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A FRACTAL PLANCHEREL THEOREM

Abstract

A measure μ on $\mathbb{R}^n$ is called locally and uniformly h-dimensional if $\mu(B_r(x)) \leq h(r)$ for all $x \in \mathbb{R}^n$ and for all $0 < r < 1$, where h is a real valued function. If $f \in L^2(\mu)$ and $\mathcal{F}_\mu f$ denotes its Fourier transform with respect to μ, it is not true (in general) that $\mathcal{F}_\mu f \in L^2$ (e.g. [10]). However in this paper we prove that, under certain hypothesis on h, for any $f \in L^2(\mu)$ the L^2-norm of its Fourier transform restricted to a ball of radius r has the same order of growth as $r^n h(r^{-1})$ when $r \to \infty$. Moreover we prove that the ratio between these quantities is bounded by the $L^2(\mu)$-norm of f (Theorem 3.2). By imposing certain restrictions on the measure μ, we can also obtain a lower bound for this ratio (Theorem 4.3). These results generalize the ones obtained by Strichartz in [10] where he considered the particular case in which $h(x) = x^\alpha$.

1 Introduction.

We will say that a measure μ is locally and uniformly h-dimensional (or shortly μ is an h-dimensional measure) if and only if

$$\mu(B_r(x)) \leq h(r) \qquad \forall\, x \in \mathbb{R}^n, \forall\, 0 < r < 1, \tag{1.1}$$

where $B_r(x)$ is, as usual, the ball of radius r centered at x. We consider functions $h : [0, +\infty] \to \mathbb{R}$ that are non-decreasing, continuous, and such that

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$h(0) = 0$. We further require h to be *doubling*, i. e. there exists a constant $c > 0$ such that $h(2x) < ch(x)$. Such a function h will be called *dimension function.* A particular example is $h(x) = x^\alpha$, which was analyzed by Strichartz in [10]. In that case we will say indistinctly that μ is h-dimensional or that μ is α-dimensional. Allowing h to be more general has already proven to be useful (see for example [9],[8], [3]) and it enables us to obtain a lower bound on measures which were not included in previous results (see Section 5).

If μ is locally and uniformly 0-dimensional, meaning that the measure of any ball of radius one is bounded, then each $f \in L^2(\mu)$ defines a tempered distribution, mapping each test function φ in the Schwartz space $\mathscr{S}$ into $\int f\varphi d\mu$. Therefore its Fourier transform is also a tempered distribution defined by $\varphi \mapsto \int \hat{\varphi} f d\mu$ for $\varphi \in \mathscr{S}$, where $\hat{\varphi}$ is the usual Lebesgue Fourier transform. We will denote by $\mathcal{F}_\mu f$ this 'distributional' Fourier transform of an $f \in L^2(\mu)$. If $f \in L^1(\mu) \cap L^2(\mu)$ then it is easy to see that $\mathcal{F}_\mu f(\xi) = \int f(x)e^{i\xi x}d\mu(x)$. See for example [2].

Strichartz proved in [10] that if $f \in L^2(\mu)$ and μ is zero dimensional then $\mathcal{F}_\mu f$ belongs to $L^2(e^{-t|\xi|^2})$ for any $t > 0$ and therefore to $L^2_{loc}(\mathbb{R}^d)$. Note that if h is one of our dimension functions, we have immediately that μ is 0-dimensional.

In this paper, our goal is to prove an analogue to Plancherel's Theorem (in $L^2(\mathbb{R}^n)$ with the Lebesgue measure) for any h-dimensional measure μ. In fact we are going to show the existence of upper and lower bounds for the ratio between $r^n h(r^{-1})$ and the norm of the Fourier transform of a function f in $L^2(\mu)$ restricted to the ball of radius r. The hypotheses under which we obtain the existence of the upper bound are more general than the ones we need for the existence of the lower bound.

The h-dimensional Hausdorff measure is defined as (see for example [9])

$$\mathcal{H}^h(E) = \lim_{\delta \to 0} \left(\inf \left\{ \sum_{i=1}^{\infty} h(|U_i|) : E \subset \bigcup_{i \geq 1} U_i \text{ and } |U_i| \leq \delta \right\} \right).$$

$\mathcal{H}^h_{\llcorner E}$ will denote its restriction to a set E.

The h-lower density of a set E in x is (see for example [4])

$$\underline{D}(\mathcal{H}^h{}_{\llcorner E}, x) = \liminf_{r \to 0} \frac{\mathcal{H}^h(E \cap B_r(x))}{h(2r)}. \tag{1.2}$$

The upper density is defined by taking lim sup in the above equation. We will introduce one additional definition.

Definition 1.1. A set E will be said to be an *h-regular set* if both upper and lower densities are equal to one in $\mathcal{H}^h$-almost every point of E. In symbols

$\underline{D}(\mathcal{H}^h\llcorner_E, x) = \overline{D}(\mathcal{H}^h\llcorner_E, x) = 1$ for $\mathcal{H}^h$-almost every point of E. If the lower density is greater than a positive constant for $\mathcal{H}^h$-almost every point of E we will say that E is an *h-quasi regular set*.

It is evident that the requirement for a set to be regular is more restrictive than the one to be quasi regular. Actually it has already been proven (see [8]) that there only exist regular sets for functions of the form x^k with k integer. On the other hand, there are h-regular sets for any dimensional function h.

The lower bound that we obtain (see Theorem 4.2) will be stated for the measure $\mathcal{H}^h$ restricted to an h-dimensional and quasi regular set. In section 5 we will show an example of a set E and a function h such that $\mathcal{H}^h\llcorner_E$ is h-dimensional and E is quasi regular. Additionally we will prove that there does not exist any α such that $\mathcal{H}^\alpha\llcorner_E$ is x^α-dimensional and E is quasi regular simultaneously. This example satisfies the hypothesis of our Theorem 4.1 but does not satisfy the hypothesis of the analogous Theorem 5.5 in [10].

2 Some Technical Results.

Any h-dimensional measure μ is locally finite, which means that for μ-almost every x there exists an $r > 0$ such that $0 < \mu(B_r(x)) < \infty$. Therefore, as Strichartz proved in [10], the strong (p, p) estimate (for $p > 1$) and the weak $(1, 1)$ estimate hold for the maximal operator, defined for each $f \in L^1_{loc}(\mu)$ as

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f| \, d\mu. \tag{2.1}$$

More precisely, we have the following theorem.

Theorem 2.1. *Let μ be a locally finite measure on $\mathbb{R}^n$. For each locally integrable function f we have:*
1. $\mu(\{x : M_\mu f(x) > s\}) \leq \frac{c_n}{s} \|f\|_1 \quad \forall f \in L^1(\mu)$.
2. *For* $1 < p \leq \infty$, $\|M_\mu f\|_p \leq c_p \|f\|_p \quad \forall f \in L^p(\mu)$.

This theorem has many consequences which will be useful for our work. In particular, we have the following two corollaries.

Corollary 2.2. *Let $f \in L^1(\mu)$. For μ-almost every x, $\lim\limits_{r\to 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f d\mu = f(x)$.*

Corollary 2.3. *Let E be a h-regular set and let $f \in L^1(\mu)$. For $\mathcal{H}^h$-almost every $y \in E$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\Big| \int_{B_r(y)} f d\mu - h(r) f(y) \Big| \leq \varepsilon h(r) \qquad \forall r \leq \delta.$$

The proofs of the theorem and these corollaries are straightforward applications of Besicovitch's Covering Theorem and can be found in [10].

We also need the following quite technical Lemma, which will allow us to bound the ratio between h and its dilation by r $(h(rt)/h(t))$ by a function in the weighted space $L^1(e^{-cr^2})$.

Lemma 2.4. *Let $h : [0, \infty) \to \mathbb{R}$ be a continuous, non-decreasing, and doubling function ($h(2x) \leq c_d h(x)$). Then there exists a constant $\kappa > 0$ such that $h(rt) \leq c_d h(t) \max\{1, r^\kappa\}$ $\forall r, t > 0$.*

Proof. First note that $c_d \geq 1$, since in fact the doubling condition can be restated as $c_d \geq h(2x)/h(x)$ and this quantity is not smaller than 1 because h is a non decreasing function.

If $r < 1$, since h is non-increasing, we have $h(rt) \leq h(t)$. If $r \geq 1$ we choose the only non-negative integer k such that $2^{k-1} < r \leq 2^k$. So $h(rt) \leq h(2^k t) \leq c_d^k h(t)$. Observe that k was chosen such that $k \leq \frac{\log r}{\log 2} + 1$ and therefore it then follows that $c_d^k \leq c_d . r^{\log c_d / \log 2}$. The proof is complete by taking $\kappa = \log c_d / \log 2$. □

Recall that we are dealing with h-dimensional measures, which means that the measure of the balls of radius $r < 1$ is bounded. The next lemma provides a control of the measure of the "large" balls, i.e. those balls of radius greater than one for which the estimate (1.1) does not hold.

Lemma 2.5. *Let μ be a locally h-dimensional measure on $\mathbb{R}^n$. If $r > 1$, then $\mu(B_r(x)) \leq Cr^n$ for some C independent of x.*

Proof. Denote by Q the minimal cube centered at x that contains the ball $B_r(x)$, i.e. $Q = Q(x, r) = \{y \in \mathbb{R}^n : \|x - y\|_\infty < r\} \supset B_r(x)$. Let k be the (unique) integer such that $k - 1 < r\sqrt{n} \leq k$. Q can be divided into k^n smaller cubes of half-side $\frac{r}{k}$. Each of these cubes is contained in a ball of radius $r_0 = \sqrt{n}\frac{r}{k} \leq 1$. So we obtain $\mu(B_r(x)) \leq \mu(Q) \leq k^n \mu(B_{r_0}(x')) \leq k^n\ h(\frac{\sqrt{n}r}{k})$. Since $\sqrt{n}\frac{r}{k} \leq 1$, it follows that $h(\sqrt{n}\frac{r}{k}) \leq h(1)$. On the other hand, by the choice of k, $k^n < (r\sqrt{n} + 1)^n \leq r^n(\sqrt{n} + 1)^n$ and we obtain $\mu(B_r(x)) \leq (\sqrt{n} + 1)^n h(1) r^n$. □

3 Upper Bounds.

Our first result is an upper estimate for the L^2-norm of the Fourier transform of a function $f \in L^2(\mu)$.

Theorem 3.1. *Let μ be a locally and uniform h-dimensional measure, where h is a dimension function. Suppose that h defines a dimension not greater than n in the sense that $\lim_{t\to 0} t^n/h(t) = 0$. Then*

$$\sup_{0\le t\le 1} \frac{\sqrt{t^n}}{h(\sqrt{t})} \int e^{-t|\xi|^2} |\mathcal{F}_\mu f(\xi)|^2 \, d\xi \le c \, \|f\|_2^2 := c \int |f|^2 \, d\mu \qquad \forall f \in L^2(\mu).$$

Proof. *First Step.* We will prove that

$$\sqrt{t^n} \int |\mathcal{F}_\mu f(\xi)|^2 \, e^{-t|\xi|^2} d\xi = \pi^{n/2} \iint e^{-|x-y|^2/4t} f(x)\overline{f(y)} d\mu(x) d\mu(y). \quad (3.1)$$

Recall the inverse Fourier transform for the gaussian function $\int e^{-t|\xi|^2} e^{ix\xi} d\xi = \sqrt{t^{-n}} \pi^{n/2} e^{-|x|^2/4t}$. If f is integrable then equation (3.1) follows from Fubini's theorem since

$$\begin{aligned} \sqrt{t^n} \int |\mathcal{F}_\mu f(\xi)|^2 \, e^{-t|\xi|^2} d\xi &= \sqrt{t^n} \iiint f(x)\overline{f(y)} e^{i(x-y)\cdot\xi} e^{-t|\xi|^2} d\mu(x) d\mu(y) d\xi \\ &= \pi^{n/2} \iint e^{-|x-y|^2/4t} f(x)\overline{f(y)} d\mu(x) d\mu(y). \quad (3.2) \end{aligned}$$

Now consider any $f \in L^2(\mu)$ (not necessarily integrable). Let us define $f_k(x) = f(x)\chi_{\{|x|\le k\}}(x)\chi_{\{|f(x)|\le k\}}(x)$. This sequence converges to f in $L^2(\mu)$. Also since each f_k is in $L^1(\mu)$, it satisfies (3.2). Using Beppo Levi's Theorem, we get

$$\iint e^{-|x-y|^2/4t} f_k(x)\overline{f_k(y)} d\mu(x) d\mu(y) \to \iint e^{-|x-y|^2/4t} f(x)\overline{f(y)} d\mu(x) d\mu(y).$$

Since $f \in L^2(\mu)$, it follows that $\mathcal{F}_\mu f \in L^2(e^{-t|\xi|^2} d\xi)$. Hence we can apply the dominated convergence theorem to obtain $\sqrt{t^n} \int |\mathcal{F}_\mu f_k(\xi)|^2 \, e^{-t|\xi|^2} d\xi \to \sqrt{t^n} \int |\mathcal{F}_\mu f(\xi)|^2 \, e^{-t|\xi|^2} d\xi$, which yields 3.1.

Second Step. We will prove that for any $y \in \mathbb{R}^n$ and $f \in L^2(\mu)$,

$$\frac{1}{h(\sqrt{t})} \int e^{-|x-y|^2/4t} f(x) d\mu(x) \le C M_\mu f(y).$$

Using Fubini on the left hand side of the inequality, we obtain

$$\int e^{-|x-y|^2/4t} f(x) d\mu(x) = \int_0^\infty \frac{r}{2t} e^{-r^2/4t} \int_{B_r(y)} f(x) d\mu(x).$$

Since $\int_{B_r(y)} f(x)d\mu(x) \le \mu(B_r(y))M_\mu f(y)$, it follows that

$$\int e^{-|x-y|^2/4t} f(x)d\mu(x) \le M_\mu f(y) \int_0^\infty e^{-r^2/4t} \frac{r}{2t} \mu(B_r(y))dr. \tag{3.3}$$

We need to prove that the last integral is finite. To establish that, we split the integral into two parts, the first for $r < 1$ (where (1.1) is valid) and the second for $r \ge 1$ (where lemma 2.5 can be applied). For $r < 1$ we use the hypothesis to obtain

$$\int_0^1 e^{-r^2/4t} \frac{r}{2t} \mu(B_r(y))dr \le \int_0^1 e^{-r^2/4t} \frac{r}{2t} h(r)dr = \frac{1}{2} \int_0^{1/\sqrt{t}} e^{-r^2/4} r h(r\sqrt{t})dr$$

or equivalently

$$\frac{1}{h(\sqrt{t})} \int_0^1 e^{-r^2/4t} \frac{r}{2t} \mu(B_r(y)dr \le \frac{1}{2} \int_0^{1/\sqrt{t}} e^{-r^2/4} r \frac{h(r\sqrt{t})}{h(\sqrt{t})} dr.$$

This integral is finite by Lemma 2.4.

For $r \ge 1$,

$$\int_1^\infty e^{-r^2/4t} \frac{r}{2t} \mu(B_r(y)dr \le \int_1^\infty e^{-r^2/4t} \frac{r}{2t} r^n dr = \frac{1}{2} \sqrt{t^n} \int_{1/\sqrt{t}}^\infty e^{-r^2/4} \; r^{n+1} dr.$$

Since $\lim_{t\to 0} t^n/h(t) = 0$, we deduce that $\sqrt{t^n}/h(\sqrt{t}) \le C$ and therefore $\frac{1}{h(\sqrt{t})} \int_1^\infty e^{-r^2/4t} \frac{r}{2t} \mu(B_r(y)dr \le C$, with C independent of t. This completes the second step of our proof.

Third (and Final) Step. We will now prove the thesis. Using the first and second steps we obtain

$$\begin{aligned} \frac{\sqrt{t^n}}{h(\sqrt{t})} \int |\mathcal{F}_\mu f(\xi)|^2 e^{-t|\xi|^2} d\xi &= \pi^{n/2} \int \Big(\int e^{-|x-y|^2/4t} f(x)d\mu(x) \Big) f(y)d\mu(y) \\ &\le C \int M_\mu f(y) \, |f(y)| \, d\mu(y). \end{aligned} \tag{3.4}$$

The last term is the inner product in the Hilbert space $L^2(\mu)$. Thus we can bound it using Cauchy-Schwartz. The $L^2(\mu)$ norm of $M_\mu f$ can be bounded by using the $(2, 2)$ estimate in (2.1). Then $\int M_\mu f(y) \, |f(y)| \, d\mu(y) \le C \, \|f\|_2^2$ and this, together with (3.4), gives the desired result. □

Theorem 3.2. *Under the hypothesis of Theorem 3.1, for each $f \in L^2(\mu)$ we have*

$$\sup_{x\in\mathbb{R}^n} \sup_{r\ge 1} \frac{1}{r^n h(r^{-1})} \int_{B_r(x)} |\mathcal{F}_\mu f(\xi)|^2 \, d\xi \le C \, \|f\|_2^2 \, .$$

PROOF. We need to show that for each $x \in \mathbb{R}^n$,

$$\sup_{r\geq 1} \frac{1}{r^n h(r^{-1})} \int_{B_r(x)} |\mathcal{F}_\mu f(\xi)|^2 \, d\xi \leq C \, \|f\|_2^2 \, . \tag{3.5}$$

Making the substitution $t = r^{-2}$ in Theorem 3.1, we obtain exactly (3.5) for $B_r(0)$. Furthermore $\int_{B_r(x)} |\mathcal{F}_\mu f(\xi)|^2 \, d\xi = \int_{B_r(0)} \left|\mathcal{F}_\mu(e^{ix\xi} f)\right|^2 d\xi \leq C \left\|e^{ix\xi} f\right\|_2^2$ $= C \, \|f\|_2^2$, which yields the theorem. □

This theorem provides an upper bound but does not tell us whether the limit for $r \to \infty$ exists or not. With our definition, if a measure is h-dimensional it is also g-dimensional for any $h \leq g$. For example, if $h(x) \geq x^n$, then the measure $\mu = \mathcal{H}^h{\llcorner_E} + \mathcal{L}$ is an h-dimensional measure. Here $\mathcal{L}$ is the n-dimensional Lebesgue measure and E is a set of $\mathcal{H}^h$ finite measure. However in this case it is clear that μ has two distinct parts. One 'truly' is h-dimensional ($\mathcal{H}^h_{\llcorner E}$), but the other ($\mathcal{L}$), while by the previous remark can be considered as h-dimensional, is in fact n-dimensional.

The next theorem will allow us to split up our measure in order to separate the part of the measure that is 'exactly' h-dimensional from the one that can also be seen as having bigger dimension.

Definition 3.3. We say that a measure ν is null with respect to (another measure) μ if and only if $\mu(E) < \infty \Rightarrow \nu(E) = 0$. We will denote this by $\nu \lll \mu$.

Now we will prove a theorem that is analogous to the Radon-Nikodym Theorem.

Theorem 3.4. *Let μ a measure on $\mathbb{R}^n$ without infinitely many atoms and let ν be a σ-finite measure on $\mathbb{R}^n$ absolutely continuous (null) with respect to μ. There exists a unique decomposition of ν: $\nu = \nu_1 + \nu_2$, where $\nu_1(E) = \int_E f d\mu$ for some measurable and nonnegative function f, with $\nu_2 \lll \mu$.*

PROOF. *Uniqueness.* Let us suppose we have a decomposition $\nu = \nu_1 + \nu_2$ with $\nu_1(E) = \int_E f d\mu$ and $\nu_2 \lll \mu$. Consider $E \subset \mathbb{R}^n$. Let us analyze separately both cases, i.e. when E is σ-finite for μ and when it is not.

If E is σ-finite for μ then $E = \cup_{j\geq 1} E_j$ with $\mu(E_j) < \infty$. Since $\nu_2 \lll \mu$ we have $\nu_2(E_j) = 0$ for all $j \geq$ and therefore $\nu_2(E) = 0$, which gives $\nu_1(E) = \nu(E)$. If we have any other decomposition $\nu = \nu_1' + \nu_2'$, then $\nu_2'(E) = 0 = \nu_2(E)$ and $\nu_1'(E) = \nu(E) = \nu_1(E)$.

If E is not σ-finite for μ, then ν_2 may be positive. However by hypothesis ν is still σ-finite and then $E = \cup_{j\geq 1} \tilde{E}_j$ with $\nu(\tilde{E}_j) < \infty$, where $\tilde{E}_j$ may be chosen disjoint if necessary. Suppose we have another decomposition $\nu = \nu_1' + \nu_2'$ with $\nu_1'(E) = \int_E g \, d\mu$ and $\nu_2' \lll \mu$. In particular, $\nu_1 - \nu_1' = \nu_2' - \nu_2$. We have that $(\nu_1 - \nu_1')(\{x \in \tilde{E}_j : f(x) > g(x)\}) < \infty$, which by the definition of ν_1 and $\nu_1\prime$ implies that $\mu(\{x \in \tilde{E}_j : f(x) > g(x)\}) < \infty$. Since ν_2 and ν_2' are both null with respect to μ we have $\nu_2(\{x \in \tilde{E}_j : f(x) > g(x)\}) = \nu_2'(\{x \in \tilde{E}_j : f(x) > g(x)\}) = 0$. We can do the same calculation for the complementary set for which $f(x) < g(x)$ and conclude that $\nu_2(\tilde{E}_j') := \nu_2(\{x \in \tilde{E}_j : f(x) \neq g(x)\}) = \nu_2'(\tilde{E}_j') = 0$ and therefore $\nu_1(\tilde{E}_j') = \nu(\tilde{E}_j') = \nu_1'(\tilde{E}_j')$. In $\tilde{E}_j \setminus \tilde{E}_j'$, f and g coincide and so $\nu_1(\tilde{E}_j \setminus \tilde{E}_j') = \nu_1(\tilde{E}_j \setminus \tilde{E}_j')$. Since $\tilde{E}_j = \tilde{E}_j' \cup (\tilde{E}_j \setminus \tilde{E}_j')$, it follows that ν_1 and ν_1' coincide on each $\tilde{E}_j$ and therefore on E if the $\tilde{E}_j$ were chosen disjoint. Now it follows that $\nu_2 = \nu_2'$.

Existence. Let us consider first the case when ν is finite. We define the set $\mathscr{A} = \{A \subset \mathbb{R}^n : A \text{ is measurable}, \nu(A) > 0, \ \mu_{\llcorner A} \text{ is } \sigma\text{-finite.}\}$. If $\mathscr{A} = \emptyset$, then the theorem follows taking $\nu_2 = \nu$ and $\nu_1 = 0$. If $\mathscr{A} \neq \emptyset$, define $a := \sup_{A \in \mathscr{A}} \nu(A)$. We have that a is finite, since ν is finite. Consider the set sequence $(A_j)_{j \in \mathbb{N}} \subset \mathscr{A}$ such that $\nu(A_j) \to a$. Let $B := \bigcup_{j=1}^{\infty} A_j$. We are going to see that we can take $\nu_1 = \nu_{\llcorner B}$ and $\nu_2 = \nu_{\llcorner B^c}$. In fact, since $\mu_{\llcorner B}$ is σ-finite, we have f, the Radon-Nykodim derivative of ν with respect to $\mu_{\llcorner B}$. Now we take a set E such that $\mu(E) < \infty$. If $\nu_2(E) > 0$, then $\nu(E \cup B) > a$ which is a contradiction. Therefore $\nu_2(E) = 0$ and so $\nu_2 \lll \mu$.

Let us analyze now the case when ν is not finite (but still σ-finite). Let (E_j) be a collection of measurable sets with $\nu(E_j) < \infty$ such that $\cup E_j = E$. Without loss of generality, we can assume that E_j are pairwise disjoint. We define $\nu^j = \nu_{\llcorner E_j}$ and $\mu^j = \nu_{\llcorner E_j}$. Then ν^j is finite and regarding the previous case, we can decompose $\nu^j = \nu_1^j + \nu_2^j$. Now $\nu_1 = \sum_j \nu_1^j$ and $\nu_2 = \sum_j \nu_2^j$ satisfy the desired property. $\square$

Corollary 3.5. *If μ is an h-dimensional measure, then there exists $\varphi \geq 0$ and $\nu \lll \mathcal{H}^h$ such that $\mu = \varphi d\mathcal{H}^h + \nu$.*

PROOF. In view of the previous theorem, we only need to prove that μ is absolutely continuous respect to $\mathcal{H}^h$. Let us take a set E with $\mathcal{H}^h(E) = 0$. Then for any $\varepsilon > 0$, there is a covering $(U_i)_{i \geq 1}$ of E with $\sum_{i=1}^{\infty} h(|U_i|) < \varepsilon$, where $|U_i|$ is the diameter of U_i. Then

$$\mu(E) \leq \sum_{i=1}^{\infty} \mu(U_i) \leq \sum_{i=1}^{\infty} \mu(B_{|U_i|}(x_i)),$$

picking any $x_i \in U_i$. Using that μ is h-dimensional and the previous estimate, we have $\mu(E) \leq \sum_{i=1}^{\infty} h(|U_i|) < \varepsilon$. Since ε is arbitrary, $\mu(E) = 0$ and the proof is complete. $\square$

The next technical lemma will be necessary for our construction.

Lemma 3.6. *If ν is a locally finite measure on $\mathbb{R}^n$ and $\nu \lll \mathcal{H}^h$, then $\overline{D}_h(\nu, x) := \limsup_{r\to 0} \frac{\nu(B_r(x))}{h(2r)} = 0$ for $\mathcal{H}^h$- almost every x.*

PROOF. For each $k \in \mathbb{N}$ we define the sets $E_k = \{x \in \mathbb{R}^n : \forall \varepsilon > 0, \exists\, r \le \varepsilon$ with $\frac{\nu(B_r(x))}{h(2r)} \ge \frac{1}{k}\}$. Since $\{x \in \mathbb{R}^n : \overline{D}_h(\nu, x) > 0\} = \bigcup_{k\ge 1} E_k$, it is enough to prove that $\mathcal{H}^h(E_k) = 0$ for all k.

We can suppose that $\nu(E_k)$ is finite since $E_k = \bigcup_{l\ge 1} (E_k \cap B_l(0))$.

Let k be fixed and let $\varepsilon > 0$. For each $x \in E_k$, we can pick an $r(x) \le \varepsilon$ such that $h(2r(x)) \le k\nu(B_{r(x)}(x))$. $\{B_{r(x)}(x)\}_{x\in E_k}$ is a family of balls with uniformly bounded radii. Therefore by Besicovitch's Covering Theorem ([8]) we can take a countable subcover $\{B_{r_j}(x_j)\}_{j\ge 1}$ of E_k such that at most $c(n)$ of the balls intersect at once (i.e. $\sum \chi_{B_{r_j}} \le c(n)$).

Since $r_j \le \varepsilon$, it follows that $B_{r_j} \subset E_{k,\varepsilon} := \{x \in \mathbb{R}^n : dist(x, E_k) \le \varepsilon\}$. So we have $\sum_{j=1}^{\infty} h(2r_j) \le k \sum_{j=1}^{\infty} \nu(B_{r_j}(x_j)) \le kc(n)\nu(E_{k,\varepsilon})$ and therefore $\mathcal{H}^h(E_k) \le c(n)k\nu(E_{k,\varepsilon})$.

However since $E_k \subset \cap_{\varepsilon>0} E_{k,\varepsilon}$ and $\nu(E_k)$ is finite, we have that $\mathcal{H}^h(E_k) \le c(n)k\nu(E_k)$. In particular, $\mathcal{H}^h(E_k)$ is finite, which implies $\nu(E_k) = 0$ by the hypothesis on ν.

Using again that $\mathcal{H}^h(E_k) \le c(n)k\nu(E_k)$, we obtain the desired result. □

We are now able to establish a finer bound for certain h-dimensional measures (compare with Theorem 3.1 and Theorem 3.2).

Theorem 3.7. *Let μ be any h-dimensional measure and let $\mu = \varphi d\mathcal{H}^h + \nu$ (with $\nu \lll \mathcal{H}^h$) be the decomposition of Theorem 3.4. If $f \in L^2(\mu)$ then*

$$\limsup_{t\to 0} \frac{\sqrt{t^n}}{h(\sqrt{t})} \int e^{-t|\xi|^2} |\mathcal{F}_\mu f(\xi)|^2 d\xi \le c \int |f(x)|^2 \varphi(x) d\mathcal{H}^h(x)$$

and

$$\sup_{y\in\mathbb{R}^n} \limsup_{r\to\infty} \int_{B_r(y)} |\mathcal{F}_\mu f(\xi)|^2 d\xi \le c \int |f(x)|^2 \varphi(x) d\mathcal{H}^h(x).$$

PROOF. It suffices to prove that $\lim_{t\to 0} \frac{\sqrt{t^n}}{h(\sqrt{t})} \int e^{-t|\xi|^2} |\mathcal{F}_\nu f(\xi)|^2 d\xi = 0$ and $\lim_{t\to 0} \frac{\sqrt{t^n}}{h(\sqrt{t})} \int e^{-t|\xi|^2} \mathcal{F}_\nu f(\xi) \mathcal{F}_{\mathcal{H}^h} f(\xi) d\xi = 0$.

For this proof we will use the maximal operator M_μ as defined in (2.1). Doing the same type of computations as the ones used to obtain (3.3), we have

$$\frac{1}{h(\sqrt{t})} \int e^{-|x-y|^2/4t} f(x) d\nu(x) \le \frac{M_\nu f(y)}{h(\sqrt{t})} \int_0^\infty e^{-r^2/4t} \frac{r}{2t} \nu(B_r(y)) dr. \quad (3.6)$$

On the other hand by Lemma 3.6, for $\mathcal{H}^h$-almost every y

$$\overline{D}_h(\nu, y) = \limsup \frac{\nu(B_r(y))}{h(2r)} = 0$$

and therefore for all $\varepsilon > 0$ we can choose $0 < \delta < 1$ such that $\nu(B_r(y)) \leq \varepsilon h(r)$.

We split the integral on the right of (3.6) into two parts, $\int_0^\delta + \int_\delta^\infty$. For the first one, using that $\nu(B_r(x)) \leq \mu(B_r(x))$ and therefore ν is h-dimensional, we obtain

$$\begin{aligned}\frac{M_\nu f(y)}{h(\sqrt{t})} \int_0^\delta e^{-r^2/4t} \frac{r}{2t} \nu(B_r(x)) dr &\leq M_\nu f(y) \varepsilon \int_0^{\delta/\sqrt{t}} e^{-r^2/4} r \frac{h(r\sqrt{t})}{h(\sqrt{t})} dr \\ &\leq c \varepsilon M_\nu f(y)\end{aligned}$$

by hypothesis.

For the second one, we split again yielding

$$\begin{aligned}&\frac{M_\nu f(y)}{h(\sqrt{t})} \int_\delta^\infty e^{-r^2/4t} \frac{r}{2t} \nu(B_r(y)) dr \\ &\leq M_\nu f(y) c \left(\int_{\delta/\sqrt{t}}^{1/\sqrt{t}} e^{-r^2/4} r \frac{h(r\sqrt{t})}{h(\sqrt{t})} dr + \int_{1/\sqrt{t}}^\infty e^{-r^2/4} r^{n+1} \frac{\sqrt{t^n}}{h(\sqrt{t})} dr \right) \xrightarrow[t\to 0]{} 0.\end{aligned}$$

So if we set

$$H(t, y) := \frac{1}{h(\sqrt{t})} \int e^{-|x-y|^2/4t} f(x) d\nu(x),$$

we showed that $\lim_{t\to 0} H(t, y) = 0$. Using dominated convergence in the same way than it was used in the first step of the proof of Theorem 3.1, we obtain

$$\begin{aligned}0 = \int \lim_{t\to 0} H(t, y) \overline{f(y)} d\nu(y) &= \lim_{t\to 0} \frac{1}{h(\sqrt{t})} \int e^{-|x-y|^2/4t} f(x) d\nu(x) \overline{f(y)} d\nu(y) \\ &= \lim_{t\to 0} \frac{\sqrt{t^n}}{h(\sqrt{t})} \int e^{-t|\xi|^2} |\mathcal{F}_\nu f(\xi)|^2 d\xi.\end{aligned}$$

In the same way, if we integrate with respect to μ, we obtain

$$\lim_{t\to 0} \frac{\sqrt{t^n}}{h(\sqrt{t})} \int e^{-t|\xi|^2} \mathcal{F}_\nu f(\xi) \mathcal{F}_\mu f(\xi) d\xi = 0.$$

Now the thesis is a consequence of Theorems 3.1 and 3.2. $\square$

4 Lower Estimate.

In this section we estimate the lower bound for the μ-Fourier transform. We start by the following theorem.

Theorem 4.1. *Let $\mu = \mathcal{H}^h \llcorner_E$ for an h-regular set E (see 1.2). Suppose that the function h satisfies both $h(t) \leq t^n$ for $t \geq 1$ and $\lim_{t\to 0} \frac{t^n}{h(t)} = 0$. Also suppose that the limit $\lim_{t\to 0} \frac{h(rt)}{h(t)} := p(r)$ exists. Then for $f \in L^2(\mu)$,*

$$\lim_{t\to 0} \frac{\sqrt{t^n}}{h(\sqrt{t})} \int e^{-t|\xi|^2} |\mathcal{F}_\mu f(\xi)|^2 \, d\xi = C_{n,h} \int |f|^2 \, d\mu, \tag{4.1}$$

where $C_{n,h} = \int_0^\infty e^{-r^2/2} r p(r) dr$.

Proof. In view of (3.1), we will estimate

$$\frac{\sqrt{t^n}}{h(\sqrt{t})} \int_0^\infty e^{-r^2/4t} \frac{r}{2t} \int_{B_r(y)} f(x) d\mu(x) dr.$$

We write the first integral as sum $\int_0^\delta + \int_\delta^\infty$. For any δ the second term tends to zero, since

$$\begin{aligned} &\frac{1}{h(\sqrt{t})} \int_\delta^\infty e^{-r^2/4t} \frac{r}{2t} \int_{B_r(y)} f(x) d\mu(x) dr \qquad (4.2)\\ &\leq \frac{1}{h(\sqrt{t})} \int_\delta^\infty e^{-r^2/4t} \frac{r}{2t} \mu(B_r(y)) M_\mu f(y) dr \\ &\leq \frac{1}{h(\sqrt{t})} M_\mu f(y) \left(\int_\delta^1 e^{-r^2/4t} \frac{r}{2t} h(r) dr + \int_1^\infty e^{-r^2/4t} \frac{r}{2t} r^n dr \right) \\ &= \frac{\sqrt{t^n}}{h(\sqrt{t})} M_\mu f(y) \left(\int_{\delta/\sqrt{t}}^{1/\sqrt{t}} r h(r) e^{-r^2/4} dr + \int_{1/\sqrt{t}}^\infty r^{n+1} e^{-r^2/4} dr \right) \xrightarrow[t\to 0]{} 0, \end{aligned}$$

using that $\lim_{t\to 0} \frac{t^n}{h(t)} = 0$.

To analyze the other integral, note first that since E is regular by Corollary 2.3, we have that, for $\mathcal{H}^h$-almost every $y \in E$ (fixed) and for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \int_{B_r(y)} f d\mu - h(r) f(y) \right| \leq \varepsilon h(r) \qquad \forall r \leq \delta. \tag{4.3}$$

On the other hand,

$$\int_0^\delta \frac{1}{h(\sqrt{t})} e^{-r^2/4t} \frac{r}{2t} h(r) f(y) dr = f(y) \int_0^{2\delta/\sqrt{t}} e^{-r^2} r \frac{h(r\sqrt{t})}{h(\sqrt{t})} dr$$

and so, since $e^{-r^2} r \frac{h(r\sqrt{t})}{h(\sqrt{t})}$ is dominated by $e^{-r^2} r^{1+\kappa}$ (see Lemma 2.4), we have

$$\int_0^\delta \frac{1}{h(\sqrt{t})} e^{-r^2/4t} \frac{r}{2t} h(r) f(y) dr \xrightarrow[t\to 0]{} f(y) \int_0^\infty e^{-r^2} r p(r) dr.$$

We conclude that

$$\int_0^\delta \frac{1}{h(\sqrt{t})} e^{-r^2/4t} \frac{r}{2t} \int_{B_r(y)} f(x) d\mu(x) dr \xrightarrow[t\to 0]{} C_{n,h} f(y). \tag{4.4}$$

Combining (4.2) and (4.4) we obtain that

$$H(t,y) := \frac{1}{h(\sqrt{t})} \int_0^\infty e^{-r^2/4t} \frac{r}{2t} \int_{B_r(y)} f d\mu dr \xrightarrow[t\to 0]{} C_{n,h} f(y).$$

Since $H(t,y)$ is dominated by $f(y) \int_0^\infty e^{-r^2} r p(r) dr$ and $f \in L^2(\mu)$, it follows that

$$\lim_{t\to 0} \int H(t,y) \overline{f(y)} d\mu(y) = \int \lim_{t\to 0} H(t,y) \overline{f(y)} d\mu(y) = C_{n,h} \int_E |f|^2 \, d\mu.$$

□

Note that equation (4.3), which was very important in our proof, is a reformulation of Corollary 2.2 substituting $\mu(B_r(y))$ by $h(r)$. We are allowed to make this substitution only because E is a regular set. However this hypothesis on E is too restrictive.

Actually it has already been proven (see [8]) that there only exist regular sets for functions of the form x^k with k integer. So in order for the last theorem to be meaningful, it will be necessary to obtain a result with a weaker hypothesis. We will therefore consider h-quasi regular sets, meaning that there exists a constant $\theta > 0$ such that for $\mathcal{H}^h$-almost every $x \in E$,

$$\liminf_{r\to 0} \frac{\mathcal{H}^h(B_r(x) \cap E)}{h(r)} \geq \theta. \tag{4.5}$$

For this case, instead of the equality in (4.1), we obtain a lower bound.

Theorem 4.2. *Let $\mu = \mathcal{H}^h_E + \nu$. If $\nu \lll \mathcal{H}^h$ and E is h-quasi regular, then*

$$\liminf_{t\to 0} \int e^{-t|\xi|^2} \left|\mathcal{F}_\mu f(\xi)\right|^2 d\xi \geq c \int_E |f|^2 \, d\mathcal{H}^h. \tag{4.6}$$

PROOF. By the proof of Theorem 3.7, we can suppose $\mu = \mathcal{H}^h \llcorner_E$.

Since E is quasi regular, there exists $\delta_1 > 0$ such that if $r < \delta_1$ then

$$\mu(B_r(x)) \geq ch(r). \tag{4.7}$$

On the other hand, there exists $\delta_2 > 0$ such that if $r < \delta_2$ (and $f(y) \neq 0$), then

$$\left| \frac{1}{\mu(B_r(y))} \int_{B_r(y)} f(x) d\mu(x) - f(y) \right| < \varepsilon \, |f(y)| \, . \tag{4.8}$$

Taking $\delta = \delta_{y,\varepsilon}$ (satisfying both estimates), we may write

$$\frac{1}{h(\sqrt{t})} \iint e^{-|x-y|^2/4t} f(x)\overline{f(y)} d\mu(x) d\mu(y) = \int |f(y)|^2 \, H(y,t,\varepsilon) d\mu(y) + R(t,\varepsilon),$$

where $H(y,t,\varepsilon) = \int_0^{\delta_{y,\varepsilon}} \frac{1}{h(\sqrt{t})} e^{-r^2/4t} \frac{r}{2t} \mu(B_r(y)) dr$ and

$$R(t,\varepsilon) = \int \overline{f(y)} \int_0^{\delta_{y,\varepsilon}} \frac{e^{-r^2/4t}}{h(\sqrt{t})} \frac{r}{2t} \left(\int_{B_r(y)} f(x) d\mu(x) - f(y) \mu(B_r(y)) \right) dr d\mu(y)$$
$$+ \int \overline{f(y)} \int_{\delta_{\varepsilon,y}}^{\infty} \frac{1}{h(\sqrt{t})} e^{-r^2/4t} \frac{r}{2t} \int_{B_r(y)} f(x) d\mu(x) dr d\mu(y).$$

We are now going to bound $|R(t,\varepsilon)|$. Using (4.8) and the fact that there exist a bound independent of t for $\int_0^\delta \frac{e^{-r^2/4t}}{h(\sqrt{t})} \frac{r}{2t} \mu(B_r(y)) dr$, we can bound the first term by $C_1 \varepsilon \|f\|^2$. The second term is bounded by

$$\int |f(y)| \int_{\delta_{\varepsilon,y}}^{\infty} \frac{1}{h(\sqrt{t})} e^{-r^2/4t} \frac{r}{2t} \mu(B_r(y)) dr M_\mu f(y) d\mu(y).$$

Recalling a previous calculation, the integral $\int_\delta^\infty \frac{e^{-r^2/4t}}{h(\sqrt{t})} \frac{r}{2t} \mu(B_r(y)) dr$ can be bounded by ε if we take t small enough. Therefore by Cauchy Schwartz and the (2,2) strong estimate, the second term is bounded by $C_2 \|f\|^2$. So both estimates tell us that $|R(t,\varepsilon)| \leq C \|f\|^2$ for small enough t.

On the other hand, $H(y,t,\varepsilon)$ is bounded bellow by $\int_0^\delta \frac{e^{-r^2/4t}}{h(\sqrt{t})} \frac{r}{2t} h(r) dr$, using (4.7). Substituting and using that $\liminf_{t\to 0} \frac{h(r\sqrt{t})}{h(\sqrt{t}} < \infty$, we conclude that $\liminf_{t\to 0} H(y,t,\varepsilon) \geq C_3$.

Therefore by Fatou's Lemma

$$\liminf_{t\to 0} \frac{1}{h(\sqrt{t})} \iint e^{-|x-y|^2/4t} f(x)f(y)d\mu(x)d\mu(y) \geq c \int \ |f|^2\, d\mu.$$

$\square$

Theorem 4.3. *Let μ an h-dimensional measure such that $\mu = \mathcal{H}^h\llcorner_E + \nu$ with $\nu \lll \mathcal{H}^h$ being E h-quasi regular. Then the following inequality holds:*

$$\liminf_{r\to\infty} \frac{1}{r^n h(r^{-1})} \int_{B_r(y)} |\mathcal{F}_\mu f(\xi)|^2\, d\xi \geq c \int_E |f|^2 d\mathcal{H}^h,$$

where the constant c does not depend on y.

Proof. For any $\lambda > 0$ such that $\lambda \leq t|\xi|^2$, we have $e^{-t|\xi|^2} \leq e^{-\lambda/2} e^{-(1/2)t|\xi|^2}$. Then

$$\begin{aligned}
&\frac{\sqrt{t}^n}{h(\sqrt{t})} \int_{\{\xi : t|\xi|^2 \geq \lambda\}} e^{-t|\xi|^2} |\mathcal{F}_\mu f(\xi)|^2 d\xi \\
&\qquad \leq 2^{n/2} \frac{h((t/2)^{1/2})}{h(\sqrt{t})} \frac{(t/2)^{n/2}}{h((t/2)^{1/2})} e^{-\lambda/2} \int e^{-(1/2)t|\xi|^2} |\mathcal{F}_\mu(\xi)|^2 d\xi \\
&\qquad \leq c\; e^{-\lambda/2} \int_E |f|^2 d\mathcal{H}^h
\end{aligned}$$

by Lemma 2.4 and Theorem 3.7. Using 4.6 and picking λ big enough, we obtain

$$\liminf_{t\to 0} \frac{\sqrt{t^n}}{h(\sqrt{t})} \int_{\{\xi : t|\xi|^2 \leq \lambda\}} e^{-t|\xi|^2} |\mathcal{F}_\mu(\xi)|^2 d\xi \geq \tilde{c} \int_E |f|^2 d\mu,$$

picking the constant c smaller if needed. Now taking $t = \lambda/r^2$, we obtain

$$\frac{h(\lambda^{1/2})}{\lambda^{n/2}} \frac{\sqrt{t^n}}{h(\sqrt{t})} \int_{\{\xi : t|\xi|^2 \leq \lambda\}} e^{-t|\xi|^2} |\mathcal{F}_\mu(\xi)|^2 d\xi \leq c_\lambda \frac{1}{r^n h(r^{-1})} \int_{B_r(0)} |\mathcal{F}_\mu(\xi)|^2 d\xi,$$

where c_λ is such that $h(r^{-1})/h(\lambda^{1/2} r^{-1}) \leq c_\lambda$. This completes the proof. $\square$

5 An Example.

We conclude the paper by exhibiting an example of a function h and a set E such that $\mathcal{H}^h_{\llcorner E}$ is h-dimensional and E is quasi regular. For this example Theorem 4.3 holds. However, since E is α-dimensional but with zero $\mathcal{H}^\alpha$ measure the results of Strichartz in [10] do not apply. This shows that by considering more general dimension functions we obtained a useful generalization.

Let $h : [0, \infty) \to \mathbb{R}$ be a dimension function such that $h(2x) < 2h(x)$. Let s_k be such that $h(s_k) = 2^{-k}$. We will construct a set of Cantor type. Consider the two (closed) subintervals of $[0, 1]$, $I_{1,1}$ and $I_{1,2}$, of length s_1 obtained by suppressing the central open interval of length $1 - 2s_1$. In each of these intervals we take the two closed subinterval of length s_2 obtained by removing the central interval of length $s_1 - 2s_2$ this time. (Note that this number is positive because $h(2x) < 2h(x)$.) We obtain four intervals denoted by $I_{2,1}$, $I_{2,2}$, $I_{2,3}$, $I_{2,4}$. These intervals will be called intervals of step 2. Following in the same manner at each step, we obtain 2^k closed intervals of length s_k. Our Cantor set will be

$$E = \bigcap_{k \geq 1} \bigcup_{j=1}^{2^k} I_{k,j}.$$

We assign to each interval $I_{k,j}$ measure 2^{-k} obtaining a probability measure μ supported on E. We can see ([4]) that this measure is $\mathcal{H}^h\llcorner_E$.

We are going to show that this set satisfies the hypothesis of the Theorem 4.2, which means essentially that it is h-quasi regular. It suffices to see that $\frac{\mu(B(x,\rho))}{h(2\rho)} \geq c$ (where c is a positive constant) for all $x \in E$ and for all $\rho > 0$.

Given $x \in E$ and $\rho > 0$, denote by k the minimum integer such that there exists j between 1 and 2^k satisfying $I_{k,j} \subset B(x, \rho)$. By minimality $s_{k-1} \geq \rho$. Then

$$\frac{\mu(B(x,\rho))}{h(2\rho)} \geq \frac{\mu(I_{k,j})}{h(2\rho)} = \frac{2^{-k}}{h(2\rho)} \geq \frac{c_d}{2} \frac{1}{2^{k-1}h(\rho)} \geq \frac{c_d}{2} \frac{1}{2^{k-1}h(s_{k-1})} = \frac{c_d}{2},$$

using that $I_{k,j} \subset B(x, \rho)$, the definition of μ , Lemma 2.4, the minimality of k, and the definition of s_k. Therefore (4.5) follows.

We also need to prove that $\mu = \mathcal{H}^h\llcorner_E$ is an h- dimensional measure. In fact, $E \cap B_\rho(x) \subset I_{k-1,j}$ for some j. Consequently $\mu(B_\rho(x)) \leq \mu(I_{k-1,j}) = 2^{-(k-1)} = 2h(s_k) \leq h(\rho)$.

If we take $h(x) = x^\alpha \log(1/x)$, then we obtain a set E of dimension α but such that $\mathcal{H}^\alpha(E) = 0$. Therefore for any α, E will not be α-quasi regular and hence we cannot apply Strichartz's Theorem.

However since E is h-quasi regular for $h(x) = x^\alpha \log(1/x)$, we can apply Theorem 4.3.

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