0|^2 + 2\chi_0 + \frac{u_0^2}{2t} \right) \\
&\quad \times \left(G(x-\bar{x},-t) - G(x,-t)\right) \mathrm{d}x\mathrm{d}t \\
&= \frac{1}{R^2} \int_{-4R^2}^{-R^2} \int_{\mathbb{R}^k} \left(\left|\nabla u_0\left(\frac{x}{R}\right)\right|^2 + 2\chi_0(x) + \frac{u_0^2\left(\frac{x}{R}\right)}{2\frac{t}{R^2}} \right) \left(G(x-\bar{x},-t) - G(x,-t)\right) \mathrm{d}x\mathrm{d}t \\
&= \int_{-4}^{-1} \int_{\mathbb{R}^k} \left(|\nabla u_0(y)|^2 + 2\chi_0(Ry) + \frac{u_0^2(y)}{2s} \right) \left(G\left(y - \frac{\bar{x}}{R}, -s\right) - G(y,-s) \right) \mathrm{d}y\mathrm{d}s \\
&\le \int_{-4}^{-1} \int_{\mathbb{R}^k} \left(L^2(1+|y|^2) + 2M \right) \frac{\left| \mathrm{e}^{\frac{|y-\frac{\bar{x}}{R}|^2}{4s}} - \mathrm{e}^{\frac{|y|^2}{4s}} \right|}{(-4\pi s)^{k/2}} \mathrm{d}y\mathrm{d}s \\
&= \int_{-4}^{-1} \int_{\mathbb{R}^k} F_R(y,s)\mathrm{d}y\mathrm{d}s,
\end{aligned}$$

with $F_R(y,s) \to 0$ as $R \to \infty$ and $|F_R(y,s)| \le C(1+|y|^2)\,\mathrm{e}^{-C'|y|^2}$ if R is large. Thus,

$$\int_{-4}^{-1}\int_{\mathbb{R}^k} F_R(y,s)\mathrm{d}y\mathrm{d}s \;\to\; 0 \quad \text{as } R \to \infty$$

and, passing to the limit as $R \to \infty$ in (3.4), we deduce that for a.e. $r > 0$

$$\int_{-4}^{-1}\int_{\mathbb{R}^k} (\partial_r(u_0)_r)^2 \;\frac{rG(x,-t)}{-t}\mathrm{d}x\mathrm{d}t = 0.$$

Therefore, $\partial_r(u_0)_r = 0$ so that, for a.e. x,

$$0 = \partial_r\left(\frac{u_0(\bar{x}+rx)}{r}\right) = -\frac{u_0(\bar{x}+rx)}{r^2} + \frac{x}{r}\cdot\nabla u_0(\bar{x}+rx)$$

or equivalently,

$$-\frac{u_0(x)}{r^2} + \frac{x-\bar{x}}{r^2}\cdot\nabla u_0(x) \equiv 0.$$

But, since $u_0(rx) = ru_0(x)$, there holds that

$$\nabla u_0(x)\cdot x - u_0(x) \equiv 0.$$

Therefore,

$$\nabla u_0(x)\cdot\bar{x} \equiv 0.$$

Now, if $\nu = \frac{\bar{x}}{|\bar{x}|}$, there holds that u_0 is independent of the direction ν and the theorem is proved. □

We next obtain

Theorem 3.2 *Let u^{ε_j} be stationary solutions to $P_{\varepsilon_j}(f_{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ with $\|f_{\varepsilon_j}\|_{L^\infty(\Omega)} \le C$, $\|u^{\varepsilon_j}\|_{L^\infty(\Omega)} \le C'$ and $\varepsilon_j \to 0$. Let $u = \lim u^{\varepsilon_j}$ uniformly on compact subsets of Ω, and $\chi = \lim B_{\varepsilon_j}(u^{\varepsilon_j})$ $*$-weakly in $L^\infty(\Omega)$. Let $x_0 \in \Omega\cap\partial\{u>0\}$ and $\delta(x_0)$ as in (3.1).*

Assume $\delta(x_0) = 3M$. Let $\lambda_n \to 0$ be such that there exists $u_0(x) = \lim_{n\to\infty}\frac{1}{\lambda_n}u(x_0+\lambda_n x)$ uniformly on compact sets of $\mathbb{R}^N$. Then, there exist $\alpha > 0$ and $\gamma \ge 0$ such that, in a certain coordinate system,

$$u_0(x) = \alpha x_1^+ - \gamma x_1^-. \tag{3.5}$$

Proof As in the proof of Theorem 3.1, we will first show that the theorem holds when u_0 depends only on one variable, and then proceed by a dimension reduction argument.

In fact, we may assume that $\chi^{\lambda_n}(x) = \chi(x_0+\lambda_n x) \to \chi_0(x)$ $*$-weakly in $L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, with $\chi_0(x) \in \{0, M\}$ for almost every $x \in \mathbb{R}^N$, $\chi_0 \equiv M$ in $\{u_0 > 0\}$, $\chi_0 \equiv 0$ in $\{u_0 < 0\}$, χ_0 constant (either 0 or M) on any connected component of $\{u_0 \le 0\}^\circ$. In addition, the bounds in the proof of Lemma 3.1 in [8] imply that $\chi_0 \in BV_{\mathrm{loc}}(\mathbb{R}^N)$. In particular, $\chi_0 = M\chi_{\{\chi_0>0\}}$ and thus, $\{\chi_0 > 0\}$ is a set of locally finite perimeter (see, for instance, [5]).

So let us now show that, if u_0 depends only on one variable ($u_0 = u_0(x_1)$ in a certain coordinate system) then, the result follows. In fact, since u_0 is homogeneous, we only have one of the following:

(1) $u_0 = 0$ in $\mathbb{R}$.
(2) $u_0 < 0$ in $\{x_1 > 0\}$ and $u_0 < 0$ in $\{x_1 < 0\}$.
(3) $u_0 < 0$ in $\{x_1 > 0\}$ and $u_0 = 0$ in $\{x_1 < 0\}$.
(4) $u_0 > 0$ in $\{x_1 > 0\}$ and $u_0 > 0$ in $\{x_1 < 0\}$.
(5) $u_0 > 0$ in $\{x_1 > 0\}$ and $u_0 < 0$ in $\{x_1 < 0\}$.
(6) $u_0 > 0$ in $\{x_1 > 0\}$ and $u_0 = 0$ in $\{x_1 < 0\}$.

Actually, (1), (2), (3) and (4) are not possible. In fact, if (1), (2) or (3) hold, we have that $u_0 \le 0$ in $\mathbb{R}$ and then, χ_0 is constant equal to 0 or M in $\mathbb{R}$. If (4) holds, we have that $\chi_0 = M$ in $\mathbb{R}$. Thus, in any of these cases we either have $\delta(x_0) = 0$ or $\delta(x_0) = 6M$, a contradiction.

Therefore, under the hypotheses of this theorem only (5) and (6) are possible.

There holds that u_0 is harmonic where positive and where negative, u_0 uniformly Lipschitz in $\mathbb{R}$ and $u_0(0) = 0$. Therefore, if we have (5) or (6), there exists $\alpha > 0$ such that $u_0 = \alpha x_1$ in $\{x_1 > 0\}$. If we have (5), by the same argument there exists $\gamma > 0$ such that $u_0 = \gamma x_1$ in $\{x_1 < 0\}$. In any of the cases (5) or (6), there holds (3.5) with $\alpha > 0$ and $\gamma \ge 0$. Thus, the theorem is true if u_0 depends on 1 variable.

Now, as in the proof of Theorem 3.1, we will show by a dimension reduction argument, that u_0 depends only on 1 variable. Following the arguments in that proof, assuming u_0 depends on k variables with $k \ge 2$, we may suppose that we are in $\mathbb{R}^k$ and therefore, it is enough to show that there exists in $\mathbb{R}^k$ a point $0 \ne \bar{x} \in \partial\{u_0 > 0\}$ such that $\mathcal{W}_{(\bar{x},0)}(0^+) = 3M$.

More precisely, we will see that there exists a point $0 \ne \bar{x} \in \partial_{red}\{\chi_0 > 0\} \subset \partial\{u_0 > 0\}$, where ∂_{red} denotes reduced boundary. Recall that

$$\mathcal{W}_{(\bar{x},0)}(0^+) = \int_{-4}^{-1}\int_{\mathbb{R}^k} 2\chi_{00}(x)\, G(x, -t)\mathrm{d}x\mathrm{d}t,$$

for $\chi_{00}(x) = \lim_{j\to\infty} \chi_0(\bar{x} + \mu_j x)$ with $\mu_j \to 0$. If $\bar{x} \in \partial_{red}\{\chi_0 > 0\}$, then $\chi_{00}(x) = M\chi_{\{\langle x,\nu\rangle>0\}}$ for ν the unit interior normal to $\{\chi_0 > 0\}$ at $\bar{x}$ in the measure theoretic sense. Therefore, $\mathcal{W}_{(\bar{x},0)}(0^+) = 3M$ and, from here the proof follows as that of Theorem 3.1.

So, let $E = \{\chi_0 > 0\}$. Then, $|E| > 0$ and $|E^c| > 0$. If not, either $\chi_0 = 0$ a.e. or $\chi_0 = M$ a.e., contradicting the fact that $\delta(x_0) = 3M$. Thus, there exists $R_0 > 0$ such that $|E \cap B_{R_0}| > 0$ and $|E^c \cap B_{R_0}| > 0$, where we denote $B_{R_0} = B_{R_0}(0)$. We claim that $0 < Per(\partial E; B_{R_0}) < \infty$. In fact, the perimeter is finite in B_{R_0} since $\chi_0 = M\chi_E$ and $\chi_0 \in BV(B_{R_0})$. Now, by the isoperimetric inequality,

$$Per(\partial E; B_{R_0}) \ge C_k \min\left\{|E \cap B_{R_0}|, |E^c \cap B_{R_0}|\right\}^{(k-1)/k} > 0.$$

On the other hand,

$$Per(\partial E; B_{R_0}) = \mathcal{H}^{k-1}(B_{R_0} \cap \partial_{red} E).$$

Therefore, there exists $0 \ne \bar{x} \in \partial_{red}\{\chi_0 > 0\}$ as claimed, and the theorem is proved. □

In the remainder of the section we will let u^{ε_j} be stationary solutions to $P_{\varepsilon_j}(f_{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ with $\|f_{\varepsilon_j}\|_{L^\infty(\Omega)} \le C$, $\|u^{\varepsilon_j}\|_{L^\infty(\Omega)} \le C'$ and $\varepsilon_j \to 0$ and we will let $u = \lim u^{\varepsilon_j}$ uniformly on compact subsets of Ω, and $\chi = \lim B_{\varepsilon_j}(u^{\varepsilon_j})$ $*$-weakly in $L^\infty(\Omega)$. For points $x \in \Omega \cap \partial\{u > 0\}$ we will consider $\delta(x)$ as defined in (3.1).

We prove

Lemma 3.1 *Let* $x_0 \in \Omega \cap \partial\{u > 0\}$. *Then,*

$$\limsup_{\substack{x\to x_0\\ x\in\Omega\cap\partial\{u>0\}}} \delta(x) \le \delta(x_0).$$

Proof Let $\sigma > 0$ such that $B_{2\sigma}(x_0) \subset\subset \Omega$ and $\varphi \in C_0^\infty(B_\sigma(0))$, with $0 \le \varphi \le 1$ and $\varphi \equiv 1$ in $B_{\sigma/2}(0)$. For $\bar{x} \in B_\sigma(x_0) \cap \partial\{u > 0\}$ we denote $\psi^{\bar{x}}(x) = \varphi(x - \bar{x})$ and define

$$\mathcal{W}_{(\bar{x},0)}(r) = \mathcal{W}_{(\bar{x},0)}(r, u, \psi^{\bar{x}}, \chi)$$
$$= \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(\left|\nabla(\psi^{\bar{x}} u)\right|^2 + 2(\psi^{\bar{x}})^2 \chi + \frac{(\psi^{\bar{x}} u)^2}{2t} \right) G(x - \bar{x}, -t) \mathrm{d}x \mathrm{d}t.$$

Now fix $\eta > 0$. Then, by Theorem 2.2 in [7], there exists $r_0 = r_0(\eta)$ such that

$$\delta(\bar{x}) = \mathcal{W}_{(\bar{x},0)}(0^+) \le \mathcal{W}_{(\bar{x},0)}(r) + \frac{\eta}{2} \quad \text{if } r \le r_0, \tag{3.6}$$

where r_0 can be taken independent of the point $\bar{x} \in B_\sigma(x_0)$ if the constants in [7] are suitably chosen.

On the other hand, there exists $\theta = \theta(r, \eta) \le \sigma$ such that

$$\mathcal{W}_{(\bar{x},0)}(r) \le \mathcal{W}_{(x_0,0)}(r) + \frac{\eta}{2} \quad \text{if } \bar{x} \in B_\theta(x_0). \tag{3.7}$$

In fact, since in $\mathbb{R}^N \times [-4r^2, -r^2]$,

$$\left| |\nabla(\psi^{\bar{x}} u)|^2 + 2(\psi^{\bar{x}})^2 \chi + \frac{(\psi^{\bar{x}} u)^2}{2t} \right| \le C_r,$$
$$\left| \left(|\nabla(\psi^{\bar{x}} u)|^2 + 2(\psi^{\bar{x}})^2 \chi + \frac{(\psi^{\bar{x}} u)^2}{2t} \right) - \left(|\nabla(\psi^{x_0} u)|^2 + 2(\psi^{x_0})^2 \chi + \frac{(\psi^{x_0} u)^2}{2t} \right) \right|$$
$$\le C_r |\bar{x} - x_0|,$$

with C_r independent of $\bar{x}$, we get

$$|\mathcal{W}_{(\bar{x},0)}(r) - \mathcal{W}_{(x_0,0)}(r)|$$
$$\le \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} C_r \ |G(x - \bar{x}, -t) - G(x - x_0, -t)| \, \mathrm{d}x \mathrm{d}t$$
$$+ \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} C_r |\bar{x} - x_0| G(x - x_0, -t) \mathrm{d}x \mathrm{d}t$$
$$\le C_r' |\bar{x} - x_0|,$$

which implies (3.7).

Finally, from (3.6) and (3.7) we get, for $r \le r_0(\eta)$,

$$\limsup_{\substack{x \to x_0 \\ x \in \Omega \cap \partial\{u>0\}}} \delta(x) \le \mathcal{W}_{(x_0,0)}(r) + \eta,$$

and the result follows by letting $r \to 0$ first, and then $\eta \to 0$. □

Lemma 3.2 *Let $x_0 \in \Omega \cap \partial\{u > 0\}$ be such that $x_0 \notin \partial_*\{\chi > 0\}$, where we denote $\partial_*\{\chi > 0\}$ the set of points $x \in \mathbb{R}^N$ such that*

$$\limsup_{r \to 0} \frac{|B_r(x) \cap \{\chi > 0\}|}{|B_r(x)|} > 0, \quad \limsup_{r \to 0} \frac{|B_r(x) \cap \{\chi > 0\}^c|}{|B_r(x)|} > 0.$$

Then, $\delta(x_0) = 0$ or $\delta(x_0) = 6M$.

Proof Let $\lambda_n \to 0$ be such that there exist $u_0 = \lim_{n\to\infty} u_{\lambda_n}$ uniformly on compact sets of $\mathbb{R}^N$ and $\chi_0 = \lim_{n\to\infty} \chi^{\lambda_n}$ $*$-weakly in $L^\infty_{\text{loc}}(\mathbb{R}^N)$ (here $u_{\lambda_n}(x) = \frac{1}{\lambda_n} u(x_0 + \lambda_n x)$ and $\chi^{\lambda_n}(x) = \chi(x_0 + \lambda_n x)$).

If

$$\limsup_{r\to 0} \frac{|B_r(x_0) \cap \{\chi > 0\}|}{|B_r(x_0)|} = 0,$$

then for any $R > 0$,

$$\lim_{n\to\infty} \frac{|B_R(0) \cap \{\chi^{\lambda_n} > 0\}|}{|B_R(0)|} = 0$$

and therefore, $\chi_0 \equiv 0$. Recalling (3.2) we obtain that $\delta(x_0) = 0$.

Now, if

$$\limsup_{r\to 0} \frac{|B_r(x_0) \cap \{\chi > 0\}^c|}{|B_r(x_0)|} = 0,$$

we argue similarly and deduce that $\chi_0 \equiv M$, which implies that $\delta(x_0) = 6M$ and proves the lemma. □

As a consequence of the previous lemmas we get

Proposition 3.1 *Let $x_0 \in \Omega \cap \partial\{u > 0\}$. Then, $\delta(x_0) \in [3M, 6M]$.*

Proof We first recall that the arguments in the proof of Theorem 3.1 imply that $\delta(x_0) \in [0, 6M]$.

Next let $\rho > 0$ such that $B_\rho(x_0) \subset\subset \Omega$. Since $x_0 \in \Omega \cap \partial\{u > 0\}$, there exists $\hat{x} \in B_{\rho/2}(x_0)$ such that $u(\hat{x}) > 0$. Let us take $0 < \mu \leq \rho/2$ such that there exists $\bar{x} \in \partial B_\mu(\hat{x}) \cap \partial\{u > 0\}$ with $B_\mu(\hat{x}) \subset \{u > 0\}$.

Since $u > 0$ in $B_\mu(\hat{x})$, there holds that $\chi \equiv M$ in $B_\mu(\hat{x})$.

Now let $\lambda_n \to 0$ be such that there exist $u_0(x) = \lim_{n\to\infty} \frac{1}{\lambda_n} u(\bar{x} + \lambda_n x)$ uniformly on compact sets of $\mathbb{R}^N$ and $\chi_0(x) = \lim_{n\to\infty} \chi(\bar{x} + \lambda_n x)$ $*$-weakly in $L^\infty_{\text{loc}}(\mathbb{R}^N)$. Then $\chi_0 \equiv M$ in $\{\langle x, \nu\rangle > 0\}$ with $\nu = \frac{\hat{x} - \bar{x}}{|\hat{x} - \bar{x}|}$.

Recalling again (3.2) we obtain that $\delta(\bar{x}) \geq 3M$. Since $\bar{x} \in B_\rho(x_0) \cap \partial\{u > 0\}$, where $\rho > 0$ can be chosen arbitrarily small, we deduce from Lemma 3.1 that $\delta(x_0) \geq 3M$ and this completes the proof. □

Proposition 3.2 *There holds that $\delta(x_0) = 3M$ or $\delta(x_0) = 6M$ for $\mathcal{H}^{N-1}$-almost every $x_0 \in \Omega \cap \partial\{u > 0\}$.*

Proof We first observe that the bounds in the proof of Lemma 3.1 in [8] imply that $\chi \in BV_{\text{loc}}(\Omega)$. In particular, $\chi = M\chi_{\{\chi>0\}}$ and thus, $\{\chi > 0\}$ is a set of locally finite perimeter.

Let $x_0 \in \Omega \cap \partial\{u > 0\}$.

Assume $x_0 \in \partial_{red}\{\chi > 0\} \subset \partial_*\{\chi > 0\}$. Then, if $\lambda_n \to 0$, we have $\chi_0(x) = \lim_{n\to\infty} \chi(x_0 + \lambda_n x) = M\chi_{\{\langle x,\nu\rangle>0\}}$ for ν the unit interior normal to $\{\chi > 0\}$ at x_0 in the measure theoretic sense and therefore, $\delta(x_0) = 3M$.

Now assume $x_0 \notin \partial_*\{\chi > 0\}$. Then, it follows from Proposition 3.1 and Lemma 3.2 that $\delta(x_0) = 6M$.

Finally, we obtain the desired result observing that, by Lemma 1 in [5, Section 5.8], there holds that $\mathcal{H}^{N-1}(\partial_*\{\chi > 0\} \setminus \partial_{red}\{\chi > 0\}) = 0$. □

Proposition 3.3 *Assume $N = 2$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$. Then, $\delta(x_0) = 3M$ or $\delta(x_0) = 6M$.*

Proof Let $\lambda_n \to 0$ be such that there exist $u_0(x) = \lim_{n\to\infty} \frac{1}{\lambda_n} u(x_0 + \lambda_n x)$ uniformly on compact sets of $\mathbb{R}^N$ and $\chi_0(x) = \lim_{n\to\infty} \chi(x_0 + \lambda_n x)$ $*$-weakly in $L^\infty_{\rm loc}(\mathbb{R}^N)$.

If $u_0 \le 0$, then $\chi_0 \equiv 0$ or $\chi_0 \equiv M$, and thus Proposition 3.1 implies that $\delta(x_0) = 6M$.

If $u_0 > 0$ somewhere, we consider $\mathcal{A}$ a connected component of $\{u_0 > 0\}$. Then, from the homogeneity of u_0 we get that, in some system of coordinates, either $\mathcal{A} \subset \{x_1 > 0\}$ or else $\{x_1 > 0\} \subset \mathcal{A}$. In the first case, Lemma A1 in [3] implies that $u_0(x) = \alpha x_1^+ + o(|x|)$ in $\{x_1 > 0\}$, with $\alpha \ge 0$ and then the homogeneity of u_0 yields

$$u_0(x) = \alpha x_1^+ \quad \text{in } \{x_1 > 0\} \quad \text{and} \quad \alpha > 0.$$

Now, with a similar analysis in $\{x_1 < 0\}$ we conclude that

$$u_0(x) = \alpha x_1^+ + \bar{\alpha} x_1^- \quad \alpha > 0,\ \bar{\alpha} \in \mathbb{R}. \tag{3.8}$$

The case in which $\{x_1 > 0\} \subset \mathcal{A}$ gives, with the same arguments, that again (3.8) holds and therefore, $\delta(x_0) = 3M$ or $\delta(x_0) = 6M$. □

Remark 3.1 In [8] we obtained results on the regularity of the boundary of $\{u > 0\}$, for $u = \lim u^{\varepsilon_j}$, with u^{ε_j} stationary solutions to $P_{\varepsilon_j}(f_{\varepsilon_j})$. In particular we dealt with the cases of energy minimizers (Theorem 10.2 in [8]) and traveling waves of a combustion model (Theorem 10.1 in [8], see also Theorem 4.2 in Sect. 4).

The results in [8] imply, for the first of these applications, that $\delta(x_0) = 3M$ for every $x_0 \in \Omega \cap \partial\{u > 0\}$, when $N = 2$ or $N = 3$, and similarly for the second one, when $N = 2$. Moreover, in both cases, $\delta(x_0) = 3M$ for $\mathcal{H}^{N-1}$-almost every $x_0 \in \Omega \cap \partial\{u > 0\}$, in any dimension.

Remark 3.2 The reciprocal results of Theorems 3.1 and 3.2 are also true.

In fact, assume that a blow up limit u_0 at $x_0 \in \Omega \cap \partial\{u > 0\}$ has the form $u_0(x) = \alpha|x_1|$ with $\alpha \ge 0$. Then $\chi_0 \equiv 0$ or $\chi_0 \equiv M$ and therefore, $\delta(x_0) = 0$ or $\delta(x_0) = 6M$. Recalling Proposition 3.1, we deduce that $\delta(x_0) = 6M$ so the reciprocal of Theorem 3.1 holds.

Now assume that a blow up limit u_0 at $x_0 \in \Omega \cap \partial\{u > 0\}$ has the form $u_0(x) = \alpha x_1^+ - \gamma x_1^-$ with $\alpha > 0$ and $\gamma \ge 0$. Then $\chi_0 = M\chi_{\{x_1 > 0\}}$ or $\chi_0 \equiv M$ and therefore, $\delta(x_0) = 3M$ or $\delta(x_0) = 6M$. But $\delta(x_0) = 6M$ would give, by Theorem 3.1, that $u_0(x) = \tilde{\alpha}|\langle x, \nu\rangle|$ for some direction ν and some $\tilde{\alpha} \ge 0$, a contradiction. Therefore $\delta(x_0) = 3M$ so that the reciprocal of Theorem 3.2 also holds.

4 Application: regularity of the free boundary

In this section we present applications of the results in Sect. 3. They deal with the regularity of the boundary of $\{u > 0\}$ $(u = \lim u^{\varepsilon_j})$ in the stationary case including, in particular, regularity results for traveling waves of a combustion model.

First, we consider a family u^{ε_j} of stationary solutions to $P_{\varepsilon_j}(f_{\varepsilon_j})$ such that u^{ε_j} and f_{ε_j} are uniformly bounded in L^∞ norm. In [8] we proved that u^{ε_j} are locally uniformly bounded in Lip norm. So that, the results of the present paper apply to this family. Let $u = \lim u^{\varepsilon_j}$ uniformly on compact subsets as $\varepsilon_j \to 0$. In [8] we proved that u is a solution to

$$\Delta u = f\chi_{\{u \not\equiv 0\}} \quad \text{in } \{u > 0\} \cup \{u \le 0\}^\circ,$$

where $f = \lim f_{\varepsilon_j}$ $*$-weakly in L^∞.

Moreover, in [8] we proved that, under suitable assumptions, $\partial\{u > 0\}$ is smooth and u is a classical solution to the following free boundary problem

$$\begin{aligned} &\Delta u = f\chi_{\{u\not\equiv 0\}} && \text{in } \{u > 0\} \cup \{u \le 0\}^\circ, \\ &|\nabla u^+|^2 - |\nabla u^-|^2 = 2M && \text{on } \partial\{u > 0\}. \end{aligned} \qquad (E(f))$$

The purpose of this section is to state some theorems on the regularity of the free boundary $\partial\{u > 0\}$ that are proved in [8] for which the results in this paper are an essential tool.

In fact, assume u is defined in $B_\sigma(x_0)$ with $x_0 \in \partial\{u > 0\}$. Let $\chi = \lim B_{\varepsilon_j}(u^{\varepsilon_j})$ $*$-weakly in $L^\infty(B_\sigma(x_0))$ and consider $\delta(x_0)$ as in (3.1).

In the next theorem, we assume that, in $B_\sigma(x_0)$, u^+ is uniformly nondegenerate. This property holds in many applications (see, for instance, Theorem 4.2 below). By uniform nondegeneracy we mean that there exists $c > 0$ such that

$$u^+(x) \ge c\,\mathrm{dist}\,(x, \{u \le 0\}).$$

As a first application, we have the following result,

Theorem 4.1 (Theorem 9.7 in [8]) *There holds that $\delta(x_0) = 3M$ if and only if the free boundary is $C^{1,\alpha}$ in a neighborhood of x_0. This implies that u is a classical solution to the free boundary problem $E(f)$ in a neighborhood of x_0.*

Theorem 3.2 is a key tool in the proof of Theorem 4.1.

We point out that there are examples in [6] that show that the free boundary condition in $E(f)$ may not hold at any free boundary point. In fact, u^+ may degenerate or the density of $\{u \le 0\}$ may be zero at a boundary point. Thus, some extra assumption is needed if one wants to show that u is a solution to $E(f)$.

The results in the present paper are also used in [8] to obtain the following regularity result for traveling waves of a combustion model. In fact, we have

Theorem 4.2 (Theorem 10.1 in [8]) *Let $x = (x_1, y) \in \Omega = \mathbb{R} \times \Sigma$, with $\Sigma \subset \mathbb{R}^{N-1}$ a smooth bounded domain, let a be a continuous positive function on $\overline{\Sigma}$ and let $0 < \widetilde{\sigma} < 1$ be given.*

Consider traveling wave solutions to the following combustion model

$$\Delta v^\varepsilon - a(y) v^\varepsilon_t = \beta_\varepsilon(v^\varepsilon),$$

where β_ε is as before with $\beta'(0) > 0$. This is, $v^\varepsilon(x,t) = u^\varepsilon(x_1 + c^\varepsilon t, y)$, with u^ε solutions to

$$\begin{gathered} \Delta u^\varepsilon - c^\varepsilon a(y) u^\varepsilon_{x_1} = \beta_\varepsilon(u^\varepsilon) \quad \text{in } \Omega, \\ u^\varepsilon(-\infty, y) = (1 - \widetilde{\sigma})^{-1}, \qquad u^\varepsilon(+\infty, y) = 0 \quad \text{in } \Sigma, \\ \frac{\partial u^\varepsilon}{\partial \eta} = 0 \qquad \text{on } \mathbb{R} \times \partial\Sigma, \end{gathered}$$

for some suitable c^ε.

Let $u = \lim u^{\varepsilon_j}$ $(\varepsilon_j \to 0)$. Then, there is a subset $\mathcal{R}$ of the free boundary $\Omega \cap \partial\{u > 0\}$ which is locally a $C^{1,\alpha}$ surface and u is a classical solution to the free boundary problem $E(f)$ in a neighborhood of $\mathcal{R}$ ($f = ca(y)u_{x_1}$ with $c = \lim c^{\varepsilon_j}$). Moreover, $\mathcal{R}$ is open and dense in $\Omega \cap \partial\{u > 0\}$ and the remainder of the free boundary has $(N-1)$–dimensional Hausdorff measure zero. In dimension 2 we have $\mathcal{R} = \Omega \cap \partial\{u > 0\}$.

In addition, in any dimension, if $a \in C^{k,\alpha}_{\text{loc}}$ (resp. analytic) then, $\mathcal{R} \in C^{k+2,\alpha}_{\text{loc}}$ (resp. analytic).

We remark that this traveling wave problem was first studied in [1], where the authors obtained existence of $(u^\varepsilon, c^\varepsilon)$, strict monotonicity in the x_1 direction, uniform Lipschitz estimates and uniform nondegeneracy of the family u^ε, as well as uniform estimates of the velocities c^ε.

The proof of Theorem 4.2 relies on the fact that the density of the zero set is positive at every free boundary point. We obtain this density property by a contradiction argument that strongly uses Theorem 3.1.

References

1. Berestycki, H., Caffarelli, L.A., Nirenberg, L.: Uniform estimates for regularization of free boundary problems. In: Sadosky, C. (ed.) Analysis and Partial Differential Equations. Lecture Notes in Pure and Applied Mathematics, vol. 122, pp. 567–619. Marcel Dekker, New York (1990)
2. Buckmaster, J.D., Ludford, G.S.S.: Theory of Laminar Flames. Cambridge University Press, Cambridge (1982)
3. Caffarelli, L.A.: A Harnack inequality approach to the regularity of free boundaries. Part II: Flat free boundaries are Lipschitz. Comm. Pure Appl. Math. **42**, 55–78 (1989)
4. Caffarelli, L.A., Lederman, C., Wolanski, N.: Uniform estimates and limits for a two phase parabolic singular perturbation problem. Indiana Univ. Math. J. **46**(2), 453–490 (1997)
5. Evans, L., Gariepy, R.: Measure Theory and Fine Properties of Functions. Studies in Advanced Mathematics. CRC Press, Boca Raton (1992)
6. Lederman, C., Wolanski, N.: Singular perturbation in a nonlocal diffusion model. Commun. PDE **31**(2), 195–241 (2006)
7. Lederman, C., Wolanski, N.: A local monotonicity formula for an inhomogeneous singular perturbation problem and applications. Ann. Math. Pura Appl. **187**(2), 197–220 (2008)
8. Lederman, C., Wolanski, N.: A two phase elliptic singular perturbation problem with a forcing term. J. Math. Pures Appl. **86**, 552–589 (2006)
9. Vazquez, J.L.: The free boundary problem for the heat equation with fixed gradient condition. In: Niezgódka, M., Strzelecki, P. (eds.) Free Boundary Problems, Theory and Applications (Zakopane, 1995), Pitman Res. Notes Math. Ser., vol. 363, pp. 277–302. Longman, Harlow (1996)
10. Weiss, G.S.: A singular limit arising in combustion theory: fine properties of the free boundary. Calc. Var. Partial Differ. Equ. **17**(3), 311–340 (2003)