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Some Refinements and Generalizations of Bohr's Inequality

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Abstract: In this article, we delve into the classic Bohr inequality for complex numbers, a fundamental result in complex analysis with broad mathematical applications. We offer refinements and generalizations of Bohr's inequality, expanding on the established inequalities of N. G. de Bruijn and Radon, as well as leveraging the class of functions defined by the Daykin–Eliezer–Carlitz inequality. Our novel contribution lies in demonstrating that Bohr's and Bergström's inequalities can be derived from one another, revealing a deeper interconnectedness between these results. Furthermore, we present several new generalizations of Bohr's inequality, along with other notable inequalities from the literature, and discuss their various implications. By providing more comprehensive and verifiable conditions, our work extends previous research and enhances the understanding and applicability of Bohr's inequality in mathematical studies.

Keywords: Bohr's inequality; Bergström's inequality; Radon's inequality

MSC: 26D15; 26D20; 26D99

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1. Introduction

Mathematical inequalities play a crucial role in various areas of mathematics. They allow us to compare and analyze different mathematical quantities, serving as powerful tools to establish limits, understand relationships, and gain insights into the behavior of mathematical objects. These inequalities find applications in fields such as optimization, analysis, probability theory, and mathematical physics. For further reading on this topic, refer to [1–5] and the references therein.

One well-known inequality in the literature is Bohr's classical inequality, introduced by Bohr [6]. It states that for complex numbers z_1, z_2 and positive numbers $r_1, r_2 > 1$ satisfying $\frac{1}{r_1} + \frac{1}{r_2} = 1$, the following inequality holds true:

$$|z_1 + z_2|^2 \leqslant r_1|z_1|^2 + r_2|z_2|^2. \tag{1}$$

Notice that the inequality (1) remains an equality if and only if z_2 is equal to $(r_1 - 1)z_1$. The elegance of Bohr's inequality is found not only in its simplicity, but also in its profound implications. It offers a beautiful geometric interpretation and has been applied in various mathematical contexts. For instance, you can explore [7–9] and the references therein for further insights.

Building upon Bohr's foundational work, Archbold [10] presented a generalization of the inequality to n complex numbers $z_1, \ldots, z_n$ and positive numbers $r_1, \ldots, r_n$ satisfying $\sum_{k=1}^{n} \frac{1}{r_k} = 1$:

$$\left| \sum_{k=1}^{n} z_k \right|^2 \leq \sum_{k=1}^{n} r_k |z_k|^2. \quad (2)$$

This generalization, commonly referred to as Bohr's inequality, has numerous extensions and generalizations (see, for example, [8,11–14]).

The inequality referred to in the literature as Bergström's inequality ([15]) states that if $x_1, \cdots, x_k$ are real numbers and $a_k > 0$ for $k = 1, \cdots, n$, then

$$\frac{\left(\sum_{k=1}^{n} x_k\right)^2}{\sum_{k=1}^{n} a_k} \leq \sum_{k=1}^{n} \frac{x_k^2}{a_k},$$

with equality if and only if $\frac{x_i}{a_i} = \frac{x_j}{a_j}$ for any $i, j = 1, \ldots, n$. A generalization of Bergström's inequality was obtained in 1913 by Radon in reference [16].

Let us move on to Radon's inequality, which states that for $p > 0$, $x_k \geq 0$, and $a_k > 0$ for any $k = 1, \ldots, n$, we have:

$$\frac{\left(\sum_{k=1}^{n} x_k\right)^{p+1}}{\left(\sum_{k=1}^{n} a_k\right)^p} \leq \sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p}. \quad (3)$$

Inspired by a previous manuscript [17], which investigates inequalities (2) and (3), the authors provide a detailed outline of a general form using a real function for these inequalities and fully characterize the class of functions involved. Furthermore, the article demonstrates that Bohr's inequality is a specific instance of Radon's inequality, with $p = 1$. Building on this foundation, our study demonstrates a novel connection between Bohr's and Bergström's inequalities, showing that one can be deduced from the other.

In our main results, we present various refinements and generalizations of the classical Bohr inequality, covering a wide range of scenarios. We start by introducing a refinement based on de Bruijn's inequality, from which we derive generalizations of Bohr's inequality for any power greater than or equal to 2. Finally, by leveraging Radon's inequality, we obtain further refinements and generalizations of (2) or of another previously obtained generalization.

2. Main Results

In this section, we present our main contributions. Prior to discussing the results, we introduce three lemmas that will be utilized throughout the manuscript. The first lemma is derived from [18], while the second one is derived from [19], Theorem 1.6.

Lemma 1. *Let $a_1, \cdots, a_n$ be a sequence of real numbers and $z_1, \cdots, z_n$ be a sequence of complex numbers, then*

$$\left| \sum_{k=1}^{n} a_k z_k \right|^2 \leq \frac{1}{2} \sum_{k=1}^{n} a_k^2 \left[\sum_{k=1}^{n} |z_k|^2 + \left| \sum_{k=1}^{n} z_k^2 \right| \right]. \quad (4)$$

Equality holds in (4) if and only if there exists $\lambda \in \mathbb{C}$ such that $a_k = \mathrm{Re}(\lambda z_k)$ for any $k = 1, \cdots, n$ and $\sum_{k=1}^{n} \lambda^2 z_k^2 \geq 0$.

Notice that (4) is known as de Bruijn's inequality and provides a refined version of the classical Cauchy–Bunyakovsky–Schwarz inequality.

Lemma 2. *Let $z_1, \cdots, z_n$ be a sequence of non-zero complex numbers, then*

$$\left| \sum_{k=1}^{n} z_k \right| = \sum_{k=1}^{n} |z_k|$$

if and only if there exists $\alpha_1, \cdots, \alpha_n$ *positive real numbers such that* $z_i = \alpha_i z_1$ *for any* $i = 1, \cdots, n$, *where* $\alpha_i = \frac{|z_i|}{|z_1|}$.

Lemma 3. *Let* $z_1, \cdots, z_n$ *be a sequence of non-zero complex numbers and* $r_1, \cdots, r_n$ *be a sequence of positive numbers. Then, the following conditions are equivalent:*

1. *The equality* $r_j|z_j| = r_i|z_i|$ *holds for any* $i, j = 1, \cdots, n$ *and there exists a sequence of positive numbers* $\alpha_1, \cdots, \alpha_n$ *such that* $z_j = \alpha_j z_1$ *for every* $j = 1, \ldots, n$.
2. *The equality* $r_j z_j = r_i z_i$ *holds for any* $i, j = 1, \cdots, n$.

Proof. The implication from condition (2) to condition (1) is straightforward, leaving us to demonstrate the converse. Assuming (1) holds true, let $i, j = 1, \cdots, n$. We observe $|z_j| = \alpha_j |z_1|$, or equivalently, $\alpha_j = \frac{|z_j|}{|z_1|}$, given $z_1 \neq 0$. Furthermore, from the other condition, we infer the following:

$$\frac{r_j}{r_i} = \frac{|z_i|}{|z_j|}.$$

Finally, this leads to

$$\frac{r_j}{r_i} = \frac{|z_i|}{|z_j|} = \frac{\frac{|z_i|}{|z_1|}}{\frac{|z_j|}{|z_1|}} = \frac{\alpha_i}{\alpha_j},$$

thereby concluding the proof. □

Now, we are able to obtain our first main result, which is a refinement of Bohr's inequality.

Theorem 1. *Let* $z_1, \cdots, z_n$ *be a sequence of complex numbers and* $r_1, \cdots, r_n$ *be a sequence of positive numbers such that* $\sum_{k=1}^{n} \frac{1}{r_k} = 1$, *then*

$$\left|\sum_{k=1}^{n} z_k\right|^2 \leq \frac{1}{2}\left[\sum_{k=1}^{n} r_k|z_k|^2 + \left|\sum_{k=1}^{n} r_k z_k^2\right|\right] \leq \sum_{k=1}^{n} r_k|z_k|^2. \tag{5}$$

The equality holds if and only if $r_i z_i = r_j z_j$ *for any* $i, j = 1, \cdots, n$.

Proof. From de Bruijn's inequality, we have $\sqrt{r_k} z_k$ instead of z_k, and $\frac{1}{\sqrt{r_k}}$ instead of a_k

$$\begin{aligned}\left|\sum_{k=1}^{n} z_k\right|^2 = \left|\sum_{k=1}^{n} \frac{1}{\sqrt{r_k}} \sqrt{r_k} z_k\right|^2 &\leq \frac{1}{2}\sum_{k=1}^{n}\frac{1}{r_k}\left[\sum_{k=1}^{n} r_k|z_k|^2 + \left|\sum_{k=1}^{n} r_k z_k^2\right|\right] \\ &= \frac{1}{2}\left[\sum_{k=1}^{n} r_k|z_k|^2 + \left|\sum_{k=1}^{n} r_k z_k^2\right|\right]. \end{aligned}\tag{6}$$

Now, using the triangle inequality for the modulus in the complex plane, we have

$$\begin{aligned}\frac{1}{2}\left[\sum_{k=1}^{n} r_k|z_k|^2 + \left|\sum_{k=1}^{n} r_k z_k^2\right|\right] &\leq \frac{1}{2}\left[\sum_{k=1}^{n} r_k|z_k|^2 + \sum_{k=1}^{n}\left|r_k z_k^2\right|\right] \\ &= \frac{1}{2}\left[\sum_{k=1}^{n} r_k|z_k|^2 + \sum_{k=1}^{n} r_k\left|z_k^2\right|\right] \\ &= \frac{1}{2}\left[\sum_{k=1}^{n} r_k|z_k|^2 + \sum_{k=1}^{n} r_k|z_k|^2\right] \\ &= \sum_{k=1}^{n} r_k|z_k|^2. \end{aligned}\tag{7}$$

Suppose that the equality holds in (5). Without a loss of generality, we can also assume that $z_k \neq 0$ for any $k = 1, \cdots, n$. Combining (6) and (7), we obtain the following equalities:

$$\left|\sum_{k=1}^{n} \frac{1}{\sqrt{r_k}} \sqrt{r_k} z_k\right|^2 = \frac{1}{2} \sum_{k=1}^{n} \frac{1}{r_k} \left[\sum_{k=1}^{n} r_k |z_k|^2 + \left|\sum_{k=1}^{n} r_k z_k^2\right|\right], \tag{8}$$

and

$$\left|\sum_{k=1}^{n} r_k z_k^2\right| = \sum_{k=1}^{n} \left|r_k z_k^2\right|. \tag{9}$$

This indicates that we achieve equality in both the standard triangle inequality for the modulus of complex numbers and in de Bruijn's inequality. Then, from (9) and Lemma 2, there exists $\alpha_1, \cdots, \alpha_n$ positive real numbers such that $z_k = \alpha_k z_1$ for any $k = 1, \cdots, n$. On the other hand, from (8), we conclude that there exists $\lambda \in \mathbb{C}$ such that $\frac{1}{\sqrt{r_k}} = Re(\lambda \sqrt{r_k} z_k)$. Combining such conditions, we have

$$\frac{1}{\sqrt{r_k}} = Re(\lambda \sqrt{r_k} z_k) = Re(\lambda \sqrt{r_k} \alpha_k z_1) = \sqrt{r_k} \alpha_k Re(\lambda z_1) = \sqrt{r_k} \alpha_k \frac{1}{r_1},$$

or equivalently $\frac{r_1}{r_k} = \alpha_k$, and this implies that for any $k = 1, \cdots, n$ it holds that

$$z_k = \frac{r_1}{r_k} z_1.$$

On the other hand, if $r_i z_i = r_j z_j$ for any $i, j = 1, \cdots, n$, it is straightforward to confirm the equality in (5). □

Remark 1. *In particular, we note that if all the z_k in Theorem 1 are real, then the refinement recently obtained coincides with the upper bound originally given by Bohr.*

From Theorem 1, we proceed to establish an extension of Bohr's inequality for powers greater than or equal to 2.

Corollary 1. *Let $z_1, \cdots, z_n$ be a sequence of complex numbers and $r_1, \cdots, r_n$ be a sequence of positive numbers such that $\sum_{k=1}^{n} \frac{1}{r_k} = 1$, and $s \geq 2$, then*

$$\left|\sum_{k=1}^{n} z_k\right|^s \leq \frac{1}{2}\left[\left(\sum_{k=1}^{n} r_k |z_k|^2\right)^{\frac{s}{2}} + \left|\sum_{k=1}^{n} r_k z_k^2\right|^{\frac{s}{2}}\right] \leq \left(\sum_{k=1}^{n} r_k |z_k|^2\right)^{\frac{s}{2}}.$$

The equality holds if and only if $r_i z_i = r_j z_j$ for any $i, j = 1, \cdots, n$.

Proof. According to Theorem 1 and the convexity of the function $f(t) = t^{\frac{s}{2}}$ on the interval $[0, +\infty)$, we have

$$\begin{aligned}
\left|\sum_{k=1}^{n} z_k\right|^s = \left(\left|\sum_{k=1}^{n} z_k\right|^2\right)^{\frac{s}{2}} &\leq \left(\frac{1}{2}\left[\sum_{k=1}^{n} r_k |z_k|^2 + \left|\sum_{k=1}^{n} r_k z_k^2\right|\right]\right)^{\frac{s}{2}} \\
&\leq \frac{1}{2}\left(\sum_{k=1}^{n} r_k |z_k|^2\right)^{\frac{s}{2}} + \frac{1}{2}\left|\sum_{k=1}^{n} r_k z_k^2\right|^{\frac{s}{2}} \\
&\leq \left(\sum_{k=1}^{n} r_k |z_k|^2\right)^{\frac{s}{2}}.
\end{aligned}$$

□

Upon examining the previous corollary, it is clear that we relied on two fundamental properties of the function $f(t) = t^{\frac{s}{2}}$, namely its monotonicity and its mid-point convexity. Interestingly, these conditions imply convexity (see Remark on page 4 of [20]). Therefore, we have the following result, the proof of which we omit since it is analogous to the one given in Corollary 1.

Corollary 2. *Let $z_1, \cdots, z_n$ be a sequence of complex numbers and $r_1, \cdots, r_n$ be a sequence of positive numbers such that $\sum_{k=1}^{n} \frac{1}{r_k} = 1$, and f an increasing, convex function on $[0, +\infty)$, then*

$$\begin{aligned} f\left(\left|\sum_{k=1}^{n} z_k\right|^2\right) &\leq f\left(\frac{1}{2}\left[\sum_{k=1}^{n} r_k|z_k|^2 + \left|\sum_{k=1}^{n} r_k z_k^2\right|\right]\right) \\ &\leq \frac{1}{2} f\left(\sum_{k=1}^{n} r_k|z_k|^2\right) + \frac{1}{2} f\left(\left|\sum_{k=1}^{n} r_k z_k^2\right|\right) \\ &\leq f\left(\sum_{k=1}^{n} r_k|z_k|^2\right). \end{aligned}$$

Inspired by Theorem 1, which provided a refinement of (2) through an enhanced version of the discrete Cauchy–Buniakowsky–Schwarz inequality, we revisit a family of functions initially defined by Daykin, Eliezer, and Carlitz in [21]. Our goal is to utilize this family of functions to derive new and improved versions of Bohr's inequality.

In the next lemma, we recall how this family of functions is defined.

Lemma 4. *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be positive real numbers and let $f(\cdot,\cdot), g(\cdot,\cdot)$ be positive functions with two variables. The inequality*

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leqslant \sum_{k=1}^{n} f(a_k, b_k) \sum_{k=1}^{n} g(a_k, b_k) \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2,$$

holds if and only if

1. $f(a,b)g(a,b) = a^2 b^2$,
2. $f(ka, kb) = k^2 f(a,b)$ *for* $k > 0$,
3. $\frac{bf(a,1)}{xf(b,1)} + \frac{af(b,1)}{bf(a,1)} \leq \frac{a}{b} + \frac{b}{a}$ *holds for any positive real numbers a and b.*

By employing the class of functions delineated by the Daykin–Eliezer–Carlitz inequality, we derive the following refinement of (2).

Theorem 2. *Let $z_1, \cdots, z_n$ be a sequence of complex numbers, $r_1, \cdots, r_n$ be a sequence of positive numbers such that $\sum_{k=1}^{n} \frac{1}{r_k} = 1$, and f, g functions satisfying (1), (2), and (3), as stated in Lemma 4, then*

$$\left|\sum_{k=1}^{n} z_k\right|^2 \leq \sum_{k=1}^{n} f\left(\frac{1}{\sqrt{r_k}}, \sqrt{r_k|z_k|^2}\right) \sum_{k=1}^{n} g\left(\frac{1}{\sqrt{r_k}}, \sqrt{r_k|z_k|^2}\right) \leq \sum_{k=1}^{n} r_k|z_k|^2.$$

Proof. Applying Lemma 4 with $a_k = \frac{1}{\sqrt{r_k}}$ and $b_k = \sqrt{r_k|z_k|^2}$ for any $k = 1, \cdots, n$, we have

$$\begin{aligned} \left|\sum_{k=1}^{n} z_k\right|^2 &\leq \left(\sum_{k=1}^{n} |z_k|\right)^2 = \left(\sum_{k=1}^{n} \frac{1}{\sqrt{r_k}} \sqrt{r_k|z_k|^2}\right)^2 \\ &\leq \sum_{k=1}^{n} f\left(\frac{1}{\sqrt{r_k}}, \sqrt{r_k|z_k|^2}\right) \sum_{k=1}^{n} g\left(\frac{1}{\sqrt{r_k}}, \sqrt{r_k|z_k|^2}\right) \\ &\leq \sum_{k=1}^{n} r_k|z_k|^2. \end{aligned}$$

□

For the particular cases of functions f and g, we obtain the following corollary.

Corollary 3. *Let $z_1, \cdots, z_n$ be a sequence of complex numbers and let $r_1, \cdots, r_n$ be a sequence of positive numbers such that $\sum_{k=1}^{n} \frac{1}{r_k} = 1$, then*

$$\left|\sum_{k=1}^{n} z_k\right|^2 \leq \sum_{k=1}^{n} \frac{1 + r_k^2|z_k|^2}{r_k} \sum_{k=1}^{n} \frac{r_k|z_k|^2}{1 + r_k^2|z_k|^2} \leq \sum_{k=1}^{n} r_k|z_k|^2,$$

and for any $t \in [0, 1]$

$$\left|\sum_{k=1}^{n} z_k\right|^2 \leq \sum_{k=1}^{n} r_k^{-t}|z_k|^{1-t} \sum_{k=1}^{n} r_k^{t}|z_k|^{1+t} \leq \sum_{k=1}^{n} r_k|z_k|^2.$$

Proof. It is enough to see that $f(a, b) = a^2 + b^2, g(a, b) = \frac{a^2b^2}{a^2+b^2}$ $f(a, b) = a^{1+t}b^{1-t}, g(a, b) = a^{1-t}b^{1+t}$ with $t \in [0, 1]$ are pairs of functions of the Daykin–Eliezer–Carlitz inequality type. □

The equivalence between various mathematical inequalities holds significant importance both logically and historically, as evidenced by a vast body of literature exploring their connections. In summary, we demonstrate in this article that two of the discussed inequalities, Bohr's and Bergström's inequalities, are interconnected and derivable from each other.

Theorem 3. *The following inequalities are equivalent:*

1. *Borh's inequality—Let $z_1, \cdots, z_n$ be a sequence of complex numbers and $r_1, \cdots, r_n$ be a sequence of positive numbers such that $\sum_{k=1}^{n} \frac{1}{r_k} = 1$, then*

$$\left|\sum_{k=1}^{n} z_k\right|^2 \leq \sum_{k=1}^{n} r_k|z_k|^2.$$

   The equality holds if and only if $r_i z_i = r_j z_j$ for any $i, j = 1, \cdots, n$.

2. *Bergström's inequality—Let $x_1, \cdots, x_n$ be a sequence of real numbers and $a_1, \cdots, a_n$ be a sequence of positive numbers, then*

$$\frac{\left(\sum_{k=1}^{n} x_k\right)^2}{\sum_{k=1}^{n} a_k} \leq \sum_{k=1}^{n} \frac{x_k^2}{a_k},$$

   with equality if and only if $\frac{x_i}{a_i} = \frac{x_j}{a_j}$ for any $i, j = 1, \cdots, n$.

Proof. We start by demonstrating that (2) is a straightforward and specific case of Bohr's inequality. To achieve this, it suffices to consider, for any $k = 1, \cdots, n$,

$$r_k = \frac{a_1 + a_2 + \ldots + a_n}{a_k}$$

and $z_k = x_k$ in Theorem 1.

Now, we only need to prove that (2) implies (1). For any $k = 1, \ldots, n$ let $z_k = e^{i\theta_k}|z_k|$; then, according to the triangle inequality, we have

$$\left|\sum_{k=1}^{n} z_k\right|^2 \leq \left(\sum_{k=1}^{n} |z_k|\right)^2. \tag{10}$$

Now, by using inequality (2) with $a_k = \frac{1}{r_k}$ for any $k = 1, \ldots, n$, then $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{1}{r_k} = 1$ and

$$\left(\sum_{k=1}^{n}|z_k|\right)^2 \leq \sum_{k=1}^{n} r_k|z_k|^2. \tag{11}$$

Finally, combining (10) and (11), we obtain the classical Borh inequality. □

Now, we turn your attention to the case $r_k = n$. Specifically, if we consider in (2), $r_k = n$ for any $k = 1, \cdots, n$, we obatin the following inequality:

$$\left|\sum_{k=1}^{n} z_k\right|^2 \leq n\sum_{k=1}^{n}|z_k|^2, \tag{12}$$

with equality if and only if all z_k's are equal.

The final inequality is also a consequence of the classical Cauchy–Buniakowsky–Schwarz inequality. For convenience, we denote the following positive number using R,

$$R = n\sum_{k=1}^{n}|z_k|^2 - \left|\sum_{k=1}^{n} z_k\right|^2.$$

On the other hand, in [22], the authors provide an explicit expression for R, more precisely

$$R = \sum_{1\leq i<j\leq n}|z_i - z_j|^2 = n\sum_{k=1}^{n}|z_k|^2 - \left|\sum_{k=1}^{n} z_k\right|^2.$$

Futhermore, Rubió-Massegú et al. obtained the following lower bound for R:

$$R = \sum_{1\leq i<j\leq n}|z_i - z_j|^2 \geq \frac{n}{2}\left(\sum_{k=1}^{n}|z_k|^2 - \left|\sum_{k=1}^{n} z_k^2\right|\right). \tag{13}$$

Notice that from Theorem 1, we obtain a lower bound for R in terms of a finite sum of positive numbers.

Proposition 1. *Let $z_1, \cdots, z_n$ be a sequence of complex numbers, then*

$$0 \leq \frac{n}{2}\left[\sum_{k=1}^{n}|z_k|^2 + \left|\sum_{k=1}^{n} z_k^2\right|\right] - \left|\sum_{k=1}^{n} z_k\right|^2 \leq n\sum_{k=1}^{n}|z_k|^2 - \left|\sum_{k=1}^{n} z_k\right|^2. \tag{14}$$

Now, we are prepared to enhance inequality (12) utilizing another renowned inequality for real numbers. Initially, let us examine a result by Pafnuty Chebyshev (see [1]). He showed that if we have two sequences of real numbers, $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, that are both increasing or both decreasing, then

$$\left(\sum_{i=1}^{n}\frac{1}{n}x_i\right)\left(\sum_{i=1}^{n}\frac{1}{n}y_i\right) \leq \sum_{i=1}^{n}\frac{1}{n}x_iy_i.$$

Then, in 2012, Nakasuji et al. developed a weighted version of Chebyshev's inequality for sequences of real numbers. They proved the following lemma (see [23], Corollary 1):

Lemma 5. *If $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are sequences of real numbers and are simultaneously monotone increasing or monotone decreasing, then*

$$\left(\sum_{i=1}^{n} w_ix_i\right)\left(\sum_{i=1}^{n} w_iy_i\right) \leqslant \sum_{i=1}^{n} w_ix_iy_i,$$

where $w_1, \ldots, w_n$ are positive numbers such that $\sum_{i=1}^{n} w_i = 1$.

Now, we state the following theorem.

Theorem 4. *Let $z_1, \cdots, z_n$ be a sequence of complex numbers and let all $w_1, ..., w_n$ be a sequence of positive numbers with $\sum_{k=1}^n w_k = 1$, then*

$$\left|\sum_{k=1}^n w_k z_k\right|^2 \leq \left(\sum_{k=1}^n w_k |z_k|\right)^2 \leq \sum_{k=1}^n w_k |z_k|^2. \tag{15}$$

Proof. We have, as a consequence of the triangle inequality and Lemma 5, that

$$\begin{aligned}\left|\sum_{k=1}^n w_k z_k\right|^2 &\leq \left(\sum_{k=1}^n w_k |z_k|\right)^2\\ &= \left(\sum_{k=1}^n w_k |z_k|\right)\left(\sum_{k=1}^n w_k |z_k|\right)\\ &\leq \sum_{k=1}^n w_k |z_k|^2.\end{aligned}$$

In the last inequality, we assume that we reorder the sequence $|z_1|, \cdots, |z_n|$ to be monotone increasing. □

Remark 2. *It is worth noting that if we set $w_k = \frac{1}{n}$ in (15), we obtain the following refinement of (12).*

Corollary 4. *Let $z_1, \cdots, z_n$ be a sequence of complex numbers, then*

$$\left|\sum_{k=1}^n z_k\right|^2 \leq \left(\sum_{k=1}^n |z_k|\right)^2 \leq n\sum_{k=1}^n |z_k|^2.$$

In particular, we have

$$0 \leq \left(\sum_{k=1}^n |z_k|\right)^2 - \left|\sum_{k=1}^n z_k\right|^2 \leq n\sum_{k=1}^n |z_k|^2 - \left|\sum_{k=1}^n z_k\right|^2 = R. \tag{16}$$

Combining inequalities (13), (14), and (16), we obtain the following lower bound for R.

Proposition 2. *Let $z_1, \cdots, z_n$ be a sequence of complex numbers, then*

$$\max\{R_1, R_2, R_3\} \leq R,$$

where

$$R_1 = \frac{n}{2}\left(\sum_{k=1}^n |z_k|^2 - \left|\sum_{k=1}^n z_k^2\right|\right),$$

$$R_2 = \frac{n}{2}\left[\sum_{k=1}^n |z_k|^2 + \left|\sum_{k=1}^n z_k^2\right|\right] - \left|\sum_{k=1}^n z_k\right|^2,$$

and

$$R_3 = \left(\sum_{k=1}^n |z_k|\right)^2 - \left|\sum_{k=1}^n z_k\right|^2.$$

Remark 3. *The following numerical examples will illustrate the incomparability of the lower bounds obtained in Proposition 2.*

Consider the following cases:

1. *If* $z_1 = 1$ *and* $z_2 = 2$*, then one may verify that* $R_2 = 1 > 0 = R_3$*.*
2. *If* $z_1 = 1$ *and* $z_2 = 2i$*, then* $R_1 - R_3 = 4$ *and this implies that* $R_1 > R_3$*.*
3. *Now, if we consider for any* $n \geq 2$*,* $z_k = e^{\frac{2(k-1)\pi i}{2n}}$ *with* $k = 1, \cdots, n$*, then* $\sum_{k=1}^{n} z_k^2 = 0$ *and*

$$R_2 - R_3 = \frac{n^2}{2} - n^2 = -\frac{n^2}{2} < 0 \Rightarrow R_2 < R_3,$$

and

$$R_1 - R_2 = \left|\sum_{k=1}^{n} z_k\right|^2 - n\left|\sum_{k=1}^{n} z_k^2\right| = \left|\sum_{k=1}^{n} z_k\right|^2 > 0,$$

thus we have $R_1 > R_2$*.*

4. *Now, if we consider for any* $n \geq 2$*,* $w_k = e^{\frac{2(k-1)\pi i}{n}}$ *with* $k = 1, \cdots, n$*, then* $\sum_{k=1}^{n} z_k = 0$ *and*

$$R_1 - R_3 = \frac{n^2}{2} - \frac{n}{2}\left|\sum_{k=1}^{n} z_k^2\right| - n^2 = -\frac{n^2}{2} - \frac{n}{2}\left|\sum_{k=1}^{n} z_k^2\right| < 0,$$

i.e., $R_1 < R_3$*. On the other hand,*

$$R_1 - R_2 = \left|\sum_{k=1}^{n} z_k\right|^2 - n\left|\sum_{k=1}^{n} z_k^2\right| = -n\left|\sum_{k=1}^{n} z_k^2\right| < 0,$$

which allows us to show that $R_1 < R_2$*.*
This illustrates that the lower bounds obtained in Proposition 2 are not directly comparable.

Now, we are able to derive a generalization of Bohr's inequality for a power different from 2. This motivation stems from the relationship between this inequality and Radon's inequality.

To achieve this generalization, it is necessary to recall the following result obtained in [24], Theorem 2.3.

Lemma 6. *Let* $x_1, \cdots, x_n$ *be a sequence of real numbers,* $a_1, \cdots, a_n$ *be a sequence positive numbers,* $p \geq 0$*, and* $m \geq 1$*, then*

$$\frac{\left(\sum_{k=1}^{n} x_k a_k^{m-1}\right)^{p+m}}{\left(\sum_{k=1}^{n} a_k^m\right)^{m+p-1}} \leq \sum_{k=1}^{n} \frac{x_k^{p+m}}{a_k^p}, \tag{17}$$

with equality if and only if $\frac{x_i}{a_i} = \frac{x_j}{a_j}$ *for any* $i, j = 1, \cdots, n$*.*

Theorem 5. *Let* $z_1, \cdots, z_n$ *be a sequence of complex numbers,* $p \geq 0$*,* $m \geq 1$*, and* $r_1, \cdots, r_n$ *be a sequence of positive numbers, then*

$$\left|\sum_{k=1}^{n} z_k r_k^{m-1}\right|^{p+m} \leq \left(\sum_{k=1}^{n} |z_k| r_k^{m-1}\right)^{p+m} \leq \left(\sum_{k=1}^{n} r_k^m\right)^{m+p-1} \sum_{k=1}^{n} r_k^{-p} |z_k|^{p+m}. \tag{18}$$

The equality

$$\left|\sum_{k=1}^{n} z_k r_k^{m-1}\right|^{p+m} = \left(\sum_{k=1}^{n} r_k^m\right)^{m+p-1} \sum_{k=1}^{n} r_k^{-p} |z_k|^{p+m},$$

holds if and only if $\frac{z_i}{r_i} = \frac{z_j}{r_j}$ *for any* $i, j = 1, \cdots, n$*.*

Proof. We note that for the usual triangle inequality for the modulus of complex numbers, we have the first inequality.

On the other, if we replace x_k by $|z_k|$ in (17), as well as a_k by r_k, then we have

$$\left(\sum_{k=1}^{n} |z_k| r_k^{m-1}\right)^{p+m} \leq \left(\sum_{k=1}^{n} r_k^m\right)^{m+p-1} \sum_{k=1}^{n} r_k^{-p} |z_k|^{p+m}.$$

Now, we assume that holds the equality

$$\left|\sum_{k=1}^{n} z_k r_k^{m-1}\right|^{p+m} = \left(\sum_{k=1}^{n} r_k^m\right)^{m+p-1} \sum_{k=1}^{n} r_k^{-p} |z_k|^{p+m}.$$

Then, according to Lemma 6, we conclude that for any $i, j = 1, \cdots, n$, we have $\frac{|z_i|}{r_i} = \frac{|z_j|}{r_j}$. On the other hand, according to Lemma 2, we conclude that there exists $\alpha_1, \cdots, \alpha_n$ such that $z_k = \alpha_k z_1$. Finally, according to Lemma 3, we obtain that $\frac{z_i}{r_i} = \frac{z_j}{r_j}$ for any $i, j = 1, \cdots, n$.

If $\frac{z_i}{r_i} = \frac{z_j}{r_j}$ for any $i, j = 1, \cdots, n$, it is straightforward to confirm the equality in (18). □

Now, we present a generalization and refinement of Bohr's inequality.

Corollary 5. *Let $z_1, \cdots, z_n$ be a sequence of complex numbers, $p > 0$, $m \geq 1$, and $r_1, \cdots, r_n$ be a sequence of positive numbers such that $\sum_{k=1}^{n} r_k^{-\frac{m}{p}} = 1$, then*

$$\left|\sum_{k=1}^{n} z_k r_k^{\frac{-(m-1)}{p}}\right|^{p+m} \leq \left(\sum_{k=1}^{n} |z_k| r_k^{\frac{-(m-1)}{p}}\right)^{p+m} \leq \sum_{k=1}^{n} r_k |z_k|^{p+m}.$$

The equality

$$\left|\sum_{k=1}^{n} z_k r_k^{\frac{-(m-1)}{p}}\right|^{p+m} = \sum_{k=1}^{n} r_k |z_k|^{p+m}$$

holds if and only if $r_i^{\frac{1}{p}} z_i = r_j^{\frac{1}{p}} z_j$ for any $i, j = 1, \cdots, n$.

Proof. To obtain this statement is enough to consider $r_k^{-\frac{1}{p}}$ instead of r_k in Theorem 5. □

Remark 4. *By considering specific values of the parameters in Theorem 5, we derive well-known inequalities and refinements that have been previously obtained by various authors. For instance, setting $m = p = 1$ yields the classical Bohr inequality.*

In the following statement, we obtain a generalization of Bohr's inequality and a new refinement. This generalization was previously obtained by Vacić and Kečkić in [25].

Corollary 6. *Let $z_1, \cdots, z_n$ be a sequence of complex numbers, $s > 1$, and $r_1, \cdots, r_n$ be a sequence of positive numbers, then*

$$\left|\sum_{k=1}^{n} z_k\right|^{s} \leq \left(\sum_{k=1}^{n} |z_k|\right)^{s} \leq \left(\sum_{k=1}^{n} r_k^{\frac{1}{(1-s)}}\right)^{s-1} \sum_{k=1}^{n} r_k |z_k|^{s}.$$

The equality

$$\left|\sum_{k=1}^{n} z_k\right|^{s} = \left(\sum_{k=1}^{n} r_k^{\frac{1}{(1-s)}}\right)^{s-1} \sum_{k=1}^{n} r_k |z_k|^{s} \tag{19}$$

holds if and only if $r_i^{\frac{1}{(s-1)}} z_i = r_j^{\frac{1}{(s-1)}} z_j$ *for any* $i, j = 1, \cdots, n$.

Proof. To obtain this series of inequalities, it is sufficient to replace r_k by $r_k^{\frac{1}{(1-s)}}$, $m = 1$, and $p = s - 1$ in (18). □

Remark 5. *In [25], it was established by the authors that the equality in (19) is attained if and only if*

(1) *For any* $i, j = 1, \cdots, n$ *hold* $r_i|z_i|^{s-1} = r_j|z_j|^{s-1}$ *and* $z_i\bar{z}_j \geq 0$.

We show that condition (1) *is equivalent to the one obtained in Corollary 6.*

Assuming that (1) *holds, if we denote* $z_k = e^{i\theta_k}|z_k|$, *where* $\theta_k \in [0, 2\pi)$ *for any* $k = 1, \cdots, n$, *then the condition* $z_i\bar{z}_j \geq 0$ *implies that* $\theta_i = \theta_j$ *for any* $i, j = 1, \cdots, n$. *Equivalently, we can say that there exists* $\alpha_1, \cdots, \alpha_n$ *such that* $z_k = \alpha_k z_1$. *From the other condition in* (1), *we deduce that*

$$r_i^{\frac{1}{(s-1)}} |z_i| = r_j^{\frac{1}{(s-1)}} |z_j|$$

for any $i, j = 1, \cdots, n$. *Thus, using Lemma 3, we have*

$$r_i^{\frac{1}{(s-1)}} z_i = r_j^{\frac{1}{(s-1)}} z_j$$

for any $i, j = 1, \cdots, n$.

It is evident that if $r_i^{\frac{1}{(s-1)}} z_i = r_j^{\frac{1}{(s-1)}} z_j$ *for any* $i, j = 1, \cdots, n$, *then condition* (1) *trivially holds as well.*

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