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A Black–Scholes option pricing model with transaction costs

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Abstract

We consider a boundary value problem for a nonlinear differential equation which arises in an option pricing model with transaction costs. We apply the method of upper and lower solutions in order to obtain solutions for the stationary problem. Moreover, we give conditions for the existence of solutions of the general evolution equation.

1. Introduction

Black–Scholes models including transaction costs have been studied by many authors [2,5,6]. In this work we assume that the costs behave as a nonincreasing linear function $h(x) = a - bx$ $(a, b > 0)$, depending on the trading stocks needed to hedge the replicating portfolio. Following the idea of Leland [6], if the value of the option is denoted by $V(S,t)$, where S is the value of the underlying asset, for $\Pi = V - \Delta S$ we have

$$d\Pi = dV - \Delta dS - \left[\left(a - b|\nu|\right)S|\nu|\right],$$

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where ν is the number of shares of the asset which are traded in order to maintain the equilibrium of the portfolio. By Ito's lemma, we conclude that

$$\nu \simeq \frac{\partial^2 V}{\partial S^2}(S,t)\,dS \simeq \frac{\partial^2 V}{\partial S^2}\sigma S\phi\sqrt{dt}$$

with $\phi \sim \mathcal{N}(0,1)$. Then, the expected value of the transaction costs is given by

$$E\big((a-b|\nu|)S|\nu|\big) = \left|\frac{\partial^2 V}{\partial S^2}\right|\sigma S^2\sqrt{\frac{2}{\pi}}\sqrt{dt}\,a - bS^3\left(\frac{\partial^2 V}{\partial S^2}\right)^2\sigma^2\,dt.$$

Hence we obtain the equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - a\left|\frac{\partial^2 V}{\partial S^2}\right|\sigma S^2\sqrt{\frac{2}{\pi\,dt}} + \frac{\partial^2 V}{\partial S^2}^2 bS^3\sigma^2 + r\left(\frac{\partial V}{\partial S}S - V\right) = 0. \tag{1.1}$$

Assuming that a is small enough we have that

$$\tilde{\sigma}^2 = \sigma^2\left(1 - \frac{a}{\sigma}\sqrt{\frac{2}{\pi\,dt}}\right) > 0.$$

If $\frac{\partial^2 V}{\partial S^2} > 0$, the stationary problem for (1.1) reads as

$$\frac{1}{2}\tilde{\sigma}^2 S^2\frac{\partial^2 V}{\partial S^2} + b\sigma^2 S^3\left(\frac{\partial^2 V}{\partial S^2}\right)^2 + r\left(\frac{\partial V}{\partial S}S - V\right) = 0. \tag{1.2}$$

In the next section we study Eq. (1.2) under Dirichlet boundary conditions, namely

$$V(c) = V_c, \qquad V(d) = V_d \tag{1.3}$$

for some fixed $d > c > 0$.

In the third section we show that a solution of (1.2) may be obtained as the limit of a nonincreasing (respectively nondecreasing) sequence of upper (lower) solutions.

In the last section we study the existence of solutions of the evolution equation (1.1).

2. The stationary problem

In this section we consider the problem (1.2) under the Dirichlet boundary conditions (1.3). Our main result is the following

Theorem 2.1. (1.2)–(1.3) *admits a convex solution* (*which is unique*) *if and only if*

$$\frac{V_d}{d} \leqslant \frac{V_c}{c}.$$

Proof. Let us introduce the change of variables given by

$$x = \log S, \qquad u(x) = V(S).$$

Then, if $y(x) = \frac{\partial u}{\partial x} - u$, we have that $y'(x) = S^2 V''(S)$ and so V is convex if and only if $y'(x) > 0$. Moreover, (1.3) can be written as

$$\frac{1}{2}\tilde{\sigma}^2 y' + b\sigma^2 e^{-x}(y')^2 + ry = 0$$

or equivalently

$$y'(x) = -\tilde{\sigma}^2/2 + \sqrt{\frac{\tilde{\sigma}^4/4 - 4ryb\sigma^2 e^{-x}}{2b\sigma^2 e^{-x}}}, \quad \bar{c} \leqslant x \leqslant \bar{d}, \tag{2.1}$$

where $\bar{c} = \log c$, $\bar{d} = \log d$. As $y' > 0$ we deduce that $y \leqslant y(\bar{d})$.

For fixed $K \leqslant 0$ let y_K be the unique solution of Eq. (2.1) with $y_K(\bar{d}) = K$. By standard results, it follows that y_K is defined on $[\bar{c}, \bar{d}]$, and the mapping $K \to y_K$ is continuous for the norm of $C([\bar{c}, \bar{d}])$. On the other hand, if $u'_K - u_K = y_K$, assuming the condition $u_K(\bar{d}) = V_d$ we obtain

$$u_K(x) = \left(e^{-\bar{d}} V_d - \int_x^{\bar{d}} y_K(t) e^{-t}\, dt\right) e^x.$$

As $y_K \leqslant K$ on $[\bar{c}, \bar{d}]$,

$$u_K(\bar{c}) \geqslant e^{\bar{c}-\bar{d}} V_d - K\left(1 - e^{\bar{c}-\bar{d}}\right) \to +\infty \quad \text{as } K \to -\infty.$$

Moreover, a simple computation shows that $\frac{\partial}{\partial K}(u_K(\bar{c})) < 0$, proving that $u_K(\bar{c})$ is strictly nonincreasing with respect to K.

Hence we have

(i) if $u_0(\bar{c}) \leqslant V_c$, then there exists a unique $K \leqslant 0$ such that $V(S) = u_K(x)$ is a solution of (1.2)–(1.3);
(ii) if $u_0(\bar{c}) > V_c$, then (1.2)–(1.3) is not solvable.

As $y_0 \equiv 0$, then $u_0(x) = e^{x-\bar{d}} V_d$ and the result follows. □

3. Upper and lower solutions

In this section we apply the method of upper and lower solutions to (1.2)–(1.3). We follow the idea of [1]. In order to find convex solutions, it suffices to find a solution of the problem

$$V'' + H(S, V, V') = 0, \quad V(c) = V_c, \quad V(d) = V_d, \tag{3.1}$$

where

$$H(S, V, V') = \frac{\tilde{\sigma}^2 S^2/2 - \left(\sqrt{\tilde{\sigma}^4 S^4/4 + 4bS^3\sigma^2 r|V'S - V|}\right)}{2b\sigma^2 S^3}$$

such that $V'(d) \leqslant V_d/d$. Indeed, in this case we have that $(V'S - V)' = V''S \geqslant 0$, proving that $V'S - V \leqslant V'(d)d - V_d \leqslant 0$ and V is a solution of the original problem.

In order to prove the main result of this section we recall that (α, β) is an ordered couple of a lower and an upper solution for (3.1) if $\alpha \leqslant \beta$ and

$$\alpha'' + H(\cdot, \alpha, \alpha') \geqslant 0 \geqslant \beta'' + H(\cdot, \beta, \beta')$$

with

$$\alpha(c) \leqslant V_c \leqslant \beta(c), \qquad \alpha(d) \leqslant V_d \leqslant \beta(d).$$

Remark 3.1. A simple computation shows that H satisfies the Lipschitz conditions

$$\begin{aligned} &\left|H(S,U,X) - H(S,V,X)\right| \leqslant K|U - V|, \\ &\left|H(S,U,X) - H(S,U,Y)\right| \leqslant K'|X - Y|, \end{aligned}$$

where $K = \frac{2r}{c^2\tilde{\sigma}^2}$, $K' = \frac{2r}{c\tilde{\sigma}^2}$. We shall assume that $K' < \frac{\pi}{d-c}$, or equivalently,

$$r < \frac{c\tilde{\sigma}^2\pi}{2(d-c)}. \tag{3.2}$$

We shall need the following auxiliary lemmas.

Lemma 3.1. *Assume that* (3.2) *holds and let* $\lambda > 0$ *be large enough. Then for any* $z, \theta \in C([c,d])$ *the equation*

$$u'' + H(S, z, u') - \lambda u = \theta(S)$$

is uniquely solvable under Dirichlet conditions. Furthermore, the application $\mathcal{K} : C([c,d])^2 \to C([c,d])$ *given by* $\mathcal{K}(z,\theta) = u$ *is compact.*

Proof. For $\tau \in [0,1]$ consider the semilinear operator given by $\mathcal{S}u = u'' + \tau H(S,z,u') - \lambda u$. Then, if $u - v \in H_0^1(c,d)$ a simple computation shows that

$$\|\mathcal{S}u - \mathcal{S}v\|_{L^2}\|u - v\|_{L^2} \geqslant \left(1 - \frac{\tau K'}{\pi}\right)\|u' - v'\|_{L^2}^2 + \lambda\|u - v\|_{L^2}^2.$$

Hence, if we define the compact operator $T : H^1(c,d) \to H^1(c,d)$ given by $T(\bar{u}) = u$, where u is the unique solution of the linear problem

$$u'' - \lambda u = \theta - H(S, z, \bar{u}'), \quad u(c) = V_c, \quad u(d) = V_d$$

existence follows from Leray–Schauder theorem.

Moreover, if $u = \mathcal{K}(z,\theta)$ and $u_0 = \mathcal{K}(z_0,\theta_0)$, then

$$(u - u_0)'' + \psi(u - u_0)' - \lambda(u - u_0) = H(S, z_0, u_0') - H(S, z, u_0') + \theta - \theta_0,$$

where

$$\psi(S) = \frac{H(S,z,u') - H(S,z,u_0')}{u' - u_0'} \in L^\infty(c,d), \quad \|\psi\|_\infty \leqslant K',$$

and the compactness of $\mathcal{K}$ follows easily using the standard a priori bound

$$\|w\|_{H_0^1} \leqslant \gamma\|w'' + \psi w' - \lambda w\|_{L^2},$$

where the constant γ depends only on K'. $\square$

Lemma 3.2. *Assume there exists an ordered couple* (α, β) *of a lower and an upper solution. Then* (3.1) *admits at least one solution* V *with* $\alpha \leqslant V \leqslant \beta$.

Proof. For $\lambda \geqslant K$ and $\bar{V} \in C([c,d])$ define $T\bar{V} = V$ as the unique solution of the problem

$$V'' + H(S, \bar{V}, V') - \lambda V = -\lambda \bar{V}, \quad V(c) = V_c, \quad V(d) = V_d.$$

Compactness of T follows easily from Lemma 3.1. Moreover, if $\bar{V} \leqslant \beta$ then

$$\begin{aligned} V'' + H(S, \bar{V}, V') + K\bar{V} - \lambda V &= (K-\lambda)\bar{V} \geqslant (K-\lambda)\beta \\ &\geqslant (K-\lambda)\beta + \beta'' + H(S, \beta, \beta'). \end{aligned}$$

Hence, setting

$$\psi(S) = \frac{H(S, \bar{V}, V') - H(S, \bar{V}, \beta')}{V' - \beta'}$$

we deduce that

$$\begin{aligned} &(V-\beta)'' + \psi(V-\beta)' - \lambda(V-\beta) \\ &\quad \geqslant \left[H(S, \beta, \beta') + K\beta\right] - \left[H(S, \bar{V}, \beta') + K\bar{V}\right] \geqslant 0. \end{aligned}$$

As $V(c) \leqslant \beta(c)$ and $V(d) \leqslant \beta(d)$, it follows from the maximum principle that $V \leqslant \beta$. In the same way, if $\bar{V} \geqslant \alpha$ we obtain that $V \geqslant \alpha$ and the proof follows from Schauder fixed point theorem. □

Theorem 3.3. *Assume there exists an ordered couple* (α, β) *of a lower and an upper solution. Set* $\lambda \geqslant K$ *and define the sequences* $\{\underline{V}_n\}$ *and* $\{\bar{V}_n\}$ *given by*

$$\underline{V}_0 = \alpha, \qquad \bar{V}_0 = \beta$$

and $\bar{V}_{n+1}, \underline{V}_{n+1}$ *the* (*unique*) *respective solutions of the problems*

$$\begin{aligned} &\bar{V}''_{n+1} + H(S, \bar{V}_n, \bar{V}'_{n+1}) - \lambda \bar{V}_{n+1} = -\lambda \bar{V}_n, \\ &\underline{V}''_{n+1} + H(S, \underline{V}_n, \underline{V}'_{n+1}) - \lambda \underline{V}_{n+1} = -\lambda \underline{V}_n \end{aligned}$$

satisfying the Dirichlet boundary conditions. Then $(\underline{V}_n\ \bar{V}_n)$ *is an ordered couple of a lower and an upper solution. Furthermore,* $\{\bar{V}_n\}$ (*respectively* $\{\underline{V}_n\}$) *is nonincreasing* (*nondecreasing*) *and converges to a solution of* (3.1).

Proof. From the previous lemma, we know that $\alpha \leqslant \bar{V}_1 \leqslant \beta$. Moreover,

$$\begin{aligned} &\bar{V}''_1 + H(S, \bar{V}_1, \bar{V}'_1) \\ &\quad = (\lambda - K)(\bar{V}_1 - \beta) + \left[H(S, \bar{V}_1, \bar{V}'_1) + K\bar{V}_1\right] - \left[H(S, \beta, \bar{V}'_1) + K\beta\right] \leqslant 0. \end{aligned}$$

Hence, $\bar{V}_1$ is an upper solution of the problem. Inductively it follows that $\bar{V}_n$ is an upper solution for every n, with $\alpha \leqslant \bar{V}_{n+1} \leqslant \bar{V}_n$. Hence, $\bar{V}_n$ converges pointwise to a function $\bar{V}$. By definition of $\bar{V}_n$,

$$\bar{V}''_{n+1} + H(S, \bar{V}_n, \bar{V}'_{n+1}) \to 0$$

pointwise. Moreover, from Lemma 3.1 we conclude that $\{\bar{V}_n\}$ is bounded in $H^1(c,d)$; hence in $H^2(c,d)$, and it follows that

$$\bar{V}'' + H(S, \bar{V}, \bar{V}') = 0.$$

Thus, $\bar{V}$ is a solution of the problem. The proof for $\underline{V}_n$ is analogous. Furthermore, if we assume as inductive hypothesis that $\underline{V}_n \leqslant \bar{V}_n$ it follows as in the previous lemma that $\underline{V}_{n+1} \leqslant \bar{V}_{n+1}$. □

Remark 3.2. In particular, we may take as upper solution any constant β such that $\beta \geqslant V_c, V_d$. On the other hand, if the lower solution α satisfies

$$\alpha(d) = V_d, \qquad \alpha'(d) \leqslant \frac{V_d}{d},$$

then any solution $V \geqslant \alpha$ of (3.1) verifies that $V'(d) \leqslant \alpha'(d)$. Hence, V is a solution of (1.2). In particular, under appropriate conditions it is possible to find a lower solution $\alpha(S) = mS^2 + nS + p$ for some positive m, p.

4. Solutions to the evolution problem

In this section we consider the nonstationary problem (1.1) under initial-Dirichlet conditions, namely

$$\begin{cases} 0 = V_t + b\sigma^2 s^3 V_{ss}^2 + \frac{1}{2}\tilde{\sigma}^2 s^2 V_{ss} + r(sV_s - V), \\ V(T,s) = f(s), \quad s \in (c,d), \\ V(t,c) = f(c), \quad V(t,d) = f(d), \end{cases} \tag{4.1}$$

for some $f \in C([c,d])$.

If we introduce the change of variables given by $W(t,x) = V(T-t, e^x)$ in the domain $\Omega = (0,T) \times (\bar{c}, \bar{d})$.

Then we have the following problem:

$$\begin{cases} 0 = -W_t + A(W_{xx} - W_x) + r(W_x - W), \\ W(0,x) = f(e^x), \quad x \in (\bar{c}, \bar{d}), \\ W(t,\bar{c}) = f(e^{\bar{c}}), \\ W(t,\bar{d}) = f(e^{\bar{d}}), \end{cases} \tag{4.2}$$

where

$$A = \frac{1}{2}\tilde{\sigma}^2 + b\sigma^2 e^{-x}(W_{xx} - W_x).$$

Setting

$$Z(t,x) = W_{xx}(t,x) - W_x(t,x), \quad P = Z_x,$$

we obtain the equation

$$0 = -Z_t + a(x,Z)Z_{xx} + d(x,Z,P) \tag{4.3}$$

under the conditions

$$Z(0,x) = Z_0(x), \quad x \in [\bar{c}, \bar{d}], \tag{4.4}$$

$$Z(t,\bar{c}) = Z_0(\bar{c}), \qquad Z(t,\bar{d}) = Z_0(d), \tag{4.5}$$

where

$$a(x,Z) = \frac{1}{2}\tilde{\sigma}^2 + 2b\sigma^2 e^{-x} Z,$$

$$d(x,Z,P) = -6b\sigma^2 e^{-x} ZP + 2b\sigma^2 e^{-x} P^2 + \left(r - \frac{1}{2}\tilde{\sigma}^2\right)P - Z(r - 2b\sigma^2 e^{-x} Z)$$

and $Z_0(x) = f''(e^x)e^{2x}$. Let us define

$$\tilde{a}(x,Z) = a\big(x, [Z]_+\big)$$

and

$$\tilde{d}(x,Z,P) = -6b\sigma^2 e^{-x} ZP + b\sigma^2 e^{-x} P^2 + \left(r - \frac{1}{2}\tilde{\sigma}^2\right)P - Z[r - 2b\sigma^2 e^{-x} Z]_+$$

and consider the problem

$$0 = -Z_t + \tilde{a}(x,Z)Z_{xx} + \tilde{d}(x,Z,Z_x) \tag{4.6}$$

under the conditions (4.4)–(4.5).

Proposition 4.1. *Given $Z_0 \in C[\bar{c}, \bar{d}]$ there exists a solution $Z \in C^{2,1}(\Omega) \cap C(\bar{\Omega})$ to* (4.6)–(4.4)–(4.5).

Proof. We have that $\tilde{a}(x,Z) \geqslant \tilde{\sigma}^2/2$, and it is clear that for every $R > 0$, α is a Lipschitz function on $[\bar{c}, \bar{d}] \times [-R, R]$. Moreover, $\tilde{d}(x,Z,Z_x)$ is a Lipschitz function on $[\bar{c}, \bar{d}] \times [-R, R] \times [-R, R]$ and satisfies

$$Z\tilde{d}(x,Z,0) \leqslant 0.$$

Moreover, for any fixed Z we have that

$$|P|\left|\frac{\partial \tilde{a}}{\partial Z}(x,Z)\right| + \left|\frac{\partial \tilde{a}}{\partial x}(x,Z)\right| + \left|\tilde{d}(x,Z,0)\right| \leqslant C|P|^2$$

when $|P| \to +\infty$. By Theorem 12.16 in [4], the proof follows. $\square$

Theorem 4.2. *For $Z_0(x) = f''(e^x)e^{2x}$, let Z be the solution given by the previous proposition, and assume that*

$$0 \leqslant f''(y) \leqslant \frac{r}{2b\sigma^2 y} \quad \text{for } y \in [c,d].$$

Then Z is a solution to (4.3)–(4.5).

Proof. From the hypothesis, it is immediate that $Z_0(x) \leqslant \frac{re^x}{2b\sigma^2}$. Thus, by the maximum principle Z satisfies

$$0 \leqslant Z(t,x) \leqslant \frac{re^x}{2b\sigma^2}, \quad (t,x) \in [0,T] \times [\bar{c}, \bar{d}].$$

Then

$$\tilde{a}(x, Z) = a(x, Z), \qquad \tilde{d}(x, Z, Z_x) = d(x, Z, Z_x)$$

and the result follows. □

Remark 4.1. If Z is a solution of (4.3)–(4.5), it is easy to obtain a solution of (4.2) from the equality $W_{xx} - W_x = Z$ and the boundary conditions.

Remark 4.2. It is clear that the coefficients $a(x, Z)$, $d(x, Z, P)$ and their derivatives with respect to Z and P are bounded on any compact subset of $[\bar{c}, \bar{d}] \times \mathbb{R}^2$. Then, problem (4.6) has no more than one solution in $C^{2,1}(\Omega) \cap C(\bar{\Omega})$ (see [3]).

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