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A SINGULAR PERTURBATION PROBLEM FOR A QUASI-LINEAR OPERATOR SATISFYING THE NATURAL GROWTH CONDITION OF LIEBERMAN*

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Abstract. In this paper we study the following problem. For $\varepsilon > 0$, take u^ε as a solution of $\mathcal{L}u^\varepsilon := \operatorname{div}\left(\frac{g(|\nabla u^\varepsilon|)}{|\nabla u^\varepsilon|}\nabla u^\varepsilon\right) = \beta_\varepsilon(u^\varepsilon)$, $u^\varepsilon \geq 0$. A solution to (P_ε) is a function $u^\varepsilon \in W^{1,G}(\Omega) \cap L^\infty(\Omega)$ such that $\int_\Omega g(|\nabla u^\varepsilon|)\frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}\nabla\varphi\,dx = -\int_\Omega \varphi\,\beta_\varepsilon(u^\varepsilon)\,dx$ for every $\varphi \in C_0^\infty(\Omega)$. Here $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta\left(\frac{s}{\varepsilon}\right)$, with $\beta \in \mathrm{Lip}(\mathbb{R})$, $\beta > 0$ in $(0,1)$ and $\beta = 0$ otherwise. We are interested in the limiting problem, when $\varepsilon \to 0$. As in previous work with $\mathcal{L} = \Delta$ or $\mathcal{L} = \Delta_p$ we prove, under appropriate assumptions, that any limiting function is a weak solution to a free boundary problem. Moreover, for nondegenerate limits we prove that the reduced free boundary is a $C^{1,\alpha}$ surface. This result is new even for Δ_p. Throughout the paper, we assume that g satisfies the conditions introduced by Lieberman in [*Comm. Partial Differential Equations*, 16 (1991), pp. 311–361].

Key words. free boundaries, Orlicz spaces, singular perturbation

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1. Introduction.

In this paper we study the following singular perturbation problem: For $\varepsilon > 0$, take u^ε as a nonnegative solution of

$$(P_\varepsilon) \qquad \mathcal{L}u^\varepsilon = \beta_\varepsilon(u^\varepsilon), \quad u^\varepsilon \geq 0,$$

where $\mathcal{L}v := \operatorname{div}\left(\frac{g(|\nabla v|)}{|\nabla v|}\nabla v\right)$.

A solution to (P_ε) is a function $u^\varepsilon \in W^{1,G}(\Omega) \cap L^\infty(\Omega)$ (see the notation for the definition of $W^{1,G}(\Omega)$) such that

$$\int_\Omega g(|\nabla u^\varepsilon|)\frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}\nabla\varphi\,dx = -\int_\Omega \varphi\,\beta_\varepsilon(u^\varepsilon)\,dx \tag{1.1}$$

for every $\varphi \in C_0^\infty(\Omega)$.

Here $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta\left(\frac{s}{\varepsilon}\right)$ for $\beta \in \mathrm{Lip}(\mathbb{R})$, positive in $(0,1)$ and zero otherwise. We call $M = \int_0^1 \beta(s)\,ds$.

We are interested in studying the uniform properties of solutions and understanding what happens in the limit as $\varepsilon \to 0$. We assume throughout the paper that the family $\{u^\varepsilon\}$ is uniformly bounded in the L^∞ norm. Our aim is to prove that, for every sequence $\varepsilon_n \to 0$, there exists a subsequence ε_{n_k} and a function $u = \lim u^{\varepsilon_{n_k}}$ and that u is a weak solution of the free boundary problem

$$\begin{cases} \mathcal{L}u := \operatorname{div}\left(\dfrac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) = 0 & \text{in } \{u > 0\} \cap \Omega, \\ |\nabla u| = \lambda^* & \text{on } \partial\{u > 0\} \cap \Omega \end{cases} \tag{1.2}$$

for some constant λ^* depending on g and M.

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This problem appears in combustion theory in the case $\mathcal{L} = \Delta$ when studying deflagration flames. Back in 1938, Zeldovich and Frank-Kamenetskii proposed the passage to the limit in this singular perturbation problem in [26] (the limit for the activation energy going to infinity in this flame propagation model). The passage to the limit was not studied in a mathematically rigorous way until 1990 when Berestycki, Caffarelli, and Nirenberg studied the case of N dimensional traveling waves (see [3]). Later, in [10], the general evolution problem in the one phase case was considered. Much research has been done on this matter ever since. (See, for instance, [7, 8, 16, 25].)

(1.2) is a very well known free boundary problem in the uniformly elliptic case $(0 < c \leq g(t)/t \leq C < \infty)$. This problem has also been studied in the two phase case. Regularity results for the free boundary in the case of the Laplacian can be found in [1] for one phase distributional solutions and in [4, 5] for two phase viscosity solutions. See also [2] for one phase distributional solutions in the nonlinear uniformly elliptic case. The results in [1, 4, 5] were used in [16] to obtain free boundary regularity results for limit solutions (that is, for $u = \lim u^{\varepsilon_k}$). See also [6, 17] for results in the inhomogeneous case and [11, 13] for viscosity solutions in the uniformly elliptic, variable coefficient case.

Recently, this singular perturbation problem in the case of the p-Laplacian ($g(t) = t^{p-1}$) was considered in [12]. As in the uniformly elliptic case, the authors find, for a uniformly bounded family of solutions u^ε, Lipschitz estimates uniform in ε and prove that the limit of u^ε is a solution of (1.2) for $\mathcal{L} = \Delta_p$ and $\lambda^* = \left(\frac{p}{p-1}M\right)^{1/p}$ in a pointwise sense at points in the reduced free boundary.

See also [24, 22] where the authors treat general elliptic equations of flame propagation type including the study of the regularity of the free boundary.

The aim of our present work is to study this singular perturbation problem—including the regularity of the free boundary—for operators that can be elliptic degenerate or singular, possibly nonhomogeneous (the p-Laplacian is homogeneous and this fact simplifies some of the proofs). Moreover, we admit functions g in the operator $\mathcal{L}$ with a different behavior at 0 and at infinity. Classically, the assumptions on the behavior of g at 0 and at infinity were similar to the case of the p-Laplacian. Here, instead, we adopt the conditions introduced by Lieberman in [19] for the study of the regularity of weak solutions of the elliptic equation (possibly degenerate or singular) $\mathcal{L}u = f$ with f bounded.

This condition ensures that the equation $\mathcal{L}u = 0$ is equivalent to a uniformly elliptic equation in nondivergence form with constants of ellipticity independent of the solution u in sets where $\nabla u \neq 0$. Furthermore, this condition does not imply any type of homogeneity on the function g and, moreover, it allows for a different behavior of $g(|\nabla u|)$ when $|\nabla u|$ is near zero or infinity. Precisely, we assume that g satisfies

$$0 < \delta \leq \frac{tg'(t)}{g(t)} \leq g_0 \qquad \forall t > 0 \tag{1.3}$$

for certain constants $0 < \delta \leq g_0$.

Let us observe that $\delta = g_0 = p - 1$ when $g(t) = t^{p-1}$, and reciprocally, if $\delta = g_0$, then g is a power.

Another example of a function that satisfies (1.3) is the function $g(t) = t^a \log(bt + c)$ with $a, b, c > 0$. In this case, (1.3) is satisfied with $\delta = a$ and $g_0 = a + 1$.

Another interesting case is the one of functions $g \in C^1([0, \infty))$ with $g(t) = c_1 t^{a_1}$ for $t \leq s$ and $g(t) = c_2 t^{a_2} + d$ for $t \geq s$. In this case, g satisfies (1.3) with $\delta = \min(a_1, a_2)$ and $g_0 = \max(a_1, a_2)$.

Furthermore, any linear combination with positive coefficients of functions satisfying (1.3) also satisfies (1.3). On the other hand, if g_1 and g_2 satisfy (1.3) with constants δ^i and g_0^i, $i=1,2$, the function $g=g_1g_2$ satisfies (1.3) with $\delta=\delta^1+\delta^2$ and $g_0=g_0^1+g_0^2$, and the function $g(t)=g_1(g_2(t))$ satisfies (1.3) with $\delta=\delta^1\delta^2$ and $g_0=g_0^1g_0^2$.

This observation shows that there is a wide range of functions g under the hypothesis of this work.

In this paper we show that the limit functions are solutions of (1.2) in the weak sense introduced in [21] where we proved that the reduced boundary of these weak solutions is a $C^{1,\alpha}$ surface. This notion of weak solution turns out to be very well suited for limit functions of this singular perturbation problem.

We state here the definition of weak solution and the main results in this paper.

Definition 1.1 (weak solution II in [21]). *We call u a weak solution of* (1.2) *if*

1. *u is continuous and nonnegative in Ω and $\mathcal{L}u=0$ in $\Omega\cap\{u>0\}$;*
2. *for $D\subset\subset\Omega$ there are constants $0<c_{\min}\le C_{\max}$, $\gamma\ge 1$, such that for balls $B_r(x)\subset D$ with $x\in\partial\{u>0\}$*

$$c_{\min}\le\frac{1}{r}\left(\fint_{B_r(x)}u^\gamma dx\right)^{1/\gamma}\le C_{\max};$$

3. *for $\mathcal{H}^{N-1}$ a.e. $x_0\in\partial_{red}\{u>0\}$, u has the asymptotic development*

$$u(x)=\lambda^*\langle x-x_0,\nu(x_0)\rangle^-+o(|x-x_0|),$$

where $\nu(x_0)$ is the unit interior normal to $\partial\{u>0\}$ at x_0 in the measure theoretic sense;

4. *for every $x_0\in\Omega\cap\partial\{u>0\}$,*

$$\limsup_{\substack{x\to x_0\\ u(x)>0}}|\nabla u(x)|\le\lambda^*.$$

If there is a ball $B\subset\{u=0\}$ touching $\Omega\cap\partial\{u>0\}$ at x_0, then

$$\limsup_{\substack{x\to x_0\\ u(x)>0}}\frac{u(x)}{\operatorname{dist}(x,B)}\ge\lambda^*.$$

Our first result is a bound of $\|\nabla u^\varepsilon\|_{L^\infty}$ independent of ε.

Theorem 1.1. *Let u^ε be a solution of*

$$\mathcal{L}u^\varepsilon=\beta_\varepsilon(u^\varepsilon)\qquad in\ \Omega,$$

with $\|u^\varepsilon\|_{L^\infty(\Omega)}\le L$. Then, for $\Omega'\subset\subset\Omega$ we have

$$|\nabla u^\varepsilon(x)|\le C\qquad in\ \Omega',$$

with $C=C(N,\delta,g_0,L,\|\beta\|_\infty,g(1),\operatorname{dist}(\Omega',\partial\Omega))$ if $\varepsilon\le\varepsilon_0(\Omega,\Omega')$.

Then, we have, via a subsequence, that there exists a limiting function u.

The next step is to prove that the function u is a weak solution in the sense of Definition 1.1 of the free boundary problem (1.2) for a constant λ^* depending on g and M. To this end, we have to prove that $\mathcal{L}u=0$ in $\{u>0\}$ and that we have an asymptotic development for u at any point on the reduced free boundary.

Here we find several technical difficulties associated with the loss of homogeneity of the operator $\mathcal{L}$ and to the fact that we are working in an Orlicz space. This is the case, for instance, when we need to prove the pointwise convergence of the gradients.

At some point we need to add the following hypothesis on g:

There exists $\eta_0 > 0$ such that

$$g'(t) \leq s^2 g'(ts) \quad \text{if} \quad 1 \leq s \leq 1 + \eta_0 \quad \text{and} \quad 0 < t \leq \Phi^{-1}\left(\frac{g_0}{\delta} M\right), \tag{1.4}$$

where $\Phi(\lambda) = \lambda g(\lambda) - G(\lambda)$.

We remark that condition (1.4) holds for all the examples of functions satisfying condition (1.3) described above (see section 4).

There holds the following theorem.

THEOREM 1.2. *Suppose that g satisfies* (1.3) *and* (1.4). *Let u^{ε_j} be a solution to (P_{ε_j}) in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ be such that $\partial\{u > 0\}$ has an inward unit normal ν in the measure theoretic sense at x_0, and suppose that u is nondegenerate at x_0 (see Definition* 5.1*). Under these assumptions, we have*

$$u(x) = \Phi^{-1}(M)\langle x - x_0, \nu\rangle^+ + o(|x - x_0|),$$

where $\Phi(\lambda) = \lambda g(\lambda) - G(\lambda)$.

Finally, we can apply the theory developed in [21]. We have that u is a weak solution in the sense of Definition 1.1 of the free boundary problem.

Then, we have the following theorem.

THEOREM 1.3. *Suppose that g satisfies* (1.3) *and* (1.4). *Let u^{ε_j} be a solution of (P_{ε_j}) in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly in compact subsets of Ω as $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ such that there is a unit inward normal ν to $\Omega \cap \partial\{u > 0\}$ in the measure theoretic sense at x_0. Suppose that u is uniformly nondegenerate at the free boundary in a neighborhood of x_0 (see Definition* 5.1*). Then, there exists $r > 0$ such that $B_r(x_0) \cap \partial\{u > 0\}$ is a $C^{1,\alpha}$ surface.*

Finally, we give two examples in which we can apply the regularity results in this paper. In both examples the nondegeneracy property is satisfied by the limiting function u. In the first example the limiting function is obtained by taking a sequence of minimal solutions of (P_ε) (see Definition 7.1). In the second one it is obtained by taking a sequence of minimizers of the functional

$$J_\varepsilon(v) = \int_\Omega [G(|\nabla v|) + B_\varepsilon(v)]\, dx,$$

where $B'_\varepsilon(s) = \beta_\varepsilon(s)$ (see section 7).

Moreover, in the second example we have that $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{red}\{u > 0\}) = 0$. Thus, in this case the set of singular points has zero $\mathcal{H}^{N-1}$-measure.

We also have—since the limiting function is a minimizer of the problem considered in [21]—that in the case of minimizers we don't need to add any new hypothesis to the function g. That is, the result holds for functions g satisfying only condition (1.3). In dimension 2 if we add to condition (1.3) that

(1.5) there exist constants $t_0 > 0$ and $k > 0$ so that $g(t) \leq kt$ for $t \leq t_0$,

then we have that the whole free boundary is a regular surface (see Corollary 2.2 in [20]).

Outline of the paper. The paper is organized as follows: In section 3 we prove the uniform Lipschitz continuity of solutions of (P_ε) (Corollary 3.1).

In section 4 we prove that if u is a limiting function, then $\mathcal{L}u$ is a Radon measure supported on the free boundary (Proposition 4.1). Then we prove Proposition 4.2, which says that if u is a half-plane, then the slope is 0 or $\Phi^{-1}(M)$, and Proposition 4.3, which says that if u is a sum of two half-planes, then the slopes must be equal and at most $\Phi^{-1}(M)$.

In section 5 we prove the asymptotic development of u at points in the reduced free boundary (Theorem 5.1) and we prove that u is a weak solution according to Definition 1.1.

In section 6 we apply the results of [21] to prove the regularity of the free boundary (Theorem 6.1).

In section 7 we give two examples where the limiting function satisfies the nondegeneracy property. The first one is given by the limit of minimal solutions (Theorem 7.2), and the second one is given by the limit of energy minimizers (Theorem 7.4).

In the appendices we state some properties of the function g, we prove the asymptotic development of $\mathcal{L}$-subsolutions, and we prove the existence of extremal solutions to P_ε.

2. Notation. Throughout the paper, N will denote the dimension and

$$\begin{aligned} B_r(x) &= \left\{x \in \mathbb{R}^N, |x-x_0| < r\right\}, \\ B_r^+(x) &= \left\{x \in \mathbb{R}^N, x_N > 0,\ |x-x_0| < r\right\}, \\ B_r^-(x) &= \left\{x \in \mathbb{R}^N, x_N < 0,\ |x-x_0| < r\right\}. \end{aligned}$$

For $v, w \in \mathbb{R}^N$, $\langle v, w\rangle$ denotes the standard scalar product.

For a scalar function f, $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$.

Furthermore, we denote

$$\begin{aligned} G(t) &= \int_0^t g(s)\, ds, \\ F(t) &= g(t)/t, \\ \Phi(t) &= g(t)t - G(t), \\ A(p) &= F(|p|)p \qquad \text{for } p \in \mathbb{R}^N, \\ a_{ij} &= \frac{\partial A_i}{\partial p_j} \qquad \text{for } 1 \le i, j \le N. \end{aligned}$$

We denote by $L^G(\Omega)$ the Orlicz space that is the linear hull of the set of measurable functions such that $\int_\Omega G(|u|)\, dx < \infty$ with the norm of Luxemburg. That is,

$$\|u\|_{L^G(\Omega)} = \inf\left\{\lambda > 0 \Bigg/ \int_\Omega G\left(\frac{|u|}{\lambda}\right) dx \le 1\right\}.$$

The set $W^{1,G}(\Omega)$ is the Sobolev–Orlicz space of functions in $W^{1,1}_{loc}(\Omega)$ such that both $\|u\|_{L^G(\Omega)}$ and $\||\nabla u|\|_{L^G(\Omega)}$ are finite equipped, with the norm

$$\|u\|_{W^{1,G}(\Omega)} = \max\left\{\|u\|_{L^G(\Omega)}, \||\nabla u|\|_{L^G(\Omega)}\right\}.$$

3. Uniform bound of the gradient. We begin by proving that solutions of the perturbation problem are locally uniformly Lipschitz. That is, the u^ε's are locally Lipschitz, and the Lipschitz constant is bounded independently of ε. In order to prove this result, we first need to prove a couple of lemmas.

LEMMA 3.1. *Let u^ε be a solution of*

$$\mathcal{L}u^\varepsilon = \beta_\varepsilon(u^\varepsilon) \quad \text{in } B_{r_0}(x_0)$$

such that $u^\varepsilon(x_0) \le 2\varepsilon$. Then, there exists $C = C(N, r_0, \delta, g_0, \|\beta\|_\infty, g(1))$ such that if $\varepsilon \le 1$,

$$|\nabla u^\varepsilon(x_0)| \le C.$$

Proof. Let $v(x) = \frac{1}{\varepsilon}u^\varepsilon(x_0 + \varepsilon x)$. Then, if $\varepsilon \le 1$, $\mathcal{L}v = \beta(v)$ in B_{r_0} and $v(0) \le 2$. By Harnack's inequality (see [19]) we have that $0 \le v(x) \le C_1$ in $B_{r_0/2}$ with $C_1 = C_1(N, g_0, \delta, \|\beta\|_\infty)$. Therefore, by using the derivative estimates of [19] we have that

$$|\nabla u^\varepsilon(x_0)| = |\nabla v(0)| \le C,$$

with $C = C(N, \delta, g_0, \|\beta\|_\infty, r_0, g(1))$. ∎

LEMMA 3.2. *Let u^ε be a solution of*

$$\mathcal{L}u^\varepsilon = \beta_\varepsilon(u^\varepsilon) \quad \text{in } B_1$$

and $0 \in \partial\{u^\varepsilon > \varepsilon\}$. Then, for $x \in B_{1/4} \cap \{u^\varepsilon > \varepsilon\}$,

$$u^\varepsilon(x) \le \varepsilon + C\,\mathrm{dist}(x, \{u^\varepsilon \le \varepsilon\} \cap B_1),$$

with $C = C(N, \delta, g_0, \|\beta\|_\infty, g(1))$.

Proof. For $x_0 \in B_{1/4} \cap \{u^\varepsilon > \varepsilon\}$ take $m_0 = u^\varepsilon(x_0) - \varepsilon$ and $\delta_0 = \mathrm{dist}(x_0, \{u^\varepsilon \le \varepsilon\} \cap B_1)$. Since $0 \in \partial\{u^\varepsilon > \varepsilon\} \cap B_1$, $\delta_0 \le 1/4$. We want to prove that $m_0 \le C(N, \delta, g_0, \|\beta\|_\infty, g(1))\delta_0$.

Since $B_{\delta_0}(x_0) \subset \{u^\varepsilon > \varepsilon\} \cap B_1$, we have that $u^\varepsilon - \varepsilon > 0$ in $B_{\delta_0}(x_0)$ and $\mathcal{L}(u^\varepsilon - \varepsilon) = 0$. By Harnack's inequality there exists $c_1 = c_1(N, g_0, \delta)$ such that

$$\min_{B_{\delta_0/2}(x_0)} (u^\varepsilon - \varepsilon) \ge c_1 m_0.$$

Let us take $\varphi = e^{-\mu|x|^2} - e^{-\mu\delta_0^2}$ with $\mu = 2K \,/\, \delta\delta_0^2$, where $K = 2N$ if $g_0 < 1$ and $K = 2(g_0 - 1) + 2N$ if $g_0 \ge 1$. Then, we have that $\mathcal{L}\varphi > 0$ in $B_{\delta_0} \setminus B_{\delta_0/2}$ (see the proof of Lemma 2.9 in [21]).

Let now $\psi(x) = c_2 m_0 \varphi(x - x_0)$ for $x \in \overline{B_{\delta_0}(x_0)} \setminus B_{\delta_0/2}(x_0)$. Then, again by Lemma 2.9 in [21], we have that if we choose c_2 conveniently depending on N, δ, and g_0,

$$\begin{cases} \mathcal{L}\psi(x) > 0 & \text{in } B_{\delta_0}(x_0) \setminus \overline{B_{\delta_0/2}(x_0)}, \\ \psi = 0 & \text{on } \partial B_{\delta_0}(x_0), \\ \psi = c_1 m_0 & \text{on } \partial B_{\delta_0/2}(x_0). \end{cases}$$

By the comparison principle (see Lemma 2.8 in [21]) we have

$$\psi(x) \le u^\varepsilon(x) - \varepsilon \quad \text{in } \overline{B_{\delta_0}(x_0)} \setminus B_{\delta_0/2}(x_0). \tag{3.1}$$

Take $y_0 \in \partial B_{\delta_0}(x_0) \cap \partial\{u^\varepsilon > \varepsilon\}$. Then, $y_0 \in \overline{B_{1/2}}$ and

$$\psi(y_0) = u^\varepsilon(y_0) - \varepsilon = 0. \tag{3.2}$$

Let $v^\varepsilon = \frac{1}{\varepsilon} u^\varepsilon(y_0 + \varepsilon x)$. Then, if $\varepsilon < 1$, we have that $\mathcal{L}v^\varepsilon = \beta(v^\varepsilon)$ in $B_{1/2}$ and $v^\varepsilon(0) = 1$. Therefore, by Harnack's inequality (see [19]) we have that $\max_{\overline{B}_{1/4}} v^\varepsilon \leq \widetilde{c}$ and

$$|\nabla u^\varepsilon(y_0)| = |\nabla v^\varepsilon(0)| \leq \widetilde{c} \max_{\overline{B}_{1/4}} v^\varepsilon \leq c_3. \tag{3.3}$$

Finally, by (3.1), (3.2), and (3.3) we have that $|\nabla\psi(y_0)| \leq |\nabla u^\varepsilon(y_0)| \leq c_3$. Observe that $|\nabla\psi(y_0)| = c_2 m_0 e^{-\mu\delta_0^2} 2\mu\delta_0 \leq c_3$. Therefore,

$$m_0 \leq \frac{c_3 e^{\mu\delta_0^2}}{c_2 2\mu\delta_0} = \frac{c_3 \delta e^{2K/\delta}}{c_2 4K}\delta_0$$

and the result follows. ∎

Now, we can prove the main result of this section.

PROPOSITION 3.1. *Let u^ε be a solution of $\mathcal{L}u^\varepsilon = \beta_\varepsilon(u^\varepsilon)$ in B_1. Assume that $0 \in \partial\{u^\varepsilon > \varepsilon\}$. Then, we have for $x \in B_{1/8}$,*

$$|\nabla u^\varepsilon(x)| \leq C,$$

with $C = C(N, \delta, g_0, \|\beta\|_\infty, g(1))$.

Proof. By Lemma 3.1 we know that if $x_0 \in \{u^\varepsilon \leq 2\varepsilon\} \cap B_{3/4}$, then

$$|\nabla u^\varepsilon(x_0)| \leq C_0,$$

with $C_0 = C_0(N, \delta, g_0, \|\beta\|_\infty, g(1))$.

Let $x_0 \in B_{1/8} \cap \{u^\varepsilon > \varepsilon\}$ and $\delta_0 = \text{dist}(x_0, \{u^\varepsilon \leq \varepsilon\})$.

As $0 \in \partial\{u^\varepsilon > \varepsilon\}$ we have that $\delta_0 \leq 1/8$. Therefore, $B_{\delta_0}(x_0) \subset \{u^\varepsilon > \varepsilon\} \cap B_{1/4}$ and then $\mathcal{L}u^\varepsilon = 0$ in $B_{\delta_0}(x_0)$ and, by Lemma 3.2,

$$u^\varepsilon(x) \leq \varepsilon + C_1 \text{dist}(x, \{u^\varepsilon \leq \varepsilon\}) \quad \text{in } B_{\delta_0}(x_0). \tag{3.4}$$

1. Suppose that $\varepsilon < \bar{c}\delta_0$ with $\bar{c}$ to be determined. Let $v(x) = \frac{1}{\delta_0} u^\varepsilon(x_0 + \delta_0 x)$. Then, $\mathcal{L}v = \delta_0 \beta_\varepsilon(u^\varepsilon(x_0 + \delta_0 x)) = 0$ in B_1. Therefore, by the results of [19]

$$|\nabla v(0)| \leq \widetilde{C} \sup_{B_1} v,$$

with $\widetilde{C} = \widetilde{C}(N, g_0, \delta, g(1))$. We obtain

$$|\nabla u^\varepsilon(x_0)| \leq \frac{\widetilde{C}}{\delta_0} \sup_{B_{\delta_0}(x_0)} u^\varepsilon \leq \frac{\widetilde{C}}{\delta_0}(\varepsilon + C\delta_0) \leq \widetilde{C}(\bar{c} + C).$$

2. Suppose that $\varepsilon \geq \bar{c}\delta_0$. By (3.4) we have

$$u^\varepsilon(x_0) \leq \varepsilon + C_1\delta_0 \leq \left(1 + \frac{C_1}{\bar{c}}\right)\varepsilon < 2\varepsilon$$

if we choose $\bar{c}$ big enough. By Lemma 3.1, we have $|\nabla u^\varepsilon(x_0)| \leq C$, with $C = C(N, g_0, \delta, \|\beta\|_\infty, g(1))$.

The result follows. ∎

With these lemmas we obtain the following.

COROLLARY 3.1. *Let u^ε be a solution of*

$$\mathcal{L}u^\varepsilon = \beta_\varepsilon(u^\varepsilon) \quad \text{in } \Omega,$$

with $\|u^\varepsilon\|_{L^\infty(\Omega)} \leq L$. Then, we have, for $\Omega' \subset\subset \Omega$, that there exists $\varepsilon_0(\Omega, \Omega')$ such that if $\varepsilon \leq \varepsilon_0(\Omega, \Omega')$,

$$|\nabla u^\varepsilon(x)| \leq C \quad \text{in } \Omega',$$

with $C = C(N, \delta, g_0, L, \|\beta\|_\infty, g(1), \text{dist}(\overline{\Omega', \partial\Omega}))$.

Proof. Let $\tau > 0$ such that $\forall x \in \Omega'$, $\overline{B_\tau(x)} \subset \Omega$ and $\varepsilon \leq \tau$. Let $x_0 \in \Omega'$.

1. If $\delta_0 = \text{dist}(x_0, \partial\{u^\varepsilon > \varepsilon\}) \leq \tau/8$, let $y_0 \in \partial\{u^\varepsilon > \varepsilon\}$ such that $|x_0 - y_0| = \delta_0$. Let $v(x) = \frac{1}{\tau}u^\varepsilon(y_0 + \tau x)$ and $\bar{x} = \frac{x_0 - y_0}{\tau}$, and then $|\bar{x}| < 1/8$. As $0 \in \partial\{v > \varepsilon/\tau\}$ and $\mathcal{L}v = \beta_{\varepsilon/\tau}(v)$ in B_1, we have by Proposition 3.1

$$|\nabla u^\varepsilon(x_0)| = |\nabla v(\bar{x})| \leq C.$$

2. If $\delta_0 = \text{dist}(x_0, \partial\{u^\varepsilon > \varepsilon\}) \geq \tau/8$, there holds that
(i) $B_{\tau/8}(x_0) \subset \{u^\varepsilon > \varepsilon\}$ or
(ii) $B_{\tau/8}(x_0) \subset \{u^\varepsilon \leq \varepsilon\}$.

In the first case, $\mathcal{L}u^\varepsilon = 0$ in $B_{\tau/8}(x_0)$. Therefore,

$$|\nabla u^\varepsilon(x_0)| \leq C(N, g_0, \delta, \tau, g(1), L).$$

In the second case, we can apply Lemma 3.1 and we have

$$|\nabla u^\varepsilon(x_0)| \leq C(N, g_0, \delta, \tau, g(1), 2\|\beta\|_\infty).$$

The result is proved. ∎

4. Passage to the limit. Since we have that $|\nabla u^\varepsilon|$ is locally bounded by a constant independent of ε, we have that there exists a function $u \in Lip_{loc}(\Omega)$ such that, for a subsequence $\varepsilon_j \to 0$, $u^{\varepsilon_j} \to u$. In this section we will prove some properties of the function u.

We start with some technical results.

PROPOSITION 4.1. *Let $\{u^\varepsilon\}$ be a uniformly bounded family of nonnegative solutions of (P_ε). Then, for any sequence $\varepsilon_j \to 0$ there exists a subsequence $\varepsilon'_j \to 0$ and $u \in Lip_{loc}(\Omega)$ such that the following hold*

1. *$u^{\varepsilon'_j} \to u$ uniformly in compact subsets of Ω,*
2. *$\mathcal{L}u = 0$ in $\Omega \cap \{u > 0\}$*
3. *There exists a locally finite measure μ such that $\beta_{\varepsilon'_j}(u^{\varepsilon'_j}) \rightharpoonup \mu$ as measures in Ω' for every $\Omega' \subset\subset \Omega$,*
4. *Assume $g_0 \geq 1$. Then, $\nabla u^{\varepsilon'_j} \to \nabla u$ in $L^{g_0+1}_{loc}(\Omega)$,*
5.

$$\int_\Omega F(|\nabla u|)\nabla u \nabla\varphi = -\int_\Omega \varphi \, d\mu$$

for every $\varphi \in C_0^\infty(\Omega)$. Moreover, μ is supported on $\Omega \cap \partial\{u > 0\}$.

Remark 4.1. We can always assume that $g_0 \geq 1$. If we don't want to assume it, we can change the statement in item (3) by $\nabla u^{\varepsilon'_j} \to \nabla u$ in $L^{g_1+1}_{loc}(\Omega)$, where $g_1 = \max(1, g_0)$.

Proof. (1) follows by Corollary 3.1.

In order to prove (2), take $E \subset\subset E' \subset\subset \{u > 0\}$. Then, $u \geq c > 0$ in E'. Therefore, $u^{\varepsilon'_j} > c/2$ in E' for ε'_j small. If we take $\varepsilon'_j < c/2$—as $\mathcal{L}u^{\varepsilon'_j} = 0$ in $\{u^{\varepsilon'_j} > \varepsilon'_j\}$—we have that $\mathcal{L}u^{\varepsilon'_j} = 0$ in E'. Therefore, by the results in [19], $\|u^{\varepsilon'_j}\|_{C^{1,\alpha}(E)} \leq C$.

Thus, for a subsequence we have

$$\nabla u^{\varepsilon'_j} \to \nabla u \quad \text{uniformly in } E.$$

Therefore, $\mathcal{L}u = 0$.

In order to prove (3), let us take $\Omega' \subset\subset \Omega$ and $\varphi \in C_0^\infty(\Omega)$ with $\varphi = 1$ in Ω' as a test function in (P_{ε_j}). Since $\|\nabla u^{\varepsilon'_j}\| \leq C$ in Ω', there holds that

$$C(\varphi) \geq \int_\Omega \beta_{\varepsilon'_j}\left(u^{\varepsilon'_j}\right)\varphi\, dx \geq \int_{\Omega'} \beta_{\varepsilon'_j}\left(u^{\varepsilon'_j}\right) dx.$$

Therefore, $\beta_{\varepsilon'_j}(u^{\varepsilon'_j})$ is bounded in $L^1_{loc}(\Omega)$ so that there exists a locally finite measure μ such that

$$\beta_{\varepsilon'_j}\left(u^{\varepsilon'_j}\right) \rightharpoonup \mu \quad \text{as measures;}$$

that is, for every $\varphi \in C_0(\Omega)$,

$$\int_\Omega \beta_{\varepsilon'_j}\left(u^{\varepsilon'_j}\right)\varphi\, dx \to \int_\Omega \varphi\, d\mu.$$

We divide the proof of (4) into several steps.

Let $\Omega' \subset\subset \Omega$; then by Corollary 3.1, $|\nabla u^{\varepsilon_j}| \leq C$ in Ω'. Therefore, for a subsequence ε'_j we have that there exists $\xi \in (L^\infty(\Omega'))^N$ such that

$$\begin{aligned} &\nabla u^{\varepsilon'_j} \rightharpoonup \nabla u && * - \text{weakly in } (L^\infty(\Omega'))^N, \\ &A(\nabla u^{\varepsilon'_j}) \rightharpoonup \xi && * - \text{weakly in } (L^\infty(\Omega'))^N, \\ &u^{\varepsilon'_j} \to u && \text{uniformly in } \Omega', \end{aligned} \tag{4.1}$$

where $A(p) = F(|p|)p$. For simplicity, we call $\varepsilon'_j = \varepsilon$.

Step 1. Let us first prove that for any $v \in W_0^{1,G}(\Omega')$ there holds that

$$\int_{\Omega'} (\xi - A(\nabla u))\nabla v\, dx = 0. \tag{4.2}$$

In fact, as A is monotone (i.e., $(A(\eta) - A(\zeta)) \cdot (\eta - \zeta) \geq 0\ \forall \eta, \zeta \in \mathbb{R}^N$) we have that, for any $w \in W^{1,G}(\Omega')$,

$$I = \int_{\Omega'} \big(A(\nabla u^\varepsilon) - A(\nabla w)\big)(\nabla u^\varepsilon - \nabla w)\, dx \geq 0. \tag{4.3}$$

Therefore, if $\psi \in C_0^\infty(\Omega')$,

$$
\begin{aligned}
&-\int_{\Omega'} \beta_\varepsilon(u^\varepsilon)u^\varepsilon\,dx - \int_{\Omega'} A(\nabla u^\varepsilon)\nabla w\,dx - \int_{\Omega'} A(\nabla w)(\nabla u^\varepsilon - \nabla w)\,dx \\
&= -\int_{\Omega'} \beta_\varepsilon(u^\varepsilon)u^\varepsilon\,dx - \int_{\Omega'} A(\nabla u^\varepsilon)\nabla u^\varepsilon\,dx + I \\
&= -\int_{\Omega'} \beta_\varepsilon(u^\varepsilon)u\,dx - \int_{\Omega'} \beta_\varepsilon(u^\varepsilon)(u^\varepsilon - u)\psi\,dx - \int_{\Omega'} \beta_\varepsilon(u^\varepsilon)(u^\varepsilon - u)(1-\psi)\,dx \\
&\quad - \int_{\Omega'} A(\nabla u^\varepsilon)\nabla u^\varepsilon\,dx + I \\
&\geq -\int_{\Omega'} \beta_\varepsilon(u^\varepsilon)u\,dx + \int_{\Omega'} A(\nabla u^\varepsilon)\nabla(u^\varepsilon - u)\psi\,dx + \int_{\Omega'} A(\nabla u^\varepsilon)(u^\varepsilon - u)\nabla\psi\,dx \\
&\quad - \int_{\Omega'} \beta_\varepsilon(u^\varepsilon)(u^\varepsilon - u)(1-\psi)\,dx - \int_{\Omega'} A(\nabla u^\varepsilon)\nabla u^\varepsilon\,dx,
\end{aligned} \tag{4.4}
$$

where in the last inequality we are using (4.3) and (1.1).

Now, take $\psi = \psi_j \to \chi_{\Omega'}$. If Ω' is smooth, we may assume that $\int |\nabla\psi_j|\,dx \to \operatorname{Per}\Omega'$. Therefore,

$$
\left|\int_{\Omega'} A(\nabla u^\varepsilon)(u^\varepsilon - u)\nabla\psi_j\,dx\right| \leq C\|u^\varepsilon - u\|_{L^\infty(\Omega')}\int_{\Omega'} |\nabla\psi_j|\,dx \leq C\|u^\varepsilon - u\|_{L^\infty(\Omega')}
$$

so that with this choice of $\psi = \psi_j$ in (4.4) we obtain

$$
\begin{aligned}
&\int_{\Omega'} \beta_\varepsilon(u^\varepsilon)u^\varepsilon\,dx \int_{\Omega'} A(\nabla u^\varepsilon)\nabla w\,dx \int_{\Omega'} A(\nabla w)(\nabla u^\varepsilon - \nabla w)\,dx \\
&\leq \int_{\Omega'}\beta_\varepsilon(u^\varepsilon)u\,dx - \int_{\Omega'}A(\nabla u^\varepsilon)\nabla(u^\varepsilon - u)\,dx + C\|u^\varepsilon - u\|_{L^\infty(\Omega')} + \int_{\Omega'}A(\nabla u^\varepsilon)\nabla u^\varepsilon\,dx \\
&= \int_{\Omega'} \beta_\varepsilon(u^\varepsilon)u\,dx + \int_{\Omega'} A(\nabla u^\varepsilon)\nabla u dx + C\|u^\varepsilon - u\|_{L^\infty(\Omega')}.
\end{aligned}
$$

Therefore, letting $\varepsilon \to 0$ we get by using (4.1) and (3) that

$$
-\int_{\Omega'} u\,d\mu - \int_{\Omega'} \xi\nabla w\,dx - \int_{\Omega'} A(\nabla w)(\nabla u - \nabla w)\,dx \geq -\int_{\Omega'} u\,d\mu - \int_{\Omega'} \xi\nabla u\,dx
$$

and then

$$
\int_{\Omega'} (\xi - A(\nabla w))(\nabla u - \nabla w)\,dx \geq 0. \tag{4.5}
$$

Take now $w = u - \lambda v$ with $v \in W_0^{1,G}(\Omega')$. Dividing by λ and taking $\lambda \to 0^+$ in (4.5) we obtain

$$
\int_{\Omega'} (\xi - A(\nabla u))\nabla v\,dx \geq 0.
$$

Replacing v by $-v$ we obtain (4.2).

Step 2. Let us prove that $\int_{\Omega'} A(\nabla u^\varepsilon)\nabla u^\varepsilon \to \int_{\Omega'} A(\nabla u)\nabla u$.

By passing to the limit in the equation

$$
0 = \int_{\Omega'} A(\nabla u^\varepsilon)\nabla\phi + \int_{\Omega'} \beta_\varepsilon(u^\varepsilon)\phi\,dx, \tag{4.6}
$$

we have, by Step 1, that for every $\phi \in C_0^\infty(\Omega')$,

$$0 = \int_{\Omega'} A(\nabla u)\nabla\phi + \int_{\Omega'} \phi\, d\mu. \tag{4.7}$$

On the other hand, taking $\phi = u^\varepsilon \psi$ in (4.6) with $\psi \in C_0^\infty(\Omega')$ we have that

$$0 = \int_{\Omega'} A(\nabla u^\varepsilon)\nabla u^\varepsilon \psi\, dx + \int_{\Omega'} A(\nabla u^\varepsilon) u^\varepsilon \nabla\psi\, dx + \int_{\Omega'} \beta_\varepsilon(u^\varepsilon) u^\varepsilon \psi\, dx.$$

Using that

$$\int_{\Omega'} A(\nabla u^\varepsilon) u^\varepsilon \nabla\psi\, dx \to \int_{\Omega'} A(\nabla u) u \nabla\psi\, dx,$$
$$\int_{\Omega'} \beta_\varepsilon(u^\varepsilon) u^\varepsilon \psi\, dx \to \int_{\Omega'} u\psi d\mu$$

we obtain

$$0 = \lim_{\varepsilon\to 0} \left(\int_{\Omega'} A(\nabla u^\varepsilon)\nabla u^\varepsilon \psi\, dx \right) + \int_{\Omega'} A(\nabla u) u \nabla\psi\, dx + \int_{\Omega'} u\psi d\mu.$$

Now taking $\phi = u\psi$ in (4.7) we have

$$0 = \int_{\Omega'} A(\nabla u)\nabla u\psi\, dx + \int_{\Omega'} A(\nabla u) u\nabla\psi\, dx + \int_{\Omega'} u\psi\, d\mu.$$

Therefore,

$$\lim_{\varepsilon\to 0} \int_{\Omega'} A(\nabla u^\varepsilon)\nabla u^\varepsilon \psi\, dx = \int_{\Omega'} A(\nabla u)\nabla u\psi\, dx.$$

Then,

$$\begin{aligned}
&\left| \int_{\Omega'} (A(\nabla u^\varepsilon)\nabla u^\varepsilon - A(\nabla u)\nabla u)\, dx \right| \\
&\le \left| \int_{\Omega'} (A(\nabla u^\varepsilon)\nabla u^\varepsilon - A(\nabla u)\nabla u)\psi\, dx \right| + \left| \int_{\Omega'} (A(\nabla u^\varepsilon)\nabla u^\varepsilon)(1-\psi)\, dx \right| \\
&\quad + \left| \int_{\Omega'} A(\nabla u)\nabla u(1-\psi)\, dx \right| \\
&\le \left| \int_{\Omega'} (A(\nabla u^\varepsilon)\nabla u^\varepsilon - A(\nabla u)\nabla u)\psi\, dx \right| + C\int_{\Omega'} |1-\psi|\, dx
\end{aligned}$$

so that taking $\varepsilon \to 0$ and then $\psi \to 1$ a.e. with $0 \le \psi \le 1$ we obtain

$$\int_{\Omega'} A(\nabla u^\varepsilon)\nabla u^\varepsilon dx \to \int_{\Omega'} A(\nabla u)\nabla u\, dx. \tag{4.8}$$

With similar ideas we can prove that

$$\int_{\Omega'} A(\nabla u^\varepsilon)\nabla u\, dx \to \int_{\Omega'} A(\nabla u)\nabla u\, dx. \tag{4.9}$$

Step 3. Let us prove that

$$\int_{\Omega'} G(|\nabla u^\varepsilon|)\,dx \to \int_{\Omega'} G(|\nabla u|)\,dx. \tag{4.10}$$

First, by the monotonicity of A we have

$$\begin{aligned}\int_{\Omega'} G(|\nabla u^\varepsilon|)\,dx - \int_{\Omega'} G(|\nabla u|)\,dx &= \int_{\Omega'}\int_0^1 A(\nabla u + t(\nabla u^\varepsilon - \nabla u))\nabla(u^\varepsilon - u)\,dx\\ &\ge \int_{\Omega'} A(\nabla u)\nabla(u^\varepsilon - u)\,dx.\end{aligned}$$

Therefore, we have

$$\liminf_{\varepsilon\to 0} \int_{\Omega'} G(|\nabla u^\varepsilon|)\,dx - \int_{\Omega'} G(|\nabla u|)\,dx \ge 0.$$

Now, by Step 2 we have

$$\begin{aligned}\int_{\Omega'} G(|\nabla u^\varepsilon|)\,dx - \int_{\Omega'} G(|\nabla u|)\,dx &= \int_{\Omega'}\int_0^1 A(\nabla u + t(\nabla u^\varepsilon - \nabla u))\nabla(u^\varepsilon - u)\,dx\\ &\le \int_{\Omega'} A(\nabla u^\varepsilon)\nabla(u^\varepsilon - u)\,dx \to 0.\end{aligned}$$

Thus, we have that (4.10) holds.

Step 4. This is the end of the proof of (4).

Let $u^s = su + (1-s)u^\varepsilon$. Then,

$$\begin{aligned}\int_{\Omega'} G(|\nabla u|)\,dx - \int_{\Omega'} G(|\nabla u^\varepsilon|)\,dx &= \int_{\Omega'}\int_0^1 A(\nabla u^s)\nabla(u - u^\varepsilon)\,ds\,dx\\ &= \int_{\Omega'}\int_0^1 (A(\nabla u^s) - A(\nabla u^\varepsilon))\nabla(u^s - u^\varepsilon)\,\frac{ds}{s}\,dx\\ &\quad + \int_{\Omega'} A(\nabla u^\varepsilon)\nabla(u - u^\varepsilon)\,dx.\end{aligned} \tag{4.11}$$

As in the proof of Theorem 4.1 in [21], we have that

$$\begin{aligned}&\int_{\Omega'}\int_0^1 (A(\nabla u^s) - A(\nabla u^\varepsilon))\nabla(u^s - u^\varepsilon)\,ds\,dx\\ &\qquad \ge C\left(\int_{A_2} G(|\nabla u - \nabla u^\varepsilon|)\,dx + \int_{A_1} F(|\nabla u|)|\nabla u - \nabla u^\varepsilon|^2\,dx\right),\end{aligned}$$

where

$$A_1 = \{x \in \Omega' : |\nabla u - \nabla u^\varepsilon| \le 2|\nabla u|\}, \quad A_2 = \{x \in \Omega' : |\nabla u - \nabla u^\varepsilon| > 2|\nabla u|\}.$$

Therefore, by (4.8), (4.9), (4.10), and (4.11) we have

$$\left(\int_{A_2} G(|\nabla u - \nabla u^\varepsilon|)\,dx + \int_{A_1} F(|\nabla u|)|\nabla u - \nabla u^\varepsilon|^2\,dx\right) \to 0.$$

Then, if we prove that

$$\left(\int_{A_2} G(|\nabla u - \nabla u^\varepsilon|)\,dx + \int_{A_1} F(|\nabla u|)|\nabla u - \nabla u^\varepsilon|^2\,dx\right) \ge C\int_{\Omega'} |\nabla u - \nabla u^\varepsilon|^{g_0+1}\,dx,$$

the result follows.

In fact, for every $C_0 > 0$ there exists $C_1 > 0$ such that $g(t) \geq C_1 t^{g_0}$ if $t \leq C_0$. Let C_0 be such that $|\nabla u| \leq C_0$ and $|\nabla u - \nabla u^\varepsilon| \leq C_0$. Then, by Lemma A.1,

$$G(|\nabla u^\varepsilon - \nabla u|) \geq C|\nabla u^\varepsilon - \nabla u|^{g_0+1},$$
$$F(|\nabla u|) \geq C_1 |\nabla u|^{g_0-1} \geq C|\nabla u^\varepsilon - \nabla u|^{g_0-1} \quad \text{in } A_1,$$

and the claim follows.

Finally, (5) holds by (4), (3), and (2). ∎

Lemma 4.1. *Let $\{u^{\varepsilon_j}\}$ be a uniformly bounded family of solutions of (P_{ε_j}) in Ω such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Let $x_0,\ x_n \in \Omega \cap \partial\{u > 0\}$ be such that $x_n \to x_0$ as $n \to \infty$. Let $\lambda_n \to 0$, $u_{\lambda_n}(x) = \frac{1}{\lambda_n} u(x_n + \lambda_n x)$, and $(u^{\varepsilon_j})_{\lambda_n}(x) = \frac{1}{\lambda_n} u^{\varepsilon_j}(x_n + \lambda_n x)$. Suppose that $u_{\lambda_n} \to U$ as $n \to \infty$ uniformly on compact sets of $\mathbb{R}^N$. Then, there exists $j(n) \to \infty$ such that for every $j_n \geq j(n)$ there holds that $\varepsilon_{j_n}/\lambda_n \to 0$ and*

1. *$(u^{\varepsilon_{j_n}})_{\lambda_n} \to U$ uniformly in compact subsets of $\mathbb{R}^N$,*
2. *$\nabla (u^{\varepsilon_{j_n}})_{\lambda_n} \to \nabla U$ in $L^{g_0+1}_{loc}(\mathbb{R}^N)$,*
3. *$\nabla u_{\lambda_n} \to \nabla U$ in $L^{g_0+1}_{loc}(\mathbb{R}^N)$.*

Proof. The proof follows from Proposition 4.1 as the proof of Lemma 3.2 follows from Lemma 3.1 in [7]. ∎

Now we prove a technical lemma that is the basis of our main results.

Lemma 4.2. *Let u^ε be solutions to*

$$\mathcal{L} u^\varepsilon = \beta_\varepsilon(u^\varepsilon)$$

in Ω. Then, for any $\psi \in C_0^\infty(\Omega)$ we have

$$-\int_\Omega G(|\nabla u^\varepsilon|)\psi_{x_1}\,dx + \int_\Omega F(|\nabla u^\varepsilon|)\nabla u^\varepsilon \nabla \psi\, u^\varepsilon_{x_1}\,dx = \int_\Omega B_\varepsilon(u^\varepsilon)\psi_{x_1}, \tag{4.12}$$

where $B_\varepsilon(s) = \int_0^s \beta_\varepsilon(\tau)\,d\tau$.

Proof. For simplicity, since ε will be fixed throughout the proof, we will denote $u^\varepsilon = u$.

We know that $|\nabla u| \leq C$ for some constant C. Take $g_n(t) = g(t) + \frac{t}{n}$, and then

$$\min\{1, \delta\} \leq \frac{g_n'(t)t}{g_n(t)} \leq \max\{1, g_0\}. \tag{4.13}$$

Take $A_n(p) = \frac{g_n(|p|)}{|p|}p$ and $\mathcal{L}_n(v) = \operatorname{div}(A_n(\nabla v))$. For $\Omega' \subset\subset \Omega$ let us take u_n as the solution of

$$\begin{cases} \mathcal{L}_n u_n = \beta_\varepsilon(u) & \text{in } \Omega', \\ u_n = u & \text{on } \partial\Omega'. \end{cases} \tag{4.14}$$

By (4.13), we have that all the $g_n' s$ belong to the same class, and then, by the results of [19], we have that for every $\Omega'' \subset\subset \Omega'$ there exists a constant C independent of n such that $\|u_n\|_{C^{1,\alpha}(\Omega'')} \leq C$.

Therefore, there exists u_0 such that, for a subsequence,

$$u_n \to u_0 \quad \text{uniformly on compact subsets of } \Omega',$$
$$\nabla u_n \to \nabla u_0 \quad \text{uniformly on compact subsets of } \Omega'.$$

On the other hand, $A_n(p) \to A(p)$ uniformly in compact sets of $\mathbb{R}^N$. Thus, $\mathcal{L}u_0 = \beta_\varepsilon(u)$ and $u_0 - u \in W^{1,G}(\Omega')$. Since $\mathcal{L}u = \beta_\varepsilon(u)$, it holds that $u_0 = u$ in Ω'. (Observe that, in the proof of the comparison principle, in Lemma 2.8 of [21] we can change the equation $\mathcal{L}u = 0$ by $\mathcal{L}u = f(x)$ with $f \in L^\infty(\Omega)$ to prove uniqueness of the solution of the Dirichlet problem.)

Now, let us prove that the following equality holds:

$$-\int_\Omega G_n(|\nabla u_n|)\psi_{x_1}\,dx + \int_\Omega F_n(|\nabla u_n|)\nabla u_n \nabla\psi\, u_{n x_1}\,dx = -\int_\Omega \beta_\varepsilon(u) u_{n x_1}\psi.$$

In fact, for n fixed we have that $F_n(t) = g_n(t)/t \geq 1/n$ and then by the uniform estimates of [14], $u_n \in W^{2,2}(\Omega)$. As u_n is a weak solution of (4.14) and as $u_n \in W^{2,2}(\Omega)$, taking as test function in the weak formulation of (4.14) the function $\psi u_{n x_1}$, we have that

$$\int_\Omega F_n(|\nabla u_n|)\nabla u_n \nabla(\psi u_{n x_1})\,dx = -\int_\Omega \beta_\varepsilon(u) u_{n x_1}\psi\,dx.$$

As $(G_n(|\nabla u_n|))_{x_1} = g_n(|\nabla u_n|)\frac{\nabla u_n}{|\nabla u_n|}(\nabla u_n)_{x_1} = F(|\nabla u_n|)\nabla u_n (\nabla u_n)_{x_1}$ we have that

$$-\int_\Omega G_n(|\nabla u_n|)\psi_{x_1}\,dx + \int_\Omega F_n(|\nabla u_n|)\nabla u_n \nabla\psi\, u_{n x_1}\,dx = -\int_\Omega \beta_\varepsilon(u) u_{n x_1}\psi\,dx.$$

Passing to the limit as $n \to \infty$ and then integrating by parts on the right-hand side we get

$$-\int_\Omega G(|\nabla u|)\psi_{x_1}\,dx + \int_\Omega F(|\nabla u|)\nabla u \nabla\psi\, u_{x_1}\,dx = \int_\Omega B_\varepsilon(u)\psi_{x_1}\,dx. \qquad \square$$

Now, we characterize some special global limits.

PROPOSITION 4.2. *Let $x_0 \in \Omega$, and let u^{ε_k} be solutions to*

$$\mathcal{L}u^{\varepsilon_k} = \beta_{\varepsilon_k}(u^{\varepsilon_k})$$

in Ω. If u^{ε_k} converge to $\alpha(x - x_0)_1^+$ uniformly in compact subsets of Ω, with $\varepsilon_k \to 0$ as $k \to \infty$ and $\alpha \in \mathbb{R}$, there holds that

$$\alpha = 0 \quad \text{or} \quad \alpha = \Phi^{-1}(M),$$

where $\Phi(t) = g(t)t - G(t)$.

Proof. Assume, for simplicity, that $x_0 = 0$. Since $u^{\varepsilon_k} \geq 0$, we have that $\alpha \geq 0$. If $\alpha = 0$, there is nothing to prove. So, let us assume that $\alpha > 0$. Let $\psi \in C_0^\infty(\Omega)$. By Lemma 4.2 we have

$$-\int_\Omega G(|\nabla u^{\varepsilon_k}|)\psi_{x_1}\,dx + \int_\Omega F(|\nabla u^{\varepsilon_k}|)\nabla u^{\varepsilon_k}\nabla\psi\, u^{\varepsilon_k}_{x_1}\,dx = \int_\Omega B_{\varepsilon_k}(u^{\varepsilon_k})\psi_{x_1}. \tag{4.15}$$

Since $0 \leq B_{\varepsilon_k}(s) \leq M$, there exists $M(x) \in L^\infty(\Omega)$, $0 \leq M(x) \leq M$, such that $B_{\varepsilon_k} \to M$ $*-$ weakly in $L^\infty(\Omega)$.

If $y \in \Omega \cap \{x_1 > 0\}$, then $u^{\varepsilon_k} \geq \frac{\alpha y_1}{2}$ in a neighborhood of y for k large. Thus, $u^{\varepsilon_k} \geq \varepsilon_k$ and we have

$$B_{\varepsilon_k}(u^{\varepsilon_k})(x) = \int_0^{u^{\varepsilon_k}/\varepsilon_k} \beta(s)\,ds = M.$$

On the other hand, if we let $K \subset\subset \Omega \cap \{x_1 < 0\}$, since by Proposition 4.1 $\beta_{\varepsilon_k}(u^{\varepsilon_k}) \to 0$ in $L^1(K)$, we have that $\int_K \left|\nabla B_{\varepsilon_k}(u^{\varepsilon_k})\right| dx = \int_K \beta_{\varepsilon_k}(u^{\varepsilon_k})|\nabla u^{\varepsilon_k}|\, dx \to 0$. Therefore, we may assume that $B_{\varepsilon_k} \to \overline{M}$ in $L^1_{loc}(\{x_1 < 0\})$ for a constant $\overline{M} \in [0, M]$.

Passing to the limit in (4.15), using the strong convergence result in Proposition 4.1 we have

$$-\int_{\{x_1>0\}} G(\alpha)\psi_{x_1}\, dx + \int_{\{x_1>0\}} F(\alpha)\, \alpha^2 \psi_{x_1}\, dx = M \int_{\{x_1>0\}} \psi_{x_1} + \overline{M} \int_{\{x_1<0\}} \psi_{x_1}.$$

Then,

$$(-G(\alpha) + g(\alpha)\alpha) \int_{\{x_1>0\}} \psi_{x_1}\, dx = M \int_{\{x_1>0\}} \psi_{x_1}\, dx + \overline{M} \int_{\{x_1<0\}} \psi_{x_1}\, dx.$$

And, integrating by parts, we obtain

$$(-G(\alpha) + g(\alpha)\alpha) \int_{\{x_1=0\}} \psi\, dx' = M \int_{\{x_1=0\}} \psi\, dx' - \overline{M} \int_{\{x_1=0\}} \psi\, dx'.$$

Thus, $(-G(\alpha) + g(\alpha)\alpha) = M - \overline{M}$.

In order to see that $\alpha = \Phi^{-1}(M)$ let us show that $\overline{M} = 0$.

In fact, let $K \subset\subset \{x_1 < 0\} \cap \Omega$. Then for any $\eta > 0$ there exists $0 < \delta < 1$ such that

$$\begin{aligned} \left|K \cap \{\eta < B_{\varepsilon_j}(u^{\varepsilon_j}) < M - \eta\}\right| &\le \left|K \cap \{\delta < u^{\varepsilon_j}/\varepsilon_j < 1 - \delta\}\right| \\ &\le \left|K \cap \{\beta_{\varepsilon_j}(u^{\varepsilon_j}) \ge a/\varepsilon_j\}\right| \to 0 \end{aligned}$$

as $j \to \infty$, where $a = \inf_{[\delta,1-\delta]} \beta > 0$, and we are using that $\beta_{\varepsilon_j}(u^{\varepsilon_j})$ is bounded in $L^1(K)$ uniformly in j.

Now, as $B(u^{\varepsilon_j}) \to \overline{M}$ in $L^1(K)$, we conclude that

$$\left|K \cap \{\eta < \overline{M} < M - \eta\}\right| = 0$$

for every $\eta > 0$. Hence, $\overline{M} = 0$ or $\overline{M} = M$ and, since $\alpha > 0$, we must have $\overline{M} = 0$. ∎

PROPOSITION 4.3. *Let $x_0 \in \Omega$, and let u^{ε_k} be a solution to $\mathcal{L}u^{\varepsilon_k} = \beta_{\varepsilon_k}(u^{\varepsilon_k})$ in Ω. Assume g' satisfies* (4.19) *below. If u^{ε_k} converges to $\alpha(x - x_0)_1^+ + \gamma(x - x_0)_1^-$ uniformly in compact subsets of Ω, with $\alpha, \gamma > 0$ and $\varepsilon_k \to 0$ as $k \to \infty$, then*

$$\alpha = \gamma \le \Phi^{-1}(M).$$

Proof. We can assume that $x_0 = 0$.

As in the proof of Proposition 4.2 we see that $B_{\varepsilon_k}(u^{\varepsilon_k}) \to M$ uniformly on compact sets of $\{x_1 > 0\}$ and $\{x_1 < 0\}$. Since u^{ε_k} satisfies (4.12) we get, after passing to the limit, for any $\psi \in C_0^\infty(\Omega)$,

$$-\int_{\{x_1>0\}} \Phi(\alpha)\psi_{x_1}\, dx - \int_{\{x_1<0\}} \Phi(\gamma)\psi_{x_1}\, dx = \int_\Omega M\psi_{x_1}.$$

Integrating by parts we obtain

$$\int_{\{x_1=0\}} \Phi(\alpha)\psi\, dx' - \int_{\{x_1=0\}} \Phi(\gamma)\psi\, dx' = 0$$

and then $\alpha = \gamma$.

Now assume that $\alpha > \Phi^{-1}(M)$. We will prove that this is a contradiction.

Step 1. Let $\mathcal{R}_2 = \{x = (x_1, x') \in \mathbb{R}^N : |x_1| < 2, |x'| < 2\}$. From the scaling invariance of the problem, we can assume that $\mathcal{R}_2 \subset \Omega$.

We will construct a family $\{v^{\varepsilon_j}\}$ of solutions of (P_{ε_j}) in $\mathcal{R}_2$ satisfying $v^{\varepsilon_j}(x_1, x') = v^{\varepsilon_j}(-x_1, x')$ in $\mathcal{R}_2$ and such that $v^{\varepsilon_j} \to u$ uniformly on compact subsets of $\mathcal{R}_2$, where $u(x) = \alpha|x_1|$.

To this end, we take $b_{\varepsilon_j} = \sup_{\mathcal{R}_2} |u^{\varepsilon_j} - u|$ and v^{ε_j} the least solution constructed in Theorem C.1 with $\Omega = \mathcal{R}_2$ and boundary values $v^{\varepsilon_j} = u - b_{\varepsilon_j}$ on $\partial\mathcal{R}_2$. The supersolution that we are taking when applying Theorem C.1 is a constant $A \geq 1$ such that $u^\varepsilon \leq A$, and as subsolution we take a negative constant c such that $c \leq u - b_\varepsilon$. Then, we have that $c \leq u^\varepsilon \leq A$, and by this theorem we obtain that $v^\varepsilon \leq u^\varepsilon$.

We may apply the uniform estimates of the previous section in order to pass to the limit for a subsequence that we still call v^{ε_j}, and we get $v = \lim v^{\varepsilon_j} \leq u$.

Another property that we obtain by using the extremality of v^{ε_j} is that it is symmetric with respect to the variable x_1. In fact, if we take the function $\bar{v}(x_1, x') = v^{\varepsilon_j}(-x_1, x')$, this is again a solution. Therefore, by Theorem C.1, $v^{\varepsilon_j}(-x_1, x') = \bar{v}(x_1, x') \leq v^{\varepsilon_j}(x_1, x')$. Changing x_1 to $-x_1$ we reverse the inequality, thus, obtaining the desired symmetry result.

In order to prove that $u \leq v$, we considered two cases.

First, suppose that $\alpha > \Phi^{-1}(\frac{g_0}{\delta} M)$. Let $w \in C^{1,\beta}(\mathbb{R})$ be the weak solution to

$$(F(|w'|)w')' = \frac{g_0}{\delta}\beta(w) \quad \text{in } \mathbb{R}, \quad w(0) = 1, \quad w'(0) = \alpha.$$

Observe that when $w'(s) > 0$, the equation is locally uniformly elliptic so that, as long as $w' > 0$, there holds that $w \in C^2$ and a solution to

$$\big(g(w')\big)' = \frac{g_0}{\delta}\beta(w).$$

Suppose that there exists an $s \in \mathbb{R}$ such that $w'(s) = 0$. Take s_1 as the supremum of the s's such that this happens. Then, $s_1 < 0$ and, in $(s_1, 0]$, $w' > 0$ and $F(|w'|)\, w' = g(w')$. Multiplying the equation by w' and integrating in this interval we get

$$-\int_{s_1}^{0} g(w')w'' + g(w')w'\Big|_{s_1}^{0} = \frac{g_0}{\delta} B(w)\Big|_{s_1}^{0}.$$

Since $g(w')w'' = \big(G(w')\big)'$, we get

$$\Phi(\alpha) = \frac{g_0}{\delta} M - \frac{g_0}{\delta} B\big(w(s_1)\big) \leq \frac{g_0}{\delta} M,$$

which is a contradiction.

Then, $w' > 0$ everywhere. By the same calculation as before, we obtain that for any $s \in \mathbb{R}$ we have

$$\Phi(w'(s)) = \Phi(\alpha) + \frac{g_0}{\delta} B(w(s)) - \frac{g_0}{\delta} M \leq \Phi(\alpha)$$

and

$$\Phi(w'(s)) = \Phi(\alpha) + \frac{g_0}{\delta} B(w(s)) - \frac{g_0}{\delta} M \geq \Phi(\alpha) - \frac{g_0}{\delta} M = \Phi(\bar{\alpha}) \tag{4.16}$$

for some $\alpha > \bar{\alpha} > 0$. Thus, $\bar{\alpha} \leq w'(s) \leq \alpha$.

Therefore, $w'(s) = \alpha$ for $s \geq 0$ and there exists $\bar{s} < 0$ such that $w(\bar{s}) = 0$. This implies, by (4.16), that $w'(\bar{s}) = \bar{\alpha}$, and then $w'(s) = \bar{\alpha}$ for all $s \leq \bar{s}$. Therefore,

$$w(s) = \begin{cases} 1 + \alpha s, & s > 0, \\ \bar{\alpha}(s - \bar{s}), & s \leq \bar{s}. \end{cases}$$

Let $w^{\varepsilon_j}(x_1) = \varepsilon_j w(\frac{x_1}{\varepsilon_j} - \frac{b_{\varepsilon_j}}{\bar{\alpha}\varepsilon_j} + \bar{s})$; then

$$w^{\varepsilon_j}(0) = \varepsilon_j w\left(-\frac{b_{\varepsilon_j}}{\bar{\alpha}\varepsilon_j} + \bar{s}\right) = \varepsilon_j \bar{\alpha}\left(\bar{s} - \frac{b_{\varepsilon_j}}{\bar{\alpha}\varepsilon_j} - \bar{s}\right) = -b_{\varepsilon_j}$$

and ${w^{\varepsilon_j}}'(s) \leq \alpha$. Therefore, $w^{\varepsilon_j} \leq u - b_{\varepsilon_j}$ in $\mathbb{R}$ so that $w^{\varepsilon_j} \leq v^{\varepsilon_j}$ on $\partial\mathcal{R}_2$.

Then, by the comparison principle below (Lemma 4.3), we have that $w^{\varepsilon_j} \leq v^{\varepsilon_j}$ in $\mathcal{R}_2$.

Take $x_1 > 0$. Then, for j large $x_1 - \frac{b_{\varepsilon_j}}{\bar{\alpha}} > \frac{x_1}{2}$. Thus, $\frac{1}{\varepsilon_j}(x_1 - \frac{b_{\varepsilon_j}}{\bar{\alpha}}) + \bar{s} > \frac{x_1}{2\varepsilon_j} + \bar{s} > 0$ for j large.

Therefore, $w^{\varepsilon_j}(x) = \varepsilon_j + \alpha x_1 - \frac{\alpha}{\bar{\alpha}} b_{\varepsilon_j} + \alpha\varepsilon_j\bar{s}$. Hence, $w^{\varepsilon_j} \to u$ uniformly on compact sets of $\{x_1 > 0\}$.

Passing to the limit, we get that $u \leq v$ in $\mathcal{R}_2 \cap \{x_1 > 0\}$. Observe that since $v^{\varepsilon_j}(x_1, x') = v^{\varepsilon_j}(-x_1, x')$, we obtain that $u \leq v$ in $\mathcal{R}_2$.

This completes the first case.

Now, suppose that $\alpha \leq \Phi^{-1}\big(\frac{g_0}{\delta}M\big)$. Let $w \in C^{1,\beta}(\mathbb{R})$, satisfying

$$(F(|w'|)w')' = \beta(w) \quad \text{in } \mathbb{R}, \quad w(0) = 1, \quad w'(0) = \alpha.$$

Again, when $w'(s) > 0$, the equation is locally uniformly elliptic and then $w \in C^2$.

Proceeding as in the first case we see that $\bar{\alpha} \leq w'(s) < \alpha$ in $\mathbb{R}$ where, in the present case, $\Phi(\bar{\alpha}) = \Phi(\alpha) - M$.

$$w(s) = \begin{cases} 1 + \alpha s, & s > 0, \\ \bar{\alpha}(s - \bar{s}), & s \leq \bar{s}. \end{cases}$$

Let $w^{\varepsilon_j}(x_1) = \varepsilon_j w(\frac{x_1}{\varepsilon_j} - \frac{b_{\varepsilon_j}}{\bar{\alpha}\varepsilon_j} + \bar{s})$; then

$$w^{\varepsilon_j}(0) = \varepsilon_j w\left(-\frac{b_{\varepsilon_j}}{\bar{\alpha}\varepsilon_j} + \bar{s}\right) = \varepsilon_j \bar{\alpha}\left(\bar{s} - \frac{b_{\varepsilon_j}}{\bar{\alpha}\varepsilon_j} - \bar{s}\right) = -b_{\varepsilon_j}$$

and ${w^{\varepsilon_j}}'(s) \leq \alpha$. Therefore, $w^{\varepsilon_j} \leq u - b_{\varepsilon_j}$ in $\mathbb{R}$ so that $w^{\varepsilon_j} \leq v^{\varepsilon_j}$ on $\partial\mathcal{R}_2$, and since ${w^{\varepsilon_j}}' \leq \alpha \leq \Phi^{-1}(\frac{g_0}{\delta}M)$, we have, by the comparison principle below (Lemma 4.3), that $w^{\varepsilon_j} \leq v^{\varepsilon_j}$ in $\mathcal{R}_2$. We can conclude as in the previous case that $u \leq v$ in $\mathcal{R}_2$.

Step 2. Let $\mathcal{R}^+ = \{x : 0 < x_1 < 1, |x'| < 1\}$. Define

$$F_j = \int_{\partial\mathcal{R}^+ \cap \{x_1 = 1\}} F(|\nabla v^{\varepsilon_j}|)(v^{\varepsilon_j}_{x_1})^2 \, dx' + \int_{\partial\mathcal{R}^+ \cap \{|x'| = 1\}} F(|\nabla v^{\varepsilon_j}|) v^{\varepsilon_j}_n v^{\varepsilon_j}_{x_1} \, dS,$$

where ${v_n}^{\varepsilon_j}$ is the exterior normal of v^{ε_j} on $\partial\mathcal{R}^+ \cap \{|x'| = 1\}$. We first want to prove that

$$F_j \leq \int_{\partial\mathcal{R}^+ \cap \{x_1 = 1\}} \big(G(|\nabla v^{\varepsilon_j}|) + B_{\varepsilon_j}(v^{\varepsilon_j})\big) \, dx'.$$

In order to prove it, we proceed as in the proof of Lemma 4.2. That is, we can suppose that $F(s) \geq c > 0$ by using an approximation argument. Therefore, we can suppose that $v^{\varepsilon_j} \in W^{2,2}(\mathcal{R}_2)$. Multiplying equation (P_{ε_j}) by $v^{\varepsilon_j}_{x_1}$ in $\mathcal{R}^+$ and using the definitions of G and F we have

$$\begin{aligned} E_j &:= \int\int_{\mathcal{R}^+} \frac{\partial}{\partial x_1} \left(G(|\nabla v^{\varepsilon_j}|)\right) dx = \int\int_{\mathcal{R}^+} F(|\nabla v^{\varepsilon_j}|) \nabla v^{\varepsilon_j} \nabla v^{\varepsilon_j}_{x_1} \, dx \\ &= \int\int_{\mathcal{R}^+} \operatorname{div}(F(|\nabla v^{\varepsilon_j}|) \nabla v^{\varepsilon_j} v^{\varepsilon_j}_{x_1}) \, dx - \int\int_{\mathcal{R}^+} \beta_{\varepsilon_j}(v^{\varepsilon_j}) v^{\varepsilon_j}_{x_1} =: H_j - G_j. \end{aligned}$$

Using the divergence theorem and the fact that $v^{\varepsilon_j}_{x_1}(0, x') = 0$ (by the symmetry in the x_1 variable) we find that $H_j = F_j$.

From the convergence of $v^{\varepsilon_j} \to u = \alpha |x_1|$ in $\mathcal{R}_2$ and Proposition 4.1 we have that

$$\nabla v^{\varepsilon_j} \to \alpha e_1 \quad \text{a.e. in } \mathcal{R}_2^+ = \mathcal{R}_2 \cap \{x_1 > 0\}.$$

Since $|\nabla v^{\varepsilon_j}|$ are uniformly bounded, from the dominate convergence theorem we deduce that

$$\lim_{j \to \infty} F_j = \int_{\partial \mathcal{R}^+ \cap \{x_1 = 1\}} g(\alpha) \alpha \, dx' \tag{4.17}$$

and

$$\begin{aligned} F_j = E_j + G_j &= \int\int_{\mathcal{R}^+} \frac{\partial}{\partial x_1} \left(G(|\nabla v^{\varepsilon_j}|) + B_{\varepsilon_j}(v^{\varepsilon_j})\right) dx \\ &= \int_{\partial \mathcal{R}^+ \cap \{x_1 = 0\}} - \left(G(|\nabla v^{\varepsilon_j}|) + B_{\varepsilon_j}(v^{\varepsilon_j})\right) dx' \\ &\quad + \int_{\partial \mathcal{R}^+ \cap \{x_1 = 1\}} \left(G(|\nabla v^{\varepsilon_j}|) + B_{\varepsilon_j}(v^{\varepsilon_j})\right) dx' \\ &\leq \int_{\partial \mathcal{R}^+ \cap \{x_1 = 1\}} \left(G(|\nabla v^{\varepsilon_j}|) + B_{\varepsilon_j}(v^{\varepsilon_j})\right) dx'. \end{aligned}$$

Using again that $v^{\varepsilon_j} \to u = \alpha |x_1|$ uniformly on compact subsets of $\mathcal{R}_2$, we have that $|\nabla v^{\varepsilon_j}| \to \alpha$ uniformly on $\partial \mathcal{R}^+ \cap \{x_1 = 1\}$ and $B_{\varepsilon_j}(v^{\varepsilon_j}) = M$ on this set for j large. Therefore,

$$\limsup_{j \to \infty} F_j \leq \int_{\partial \mathcal{R}^+ \cap \{x_1 = 1\}} (G(\alpha) + M) \, dx'. \tag{4.18}$$

Thus, from (4.17) and (4.18) we obtain $\Phi(\alpha) \leq M$, which is a contradiction. □

Now, we prove the comparison principle needed in the proof of the lemma above. This is the step where we need an additional hypothesis: There exist $\eta_0 > 0$ such that

$$g'(t) \leq s^2 g'(ts) \quad \text{if} \quad 1 \leq s \leq 1 + \eta_0 \quad \text{and} \quad 0 < t \leq \Phi^{-1}\Big(\frac{g_0}{\delta} M\Big). \tag{4.19}$$

Remark 4.2. We remark that condition (4.19) holds for all the examples of functions g satisfying condition (1.3) considered in the introduction.

This is immediate when g is a positive power or the sum of positive powers.

If $g(t) = t^a \log(b + ct)$, we have for $s \geq 1$,

$$s^2 g'(ts) = s^{a+1} a t^{a-1} \log(b + cts) + s^{a+2} \frac{ct^a}{b + cts} \geq \Big[a t^{a-1} \log(b + ct) + \frac{sct^a}{b + cts}\Big].$$

Since

$$g'(t) = at^{a-1}\log(b+ct) + \frac{ct^a}{b+ct},$$

condition (4.19) holds if

$$\frac{s}{b+cts} \geq \frac{1}{b+ct}.$$

Or, equivalently,

$$sb + cst \geq b + cst,$$

and this last inequality holds for $s \geq 1$.

Finally, if $g \in C^1(\mathbb{R})$, $g(t) = c_1 t^{a_1}$ for $t \leq k$, and $g(t) = c_2 t^{a_2} + c_3$ for $t > k$, we have

$$s^2 g'(ts) = \begin{cases} s^{a_1+1} a_1 c_1 t^{a_1-1} & \text{if } st \leq k, \\ s^{a_2+1} a_2 c_2 t^{a_2-1} & \text{if } st \geq k \end{cases}$$

so that

1. if $t \geq k$, then $ts \geq k$ and

$$s^2 g'(ts) = s^{a_2+1} a_2 c_2 t^{a_2-1} \geq a_2 c_2 t^{a_2-1} = g'(t);$$

2. if $ts \leq k$ (i.e., $t \leq k/s$), we have, in particular, that $t \leq k$ and

$$s^2 g'(ts) = s^{a_1+1} a_1 c_1 t^{a_1-1} \geq a_1 c_1 t^{a_1-1} = g'(t);$$

3. if $k/s < t < k$, there holds that $s^2 g'(ts) = s^{a_2+1} a_2 c_2 t^{a_2-1}$ and $g'(t) = a_1 c_1 t^{a_1-1}$. Therefore, condition (4.19) is equivalent to

$$s^{a_2+1} \geq \frac{a_1 c_1}{a_2 c_2} t^{a_1-a_2}. \tag{4.20}$$

Observe that the condition that g' be continuous implies that $\frac{a_1 c_1}{a_2 c_2} = k^{a_2-a_1}$. Thus, (4.20) is equivalent to

$$s^{a_2+1} \geq \left(\frac{t}{k}\right)^{a_1-a_2}. \tag{4.21}$$

We consider two cases.

(i) If $a_1 \geq a_2$, (4.21) holds since $t < k$ and $s \geq 1$.

(ii) If $a_1 < a_2$, as $t > k/s$ there holds that

$$\left(\frac{t}{k}\right)^{a_1-a_2} < \frac{1}{s^{a_1-a_2}} \leq s^{a_2+1},$$

because $\frac{1}{s^{a_1}} \leq s$ since $s \geq 1$.

Let us now prove the comparison lemma used in the proof of Proposition 4.3.

LEMMA 4.3. *Let $w^\varepsilon(x_1)$ in $C^2(\mathbb{R})$ be such that ${w^\varepsilon}'(x_1) \geq \bar{\alpha} > 0$, and let $v^\varepsilon(x) \geq 0$ be a solution of $\mathcal{L}v^\varepsilon = \beta_\varepsilon(v^\varepsilon)$ in $\mathcal{R} = \{x = (x_1, x') : a < x_1 < b, |x'| < r\}$, continuous up to $\partial\mathcal{R}$. Then, the following comparison principle holds: If $v^\varepsilon(x) \geq w^\varepsilon(x_1)$ for all $x \in \partial\mathcal{R}$ and if*

1. *$\mathcal{L}(w^\varepsilon) \geq \frac{g_0}{\delta}\beta_\varepsilon(w^\varepsilon)$ on $\mathbb{R}$*

or

2. *$\mathcal{L}w^\varepsilon \geq \beta_\varepsilon(w^\varepsilon)$, $w^{\varepsilon\prime} \leq \Phi^{-1}(\frac{g_0}{\delta}M)$, and g' satisfies condition* (4.19),

then $v^\varepsilon(x) \geq w^\varepsilon(x_1)$ for all $x \in \mathcal{R}$.

Proof. Since $w^{\varepsilon\prime}(x_1) \geq \bar{\alpha}$ there exists x_0 such that $w^\varepsilon(x_0) = 0$. Let us suppose that $x_0 = 0$.

Since $v^\varepsilon(x) \geq 0$, we can find τ such that

$$w^\varepsilon(x_1 - \tau) < v^\varepsilon(x) \quad \text{on } \bar{\mathcal{R}}.$$

For $\eta > 0$ sufficiently small define

$$w^{\varepsilon,\eta}(x_1) := w^\varepsilon(\varphi_\eta(x_1 - c_\eta)),$$

where $\varphi_\eta(s) = s + \eta s^2$ and $c_\eta > 0$ is the smallest constant such that $\varphi_\eta(s - c_\eta) \leq s$ on $[-2\tau, 2\tau]$ (observe that $c_\eta \to 0$ when $\eta \to 0$). If $c_\eta - \frac{1}{\eta} \leq -2\tau$, then $\varphi_\eta(s - c_\eta) \leq 0$ for $s \leq c_\eta$. Observe that, in $[-2\tau, 2\tau]$, $w^{\varepsilon,\eta} \leq w^\varepsilon$ and, as $\eta \to 0$, $w^{\varepsilon,\eta} \to w^\varepsilon$ uniformly.

If we call $\widetilde{\varphi}_\eta(s) = \varphi_\eta(s - c_\eta)$, we have

$$\mathcal{L}w^{\varepsilon,\eta} = g'\left(w^{\varepsilon\prime}(\widetilde{\varphi}_\eta)\,\widetilde{\varphi}'_\eta\right) w^{\varepsilon\prime\prime}(\widetilde{\varphi}_\eta)\,(\widetilde{\varphi}'_\eta)^2 + g'\left(w^{\varepsilon\prime}(\widetilde{\varphi}_\eta)\,\widetilde{\varphi}'_\eta\right) w^{\varepsilon\prime}(\widetilde{\varphi}_\eta)\,\widetilde{\varphi}''_\eta. \tag{4.22}$$

In the first case, we use that, by condition (1.3), we have for $s \geq 1$,

$$g'(ts) \geq \delta\frac{g(ts)}{ts} \geq \delta\frac{g(t)}{ts} \geq \frac{\delta g'(t)}{g_0 s}.$$

Therefore,

$$s^2 g'(ts) \geq \frac{\delta}{g_0} s g'(t). \tag{4.23}$$

Taking $s = \widetilde{\varphi}'_\eta$ and $t = w^{\varepsilon\prime}(\widetilde{\varphi}'_\eta)$ and using (4.22), (4.23), and the fact that $\varphi_\eta{}'' > 0$ and $w^{\varepsilon\prime} > 0$, we have

$$\mathcal{L}w^{\varepsilon,\eta} > \frac{\delta}{g_0} g'\left(w^{\varepsilon\prime}(\widetilde{\varphi}_\eta)\right) w^{\varepsilon\prime\prime}(\widetilde{\varphi}_\eta)\widetilde{\varphi}'_\eta = \frac{\delta}{g_0}\mathcal{L}w^\varepsilon(\widetilde{\varphi}_\eta)\widetilde{\varphi}'_\eta \geq \beta_\varepsilon(w^{\varepsilon,\eta})\widetilde{\varphi}'_\eta.$$

Since $\beta_\varepsilon(w^{\varepsilon,\eta}) = 0$ when $x_1 \leq c_\eta$ and $\widetilde{\varphi}'_\eta \geq 1$ when $x_1 \geq c_\eta$, we have that $\mathcal{L}w^{\varepsilon,\eta} > \beta_\varepsilon(w^{\varepsilon,\eta})$.

For the second case, choose η small enough so that $0 < c_\eta \leq 1$ and $\widetilde{\varphi}'_\eta(r) \leq 1 + \eta_0$ for $a < r < b$.

If $x_1 < c_\eta$, we proceed as in the previous case and deduce that $\mathcal{L}(w^{\varepsilon,\eta}) > 0 = \beta_\varepsilon(w^{\varepsilon,\eta})$.

If $x_1 \geq c_\eta$, we can apply condition (4.19) with $s = \widetilde{\varphi}'_\eta$ and $t = w^{\varepsilon\prime}(\widetilde{\varphi}'_\eta)$ since $w^{\varepsilon\prime} \leq \Phi^{-1}(\frac{g_0}{\delta}M)$.

Then, using that $\varphi_\eta{}'' > 0$ and $w^{\varepsilon\prime} > 0$ and (4.22) we have

$$\mathcal{L}w^{\varepsilon,\eta} > g'\left(w^{\varepsilon\prime}(\widetilde{\varphi}_\eta)\right) w^{\varepsilon\prime\prime}(\widetilde{\varphi}_\eta) = \mathcal{L}w^\varepsilon(\widetilde{\varphi}_\eta) \geq \beta_\varepsilon(w^{\varepsilon,\eta}).$$

Summarizing, in both cases we have

$$\mathcal{L}w^{\varepsilon,\eta} > \beta_\varepsilon(w^{\varepsilon,\eta}), \quad w^{\varepsilon,\eta} \to w^\varepsilon \text{ as } \eta \to 0, \quad \text{and } w^{\varepsilon,\eta} \leq w^\varepsilon.$$

Let now $\tau^* \geq 0$ be the smallest constant such that

$$w^{\varepsilon,\eta}(x_1 - \tau^*) \leq v^\varepsilon(x) \quad \text{ in } \overline{\mathcal{R}}.$$

We want to prove that $\tau^* = 0$. By the minimality of τ^*, there exists a point $x^* \in \overline{\mathcal{R}}$ such that $w^{\varepsilon,\eta}(x_1^* - \tau^*) = v^\varepsilon(x^*)$. If $\tau^* > 0$, then $w^{\varepsilon,\eta}(x_1 - \tau^*) < w^{\varepsilon,\eta}(x_1) \leq w^\varepsilon(x_1) \leq v^\varepsilon(x)$ on $\partial\mathcal{R}$, and hence, x^* is an interior point of $\mathcal{R}$.

At this point observe that the gradient of $w^{\varepsilon,\eta}(x_1 - \tau^*)$ does not vanish and $\mathcal{L}w^{\varepsilon,\eta}(x_1^* - \tau^*) > \beta_\varepsilon(w^{\varepsilon,\eta}(x^* - \tau^*)) = \beta_\varepsilon(v^\varepsilon(x^*)) = \mathcal{L}v^\varepsilon(x^*)$. We also have $w^{\varepsilon,\eta}(x_1 - \tau^*) \leq v^\varepsilon(x)$ in $\mathcal{R}$ and $w^{\varepsilon,\eta}(x_1^* - \tau^*) = v^\varepsilon(x^*)$. Then, also $\nabla w^{\varepsilon,\eta}(x_1^* - \tau^*) = \nabla v^\varepsilon(x^*)$.

Let

$$Lv = \sum_{i,j=1}^N a_{ij}(\nabla w^{\varepsilon,\eta}(x_1 - \tau^*))v_{x_i x_j}.$$

Since $|\nabla w^{\varepsilon,\eta}(x_1 - \tau^*)| > 0$ near x^*, L is well defined near the point x^* and, by condition (1.3), L is uniformly elliptic.

Since $\nabla w^{\varepsilon,\eta}(x_1^* - \tau^*) = \nabla v^\varepsilon(x^*)$, we have that

$$Lw^{\varepsilon,\eta}(x_1^* - \tau^*) = \mathcal{L}w^{\varepsilon,\eta}(x_1^* - \tau^*) > \mathcal{L}v^\varepsilon(x^*) = \sum_{i,j=1}^N a_{ij}(\nabla v^\varepsilon(x^*))v^\varepsilon_{x_i x_j} = Lv^\varepsilon(x^*).$$

Moreover, since v^ε is a solution to

$$\widetilde{L}v := \sum_{i,j=1}^N a_{ij}(\nabla v^\varepsilon(x))v_{x_i x_j} = \beta_\varepsilon(v),$$

$\widetilde{L}$ is uniformly elliptic in a neighborhood of x^* with Hölder continuous coefficients and $\beta_\varepsilon(v^\varepsilon) \in Lip$, there holds that $v^\varepsilon \in C^2$ in a neighborhood of x^*.

Therefore, we have for some $\eta > 0$,

$$\begin{cases} Lw^{\varepsilon,\eta}(x_1 - \tau^*) > Lv^\varepsilon(x) & \text{in } B_\eta(x^*),\\ w^{\varepsilon,\eta}(x_1^* - \tau^*) = v^\varepsilon(x^*), & \\ w^{\varepsilon,\eta}(x_1 - \tau^*) \leq v^\varepsilon(x) & \text{in } \overline{\mathcal{R}}. \end{cases}$$

But these three statements contradict the strong maximum principle. Therefore, $\tau^* = 0$ and, thus, $w^{\varepsilon,\eta} \leq v^\varepsilon$ on $\overline{\mathcal{R}}$.

Letting $\eta \to 0$ we obtain the desired result. ▯

5. Asymptotic behavior of limit solutions. Now we want to prove—for g satisfying conditions (1.3) and (1.4)—the asymptotic development of the limiting function u. We will obtain this result, under suitable assumptions on the function u. First, we give the following definition.

DEFINITION 5.1. *Let v be a continuous nonnegative function in a domain $\Omega \subset \mathbb{R}^N$. We say that v is nondegenerate at a point $x_0 \in \Omega \cap \{v = 0\}$ if there exist c, $r_0 > 0$ such that*

$$\frac{1}{r^N}\int_{B_r(x_0)} v\,dx \geq cr \quad \text{ for } 0 < r \leq r_0.$$

We say that v is uniformly nondegenerate in a set $\Omega' \subset \Omega \cap \{v = 0\}$ if the constants c and r_0 can be taken independent of the point $x_0 \in \Omega'$.

We have the following theorem.

THEOREM 5.1. *Suppose that g satisfies conditions* (1.3) *and* (1.4). *Let u^{ε_j} be a solution to (P_{ε_j}) in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ be such that $\partial\{u > 0\}$ has an inward unit normal ν in the measure theoretic sense at x_0, and suppose that u is nondegenerate at x_0. Under these assumptions, we have*

$$u(x) = \Phi^{-1}(M)\langle x - x_0, \nu\rangle^+ + o(|x - x_0|).$$

The proof of this theorem makes strong use of the following result.

THEOREM 5.2. *Suppose that g satisfies conditions* (1.3) *and* (1.4). *Let u^{ε_j} be a solution to (P_{ε_j}) in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly in compact subsets of Ω and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$. Then,*

$$\limsup_{\substack{x \to x_0 \\ u(x) > 0}} |\nabla u(x)| \le \Phi^{-1}(M).$$

Proof. Let

$$\alpha := \limsup_{\substack{x \to x_0 \\ u(x) > 0}} |\nabla u(x)|.$$

Since $u \in Lip_{loc}(\Omega)$, $\alpha < \infty$. If $\alpha = 0$, we are done. So, suppose that $\alpha > 0$. By the definition of α there exists a sequence $z_k \to x_0$ such that

$$u(z_k) > 0, \qquad |\nabla u(z_k)| \to \alpha.$$

Let y_k be the nearest point from z_k to $\Omega \cap \partial\{u > 0\}$, and let $d_k = |z_k - y_k|$.

Consider the blow up sequence u_{d_k} with respect to $B_{d_k}(y_k)$. That is, $u_{d_k}(x) = \frac{1}{d_k} u(y_k + d_k x)$. Since u is Lipschitz and $u_{d_k}(0) = 0$ for every k, there exists $u_0 \in Lip(\mathbb{R}^N)$ such that (for a subsequence) $u_{d_k} \to u_0$ uniformly in compact sets of $\mathbb{R}^N$. We also have that $\mathcal{L}u_0 = 0$ in $\{u_0 > 0\}$.

Now, set $\bar{z}_k = (z_k - y_k)/d_k \in \partial B_1$. We may assume that $\bar{z}_k \to \bar{z} \in \partial B_1$. Take

$$\nu_k := \frac{\nabla u_{d_k}(\bar{z}_k)}{|\nabla u_{d_k}(\bar{z}_k)|} = \frac{\nabla u(z_k)}{|\nabla u(z_k)|}.$$

Passing to a subsequence and after a rotation we can assume that $\nu_k \to e_1$. Observe that $B_{2/3}(\bar{z}) \subset B_1(\bar{z}_k)$ for k large, and therefore, u_0 is an $\mathcal{L}$-solution there. By interior Hölder gradient estimates (see [19]), we have $\nabla u_{d_k} \to \nabla u_0$ uniformly in $B_{1/3}(\bar{z})$, and therefore, $\nabla u(z_k) \to \nabla u_0(\bar{z})$. Thus, $\nabla u_0(\bar{z}) = \alpha\, e_1$ and, in particular, $\partial_{x_1} u_0(\bar{z}) = \alpha$.

Next, we claim that $|\nabla u_0| \le \alpha$ in $\mathbb{R}^N$. In fact, let $R > 1$ and $\delta > 0$. Then, there exists $\tau_0 > 0$ such that $|\nabla u(x)| \le \alpha + \delta$ for any $x \in B_{\tau_0 R}(x_0)$. For $|z_k - x_0| < \tau_0 R/2$ and $d_k < \tau_0/2$ we have $B_{d_k R}(z_k) \subset B_{\tau_0 R}(x_0)$ and, therefore, $|\nabla u_{d_k}(x)| \le \alpha + \delta$ in B_R for k large. Passing to the limit, we obtain $|\nabla u_0| \le \alpha + \delta$ in B_R, and since δ and R were arbitrary, the claim holds.

Since ∇u_0 is Hölder continuous in $B_{1/3}(\bar{z})$, there holds that $\nabla u_0 \neq 0$ in a neighborhood of $\bar{z}$. Thus, by the results in [18], $u_0 \in W^{2,2}$ in a ball $B_r(\bar{z})$ for some $r > 0$, and since

$$\int A(\nabla u_0) \nabla\varphi \, dx = 0 \quad \text{for every } \varphi \in C_0^\infty(B_r(\bar{z})),$$

taking $\varphi = \psi_{x_1}$ and integrating by parts we see that, for $w = \frac{\partial u_0}{\partial x_1}$,

$$\sum_{i,j=1}^{N} \int_{B_r(\bar z)} a_{ij}\big(\nabla u_0(x)\big) w_{x_j} \psi_{x_i}\, dx = 0.$$

That is, w is a solution to the uniformly elliptic equation

$$\mathcal{T} w := \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ij} \left(\nabla u_0(x) \right) w_{x_j} \right) = 0.$$

Now let $\bar w = \alpha - w$. Then, $\bar w \ge 0$ in $B_r(\bar z)$, $\bar w(\bar z) = 0$ and $\mathcal{T} \bar w = 0$ in $B_r(\bar z)$. By the Harnack inequality we conclude that $\bar w \equiv 0$. Hence, $w \equiv \alpha$ in $B_r(\bar z)$.

Now, since we can repeat this argument around any point where $w = \alpha$, by a continuation argument, we have that $w = \alpha$ in $B_1(\bar z)$.

Therefore, $\nabla u_0 = \alpha\, e_1$ and we have, for some $y \in \mathbb{R}^N$, $u_0(x) = \alpha(x_1 - y_1)$ in $B_1(\bar z)$. Since $u_0(0) = 0$, there holds that $y_1 = 0$ and $u_0(x) = \alpha x_1$ in $B_1(\bar z)$. Finally, since $\mathcal{L} u_0 = 0$ in $\{u_0 > 0\}$, by a continuation argument, we have that $u_0(x) = \alpha x_1$ in $\{x_1 \ge 0\}$.

On the other hand, as $u_0 \ge 0$, $\mathcal{L} u_0 = 0$ in $\{u_0 > 0\}$, and $u_0 = 0$ in $\{x_1 = 0\}$, we have, by Lemma B.2, that

$$u_0 = -\gamma x_1 + o(|x|) \quad \text{ in } \{x_1 < 0\}$$

for some $\gamma \ge 0$.

Now, define for $\lambda > 0$, $(u_0)_\lambda(x) = \frac{1}{\lambda} u_0(\lambda x)$. There exist a sequence $\lambda_n \to 0$ and $u_{00} \in Lip(\mathbb{R}^N)$ such that $(u_0)_{\lambda_n} \to u_{00}$ uniformly in compact subsets of $\mathbb{R}^N$. We have $u_{00}(x) = \alpha x_1^+ + \gamma x_1^-$.

By Lemma 4.1 there exists a sequence $\varepsilon'_j \to 0$ such that $u^{\varepsilon'_j}$ is a solution to $(P_{\varepsilon'_j})$ and $u^{\varepsilon'_j} \to u_0$ uniformly on compact subsets of $\mathbb{R}^N$. Applying a second time Lemma 4.1 we find a sequence $\varepsilon''_j \to 0$ and a solution $u^{\varepsilon''_j}$ to $(P_{\varepsilon''_j})$ converging uniformly in compact subsets of $\mathbb{R}^N$ to u_{00}. Now we can apply Proposition 4.2 in the case that $\gamma = 0$ or Proposition 4.3 in the case that $\gamma > 0$, and we conclude that $\alpha \le \Phi^{-1}(M)$. □

Proof of Theorem 5.1. Assume that $x_0 = 0$ and $\nu = e_1$. Take $u_\lambda(x) = \frac{1}{\lambda} u(\lambda x)$. Let $\rho > 0$ such that $B_\rho \subset\subset \Omega$, and since $u_\lambda \in Lip(B_{\rho/\lambda})$ uniformly in λ, $u_\lambda(0) = 0$, there exists $\lambda_j \to 0$ and $U \in Lip(\mathbb{R}^N)$ such that $u_{\lambda_j} \to U$ uniformly on compact subsets of $\mathbb{R}^N$. From Proposition 4.1 and Lemma 4.1, $\mathcal{L} u_\lambda = 0$ in $\{u_\lambda > 0\}$. Using the fact that e_1 is the inward normal in the measure theoretic sense, we have, for fixed k,

$$|\{u_\lambda > 0\} \cap \{x_1 < 0\} \cap B_k| \to 0 \quad \text{ as } \lambda \to 0.$$

Hence, $U = 0$ in $\{x_1 < 0\}$. Moreover, U is nonnegative in $\{x_1 > 0\}$, $\mathcal{L} U = 0$ in $\{U > 0\}$, and U vanishes in $\{x_1 \le 0\}$. Then, by Lemma B.2 we have that there exists $\alpha \ge 0$ such that

$$U(x) = \alpha x_1^+ + o(|x|).$$

By Lemma 4.1 we can find a sequence $\varepsilon'_j \to 0$ and solutions $u^{\varepsilon'_j}$ to $(P_{\varepsilon'_j})$ such that $u_{\varepsilon'_j} \to U$ uniformly on compact subsets of $\mathbb{R}^N$ as $j \to \infty$. Define $U_\lambda(x) = \frac{1}{\lambda} U(\lambda x)$; then $U_\lambda \to \alpha x_1^+$ uniformly on compact subsets of $\mathbb{R}^N$. Applying again Lemma 4.1

we find a second sequence $\sigma_j \to 0$ and u^{σ_j} solution to (P_{σ_j}) such that $u^{\sigma_j} \to \alpha x_1^+$ uniformly on compact subsets of $\mathbb{R}^N$ and

$$\nabla u^{\sigma_j} \to \alpha \chi_{\{x_1>0\}} e_1 \quad \text{in } L^{g_0+1}_{loc}\left(\mathbb{R}^N\right).$$

Now, we proceed as in the proof of Proposition 4.2. Let $\psi \in C_0^\infty(\mathbb{R}^N)$, and choose $u^{\sigma_j}_{x_1}\psi$ as test function in the weak formulation of $\mathcal{L}u^{\sigma_j} = \beta_{\sigma_j}(u^{\sigma_j})$. Then,

$$B_{\sigma_j}(u^{\sigma_j}) \to M\chi_{\{x_1>0\}} + \overline{M}\chi_{\{x_1<0\}} \quad * -\text{weakly in } L^\infty,$$

with $\overline{M} = 0$ or $\overline{M} = M$. Moreover, $\Phi(\alpha) = M - \overline{M}$.

By the nondegeneracy assumption on u we have

$$\frac{1}{r^N}\int_{B_r} u_{\lambda_j}\, dx \geq cr$$

and then

$$\frac{1}{r^N}\int_{B_r} U_{\lambda_j}\, dx \geq cr.$$

Therefore, $\alpha > 0$ so that we have that $\overline{M} = 0$. Then, $\alpha = \Phi^{-1}(M)$.

We have shown that

$$U(x) = \begin{cases} \Phi^{-1}(M)x_1 + o(|x|), & x_1 > 0, \\ 0, & x_1 \leq 0. \end{cases}$$

By Theorem 5.2, $|\nabla U| \leq \Phi^{-1}(M)$ in $\mathbb{R}^N$. As $U = 0$ on $\{x_1 = 0\}$ we have $U \leq \Phi^{-1}(M)x_1$ in $\{x_1 > 0\}$.

Since $\mathcal{L}U = 0$ in $\{x_1 > 0\}$, $U = 0$ on $\{x_1 = 0\}$, there holds that $U \in C^{1,\alpha}(\{x_1 \geq 0\})$. Thus, $|\nabla U(0)| = \Phi^{-1}(M) > 0$ so that, near zero, U satisfies a linear uniformly elliptic equation in nondivergence form and the same equation is satisfied by $w = U - \Phi^{-1}(M)x_1$ in $\{x_1 > 0\} \cap B_r(0)$ for some $r > 0$. We also have $w \leq 0$ so that by Hopf's boundary principle we have that $w = 0$ in $\{x_1 > 0\} \cap B_r(0)$ and then, by a continuation argument based on the strong maximum principle we deduce that $U(x) = \alpha x_1^+$ in $\mathbb{R}^N$. The proof is complete. ∎

Now we prove another result that is needed in order to see that u is a weak solution according to Definition 1.1.

THEOREM 5.3. *Let u^{ε_j} be a solution to (P_{ε_j}) in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly in compact subsets of Ω and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$, and suppose that u is nondegenerate at x_0. Assume there is a ball B contained in $\{u = 0\}$ touching x_0; then*

$$\limsup_{\substack{x\to x_0\\ u(x)>0}} \frac{u(x)}{\operatorname{dist}(x, B)} = \Phi^{-1}(M). \tag{5.1}$$

Proof. Let ℓ be the finite limit on the left-hand side of (5.1), and $y_k \to x_0$ with $u(y_k) > 0$ and

$$\frac{u(y_k)}{d_k} \to \ell, \quad d_k = \operatorname{dist}(y_k, B).$$

Consider the blow up sequence u_k with respect to $B_{d_k}(x_k)$, where $x_k \in \partial B$ are points with $|x_k - y_k| = d_k$, and choose a subsequence with blow up limit u_0 such that there exists

$$e := \lim_{k\to\infty} \frac{y_k - x_k}{d_k}.$$

Then, by construction, $u_0(e) = \ell = \ell\langle e, e\rangle$, $u_0(x) \le \ell\langle x, e\rangle$ for $\langle x, e\rangle \ge 0$, and $u_0(x) = 0$ for $\langle x, e\rangle \le 0$. In particular, $\nabla u_0(e) = \ell\, e$.

By the nondegeneracy assumption, we have that $\ell > 0$. Since $|\nabla u_0(e)| = \ell > 0$ and ∇u_0 is continuous, both u_0 and $\ell\langle x, e\rangle^+$ are solutions of $Lv = 0$ in $\{u_0 > 0\} \cap \{\langle x, e\rangle \ge 0\} \cap \{|\nabla u_0| > 0\}$ where

$$Lv := \sum_{i,j=1}^N b_{ij}(\nabla u_0) v_{x_1 x_j}$$

is uniformly elliptic and

$$b_{ij}(p) = \delta_{ij} + \left(\frac{g'(|p|)|p|}{g(|p|)} - 1\right)\frac{p_i p_j}{|p|^2}.$$

Now, from the strong maximum principle, we have that they must coincide in a neighborhood at the point e.

By continuation we have that $u_0 = \ell\langle x, e\rangle^+$. Thus, we have, by Proposition 4.2, that $\ell = \Phi^{-1}(M)$. ◻

6. Regularity of the free boundary. We can now prove a regularity result for the free boundary of limits of solutions to (P_ε).

THEOREM 6.1. *Assume that g satisfies conditions* (1.3) *and* (1.4). *Let u^{ε_j} be a solution to (P_{ε_j}) in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly in compact subsets of Ω and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ be such that there is an inward unit normal ν in the measure theoretic sense at x_0. Suppose that u is uniformly nondegenerate at the free boundary in a neighborhood of x_0 (see Definition* 5.1*). Then, there exists $r > 0$ such that $B_r(x_0) \cap \partial\{u > 0\}$ is a $C^{1,\alpha}$ surface.*

Proof. By Corollary 3.1, Theorems 5.1 and 5.3, and the nondegeneracy assumption we have that u is a weak solution in the sense of Definition 1.1. Therefore, Theorem 9.4 of [21] applies, and the result follows. ◻

7. Some examples. In this section we give some examples in which the nondegeneracy condition is satisfied so that in these cases $\partial_{red}\{u > 0\}$ is a $C^{1,\alpha}$ surface.

For the case of a limit of minimizers of the functionals

$$J_\varepsilon(v) = \int_\Omega G(|\nabla v|)\,dx + \int_\Omega B_\varepsilon(v)\,dx \tag{7.1}$$

with $B'_\varepsilon(s) = \beta_\varepsilon(s)$, we will also prove that $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{red}\{u > 0\}) = 0$.

The uniform nondegeneracy condition will follow from the linear growth away from the free boundary. This is a well known result for the case of the Laplacian. We prove it here for the operator $\mathcal{L}$ (Theorem 7.1). The proof is based on an iteration argument that, in the case of the proof for the Laplacian, makes use of the mean value property (see [9]). We replace it here by a blow up argument (see Lemma 7.1).

LEMMA 7.1. *Let $c_1 > 1$, and let $u^\varepsilon \in C(\Omega)$, $|\nabla u^\varepsilon| \le L$ with $\mathcal{L}u^\varepsilon = 0$ in $\{u^\varepsilon > \varepsilon\}$ be such that there exists $C > 0$ so that $u^\varepsilon(x) \ge C\,\mathrm{dist}(x, \partial\{u^\varepsilon > \varepsilon\})$ if $u^\varepsilon(x) > c_1\varepsilon$*

and $d(x) = \operatorname{dist}(x, \partial\{u^\varepsilon > \varepsilon\}) < 1/2 \operatorname{dist}(x, \partial\Omega)$. *Then, there exists* $\delta_0 > 0$ *and* $\delta_0 = \delta_0(c_1, C)$ *such that* $\forall \varepsilon > 0$ *and* $\forall x \in \{u^\varepsilon > c_1\varepsilon\}$ *with* $d(x) < 1/2 \operatorname{dist}(x, \partial\Omega)$ *we have*

$$\sup_{B_{d(x)}(x)} u^\varepsilon \geq (1 + \delta_0) u^\varepsilon(x).$$

Proof. Suppose by contradiction that there exist sequences $\delta_k \to 0$, $\varepsilon_k > 0$, and $x_k \in \{u^{\varepsilon_k} > c_1\varepsilon_k\}$ with $d_k = d(x_k) < 1/2 \operatorname{dist}(x_k, \partial\Omega)$ such that

$$\sup_{B_{d_k}(x_k)} u^{\varepsilon_k} \leq (1 + \delta_k) u^{\varepsilon_k}(x_k).$$

Take $w_k(x) = \frac{u^{\varepsilon_k}(x_k + d_k x)}{u^{\varepsilon_k}(x_k)}$. Then, $w_k(0) = 1$ and

$$\max_{\overline{B_1}} w_k \leq (1 + \delta_k), \quad w_k > 0, \quad \text{and} \quad \mathcal{L}_k w_k = 0 \quad \text{in } B_1,$$

where $\mathcal{L}_k v = \operatorname{div}(\frac{g_k(|\nabla v|)}{|\nabla v|} \nabla v)$ with $g_k(t) = g(\frac{u^{\varepsilon_k}(x_k) t}{d_k})$.

On the other hand, in B_2 we have

$$\|\nabla w_k\|_{L^\infty(B_2)} = \|\nabla u^{\varepsilon_k}(x_k + d_k x)\|_{L^\infty(B_2)} \frac{d_k}{u^{\varepsilon_k}(x_k)} \leq \frac{L}{C}.$$

Then, there exists $\overline{w} \in C(\overline{B}_1)$ such that

$$w_k \to \overline{w} \quad \text{uniformly in } \overline{B}_1.$$

Take $0 < r < 1$, and let $v_k(x) = (1 + \delta_k) - w_k(x)$. Then, since g_k satisfies (1.3), by the Harnack inequality we have

$$0 \leq v_k(x) \leq c(r) v_k(0) \quad \text{for } |x| < r.$$

By passing to the limit we have

$$0 \leq 1 - \overline{w} \leq c(r)(1 - \overline{w}(0)) = 0.$$

Therefore, $\overline{w} = 1$ in B_1.

Let $y_k \in \partial\{u_k > \varepsilon_k\}$ with $|x_k - y_k| = d_k$. Then, if $z_k = \frac{y_k - x_k}{d_k}$, we have

$$w_k(z_k) = \frac{\varepsilon_k}{u^{\varepsilon_k}(x_k)} \leq \frac{1}{c_1}$$

and we may assume that $z_k \to \bar{z} \in \partial B_1$. Thus, $1 = \overline{w}(\bar{z}) \leq \frac{1}{c_1} < 1$. This is a contradiction, and the lemma is proved. ∎

THEOREM 7.1. *Let* $c_1 > 1$, C, $L > 0$, *and* $\Omega' \subset\subset \Omega$. *There exist* c_0, $r_0 > 0$ *such that if* $u^\varepsilon \in C(\Omega)$ *is such that* $\mathcal{L}u^\varepsilon = 0$ *in* $\{u^\varepsilon > \varepsilon\}$, $\|u^\varepsilon\|_{L^\infty(\Omega')}$, $\|\nabla u^\varepsilon\|_{L^\infty(\Omega')} \leq L$, *and* $u^\varepsilon(x) \geq C \operatorname{dist}(x, \partial\{u^\varepsilon > \varepsilon\})$ *if* $x \in \{u^\varepsilon > c_1\varepsilon\} \cap \Omega'$ *and* $d(x) = \operatorname{dist}(x, \partial\{u^\varepsilon > \varepsilon\}) < 1/2 \operatorname{dist}(x, \partial\Omega')$, *then if* $x_0 \in \Omega' \cap \{u^\varepsilon > c_1\varepsilon\}$ *with* $\operatorname{dist}(x_0, \partial\{u^\varepsilon > \varepsilon\}) < 1/2 \operatorname{dist}(x, \partial\Omega')$, *it holds that*

$$\sup_{B_r(x_0)} u^\varepsilon \geq c_0 r \quad \text{for } 0 < r < r_0.$$

Proof. The proof follows as that of Theorem 1.9 in [9] by using Lemma 7.1 and the same iteration argument as in that theorem. ∎

As a corollary we get the locally uniform nondegeneracy of $u = \lim u^\varepsilon$ if u^ε are solutions to (P_ε) with linear growth. In fact, see the following corollary.

COROLLARY 7.1. *Let u^{ε_j} be uniformly bounded solutions to (P_{ε_j}) in Ω such that for every $\Omega' \subset\subset \Omega$ there exist constants $c_1 > 1$ and $C > 0$ such that $u^{\varepsilon_j}(x) \geq C\,\mathrm{dist}(x, \partial\{u^{\varepsilon_j} > \varepsilon_j\})$ if $x \in \{u^{\varepsilon_j} > c_1\varepsilon_j\} \cap \Omega'$ and $d(x) = \mathrm{dist}(x, \partial\{u^{\varepsilon_j} > \varepsilon_j\}) < 1/2\,\mathrm{dist}(x, \partial\Omega')$. Assume $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω.*

Then, there exist constants c_0 and r_0 depending on c_1 and C, the uniform bound of $\|u^{\varepsilon_j}\|_{L^\infty(\Omega)}$ and Ω', such that for every $x_0 \in \Omega' \cap \overline{\{u > 0\}}$ such that $\mathrm{dist}(x_0, \partial\{u > 0\}) < 1/2\,\mathrm{dist}(x_0, \partial\Omega')$,

$$\sup_{B_r(x_0)} u \geq c_0 r \quad \text{for } 0 < r < r_0.$$

Proof. The proof follows from Theorem 7.1 as in Chapter 1 in [9]. □

7.1. Example 1. Before we give the first example we need the following definition.

DEFINITION 7.1. *Let u^ε be a solution to (P_ε). We say that u^ε is a minimal solution to (P_ε) in Ω if whenever we have v^ε a strong supersolution to (P_ε) in $\Omega' \subset\subset \Omega$, i.e.,*

$$v^\varepsilon \in W^{1,G}(\Omega) \cap C\left(\overline{\Omega'}\right), \quad g(|\nabla v^\varepsilon|)\frac{\nabla v^\varepsilon}{|\nabla v^\varepsilon|} \in W^{1,1}(\Omega'), \quad \mathcal{L}v^\varepsilon \leq \beta_\varepsilon(v^\varepsilon) \text{ in } \Omega',$$

which satisfies

$$v^\varepsilon \geq u^\varepsilon \text{ on } \partial\Omega,$$

then

$$v^\varepsilon \geq u^\varepsilon \text{ in } \Omega'.$$

We will not discuss the existence of minimal solutions of the operator $\mathcal{L}$ in this paper. But, let us point out that the interest in considering this kind of solution is that when $\Omega = (-\infty, +\infty) \times \Sigma$, a solution u^ε of P_ε that is strictly decreasing in the x_1 variable with $\lim_{x_1 \to +\infty} u^\varepsilon(x_1, x') = 0$ uniformly for $x' \in \Sigma$ is a minimal solution. The proof of this fact follows the lines of the comparison result Lemma 4.3. See also [3, Theorem 7.1] for the proof that traveling waves of the equation $\Delta u^\varepsilon - u^\varepsilon_t = \beta_\varepsilon(u^\varepsilon)$ are minimal solutions.

We can prove for minimal solutions, as in Theorem 4.1 in [3], the following lemma.

LEMMA 7.2. *Let u^ε be minimal solutions to (P_ε) in a domain $\Omega \subset \mathbb{R}^N$. For every $\Omega' \subset\subset \Omega$, there exist C, ρ, and ε_0, depending on N, δ, g_0, $\mathrm{dist}(\Omega', \partial\Omega)$, and the function β such that, if $\varepsilon \leq \varepsilon_0$ and $x \in \Omega'$, then*

$$u^\varepsilon(x) \geq C\,\mathrm{dist}(x, \{u^\varepsilon \leq \varepsilon\})$$

if $\mathrm{dist}(x, \{u^\varepsilon \leq \varepsilon\}) \leq \rho$.

Proof. We drop the superscript ε.

The proof is similar to the one of Theorem 4.1 in [3]. We have to make a modification, since we are dealing with the operator $\mathcal{L}$ instead of the Laplacian.

Let $x_0 \in \{u > \varepsilon\}$. Without loss of generality we may suppose that x_0 is the origin, and let $d_\varepsilon(0) = \mathrm{dist}(0, \{u \leq \varepsilon\}) = 2\gamma$. In $B_{2\gamma}$, u satisfies $\mathcal{L}u = 0$, and, therefore, by the Harnack inequality, we have $u \leq Cu(0)$ in B_γ for some constant C.

We will construct a radial supersolution satisfying the hypotheses in Definition 7.1 such that $v(0) = a\varepsilon < u(0)$ for some constant $0 < a < 1$. Also $v(\gamma) \geq D\gamma$ for some constant D under control.

By our hypothesis that u is a minimal solution it follows that we cannot have $v \geq u$ everywhere on ∂B_γ. Therefore,

$$D\gamma \leq v(\gamma) \leq Cu(0),$$

and this is what we want to prove.

Let $0 < a < b < 1$, and let $v \in C^1(B_\gamma)$ be defined as

$$v(r) = \begin{cases} \varepsilon a, & 0 \leq r \leq r_0, \\ \varepsilon a + k(r - r_0)^{\frac{\delta+1}{\delta}}, & r_0 \leq r \leq \lambda, \\ H - A(\gamma - r)^{\frac{\delta+1}{\delta}}, & \lambda \leq r \leq \gamma, \end{cases}$$

with r_0, λ, k, H, and A to be chosen. Take $\lambda = r_0 + \widetilde{C}\varepsilon(b-a)$ with $\widetilde{C}$ to be chosen and $\gamma - \lambda = \mu_0\gamma$. Set $v(\lambda) = \varepsilon b$. Then,

$$k = \frac{\varepsilon(b-a)}{(\lambda - r_0)^{\frac{\delta+1}{\delta}}} = \frac{1}{\widetilde{C}^{\frac{\delta+1}{\delta}} \left(\varepsilon(b-a)\right)^{1/\delta}}.$$

Since $|\nabla v| \neq 0$, we have

$$\mathcal{L}v = \frac{g(|\nabla v|)}{|\nabla v|} \left[\Delta v + \sum_{i,j} \left(\frac{g'(|\nabla v|)|\nabla v|}{g(|\nabla v|)} - 1 \right) \frac{v_{x_i}}{|\nabla v|} \frac{v_{x_j}}{|\nabla v|} v_{x_i x_j} \right].$$

Thus, in $\lambda \leq r \leq \gamma$, since $|\nabla v| = A\frac{\delta+1}{\delta}(\gamma - r)^{1/\delta}$, we have

$$\begin{aligned} \mathcal{L}v &= g(|\nabla v|) \left[\frac{N-1}{r} - \frac{g'(|\nabla v|)|\nabla v|}{g(|\nabla v|)} \frac{1}{\delta(\gamma - r)} \right] \\ &\leq g(|\nabla v|) \left[\frac{N-1}{r} - \frac{1}{\gamma - r} \right] \leq g(|\nabla v|) \left[\frac{N-1}{\lambda} - \frac{1}{\gamma - \lambda} \right] \\ &= g(|\nabla v|) \left[\frac{(N-1)}{\gamma(1-\mu_0)} - \frac{1}{\mu_0 \gamma} \right]. \end{aligned}$$

Then, if μ_0 is sufficiently small, we have $\mathcal{L}v \leq 0$ in $\lambda \leq r \leq \gamma$.

In $r_0 \leq r \leq \lambda$ we have

$$\mathcal{L}v = g(|\nabla v|) \left[\frac{N-1}{r} + \frac{g'(|\nabla v|)|\nabla v|}{g(|\nabla v|)} \frac{1}{\delta(r - r_0)} \right] \leq g(|\nabla v|) \left[\frac{N-1}{r} + \frac{g_0}{\delta(r - r_0)} \right],$$

and then since $|\nabla v| = k\frac{\delta+1}{\delta}(r - r_0)^{1/\delta}$,

$$\mathcal{L}v \leq \frac{g\left(k\frac{\delta+1}{\delta}(r - r_0)^{1/\delta}\right)}{r - r_0} L,$$

with $L = L(g_0, \delta, N)$.

Since $k(r - r_0)^{1/\delta} \leq 1/\widetilde{C}$ if $r \leq \lambda$, we have

$$g\left(k\frac{\delta+1}{\delta}(r - r_0)^{1/\delta}\right) \leq R\left(k\frac{\delta+1}{\delta}(r - r_0)^{1/\delta}\right)^{\delta},$$

with $R = \widetilde{C}^\delta g(\frac{1}{\widetilde{C}})$. Thus,

$$\mathcal{L}v \le L \frac{R\left(k\frac{\delta+1}{\delta}(r-r_0)^{1/\delta}\right)^\delta}{r-r_0} = L\,R\left(\frac{\delta+1}{\delta}\right)^\delta k^\delta. \tag{7.2}$$

Let $\kappa = \min_{[a,b]}\beta$ and $\widetilde{C}$ large enough so that

$$\frac{L\,g\left(\frac{1}{\widetilde{C}}\right)\left(\frac{\delta+1}{\delta}\right)^\delta}{\widetilde{C}(b-a)} \le \kappa.$$

Since

$$k^\delta = \frac{1}{\widetilde{C}^{\delta+1}\varepsilon(b-a)},$$

we have that in $r_0 \le r \le \lambda$, $a\varepsilon \le v \le b\varepsilon$ and

$$\mathcal{L}v \le \frac{\kappa}{\varepsilon} \le \beta_\varepsilon(v).$$

So, with this election of $\lambda, k,$ and r_0 we have that $\mathcal{L}v \le \beta_\varepsilon(v)$ in B_γ.
On the other hand, by the continuity of v' we have that

$$k\frac{\delta+1}{\delta}(\lambda-r_0)^{1/\delta} = A\frac{\delta+1}{\delta}(\gamma-\lambda)^{1/\delta}.$$

Thus, $k(\lambda-r_0)^{1/\delta} = A(\gamma-\lambda)^{1/\delta}$ so that

$$A(\mu_0\gamma)^{1/\delta} = \frac{\left(\widetilde{C}\varepsilon(b-a)\right)^{1/\delta}}{\widetilde{C}^{\frac{\delta+1}{\delta}}(\varepsilon(b-a))^{1/\delta}} = \widetilde{C}^{-1}.$$

On the other hand, by the continuity of v, since $v(\lambda) = \varepsilon b$,

$$v(\gamma) = H = \varepsilon b + A(\gamma-\lambda)^{\frac{\delta+1}{\delta}} \ge A(\mu_0\gamma)^{\frac{\delta+1}{\delta}} = \widetilde{C}^{-1}\mu_0\gamma = D\gamma,$$

with $D = D(g_0, \delta, \kappa, a, b, N)$. We have the desired result. ∎

Then, by Theorems 6.1 and 7.1, we have the following theorem.

THEOREM 7.2. *Assume that g satisfies conditions* (1.3) *and* (1.4). *Let u^{ε_j} be uniformly bounded minimal solutions to (P_{ε_j}) in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly in compact subsets of Ω as $\varepsilon_j \to 0$. Then, $\Omega \cap \partial_{red}\{u > 0\} \in C^{1,\alpha}$.*

7.2. Example 2. We consider solutions of (P_ε) that are local minimizers of the functional

$$J_\varepsilon(v) = \int_\Omega [G(|\nabla v|) + B_\varepsilon(v)]\,dx \tag{7.3}$$

where $B'_\varepsilon(s) = \beta_\varepsilon(s)$. That is, for any $\Omega' \subset\subset \Omega$, u^ε minimizes

$$\int_{\Omega'} [G(|\nabla v|) + B_\varepsilon(v)]\,dx$$

in $u^\varepsilon + W^{1,G}(\Omega')$.

By Theorem 7.1, in order to prove the nondegeneracy we need only to prove the linear growth away from $\partial\{u^\varepsilon > \varepsilon\}$. The proof follows the lines of Corollary 1.7 in [9].

LEMMA 7.3. *Given $c_1 > 1$ there exists a constant C such that if u^ε is a local minimizer of J_ε in B_1 and $u^\varepsilon(x_0) > c_1\varepsilon$, $x_0 \in B_{1/4}$, then*

$$u^\varepsilon(x_0) \geq C \operatorname{dist}(x_0, \{u^\varepsilon \leq \varepsilon\})$$

if $\operatorname{dist}(x_0, \{u^\varepsilon \leq \varepsilon\}) \leq 1/4$.

Proof. The proof follows as in Theorem 1.6 in [9]. □

Therefore, we have that minimizers satisfy the uniform nondegeneracy condition.

Now, we want to prove that for the limiting function we have that almost every point of the free boundary belongs to the reduced free boundary. To this end, we will prove that the limiting function is a minimizer of the problem treated in [21]. We will follow the steps of Theorem 1.16 in [9]. We will give only the details when the proof parts from the one in [9].

First, we want to estimate the measure of the level sets $\partial\Omega_\lambda$ where $\Omega_\lambda = \{u^\varepsilon > \lambda\}$. Without loss of generality we may assume that $B_1 \subset\subset \Omega$.

For a given set $\mathcal{D}$ we denote by $\mathcal{N}_\delta(\mathcal{D})$ the set of points x such that $\operatorname{dist}(x, \mathcal{D}) < \delta$.

THEOREM 7.3. *Given $c_1 > 1$ there exist $c_2, c_3 > 0$ such that if $\lambda \geq c_1\varepsilon$ and $1/4 \geq \delta \geq c_2\lambda$, then, for $R < 1/4$, we have*

$$|\mathcal{N}_\delta(\partial\Omega_\lambda) \cap B_R| \leq c_3\delta R^{N-1}.$$

In order to prove this theorem, we need two lemmas.

LEMMA 7.4. *If $\lambda > \varepsilon$ and $R \leq 3/4$, then*

$$\int_{\{\lambda < u^\varepsilon < \delta\}\cap B_R} G(|\nabla u^\varepsilon|)\, dx \leq c\delta R^{N-1}.$$

Proof. First, let us prove that for $w \in W^{1,G}(B_R)$ such that $\operatorname{supp} w \subset \{u^\varepsilon \geq \lambda\}$ with $\lambda > \varepsilon$, we have

$$\int_{B_R} F(|\nabla u^\varepsilon|)\nabla u^\varepsilon \nabla w\, dx = \int_{\partial B_R} w\, F(|\nabla u^\varepsilon|)\, \frac{\partial u^\varepsilon}{\partial \nu}\, d\mathcal{H}^{N-1}. \tag{7.4}$$

We follow the ideas in the proof of Lemma 4.2. That is, we suppose first that $F(t) \geq c$ and then we use an approximation argument as in that lemma.

If $F(t) \geq c$, then, by the estimates of [14], we have that the solutions are in $W^{2,2}(\Omega)$, so (7.4) follows by integrating by parts and using the fact that $\mathcal{L}u^\varepsilon = 0$ in $\{u^\varepsilon > \varepsilon\}$. Finally, we use the approximation argument of Lemma 4.2 and the result follows.

Now, let $w = \min\{(u^\varepsilon - \lambda)^+, \delta - \lambda\}$. Then, $w \in W^{1,G}(B_R)$ $\operatorname{supp} w \subset \{u^\varepsilon \geq \lambda\}$ so that by (7.4) we have

$$\int_{\{\lambda < u^\varepsilon < \delta\}\cap B_R} G(|\nabla u^\varepsilon|)\, dx \leq C \int_{\partial B_R} w\, F(|\nabla u^\varepsilon|)\, \frac{\partial u^\varepsilon}{\partial \nu}\, d\mathcal{H}^{N-1} \leq C\delta R^{N-1}$$

and the result follows. □

LEMMA 7.5. *Given $c_1 > 1$ there exist $C_1, C_2, c_2 > 0$ such that if $\lambda \geq c_1\varepsilon$ and $1/8 > \delta \geq c_2\lambda$, we have, for $R < 1/4$,*

$$|\mathcal{N}_\delta(\partial\Omega_\lambda) \cap B_R| \leq C_2 \int_{\{\lambda < u^\varepsilon < C_1\delta\}\cap B_{R+\delta}} G(|\nabla u^\varepsilon|)\, dx.$$

Proof. First, we cover $\mathcal{N}_\delta(\partial\Omega_\lambda) \cap B_R$ with balls $B_j = B_\delta(x_j)$ with centers $x_j \in \partial\Omega_\lambda \cap B_R$ which overlap at most by n_0 (with $n_0 = n_0(N)$).

We claim that in each of these balls there exist two subballs B_j^1 and B_j^2 with radii $r_j = C\,\delta$ with C to be fixed below such that if $v = (u^\varepsilon - \lambda)^+$, then

$$v \geq \frac{c_0}{8}\delta \quad \text{in } B_j^1, \qquad v \leq \frac{c_0}{16}\delta \quad \text{in } B_j^2,$$

where c_0 is the constant of nondegeneracy for balls centered in $B_{1/4}$ with radii at most $1/8$.

In fact, take $B_j^2 = B_{r_j}(x_j)$ with $r_j = \frac{c_0}{16L}\delta$ (here $\|\nabla u^\varepsilon\|_{L^\infty(B_{3/4})} \leq L$). Observe that since $u^\varepsilon(x_j) = \lambda$, then $v(x) \leq Lr_j = \frac{c_0}{16}\delta$ if $x \in B_j^2$.

Now let $y_j \in \overline{B_{\delta/4}(x_j)}$ such that

$$u^\varepsilon(y_j) = \sup_{B_{\delta/4}(x_j)} u^\varepsilon \geq c_0\frac{\delta}{4}.$$

Let $B_j^1 = B_{r_j}(y_j)$. Then if $x \in B_j^1$,

$$u^\varepsilon(x) \geq u^\varepsilon(y_j) - Lr_j \geq c_0\frac{\delta}{4} - Lr_j.$$

Thus,

$$u^\varepsilon(x) - \lambda \geq c_0\frac{\delta}{4} - Lr_j - \lambda \geq \left(\frac{c_0}{4} - \frac{c_0}{16} - c_2^{-1}\right)\delta \geq \frac{c_0}{8}\delta$$

if $c_2^{-1} \leq \frac{c_0}{16}$.

Let $m_j = \fint_{B_j} v$. We claim that in one of the balls B_j^1 and B_j^2 we must have $|v - m_j| \geq c\delta$ for a certain constant $c > 0$.

Suppose by contradiction that there exist $x_1 \in B_j^1$ and $x_2 \in B_j^2$ with

$$|v(x_1) - m_j| < c\delta, \qquad |v(x_2) - m_j| < c\delta.$$

Then,

$$\frac{c_0}{8}\delta - \frac{c_0}{16}\delta \leq v(x_1) - v(x_2) < 2c\delta$$

which is a contradiction if we take $c_0/16 \geq 2c$.

Therefore, if k is such that $|B_j^1| = |B_j^2| = k|B_j|$, we have by the convexity of G and Poincaré inequality that

$$\begin{aligned}\frac{1}{|B_j|}\int_{B_j} G(|\nabla v|)\,dx &\geq G\left(\frac{1}{|B_j|}\int_{B_j} |\nabla v|\,dx\right)\\ &\geq G\left(\frac{C}{|B_j|}\int_{B_j} \frac{|v - m_j|}{\delta}\,dx\right) \geq G\left(\frac{C}{|B_j|}k|B_j|c\right).\end{aligned}$$

This implies that

$$\int_{B_j} G(|\nabla v|)\,dx \geq C|B_j|.$$

As

$$B_R \cap \mathcal{N}_\delta(\partial\Omega_\lambda) \subset \bigcup B_j$$

we have

$$|B_R \cap \mathcal{N}_\delta(\partial\Omega_\lambda)| \le \sum |B_j| \le \frac{1}{C} \sum \int_{B_j} G(|\nabla v|)\, dx$$
$$\le \frac{n_0}{C} \int_{\bigcup B_j} G(|\nabla v|)\, dx = \frac{n_0}{C} \int_{\bigcup B_j \cap \{u^\varepsilon > \lambda\}} G(|\nabla u^\varepsilon|)\, dx.$$

On the other hand, if $x \in B_j$, then $u^\varepsilon(x) < C_1\delta$, where $C_1 = c_2^{-1} + L$. Then, as $\bigcup B_j \subset B_{R+\delta}$, we have

$$|B_R \cap \mathcal{N}_\delta(\partial\Omega_\lambda)| \le \frac{n_0}{C} \int_{\{\lambda < u^\varepsilon < C_1\delta\} \cap B_{R+\delta}} G(|\nabla u^\varepsilon|)\, dx. \qquad \square$$

Proof of Theorem 7.3. Using Lemmas 7.4 and 7.5 we have

$$|B_{R-\delta} \cap \mathcal{N}_\delta(\partial\Omega_\lambda)| \le C_0 \int_{\{\lambda < u^\varepsilon < C_1\delta\} \cap B_R} G(|\nabla u^\varepsilon|)\, dx \le C_0 c C_1 \delta R^{N-1}.$$

As $|B_R \setminus B_{R-\delta}| \le C\delta R^{N-1}$ we obtain the conclusion of Theorem 7.3. $\square$

Now, we can pass to the limit as $\varepsilon \to 0$. There exists a subsequence u^{ε_k} converging, as $\varepsilon_k \to 0$, to a function $u_0 \in W^{1,G}(\Omega)$ strongly in $L^{\delta+1}(\Omega)$, weakly in $W^{1,G}(\Omega)$, and uniformly in every compact subset of Ω.

Let $\Omega' \subset\subset \Omega$, $x_0 \in \Omega' \cap \partial\{u_0 > 0\}$, and $\rho_0 \le 1/2\mathrm{dist}(\Omega', \partial\Omega)$. Then, by using the previous results we can prove as in Theorem 1.16 in [9] that u_0 is a local minimizer of

$$J_0(v) := \int_{B_\rho(x_0)} [G(|\nabla v|) + M\chi_{\{v>0\}}]\, dx.$$

Finally, we can apply the results of [21] and conclude that $\mathcal{H}^{N-1}$-almost every point of the free boundary belongs to the reduced free boundary. Moreover, by applying the regularity results for minimizers of J_0 from [21] (see [20] for the regularity of the whole free boundary in dimension 2) we have the following theorem.

THEOREM 7.4. *Suppose that g satisfies* (1.3). *Let u^{ε_j} be a local minimizer of* (7.3) *in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly in compact subsets of Ω and $\varepsilon_j \to 0$. Then, $\partial_{red}\{u > 0\}$ is a $C^{1,\alpha}$ surface and $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{red}\{u > 0\}) = 0$. In dimension* 2, *if there exist t_0 and k such that $g(t) \le k\,t$ for $t \le t_0$, there holds that the whole free boundary is a regular surface.*

Appendix A. Properties of *G*.

The following result is proved in [21].

LEMMA A.1. *The function g satisfies the following properties:*

(g1) $\min\{s^\delta, s^{g_0}\}g(t) \le g(st) \le \max\{s^\delta, s^{g_0}\}g(t)$,

(g2) *G is convex and C^2,*

(g3) $\frac{tg(t)}{1+g_0} \le G(t) \le tg(t) \quad \forall\ t \ge 0$.

Remark A.1. By (g1) and (g3) we have a similar inequality for G.

(G1) $\min\{s^{\delta+1}, s^{g_0+1}\}\frac{G(t)}{1+g_0} \le G(st) \le (1+g_0)\max\{s^{\delta+1}, s^{g_0+1}\}G(t)$,

and then using the convexity of G and this last inequality we have

(G2) $G(a+b) \le 2^{g_0}(1+g_0)(G(a)+G(b))\ \forall\ a, b > 0$.

As g is strictly increasing we can define g^{-1}. Now, we prove that g^{-1} satisfies a condition similar to (1.3). That is, see the following lemma.

LEMMA A.2. *The function g^{-1} satisfies the inequalities*

$$\frac{1}{g_0} \le \frac{t\big(g^{-1}\big)'(t)}{g^{-1}(t)} \le \frac{1}{\delta} \qquad \forall t > 0. \tag{A.1}$$

Moreover, g^{-1} satisfies

$$(\widetilde{g}1) \qquad \min\left\{s^{1/\delta}, s^{1/g_0}\right\} g^{-1}(t) \le g^{-1}(st) \le \max\left\{s^{1/\delta}, s^{1/g_0}\right\} g^{-1}(t),$$

and if $\widetilde{G}$ is such that $\widetilde{G}'(t) = g^{-1}(t)$, then

$$(\widetilde{g}2) \qquad \frac{\delta t g^{-1}(t)}{1+\delta} \le \widetilde{G}(t) \le t g^{-1}(t) \quad \forall\, t \ge 0,$$

$(\widetilde{G}1)$
$$\frac{(1+\delta)}{\delta} \min\left\{s^{1+1/\delta}, s^{1+1/g_0}\right\} \widetilde{G}(t) \le \widetilde{G}(st) \le \frac{\delta}{1+\delta} \max\left\{s^{1+1/\delta}, s^{1+1/g_0}\right\} \widetilde{G}(t),$$

$$(\widetilde{g}3) \qquad ab \le \varepsilon G(a) + C(\varepsilon)\widetilde{G}(b) \quad \forall\, a, b > 0 \text{ and } \varepsilon > 0 \text{ small,}$$

$$(\widetilde{g}4) \qquad \widetilde{G}(g(t)) \le g_0 G(t).$$

THEOREM A.1. *$L^{\widetilde{G}}(\Omega)$ is the dual of $L^G(\Omega)$. Moreover, $L^G(\Omega)$ and $W^{1,G}(\Omega)$ are reflexive.*

Appendix B. A result on $\mathcal{L}$-solutions with linear growth. In this section we will state some properties of $\mathcal{L}$-subsolutions.

LEMMA B.1. *Let $0 < r \le 1$. Let $u \in C(\overline{B_r^+})$ be such that $\mathcal{L}u \ge 0$ in B_r^+ and $0 \le u \le \alpha x_N$ in B_r^+, $u \le \delta_0 \alpha x_N$ on $\partial B_r^+ \cap B_{r_0}(\bar{x})$ with $\bar{x} \in \partial B_r^+$, $\bar{x}_N > 0$, and $0 < \delta_0 < 1$.*

Then, there exist $0 < \gamma < 1$ and $0 < \eta \le 1$, depending only on r and N such that

$$u(x) \le \gamma \alpha x_N \quad \text{in } B_{\eta r}^+.$$

Proof. By the invariance of the equation $\mathcal{L}u \ge 0$ under the rescaling $\bar{u}(x) = u(rx)/r$ we can suppose that $r = 1$.

Let ψ^α be a $\mathcal{L}_\alpha$-solution in B_1^+, with smooth boundary data, such that

$$\begin{cases} \psi^\alpha = x_N & \text{on } \partial B_1^+ \setminus B_{r_0}(\bar{x}), \\ \delta_0 x_N \le \psi^\alpha \le x_N & \text{on } \partial B_1^+ \cap B_{r_0}(\bar{x}), \\ \psi^\alpha = \delta_0 x_N & \text{on } \partial B_1^+ \cap B_{r_0/2}(\bar{x}), \end{cases}$$

where $\mathcal{L}_\alpha v = \operatorname{div}(\frac{g_\alpha(|\nabla v|)}{|\nabla v|}\nabla v)$ and $g_\alpha(t) = g(\alpha t)$.

Therefore, $\mathcal{L}(\alpha\psi^\alpha) = 0$ and, by the comparison principle (see [21]), $u \le \alpha\psi^\alpha$ in B_1^+.

If we see that there exist $0 < \gamma < 1$ and $\eta > 0$, independent of α, such that $\psi^\alpha \le \gamma x_N$ in B_η^+, the result follows.

First, observe that

$$\delta \leq \frac{g'_\alpha(t)t}{g_\alpha(t)} \leq g_0. \tag{B.1}$$

Then, by the results in [19], for $0 < \rho_0 < 1$ and some $0 < \beta < 1$,

(B.2)
$\psi^\alpha \in C^{1,\beta}(\overline{B^+_{\rho_0}}) \cap C^\beta(\overline{B^+_1})$.
The $C^{1,\beta}(\overline{B^+_{\rho_0}})$ and $C^\beta(\overline{B^+_1})$ norms are bounded by a constant independent of α.
The constant of the Harnack inequality is independent of α.

If $|\nabla\psi^\alpha| \geq \mu > 0$ in some open set U, we have that $\psi^\alpha \in W^{2,2}(U)$ and ψ^α is a solution of the linear uniformly elliptic equation

$$\mathcal{T}_\alpha \psi = \sum_{i,j=1}^N b^\alpha_{ij}\psi_{x_ix_j} = 0 \quad \text{in } U, \tag{B.3}$$

where

$$b^\alpha_{ij} = \delta_{ij} + \left(\frac{g'_\alpha(|\nabla\psi^\alpha|)|\nabla\psi^\alpha|}{g_\alpha(|\nabla\psi^\alpha|)} - 1\right)\frac{D_i\psi^\alpha D_j\psi^\alpha}{|\nabla\psi^\alpha|^2},$$

and the constant of ellipticity depends only on g_0 and δ.

Now, we divide the proof into several steps.

Step 1. Let $w^\alpha = x_N - \psi^\alpha$. Then, $w^\alpha \in C^{1,\beta}(\overline{B^+_{\rho_0}}) \cap C^\beta(\overline{B^+_1})$ and it is a solution of $\mathcal{T}_\alpha w^\alpha = 0$ in any open set U where $|\nabla\psi^\alpha| \geq \mu > 0$.

On the other hand, as $\psi^\alpha \leq x_N$ on ∂B^+_1 and both functions are $\mathcal{L}_\alpha$-solutions we have, by comparison, that $\psi^\alpha \leq x_N$ in B^+_1. Therefore, $w^\alpha \geq 0$ in B^+_1.

Step 2. Let us prove that there exist ρ, $\bar{c}$, and α_0 such that $|\nabla\psi^\alpha| \geq \bar{c}$ in B^+_ρ if $0 < \alpha \leq \alpha_0$.

First, let us see that there exist $c > 0$ and α_1 such that

$$\psi^\alpha(1/2e_N) \geq c \quad \text{if } 0 < \alpha \leq \alpha_1. \tag{B.4}$$

If not, there exists a sequence $\alpha_k \to 0$ such that $\psi^{\alpha_k}(1/2e_N) \to 0$. Since the constant in the Harnack inequality is independent of α (see (B.2)), we have that $\psi^{\alpha_k} \to 0$ uniformly in compact sets of B^+_1.

On the other hand, using the fact that ψ^α are uniformly bounded in $C^\beta(\overline{B^+_1})$, we have that there exists $\psi \in C^1(B^+_1) \cap C^\beta(\overline{B^+_1})$ such that, for a subsequence, $\psi^{\alpha_k} \to \psi$ uniformly in $\overline{B^+_1}$.

Therefore, $\psi = 0$ in $\overline{B^+_1}$. But we have that $\psi = \delta_0 x_N$ on $B_{r_0/2}(\bar{x}) \cap \partial B^+_1$, which is a contradiction.

Now, let $x_1 \in \{x_N = 0\} \cap B_{1/2}$. Take $x_0 = x_1 + \frac{e_N}{4}$. By (B.2) we have that there exists a constant c_1 independent of α such that $\psi^\alpha(x) \geq c_1\psi^\alpha(1/2e_N)$ for any $x \in \partial B_{1/8}(x_0)$ and, therefore, by (B.4), $\psi^\alpha \geq \tilde{c}$ on $\partial B_{1/8}(x_0)$.

Take $v = \varepsilon(e^{-\lambda|x-x_0|^2} - e^{-\lambda/16})$, and choose λ such that $\mathcal{L}_\alpha v > 0$ in $B_{1/4}(x_0) \setminus B_{1/8}(x_0)$ and ε such that $v = \tilde{c}$ on $\partial B_{1/8}(x_0)$ (observe that by Lemma 2.9 in [21] λ and ε can be chosen independent of α).

Since $\psi^\alpha \geq 0 = v$ on $\partial B_{1/4}(x_0)$ and $\psi^\alpha \geq v$ on $\partial B_{1/8}(x_0)$, we have, by comparison, that $\psi^\alpha \geq v$ in $B_{1/4}(x_0) \setminus B_{1/8}(x_0)$.

On the other hand, $v_{x_N}(x_1) = \varepsilon 2\lambda(x_0 - x_1)_N e^{-\lambda|x_1-x_0|^2} = \frac{\lambda\varepsilon}{2}e^{-\lambda/16} = \bar{c}$, and, therefore, $\psi^\alpha_{x_N}(x_1) \geq \bar{c}$.

As $\nabla\psi^\alpha$ are uniformly Hölder in $\overline{B^+_{3/4}}$, we have that there exists ρ independent of α and x_1 such that $\psi^\alpha_{x_N}(x) \geq \bar{c}$ in $B^+_\rho(x_1)$.

Step 3. Since $|\nabla\psi^\alpha| \geq \bar{c}$ in B^+_ρ, we have that $\mathcal{T}_\alpha w^\alpha = 0$ there.

Suppose that $w^\alpha(1/2\rho e_N) \geq \tilde{c}$, with $\tilde{c}$ independent of α. Then, by Hopf's principle we have that there exists σ_1 depending only on N and the ellipticity of $\mathcal{T}_\alpha$ such that $w^\alpha \geq \sigma_1 x_N$ in $B^+_{\rho/2}$. Then, taking $\gamma = 1 - \sigma_1$ we obtain the desired result.

Step 4. Finally, let us see that the assumption in Step 3 is satisfied. That is, let us see that $w^\alpha(1/2\rho e_N) \geq \tilde{c} > 0$, where $\tilde{c}$ is independent of α.

Suppose by contradiction that, for a subsequence, $w^{\alpha_k}(1/2\rho e_N) \to 0$. We know that in B^+_ρ $\mathcal{T}_\alpha w^\alpha = 0$. Therefore, applying the Harnack inequality we have that $w^{\alpha_k} \to 0$ in B^+_ρ.

On the other hand, since $\psi^\alpha \to \psi$ and $\nabla\psi^\alpha \to \nabla\psi$ uniformly in $\overline{B^+_{\rho_0}}$ for every $0 < \rho_0 < 1$, it holds that $w^{\alpha_k} \to w = x_n - \psi$ in $C^1(\overline{B^+_{\rho_0}})$. Let

$$\mathcal{A} = \{x \in B^+_1 \ / \ w = 0\},$$

and suppose that there exists a point $x_1 \in \partial A \cap B^+_1$. Then, as $w^\alpha \geq 0$ we have that w attains its minimum at this point. Therefore, $\nabla w(x_1) = 0$.

Since $\nabla w^{\alpha_k} \to \nabla w$ uniformly in a neighborhood of x_1, we have that for some $\tau > 0$ independent of α_k, $|\nabla\psi^{\alpha_k}| \geq 1/2$ in $B_\tau(x_1)$. Thus, in this ball, w^{α_k} satisfies $\mathcal{T}_{\alpha_k} w^{\alpha_k} = 0$.

Now, applying the Harnack inequality in $B_\tau(x_1)$ and then passing to the limit we obtain that $w = 0$ in $B_{\tau/2}(x_1)$, which is a contradiction.

Hence, $w = 0$ in $\overline{B^+_1}$. But, on the other hand, we have $w = x_N - \delta_0 x_N > 0$ on $\partial B_1 \cap B_{r_0/2}(\bar{x})$, which is a contradiction. ∎

With Lemma B.1 we can also prove the asymptotic development of $\mathcal{L}$-solutions.

Lemma B.2. *Let u be Lipschitz continuous in $\overline{B^+_1}$, $u \geq 0$ in B^+_1, $\mathcal{L}u = 0$ in $\{u > 0\}$, and $u = 0$ on $\{x_N = 0\}$. Then, in B^+_1, u has the asymptotic development*

$$u(x) = \alpha x_N + o(|x|),$$

with $\alpha \geq 0$.

Proof. Let

$$\alpha_j = \inf\left\{l \ / \ u \leq l x_n \text{ in } B^+_{2^{-j}}\right\}.$$

Let $\alpha = \lim_{j\to\infty} \alpha_j$.

Given $\varepsilon_0 > 0$ there exists j_0 such that for $j \geq j_0$ we have $\alpha_j \leq \alpha + \varepsilon_0$. From here, we have $u(x) \leq (\alpha + \varepsilon_0)x_N$ in $B^+_{2^{-j}}$ so that

$$u(x) \leq \alpha x_N + o(|x|) \text{ in } B^+_1.$$

Since $u \geq 0$, if $\alpha = 0$, the result follows. So, let us assume that $\alpha > 0$.

Suppose that $u(x) \neq \alpha x_N + o(|x|)$. Then there exists $x_k \to 0$ and $\bar{\delta} > 0$ such that

$$u(x_k) \leq \alpha x_{k,N} - \bar{\delta}|x_k|.$$

Let $r_k = |x_k|$ and $u_k(x) = r_k^{-1} u(r_k x)$. Then, there exists u_0 such that, for a subsequence that we still call u_k, $u_k \to u_0$ uniformly in $\overline{B_1^+}$ and

$$u_k(\bar{x}_k) \leq \alpha \bar{x}_{k,N} - \bar{\delta},$$
$$u_k(x) \leq (\alpha + \varepsilon_0) x_N \text{ in } B_1^+,$$

where $\bar{x}_k = \frac{x_k}{r_k}$, and we can assume that $\bar{x}_k \to x_0$.

In fact, $u(x) \leq (\alpha+\varepsilon_0) x_N$ in $B^+_{2^{-j_0}}$ and, therefore, $u_k(x) \leq (\alpha+\varepsilon_0)x_N$ in $B^+_{r_k^{-1}2^{-j_0}}$, and the claim follows if k is big enough so that $r_k^{-1} 2^{-j_0} \geq 1$.

If we take $\bar{\alpha} = \alpha + \varepsilon_0$, we have

$$\begin{cases} \mathcal{L} u_k \geq 0 & \text{in } B_1^+, \\ u_k = 0 & \text{on } \{x_N = 0\}, \\ 0 \leq u_k \leq \bar{\alpha} x_N & \text{on } \partial B_1^+, \\ u_k \leq \delta_0 \bar{\alpha} x_N & \text{on } \partial B_1^+ \cap B_{\bar{r}}(\bar{x}) \end{cases}$$

for some $\bar{x} \in \partial B_1^+$ and $\bar{x}_N > 0$ and some small $\bar{r} > 0$.

In fact, as the u_k's are continuous with uniform modulus of continuity, we have

$$u_k(x_0) \leq \alpha x_{0,N} - \frac{\bar{\delta}}{2} \text{ if } k \geq \bar{k}.$$

Moreover, there exists $r_0 > 0$ such that $u_k(x) \leq \alpha x_N - \frac{\bar{\delta}}{4}$ in $B_{4r_0}(x_0)$. If $x_{0,N} > 0$, we take $\bar{x} = x_0$, and if not, we take $\bar{x} \in \partial B_{3r_0}(x_0) \cap \partial B_1$. Then, $\bar{x}_N > 0$ and

$$u_k(x) \leq \alpha x_N - \frac{\bar{\delta}}{4} \text{ in } B_{r_0}(\bar{x}) \subset\subset \{x_N > 0\}.$$

Moreover, there exists $0 < \delta_0 < 1$ such that $\alpha x_N - \frac{\bar{\delta}}{4} \leq \delta_0 \alpha x_N \leq \delta_0 \bar{\alpha} x_N$ in $B_{r_0}(\bar{x})$, and the claim follows.

Now, by Lemma B.1, there exists $0 < \gamma < 1$ and $\eta > 0$ independent of ε_0 and k such that $u_k(x) \leq \gamma(\alpha + \varepsilon_0) x_N$ in B_η^+. As γ and η are independent of k and ε_0, taking $\varepsilon_0 \to 0$, we have

$$u_k(x) \leq \gamma \alpha x_N \text{ in } B_\eta^+$$

so that

$$u(x) \leq \gamma \alpha x_N \text{ in } B^+_{r_k \eta}.$$

Now, if j is big enough, we have $\gamma\alpha < \alpha_j$ and $2^{-j} \leq r_k \eta$. But this contradicts the definition of α_j. Therefore,

$$u(x) = \alpha x_N + o(|x|)$$

as we wanted to prove. □

Appendix C. Existence of extremal $\mathcal{L}$-solutions. In this section we will prove the existence of extremal solutions. First, we will give the definition of sub- and supersolution of problem (1.1) in a more general sense (for simplicity, we will omit the ε).

In [23] there is a review on this topic.

Definition C.1. *A function $\bar u \in W^{1,G}(\Omega)$ is called a supersolution if $\bar u$ is of the form*

$$\bar u = \min\{\bar u_1, \bar u_2, \dots, \bar u_m\},$$

where $\bar u_1, \bar u_2, \dots, \bar u_m \in W^{1,G}(\Omega)$ and each $\bar u_j$, $1 \le j \le m$, satisfies the condition:

$$(H_{super}) \qquad \begin{cases} \mathcal{L}\bar u_j \le \beta(\bar u_j) & \text{on } \Omega,\\ \bar u_j \ge u_0 & \text{on } \partial\Omega \end{cases}$$

in a weak sense, with $u_0 \in C^\alpha(\bar\Omega) \cap W^{1,G}(\Omega)$.

Subsolutions are defined in the same way as the maxima of a finite number of functions in $W^{1,G}(\Omega)$ satisfying condition (H_{sub}) obtained by reversing the inequalities in (H_{super}).

We will assume in this section the existence of a subsolution $\underline{u} = \max\{\underline{u}_1, \dots, \underline{u}_k\}$ and a supersolution $\bar u = \min\{\bar u_1, \dots, \bar u_m\}$ such that $\underline{u} \le \bar u$. We will also assume that there exists a constant A such that for all $i = 1, \dots, k$ and $j = 1, \dots, m$ we have $|\underline{u}_i| \le A$ and $|\bar u_j| \le A$.

Using the same technique as in Theorem 8 in [15], we will prove the following theorem.

Theorem C.1. 1. *Problem (P_ε) has a least solution u_*—with boundary data greater than or equal to u_0 on $\partial\Omega$—in the order interval $[\underline{u}, \bar u]$, i.e., $\underline{u} \le u_* \le \bar u$, and if u is any solution of (P_ε) with $u \ge u_0$ on $\partial\Omega$ such that $\underline{u} \le u \le \bar u$, then $u_* \le u$. Moreover, $u_* = u_0$ on $\partial\Omega$.*

2. *Problem (P_ε) has a greatest solution u^*—with boundary data less than or equal to u_0 on $\partial\Omega$—in the order interval $[\underline{u}, \bar u]$, i.e., $\underline{u} \le u^* \le \bar u$, and if u is any solution of (P_ε) with $u \le u_0$ on $\partial\Omega$ such that $\underline{u} \le u \le \bar u$, then $u^* \ge u$. Moreover, $u^* = u_0$ on $\partial\Omega$.*

The proof of this theorem is based on the following lemma.

Lemma C.1. *There exists a solution u of (1.1) with $u = u_0$ on $\partial\Omega$ such that $\underline{u} \le u \le \bar u$.*

Proof. We use a construction similar to the one in [15]. Here, we have to make a modification, since we are dealing with the space $W^{1,G}(\Omega)$. We define $b : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$b(x,t) = \begin{cases} g\big(10A - \bar u(x)\big) & \text{if} \quad t \ge 10A,\\ g\big(t - \bar u(x)\big) & \text{if} \quad 10A \ge t > \bar u(x),\\ 0 & \text{if} \quad \underline{u}(x) \le t \le \bar u(x),\\ -g\big(\underline{u}(x) - t\big) & \text{if} \quad -10A \le t < \underline{u}(x),\\ -g\big(\underline{u}(x) + 10A\big) & \text{if} \quad t \le -10A. \end{cases}$$

Since $|\underline{u}|, |\bar u| \le A$, there holds that $|b(x,t)| \le g(11\,A)$ for every $x \in \Omega$ and $t \in \mathbb{R}$. Let $1 \le i \le k$ and $1 \le j \le m$, and define, for each $u \in W^{1,G}(\Omega)$,

$$T_{ij}(u)(x) = \begin{cases} \underline{u}_i(x) & \text{if} \quad u(x) < \underline{u}_i(x),\\ u(x) & \text{if} \quad \underline{u}_i(x) \le u(x) \le \bar u_j(x),\\ \bar u_j(x) & \text{if} \quad u(x) > \bar u_j(x) \end{cases}$$

and

$$T(u)(x) = \begin{cases} \underline{u}(x) & \text{if} \quad u(x) < \underline{u}(x), \\ u(x) & \text{if} \quad \underline{u}(x) \leq u(x) \leq \bar{u}(x), \\ \bar{u}(x) & \text{if} \quad u(x) > \bar{u}(x) \end{cases}$$

for a.e. $x \in \Omega$.

Next, we consider the following equation:

$$\text{(C.1)} \qquad -\mathcal{L}(u) + B(u) + C(u) = 0 \text{ weakly in } \Omega, \quad u = u_0 \text{ on } \partial\Omega,$$

where

$$B(u)(x) := b(x, u(x))$$

and

$$C(u) := \beta(T(u)) - \sum_{\substack{1\leq i\leq k \\ 1\leq j\leq m}} |\beta(T_{ij}(u)) - \beta(T(u))|.$$

First, we want to show the existence of the solution of (C.1). We will use a fixed point argument.

For each $v \in C(\bar{\Omega})$ take u as the weak solution of $\mathcal{L}u = \gamma(x,v)$, where $\gamma(x,v) = [b(x,v) + C(v)]$.

Observe that since $|b(x,v)| \leq g(11\,A)$ and since β is bounded, we have $|\gamma(x,v)| \leq C_A$.

Moreover, since $u_0 \in C^{\alpha}(\overline{\Omega})$, there exist $0 < \tilde{\mu} < 1$ and a constant K_A such that $\|u\|_{C^{\tilde{\mu}}(\overline{\Omega})} \leq K_A$.

Let $0 < \mu < \tilde{\mu}$, and consider the operator $M : B_{K_A} \to B_{K_A}$, where $B_{K_A} = \{v \in C^{\mu}(\overline{\Omega}) : \|v\|_{C^{\mu}(\overline{\Omega})} \leq K_A\}$, such that $M(v) = u$, where u is the solution of $\mathcal{L}u = \gamma(x,v)$ with $u = u_0$ on $\partial\Omega$. We want to show that this operator has a fixed point, that is, a solution of $\mathcal{L}u = \gamma(x,u)$. By Schauder's fixed point theorem, this holds if the operator M is compact.

Let us see that this is the case. In fact, let $\|v_n\|_{C^{\mu}(\overline{\Omega})} \leq K_A$. Then, by the results of Lieberman [19], the corresponding solutions u_n satisfy $\|u_n\|_{C^{\tilde{\mu}}(\overline{\Omega})} \leq K_A$ and $\|\nabla u_n\|_{C^{\alpha}(\bar{\Omega}')} \leq \bar{C}_{A,\Omega'}$ for every $\Omega' \subset\subset \Omega$. Therefore, for a subsequence, $u_{n_k} \to u$ in $C^{\mu}(\overline{\Omega})$ and $\nabla u_{n_k} \to \nabla u$ uniformly on compact sets of Ω. Moreover, without loss of generality we may assume that $v_{n_k} \to v$ uniformly in Ω. Then, passing to the limit, we have that $u \in B_{K_A}$ is the weak solution of $\mathcal{L}u = \gamma(x,v)$, $u = u_0$ on $\partial\Omega$, so $u = Mv$. Thus, M is compact.

We will show that any solution u of (C.1) must satisfy

$$\text{(C.2)} \qquad \underline{u}_q \leq u \leq \bar{u}_r \quad \forall q \in \{1,\ldots,k\}, \quad r \in \{1,\ldots,m\}.$$

(C.2) implies that $\underline{u} = \max\{\underline{u}_q : 1 \leq q \leq k\} \leq u \leq \min\{\bar{u}_r : 1 \leq r \leq m\} = \bar{u}$. Then, by the definition of b we have that $b(x,u(x)) = 0$ a.e. in Ω, i.e., $B(u) = 0$. Also $T_{ij}(u) = T(u) = u$ $\forall i,j$. Thus, $C(u) = \beta(u)$. Therefore, u is a solution of (1.1) and $\underline{u} \leq u \leq \bar{u}$.

So, let us prove that $\underline{u}_q \leq u$ (similarly we can show that $u \leq \bar{u}_r$).

Since $\underline{u}_q$ satisfies (H_{sub}), we have that

$$\mathcal{L}(\underline{u}_q) \geq \beta(\underline{u}_q).$$

Subtracting (C.1), we obtain for $\phi \in W^{1,G}$, $\phi \geq 0$,

$$\begin{aligned} & -\langle \mathcal{L}(\underline{u}_q), \phi\rangle + \langle \mathcal{L}(u), \phi\rangle \\ & \quad \leq -\int_\Omega \Bigg[\beta(\underline{u}_q) - \beta(T(u)) + \sum_{\substack{1\leq i\leq k\\ 1\leq j\leq m}} |\beta(T_{ij}(u)) - \beta(T(u))|\Bigg]\phi\,dx \\ & \qquad + \int_\Omega b(x,u)\phi\,dx. \end{aligned} \tag{C.3}$$

Taking $\phi = (\underline{u}_q - u)^+$ as a test function in (C.3) we get, by the monotonicity of the operator $-\mathcal{L}$,

$$\begin{aligned} \langle -\mathcal{L}(\underline{u}_q) + \mathcal{L}(u), \phi\rangle &= \int_\Omega \left[g(|\nabla \underline{u}_q|)\frac{\nabla \underline{u}_q}{|\nabla \underline{u}_q|} - g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right]\nabla\phi\,dx \\ &= \int_{\{\underline{u}_q > u\}} \left[g(|\nabla \underline{u}_q|)\frac{\nabla \underline{u}_q}{|\nabla \underline{u}_q|} - g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right]\left[\nabla \underline{u}_q - \nabla u\right]\,dx \geq 0. \end{aligned} \tag{C.4}$$

On the other hand,

$$\begin{aligned} &\int_\Omega \Bigg[\beta(\underline{u}_q) + \beta(T(u)) + \sum_{\substack{1\leq i\leq k\\ 1\leq j\leq m}} |\beta(T_{ij}(u)) - \beta(T(u))|\Bigg](\underline{u}_q - u)^+\,dx \\ &= \int_{\{\underline{u}_q > u\}} \Bigg[\beta(\underline{u}_q) - \beta(T(u)) + \sum_{\substack{1\leq i\leq k\\ 1\leq j\leq m}} |\beta(T_{ij}(u)) - \beta(T(u))|\Bigg](\underline{u}_q - u)\,dx \geq 0. \end{aligned} \tag{C.5}$$

In fact, for $\underline{u}_q(x) > u(x)$, we have $\underline{u}(x) \geq u(x)$ and

$$T_{qj}(u)(x) = \underline{u}_q(x), \quad T(u)(x) = \underline{u}(x).$$

Hence,

$$\begin{aligned} &\beta(\underline{u}_q) - \beta(T(u)) + \sum_{\substack{1\leq i\leq k\\ 1\leq j\leq m}} \big|\beta(T_{ij}(u)) - \beta(T(u))\big| \\ &\quad \geq \beta(\underline{u}_q) - \beta(T(u)) + \big|\beta(T_{qj}(u)) - \beta(T(u))\big| = \beta(\underline{u}_q) - \beta(u) + |\beta(\underline{u}_q) - \beta(u)| \geq 0 \end{aligned}$$

and, thus, (C.5) holds.

Using (C.3), (C.4), and (C.5) and observing that $\underline{u}_q \leq \underline{u}$ we obtain

$$\begin{aligned} 0 \leq \int_\Omega b(\cdot,u)(\underline{u}_q - u)^+\,dx &= \int_{\{\underline{u}_q(x) > u(x)\}} b(\cdot,u)(\underline{u}_q - u)\,dx \\ &= \int_{\{\underline{u}_q(x) > u(x) > -10A\}} -g(\underline{u} - u)(\underline{u}_q - u)\,dx \\ &\quad + \int_{\{\underline{u}_q(x) > u(x), -10A \geq u\}} -g(\underline{u} + 10A)(\underline{u}_q - u)\,dx \leq 0, \end{aligned}$$

from where it follows that

$$0 = \int_{\{\underline{u}_q(x)>u(x)>-10A\}} g(\underline{u}-u)(\underline{u}_q - u)\,dx$$
$$\geq \int_{\{\underline{u}_q(x)>u(x)>-10A\}} g(\underline{u}_q-u)(\underline{u}_q - u)\,dx = \int_{\{u>-10A\}} g((\underline{u}_q - u)^+)(\underline{u}_q - u)^+\,dx.$$

This implies $(\underline{u}_q - u)^+ = 0$ a.e. in $\{u > -10A\}$, i.e., $\underline{u}_q \leq u$ in $\{u > -10A\}$.

On the other hand,

$$0 = \int_{\{\underline{u}_q(x)>u(x),u\leq-10A\}} g(\underline{u}+10A)(\underline{u}_q - u)\,dx$$
$$\geq \int_{\{\underline{u}_q(x)>u(x),u\leq-10A\}} g(9A)9A\,dx = g(9A)9A\left|\{\underline{u}_q(x) > u(x), u \leq -10A\}\right|.$$

This implies $|\{\underline{u}_q(x) > u(x), u \leq -10A\}| = 0$.

Then, $\underline{u}_q \leq u$ a.e. in $\{u \leq -10A\}$. (Then, $\{u \leq -10A\} = \emptyset$.)

In any case, $\underline{u}_q \leq u$. The result follows. ∎

Proof of Theorem C.1. To complete the proof we follow the lines of Theorem 8 in [15].

To prove (1), let $\tilde{C}$ be any bound of u_0 in $W^{1,G}(\Omega)$. Let $\bar{C}$ be the a priori bound in $W^{1,G}(\Omega)$ of a solution to (1.1) with boundary value bounded by $\tilde{C}$ in $W^{1,G}(\Omega)$. Define

$$T_{\bar{C}} = \big\{u \in W^{1,G}(\Omega) : \|u\|_{W^{1,G}} \leq \bar{C} \text{ and}$$
$$u \text{ is a solution of (1.1) such that } \underline{u} \leq u \leq \bar{u} \text{ and } u \geq u_0 \text{ on } \partial\Omega\big\}.$$

Then, T is not empty by the previous lemma. We have to prove that T (with the order $\leq$) has a least element. The proof is based on Zorn's lemma and a continuity argument.

Since in our case $|\bar{u}|, |\underline{u}| \leq A$, we can take $v = A - u$, $\bar{\beta}(t) = -\frac{1}{\varepsilon}\beta(\frac{A-t}{\varepsilon})$, $\bar{v} = A - \bar{u}$, $\underline{v} = A - \underline{u}$, and

$$\mathcal{L}v = \bar{\beta}(v). \tag{C.6}$$

We consider the set (with a constant C related to $\bar{C}$, A, and Ω)

$$S_C = \big\{v \in W^{1,G}(\Omega) : \|v\|_{W^{1,G}} \leq C \text{ and } v \text{ is a solution of (C.6) such that } \underline{v} \leq v \leq \bar{v}$$
$$\text{and } v \leq v_0 \text{ on } \partial\Omega\big\}$$

and prove that S has a largest element. Observe that now $v \geq 0$ for all $v \in S$. By the previous lemma $S \neq \emptyset$.

The proof of Theorem 8 in [15] uses the fact that the functions in S_C are nonnegative and a compactness argument. In our case, since the functions in S_C are uniformly bounded in $W^{1,G}(\Omega)$, any sequence in S_C has a subsequence that converges a.e. in Ω, $\mathcal{H}^{N-1}$–a.e. on the boundary, weakly in $W^{1,G}(\Omega)$, and uniformly on compact subsets of Ω together with their gradients. Therefore, the limit belongs to S_C. Using this argument, we can follow the lines of that theorem and conclude that any chain in S_C has an upper bound in S_C. Then, by Zorn's lemma, S_C has a maximal element v^* in S_C with respect to the partial order $\leq$.

Let us see that v^* is the largest element of S_C. Let $v \in S_C$. Since v and v^* are both subsolutions, $\max\{v, v^*\}$ is a subsolution of (C.6). Then, by Lemma C.1 there exists a solution w of (C.6) with $w = v_0$ on $\partial\Omega$ such that $\underline{v} \leq \max\{v, v^*\} \leq w \leq \bar{v}$. Thus, $w \in S_C$ and $w \geq v^*$. By the maximality of v^*, $w = v^*$ and then $v \leq v^*$.

Observe, in particular, the fact that $w = v^*$ implies that $v^* = v_0$ on $\partial\Omega$.

Observe that, in an analogous way, we can prove (2) by taking a set similar to S. ∎

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