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A BOUNDARY VALUE PROBLEM FOR A SEMILINEAR SECOND ORDER *ODE*

Pablo Amster, María Cristina Mariani and Juan Sabia

FCEyN - Universidad de Buenos Aires

ABSTRACT

We solve by topological methods a Dirichlet problem for the general semilinear second order *ODE*. We also prove the uniqueness of the solutions. Moreover, we develop an iterative method in order to find a solution in certain cases, for which the usual Picard iteration is not appliable.

INTRODUCTION

We consider the unidimensional boundary value problem

$$(1)\begin{cases} y'' = f(x,y,y') & \text{in } (a,b) \\ y(a) = \alpha, \quad y(b) = \beta \end{cases}$$

Particular cases of this equation have been studied by several authors. For $f = g(x) + h(y)$, with $g \in L^2(a,b)$ and $h : \mathbb{R} \to \mathbb{R}$ continuous solutions may be obtained under a growth condition on h, i.e.:

$$|h(y)| \leq c|y| + d$$

for any $y \in \mathbb{R}$ and $c < \lambda_1$, the first eigenvalue for the homogeneous Dirichlet problem of the second order linear operator $Lu := -u''$. Some results for periodic type and Sturm-Liouville conditions are also known (see e.g. [AM], [AS], [B], [Br], [FM], [M]). For a general continuous function $f : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$, we will state the existence of solutions of (1) under a growth condition on (y, y'). Furthermore, uniqueness can be also proved if f satisfies a Lipschitz condition or if f is C^1 with respect to y, y' with $\frac{\partial f}{\partial y} \geq 0$.

On the other hand, we will show that a solution of (1) can be obtained constructively by a continuation-type method. Indeed, the problem can be included as a sub-family of problems where there is a parameter t, and on starting at a solution for t_0 it is possible to find a solution for $t_0+\varepsilon$ as the limit of a recursive sequence in the Sobolev Space $H^2(a,b)$.

1. EXISTENCE BY FIXED POINT METHODS

Let $\varphi(t) = m(t-a)+\alpha$, where $m = \frac{\beta-\alpha}{b-a}$, then for $z = y-\varphi$ problem (1) is equivalent to

$$(2)\begin{cases} z'' = f(t, z+\varphi, z'+m) \\ z(a) = z(b) = 0 \end{cases}$$

A simple computation shows that the Green function for the associated linear problem is

$$G(t,s) = \begin{cases} \frac{1}{b-a}(t-a)(s-b) & \text{if } s \geq t \\ \frac{1}{b-a}(s-a)(t-b) & \text{if } s \leq t \end{cases}$$

Then we may define the operator $T : C^1([a,b]) \to C^1([a,b])$ given by

$$Tz(t) = \int_a^b G(t,s) f(s, z(s)+\varphi(s), z'(s)+m)ds.$$

The continuity of T is immediate. Furthermore, by the Arzelá-Ascoli Theorem T is compact, and on account of

$$\|G(t,\cdot)\|_1 = \frac{(b-t)(t-a)}{2} \leq \frac{(b-a)^2}{8},$$

$$\|\frac{\partial G}{\partial t}(t,\cdot)\|_1 = \frac{(t-a)^2+(b-t)^2}{2(b-a)} \leq \frac{b-a}{2}$$

it is easy to conclude that

$$\|Tz\|_{1,\infty} = \|Tz\|_\infty + \|(Tz)'\|_\infty \leq (\frac{(b-a)^2}{8} + \frac{b-a}{2}) \sup_{a\leq s\leq b} |f(s, z(s)+\varphi(s), z'(s)+m)|$$

Thus, we have the following

THEOREM 1

Let us assume that $|f(s,u,v)| \leq c(|u|+|v|)+d$ *for any* $u,v \in \mathbb{R}$ *and some constants* c,d *such that*

$$c(\frac{(b-a)^2}{8}+\frac{b-a}{2})<1.$$

Then T *has a fixed point* $z \in C^2([a,b])$ *which corresponds to a solution of (2). Furthermore, if* f *is Lipschitz in* (y,y') *with constant* c, *then (2) has a unique solution.*

<u>*Proof*</u>

Let $\|z\|_{1,\infty} \leq R$. As

$$|f(s,z(s)+\varphi(s),z'(s)+m)| \leq c(|z(s)|+|z'(s)|+|\varphi(s)|+m)+d,$$

we conclude that

$$\sup_{a\leq s\leq b} |f(s,z(s)+\varphi(s),z'(s)+m)| \leq c(R+\max\{|\alpha|,|\beta|\}+m)+d$$

Hence, taking

$$R \geq \frac{c(\max\{|\alpha|,|\beta|\}+m)+d}{1-c}$$

it follows that $T(\overline{B}_R) \subset \overline{B}_R$, and by Schauder's theorem (see e.g. [L]) T has a fixed point $z \in \overline{B}_R$. Clearly, z solves (2) and $z \in C^2([a,b])$.

Moreover, if f is Lipschitz,

$$\|Tz-Tz_0\|_{1,\infty} \leq (\frac{(b-a)^2}{8}+\frac{b-a}{2})\|f(\cdot,z+\varphi,z'+m)-f(\cdot,z_0+\varphi,z_0'+m)\|_\infty$$

$$\leq (\frac{(b-a)^2}{8}+\frac{b-a}{2})c\|z-z_0\|_{1,\infty} = \delta\|z-z_0\|_{1,\infty}$$

with $\delta<1$. This proves that T is a contraction, and by the Banach fixed point theorem (2) has a unique solution.

REMARKS

i) If f *is Lipschitz with constant* c, *then*

$$|f(s,u,v)| \leq c(|u|+|v|)+|f(s,0,0)|$$

for any $u,v \in \mathbb{R}$.

ii) The value of the constant c *may be improved by considering the fixed point operator* T *defined in* $H^1(a,b)$. *The growth condition on* f *implies that* T *is well defined, since* $|f(t,z+\varphi,z'+m)| \le c(|z+\varphi|+|z'+m|)+d \in L^2$ *for any* $z \in H^1$.

The following theorem proves uniqueness if f is nondecreasing with respect to y:

THEOREM 2

Let f *be continuously differentiable with respect to* y, y', *and assume that* $\frac{\partial f}{\partial y} \ge 0$. *Then problem (1) admits at most one solution.*

Proof

Let y_1, y_2 be solutions of (1), then $y_1, y_2 \in C^2([a,b])$. Hence,

$$(y_1-y_2)'' = f(t,y_1,y_1') - f(t,y_2,y_2') = \frac{\partial f}{\partial y}(t,\xi,\chi)(y_1-y_2) + \frac{\partial f}{\partial y'}(t,\xi,\chi)(y_1-y_2)'$$

for some mean values ξ, $\chi \in L^\infty$. Thus, if $w = y_1 - y_2$, we have that

$$\begin{cases} Lw = 0 & \text{in } (a,b) \\ w(a) = w(b) = 0 \end{cases}$$

with $Lw := w'' - \frac{\partial f}{\partial y'}(t,\xi,\chi)w' - \frac{\partial f}{\partial y}(t,\xi,\chi)w$.
As $-\frac{\partial f}{\partial y}(t,\xi,\chi) \le 0$, a standard uniqueness result for linear second order *ODEs* shows that $w = 0$.

2. SOLUTIONS BY AN ITERATIVE METHOD

In this section we add a parameter $t \in [0,1]$ to problem (1)

$$(1_t)\begin{cases} y'' = tf(x,y,y') & \text{in } (a,b) \\ y(a) = \alpha \qquad y(b) = \beta \end{cases}$$

and starting at a solution of (1_{t_0}) we will construct recursively a solution of $(1_{t_0+\varepsilon})$ for some step ε. Thus, we have solutions for $0 = t_0 < t_1 < \ldots < t_n = 1$, obtaining a solution of problem (1).

Indeed, assuming that $\frac{\partial f}{\partial y} \ge 0$, if y_0 is a solution of (1_{t_0}), we define y_{n+1} as the unique solution of the linear problem

$$\begin{cases} y_{n+1}'' = (t_0+\varepsilon)[\frac{\partial f}{\partial y}(x,y_n,y_n')(y_{n+1}-y_n) + \frac{\partial f}{\partial y'}(x,y_n,y_n')(y_{n+1}'-y_n') + f(x,y_n,y_n')] \\ y_{n+1}(a) = \alpha \qquad y_{n+1}(b) = \beta \end{cases}$$

We will also assume that f and its derivatives with respect to y and y' of first and second order are bounded, and for simplicity we write

$$\|\partial f\|_\infty = max\{\|\frac{\partial f}{\partial y}\|_\infty, \|\frac{\partial f}{\partial y'}\|_\infty\}$$

$$\|\partial^2 f\|_\infty = max\{\|\frac{\partial^2 f}{\partial y^2}\|_\infty, \|\frac{\partial^2 f}{\partial y \partial y'}\|_\infty, \|\frac{\partial^2 f}{\partial y'^2}\|_\infty\}$$

REMARK: *Let $r, s \in C([a,b])$, and $L : H^2(a,b) \to L^2(a,b)$ the linear operator given by $Lz = z'' + rz' + sz$. For $s \leq 0$, it is well known that $L|_{H^2 \cap H_0^1(a,b)}$ is invertible and onto. Hence, the sequence $\{y_n\}$ is well defined, and in order to prove its convergence we will show in the following lemma that the inferior bound for L may be choosen depending only on $\|r\|_\infty$ and $\|s\|_\infty$.*

LEMMA 3

For any $R > 0$ there exists a constant c such that if $\|r\|_\infty, \|s\|_\infty \leq R$, with $s \leq 0$, then $\|z\|_{2,2} \leq c\|Lz\|_2$ for any $z \in H^2 \cap H_0^1(a,b)$.

Proof

Let us suppose that there exist $r_k, s_k \in B_R(0) \subset C([a,b])$, with $s_k \leq 0$, and $z_k \in H^2 \cap H_0^1(a,b)$ such that $\|z_k\|_{2,2} = 1$, $\|Lz_k\|_2 \to 0$. Taking

$$p_k(x) = e^{\int_a^x r_k(s)ds}$$

we have that $p_k L z_k \to 0$ in $L^2(a,b)$. Then $\int_a^b p_k(z_k')^2 \leq \int_a^b p_k(z_k')^2 - p_k s_k z_k^2 = -\int_a^b p_k L z_k . z_k \to 0$, and being $p_k \geq e^{-R(b-a)}$ we obtain that $z_k' \to 0$ in $L^2(a,b)$. Furthermore, by Poincaré's inequality $z_k \to 0$ in $H_0^1(a,b)$, and as $Lz_k \to 0$ we conclude that $z_k \to 0$ in $H^2(a,b)$, a contradiction.

THEOREM 4

There exists $\varepsilon = \varepsilon(\|f\|_\infty, \|\partial f\|_\infty, \|\partial^2 f\|_\infty)$ such that $\{y_n\}$ converges for the norm $\|\cdot\|_{2,2}$ to a solution of $(1_{t_0+\varepsilon})$.

Proof

Let $z_n = y_{n+1} - y_n$. Then

$$Lz_n := z_n'' - (t_0 + \varepsilon)[\frac{\partial f}{\partial y}(x, y_n, y_n')z_n + \frac{\partial f}{\partial y'}(x, y_n, y_n')z_n'] =$$

$$(t_0 + \varepsilon)[f(x, y_n, y_n') - \frac{\partial f}{\partial y}(x, y_{n-1}, y_{n-1}')z_{n-1} - \frac{\partial f}{\partial y'}(x, y_{n-1}, y_{n-1}')z_{n-1}'] =$$

$$\frac{(t_0+\varepsilon)}{2}\left[\frac{\partial^2 f}{\partial y^2}(x,\xi_1,\xi_2)z_{n-1}^2 + 2\frac{\partial^2 f}{\partial y \partial y'}(x,\xi_1,\xi_2)z_{n-1}z'_{n-1} + \frac{\partial^2 f}{\partial y'^2}(x,\xi_1,\xi_2)z'^2_{n-1}\right]$$

for some mean value $(\xi_1,\xi_2) \in L^\infty((a,b),\mathbb{R}^2)$.

By lemma 3, there exists a constant c depending only on $\|f\|_\infty$ and $\|\partial f\|_\infty$ such that

$$\|z_n\|_{2,2} \leq c\|Lz_n\|_2 \leq c_0 c\frac{t_0+\varepsilon}{2}\|\partial^2 f\|_\infty \|z_{n-1}\|_{2,2}^2$$

where c_0 is the constant of the imbedding $H^2(a,b) \hookrightarrow C^1([a,b])$. Hence, by induction,

$$\|z_n\|_{2,2} \leq [c_0 c\frac{t_0+\varepsilon}{2}\|\partial^2 f\|_\infty \|z_0\|_{2,2}]^{2^n-1}\|z_0\|_{2,2}$$

Moreover, as $z_0'' - (t_0+\varepsilon)[\frac{\partial f}{\partial y}(x,y_0,y_0')z_0 + \frac{\partial f}{\partial y'}(x,y_0,y_0')z_0'] = \varepsilon f(x,y_0,y_0')$, we deduce that

$$\|z_0\|_{2,2} \leq c\varepsilon\|f(x,y_0,y_0')\|_2 \leq c\varepsilon\|f\|_\infty (b-a)^{1/2}$$

Let $A = c_0 c\frac{t_0+\varepsilon}{2}\|\partial^2 f\|_\infty \|z_0\|_{2,2}$. Then

$$\|y_{n+k}-y_n\|_{2,2} \leq \sum_{j=n+1}^{n+k}\|y_j - y_{j-1}\|_{2,2} \leq \|z_0\|_{2,2}\sum_{j=n+1}^{n+k}A^{2^j-1} \leq \|z_0\|_{2,2}\sum_{j=2^{n+1}-1}^{2^{n+k}-1}A^j$$

Then, for $A<1$, $\{y_n\}$ is a Cauchy sequence. Let $y = \lim_{n\to\infty} y_n$, then $y_n \to y$ for the C^1-norm, and

$$\frac{\partial f}{\partial y}(\cdot,y_n,y_n')(y_{n+1}-y_n) + \frac{\partial f}{\partial y'}(\cdot,y_n,y_n')(y_{n+1}'-y_n') + f(x,y_n,y_n') \to f(\cdot,y,y')$$

uniformly. As $y_{n+1}'' \to y''$ in L^2, we conclude that y is a solution of $(1_{t_0+\varepsilon})$.

Thus, it suffices to choose ε such that

$$(b-a)^{1/2}\frac{c^2 c_0}{2}\|\partial^2 f\|_\infty \|f\|_\infty \varepsilon < 1$$

EXAMPLE

Let us consider $f(t,y) = k\,arctg(y) + g(t)$ *with* g *continuous. For* $\overline{y} \in L^2(a,b)$, *as* $arctg(\overline{y}) \in L^2(a,b)$ *we may define* $y = T\overline{y}$ *as the unique solution of the problem*

$$\begin{cases} y'' = f(t,\overline{y}) & \text{in } (a,b) \\ y(a) = \alpha & y(b) = \beta \end{cases}$$

For $\overline{y}, \overline{z} \in L^2$ *we have that*

$$\|T\overline{y} - T\overline{z}\|_2 \leq \left(\frac{b-a}{\pi}\right)^2 \|(T\overline{y} - T\overline{z})''\|_2 = \frac{|k|(b-a)^2}{\pi^2}\|arctg(\overline{y}) - arctg(\overline{z})\|_2$$

$$\leq \frac{|k|(b-a)^2}{\pi^2}\|\frac{1}{1+\xi^2}\|_\infty \|\overline{y} - \overline{z}\|_2$$

for some mean value ξ *. Hence,* T *is a contraction for* $|k| < (\frac{\pi}{b-a})^2$ *.*

For $|k|$ *large,* T *is not a contraction. However, theorem 4 is still applicable, and being*

$$\|f\|_\infty \leq \|g\|_\infty + \frac{|k|\pi}{2}$$

$$\|\frac{\partial^2 f}{\partial u^2}\|_\infty \leq \frac{3\sqrt{3}}{4}|k|$$

the step ε *can be established from*

$$(b-a)^{1/2}\frac{c^2 c_0}{8} 3\sqrt{3}|k|(\|g\|_\infty + \frac{|k|\pi}{2})\varepsilon < 1$$

References

[AM] P.Amster, M.C.Mariani: Resolution of Semilinear Equations by Fixed Point Methods. To appear in the Bulletin of the Belgian Mathematical Society. Simon Stevin.

[AS] S.Ahmad, J.Salazar: On existence of periodic solutions for nonlinearly perturbed conservative systems. Differential Equations, pp. 103-114, Academic Press, Orlando, FL (1980)

[B] P.W.Bates: Solutions of nonlinear elliptic systems with meshed spectra, J.Nonlinear Anal. 4 (1980), 1023-1030

[Br] K.J.Brown, S.S.Lin: Periodically perturbed conservative systems and a global inverse function theorem. J.Nonlinear Anal. 4 (1980), 193-201

[FM] A.Fonda, J.Mawhin:Iterative and Variational Methods for the Solvability of Some Semilinear Equations in Hilbert Spaces. Journal of Differential Equations, Vol 98, No. 2 (1992).

[L] N.G.Lloyd: Degree Theory, Cambridge University Press, 1978.

[M] Mawhin, J: Boundary valuc problems for nonlinear ordinary differential equations: from successive approximations to topology.

P.Amster, M. C. Mariani and J.Sabia

Dpto. de Matemática Facultad de Ciencias Exactas y Naturales, UBA Pabellón I, Ciudad Universitaria (1428). Buenos Aires, Argentina

CONICET

E-mail: pamster@dm.uba.ar - mcmarian@dm.uba.ar - jsabia@dm.uba.ar

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