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Three-nucleon forces in the $1/N_c$ expansion

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The operator structures that can contribute to three-nucleon forces are classified in the $1/N_c$ expansion. At leading order in $1/N_c$ a spin-flavor-independent term is present, as are the spin-flavor structures associated with the Fujita-Miyazawa three-nucleon force. Modern phenomenological three-nucleon forces are thus consistent with this $\mathcal{O}(N_c)$ leading force, corrections to which are suppressed by a power series in $1/N_c^2$. A complete basis of operators for the three-nucleon force, including all independent momentum structures, is given explicitly up to next-to-leading order in the $1/N_c$ expansion.

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I. INTRODUCTION

Over the past 15 years advances in few-body methods and the steady increase in computational power have enabled numerically accurate calculations of few-nucleon scattering observables and the spectra of light nuclei. In the three-nucleon system such calculations show clear evidence for three-nucleon forces (3NFs) when compared with experimental data [1,2]. The simplest and best-known example of this is that the triton binding energy is underestimated by about 800 keV if a Hamiltonian with two-nucleon potentials alone is employed [3]. A similar underbinding occurs for other light nuclei as well [4–6]. (Although, see Ref. [7] for a study of the dependence of this conclusion on the resolution scale at which the NN potential is defined.) Indeed, the role of three-nucleon forces in the spectra of light nuclei has been a subject of intense investigation during this period (see, e.g., Refs. [8–10], as well as Ref. [2]). Recently, state-of-the-art treatments of the role of 3NFs in heavier nuclei show that they could play a role in determining the location of the neutron drip line in the oxygen and calcium isotopes [11,12] and in extending the half-life of carbon-14 [13].

Historically, 3NFs were first derived in the classic paper of Fujita and Miyazawa [14]. There, a 3NF due to the exchange of two pions was computed. This 3NF still forms a key portion of the 3NFs employed today, appearing, for example, in the Urbana three-nucleon force [15,16],

$$V_{ijk} = V_{ijk}^{2\pi} + V_{ijk}^{R}, \tag{1}$$

with [17]

$$V_{ijk}^{2\pi} = \tilde{A}_{2\pi} \frac{\sigma_i \cdot \mathbf{k}_1 \sigma_k \cdot \mathbf{k}_2}{\left(\mathbf{k}_1^2 + m_\pi^2\right)\left(\mathbf{k}_2^2 + m_\pi^2\right)} [(a + b\,\mathbf{k}_1 \cdot \mathbf{k}_2)\tau_i \cdot \tau_k + d\,\tau_i \cdot (\tau_j \times \tau_k)\sigma_j \cdot (\mathbf{k}_1 \times \mathbf{k}_2)]. \tag{2}$$

Here $\mathbf{k}_{1,2}$ are the momenta of the two pions in the exchange; σ and τ are the usual Pauli matrices for nucleon spin and isospin; and the coefficients a, b, and d represent the strength of s-wave and p-wave πN scattering. If, as was assumed by Fujita and Miyazawa, we take the p-wave pieces to arise from the spin-3/2, isospin-3/2, πN channel, where the $\Delta(1232)$ resides, we have $b = 4d$.

Meanwhile, the term V_{ijk}^R in Eq. (1) is spin and isospin independent and produces repulsion. The strength of this term and the overall strength of $V_{ijk}^{2\pi}$ are adjusted so calculations with the AV18 NN potential and this 3NF reproduce the triton binding energy and "...provide additional repulsion in hypernetted-chain variational calculations of nuclear matter near equilibrium density" [16]. The combination AV18/Urbana is quite successful in describing the spectrum of nuclei up to $A = 8$ [4,5]. But it does fail to predict the correct isospin dependence of binding in these systems and also underpredicts the spin-orbit splitting of, e.g., the $3/2^-$ and $1/2^-$ resonances in the $A = 5$ system. Consequently, the Urbana 3NF has been updated to produce a set of "Illinois" potentials, which include (phenomenologically, at least) the effect of "pion ring" diagrams and have two to three parameters that are tuned to reproduce levels in the spectra of nuclei up to $A \leqslant 8$ [18]. These potentials, when acting in concert with the AV18 NN force, do a good job of describing spectra in systems with $A = 9$ and 10 [6].

However, it is not obvious that the Urbana and Illinois potentials are grounded in QCD. Some of the structures are derived from diagrams involving pion exchange, but the coefficient functions in front of those structures are, in some cases, chosen for ease of numerical implementation and given strengths which are adjusted to reproduce data. Closer connection to the chiral symmetry of QCD was sought in, e.g., the Tucson-Melbourne $3N$ potential, which considered the role of the ρ meson, as well as the constraints of chiral symmetry on the πN amplitude which appears in the two-pion-exchange 3NF [19,20]. The Brazilian 3NF also attempted to impose constraints from chiral symmetry [21].

The advent of chiral perturbation theory (χPT) as a tool for analyzing nuclear forces resulted in the derivation of a 3NF which is in accord with the pattern of chiral-symmetry breaking in QCD [22]. If the chiral expansion is applied directly to the $3N$ potential—as was done in Ref. [22]—then three contributions occur at leading order (LO). They are as follows: a short-range, spin-isospin–independent piece [as in the V_{ijk}^R of Eq. (1)]; a piece associated with the short-range emission of a pion by an NN pair with its subsequent absorption by the third nucleon; and a two-pion-exchange

3NF. The πN amplitude that appears in the two-pion-exchange piece of the chiral 3NF involves LECs from $\mathcal{L}^{(2)}_{\pi N}$: c_1, c_3, and c_4. The LECs c_3 and c_4 encode p-wave πN scattering, so χPT has the Fujita-Miyazawa force as one of the dominant pieces of its 3NF. (Indeed, if a variant of χPT with an explicit Delta degree of freedom is employed, then the Fujita-Miyazawa 3NF occurs one order earlier than the other pieces of the chiral 3NF [22,23].) The leading χPT 3NF has been used to investigate scattering in the $3N$ system [24], and nuclear spectra in *ab initio* calculations up to $A = 13$ [9,10]. And, as mentioned above, it has, under certain approximations to the many-body physics, been shown to improve descriptions of the binding of neutron-rich nuclei [11,12]. It has also been applied to obtain an equation of state for neutron-rich matter [25].

In spite of these successes, puzzling discrepancies between theory and data persist. One example is the analyzing power A_y in neutron-deuteron scattering at low energies, with a similar issue also occurring for neutron-^{3}He scattering (see, e.g., Ref. [26]). No modification of NN potentials which is consistent with the NN data and the dominance of one-pion exchange at long range seems able to explain this discrepancy, leaving the "A_y puzzle" firmly in the realm of 3NFs to resolve. However, neither the model 3NFs on the market nor the LO chiral 3NF described in the previous paragraph can do so. Of course, extending a χPT calculation of the 3NF to higher orders in the chiral expansion might reveal the operator and mechanism (or mechanisms) which solves this problem, and work along these lines is in progress [27–30]. But, as the chiral order increases, classifying the possible 3NF operators becomes very involved. It would be interesting to have an additional tool that could help sort out the most relevant operator structures.

The $1/N_c$ expansion of QCD can be used to provide this kind of insight [31,32]. This approach to the nonperturbative regime of QCD has proven very useful in the study of baryons [33]; for reviews, see Refs. [34,35]. In the context of nuclear forces the $1/N_c$ expansion was first used to study the central part of the NN potential by Savage and Kaplan [36] and then to analyze the complete potential, classifying the relative strengths of the central, spin-orbit, and tensor forces, by Kaplan and Manohar [37]. These authors analyzed the NN potential for momenta of order N_c^0, i.e., $p \sim \Lambda_{\rm QCD}$, and found that it is an expansion in $1/N_c^2$. Furthermore, the $1/N_c^2 \approx 1/10$ (in our world) hierarchy between the different contributions to the NN potential is roughly borne out in the Nijm93 [38] NN potential. The arguments that lead to this conclusion will be recapitulated in Sec. II.

In this work we extend that analysis to the three-nucleon system, classifying the possible operator structures that can contribute to a general 3NF according to a counting in $1/N_c$. We do this by computing the energy of the $3N$ system as $N_c \to \infty$, starting with the Hartree expansion for the nuclear Hamiltonian in the large-N_c limit [37,39],

$$H = N_c \sum_{s,t,m} v_{stm} \left(\frac{S}{N_c}\right)^s \left(\frac{I}{N_c}\right)^t \left(\frac{G}{N_c}\right)^m, \qquad (3)$$

where we suppressed spin and isospin indices in the spin-flavor structures $O = \{S, I, G\}$ and vector indices in the coefficients v. These coefficients are, in fact, $\mathcal{O}(1)$ functions of the momenta. The explicit factors of $1/N_c$ ensure that an m-body interaction scales generically as $1/N_c^{m-1}$, as mandated by large-N_c QCD counting [32]. Spin, isospin, and vector indices are contracted so H is rotation and isospin invariant, as well as parity even and time-reversal even. In a quark-operator basis the spin-flavor structures are given by one-body operators

$$S^i = q^\dagger \frac{\sigma^i}{2} q, \quad I^a = q^\dagger \frac{\tau^a}{2} q, \quad G^{ia} = q^\dagger \frac{\sigma^i \tau^a}{4} q, \qquad (4)$$

where $q^\dagger$, q are creation and annihilation operators for the light quarks u, d and σ, τ are the standard SU(2) Pauli matrices acting on spin and isospin, respectively. Taken together, the 15 operators in Eq. (4) generate the SU(4) algebra,

$$\begin{aligned} &[S^i, S^j] = i\epsilon^{ijk} S^k, \quad [S^i, G^{ja}] = i\epsilon^{ijk} G^{ka}, \\ &[I^a, I^b] = i\epsilon^{abc} I^c, \quad [I^a, G^{ib}] = i\epsilon^{abc} G^{ic}, \\ &[S^i, I^a] = 0, \quad [G^{ia}, G^{ib}] = \frac{i}{4}\delta^{ij}\epsilon^{abc} I^c + \frac{i}{4}\delta^{ab}\epsilon^{ijk} S^k. \end{aligned} \qquad (5)$$

Since we are interested in taking matrix elements between nucleon states we will indicate with O_α that the operator O acts on nucleon $\alpha = 1, 2, 3$, so S, I, G in Eq. (3) can be any of $S_\alpha, I_\alpha, G_\alpha$. But products of operators acting on the same nucleon in Eq. (3) must be reduced to a single operator. As is explained in Secs. II and III B, this is achieved using the relations and reduction rules for the powers of the basic operators S, I, G that act on the same nucleon, which are discussed in Ref. [39]. The contributions to the 3NF that result after such reduction can be straightforwardly estimated, since matrix elements of S and I between nucleon states are $\mathcal{O}(1)$, which is in contrast to matrix elements of G, which are $\mathcal{O}(N_c)$. The leading force thus will be constructed out of G's and unit operators, acting on the different nucleons. In fact, the algebra Eq. (5) was derived in the one-nucleon sector for external nucleon momenta of order N_c^0, and so this conclusion holds in that kinematic regime (a similar remark applies to the NN potential derived in Ref. [37]). If results for lower momenta are desired, then the counting of operators obtained here can be modified accordingly. We present the analysis of leading and subleading 3NFs in the $1/N_c$ expansion in Sec. III and summarize our conclusions in Sec. IV.

In the large-N_c limit the mass of the nucleon tends to infinity. This provides both a problem and an opportunity for computation of the nuclear potential. The opportunity arises because, in this limit, the nuclear potential can be computed as the static energy of the system in a fixed configuration in coordinate space (for analogous studies of heavy-quark systems on the lattice see Ref. [40]). This implies that the $3N$ potential (modulo issues of exchange diagrams, see below) obtained from our argument is local, being, e.g., a function of the Jacobi coordinates r_{12} and r_3 (velocity-dependent forces arise at subleading orders in $1/N_c$ and lead to nonlocalities). The problem exists because the only measurable quantity in this infinitely-massive-nucleon limit is the total potential energy, and the large-N_c analysis gives no information on the dependence of the force on r_{12} and r_3—at least none beyond the statement that the function encoding that dependence has a size given by N_c counting. Thus, since we only "measure"

the total potential energy, and we cannot tell which pieces depend only on, say, r_{12}, we can make no *a priori* distinction between contributions to that energy from NN interactions and contributions from 3NFs. The best we can do is to identify operator structures which occur in the $3N$ energy and do not arise within the large-N_c analysis of the NN potential of Ref. [37].

One might be concerned that a 3NF derived from a large-N_c analysis cannot be in accord with the meson-exchange picture used successfully for many years to derive NN and $3N$ forces. In Refs. [41,42] Banerjee *et al.* and Belitsky and Cohen explored the relationship between this picture of the nuclear force and the large-N_c analysis of Ref. [37]. Initially, it appeared that multi-meson-exchange graphs led to violations of the large-N_c scaling of the NN potential, in particular to pieces of the NN potential that scaled with powers of N_c larger than 1. However, Ref. [43] later explained this apparent discrepancy between the meson-exchange and large-N_c pictures by pointing out that the potentials analyzed in Refs. [41,42] were energy dependent, whereas almost all NN interactions used for phenomenological purposes are energy independent. Reference [43] concluded that an energy-independent NN potential could have N_c scaling consistent with that derived in Ref. [37], and so large-N_c analysis is not inconsistent with a meson-exchange picture of nuclear forces for the NN case. An important point for a successful matching calculation is that the Hartree Hamiltonian Eq. (3) and the SU(4) algebra Eq. (5) implicitly assume the presence of the Δ resonance with $S = I = 3/2$. In our discussion of the NN and NNN potentials, when taking matrix elements, we project H to the nucleons-only piece of the Hilbert space. We have not performed a matching calculation to check the consistency with the meson-exchange picture for the $3N$ potential, but it would be a worthy subject for future study.

One might also wonder whether double counting will result if the $3N$ potential obtained from the large-N_c analysis is used in a multinucleon Schrödinger equation. To address this issue we note that another assumption made in the derivation of the algebra Eq. (5) was that meson energies are of order $\Lambda_{\rm QCD}$. This implies that the energy of the intermediate nucleon state in the 3NF (see, e.g., Fig. 1) must be order $\Lambda_{\rm QCD}$ if an analysis based on this algebra is to prevail. Having states of this energy included in the computation of the nuclear potential is consistent with the insertion of the resulting nuclear force in the $3N$ Schrödinger equation (or, equivalently, a Faddeev equation) provided a momentum cutoff is employed there. If that momentum cutoff is above $\Lambda_{\rm QCD}$, but below $\sqrt{N_c}\Lambda_{\rm QCD}$,

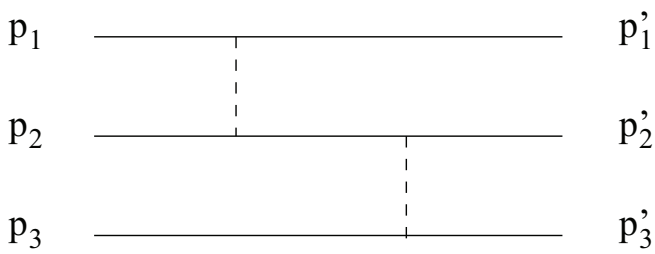


FIG. 1. Tree-level two-meson exchange contribution to the 3NF.

the intermediate nucleonic states with energies of order $\Lambda_{\rm QCD}$ (i.e., momenta $\sim \sqrt{M\Lambda_{\rm QCD}}$) will not be accounted for by the iteration of the potential via the Schrödinger/Faddeev equation and so should be included in the potential. The NN and $3N$ interactions derived here, and in Ref. [37], thus can be inserted into the quantum-mechanical equation and used to compute the wave function of nuclear systems.

With the conceptual underpinning of a 3NF in large-N_c QCD defined, and the circumstances under which it should be used in a Schrödinger equation for a multinucleon system clarified, we now turn back to the NN system in order to explain how the corresponding analysis works in that, simpler, case.

II. THE NN POTENTIAL IN THE $1/N_c$ EXPANSION: REVIEW

Here we review the $1/N_c$ analysis of Kaplan and Manohar [37] for the two-nucleon potential, setting up the notation that we will use later in Sec. III to analyze the three-nucleon force. In Ref. [37] the large-N_c expansion was used to analyze the object,

$$U^A_{NN} = (1 - P_{12})U, \tag{6}$$

where U is the sum of all direct diagrams and P_{ij} is the permutation operator that switches *all* quantum numbers of particles i and j. In nuclear physics computations it is the operator U which is inserted into the Schrödinger equation. The correct antisymmetry properties of the nuclear state are then imposed by computing matrix elements only in partial waves which are allowed by the Fermi-Dirac statistics of the nucleons.

In order to discuss the momentum dependence of the potential, we first define initial and final relative momenta as follows:

$$\mathbf{p} = \mathbf{p}_1 - \mathbf{p}_2, \quad \mathbf{p}' = \mathbf{p}'_1 - \mathbf{p}'_2, \tag{7}$$

where $\mathbf{p}_i$ ($\mathbf{p}'_i$) is the initial (final) momentum of the i-th nucleon. To simplify later analysis we also define time-reversal-odd (T-odd) and time-reversal-even (T-even) combinations of these,

$$\mathbf{p}_\pm = \mathbf{p}' \pm \mathbf{p}. \tag{8}$$

Notice that $\mathbf{p}_+$ is T-odd and $\mathbf{p}_-$ is T-even, as initial and final states are also exchanged under time reversal. Both combinations, being vectors, are odd under parity. In U only $\mathbf{p}_-$ enters at leading order in N_c since the potential is local at this order in the $1/N_c$ expansion. Powers of $\mathbf{p}_+$ indicate the presence of nonlocality. In a meson-exchange picture they arise due to the occurrence of relativistic corrections suppressed by $1/M_N$. Thus, each appearance of a power of $\mathbf{p}_+$ costs a power of $1/N_c$. Finally, energy conservation and the constraint that the external NN states in a diagram be on-shell results in

$$\mathbf{p}_+ \cdot \mathbf{p}_- = 0, \tag{9}$$

which allows us to eliminate this momentum structure. In Ref. [37] the potential U was written as a sum of products of one-body operators, including the explicit factors of $1/N_c$ as shown in the Hartree Hamiltonian, Eq. (3). Isospin invariance of the interaction requires that all isospin indices are contracted.

In general, operators acting on the same nucleon with spatial or isospin indices contracted can be simplified. For instance, $G^{ia}G^{ia}$ can be reduced to the unit operator and a subleading contribution using

$$G^{ia}G^{ia} = \tfrac{3}{16}N_c(N_c+4)\mathbb{1} - \tfrac{1}{4}I^aI^a - \tfrac{1}{4}S^iS^i, \qquad (10)$$

which is obtained from the quadratic SU(4) Casimir evaluated on the completely symmetric representation S_{N_c}. If the spatial indices are not contracted we have the more general identity

$$G^{ia}G^{ja} = \tfrac{1}{16}N_c(N_c+4)\delta^{ij}\mathbb{1} - \tfrac{1}{4}\delta^{ij}S^2 + \tfrac{1}{4}S^iS^j + \tfrac{i}{4}\epsilon^{ijk}S^k. \qquad (11)$$

The complete set of operator reduction rules can be found in Ref. [39].

It is thus sufficient to consider structures where the contracted indices are carried by operators acting on different nucleons. For instance, the leading order of the angular momentum zero ($L=0$) component of the potential is obtained from

$$U^{N_c}_{L=0} \subset N_c\sum_{n=0}^{N_c} u_n(\mathbf{p}_-^2)\left(N_c^{-2}G_1^{ia}G_2^{ia}\right)^n, \qquad (12)$$

where $u_n(\mathbf{p}_-^2)$ are arbitrary scalar functions of $\mathbf{p}_-^2$ that scale like $\mathcal{O}(N_c^0)$. This yields two strings of G's, one on each of the two nucleons, with no contracted indices among the G's which act on an individual nucleon. Each such string of G's can, nevertheless, be reduced, because the matrix element of a general m-quark operator between single-baryon states scales as [36,37,39]

$$\langle \mathrm{B}_1|N_c^{-m}O_m|\mathrm{B}_1\rangle = \frac{1}{N_c^{|I-S|}}. \qquad (13)$$

Therefore, the dominant parts in the operator resulting from each string of G's have $I=S$. If, in addition, we restrict ourselves to the case that the baryon is a nucleon only $(I,S) = (0,0),(1,1)$ contribute. But those $I=S=0$ and $I=S=1$ operators can, via the Wigner-Eckart theorem, be replaced by the $\mathcal{O}(N_c^0)$ one-body operators $\mathbb{1}$ and $N_c^{-1}G^{ia}$, up to a proportionality constant that ultimately gets absorbed in the undetermined functions of momenta that appear in the large-N_c NN potential. Thus, on a single-nucleon state, each string of G's with uncontracted indices yields a matrix element that can be written as follows:

$$\langle N|\underbrace{GG\ldots G}_{r}|N\rangle = N_c^r\langle\mathbb{1}\rangle + N_c^{r-1}\langle G\rangle + \mathcal{O}\left(N_c^{r-2}\right), \qquad (14)$$

where the spatial and isospin indices on the right-hand side of Eq. (14) are carried by Kronecker δ's and the completely antisymmetric tensor ϵ. For an example, see the appendix, in particular Eq. (A7).

Equations (14) and (11) show that it is enough to consider the one-quark operators $\mathbb{1}$ and $N_c^{-1}G_{ia}$ acting within each nucleon to construct the leading-order spin-flavor structures. With this simple rule one obtains correctly the explicit $1/N_c$ suppression factors contained in the Hartree expression, Eq. (3), for the NN interaction.

The leading-order spin-flavor structures are thus $\mathbb{1}_1\mathbb{1}_2$ and $G_1^{ia}G_2^{ja}$. The next step is to project out the different spin components of the leading-order G_1G_2 tensor, namely

$$G_1^{ia}G_2^{ia}, \quad \epsilon^{ijk}G_1^{ja}G_2^{ka}, \quad \left[G_1^{ia}G_2^{ja}\right]_2, \qquad (15)$$

where the first two correspond to $S=0$ and 1, respectively, and

$$\left[G_1^{ia}G_2^{ja}\right]_2 \equiv G_1^{ia}G_2^{ja} + G_1^{ja}G_2^{ia} - \tfrac{2}{3}\delta^{ij}G_1^{ka}G_2^{ka} \qquad (16)$$

is the $S=2$ component. The final step is the reduction of the operator G to $\sigma^i\tau^a$ when restricted to the nucleon subspace. Table I shows a complete set of independent spin-flavor structures in the NN subspace, together with their $1/N_c$ scalings, spin content, and time-reversal properties.

TABLE I. Spin-flavor structures for the two-nucleon potential. The $(\sigma_1\times\sigma_2)$ structure arises in the large-N_c analysis, but its appearance in U is precluded by permutation symmetry.

O	Order	$O_{\tau\tau}$	Order	S	T
$\mathbb{1}$	1	$\tau_1\cdot\tau_2$	$1/N_c^2$	0	+
$\sigma_1\cdot\sigma_2$	$1/N_c^2$	$\sigma_1\cdot\sigma_2\tau_1\cdot\tau_2$	1	0	+
σ_1^i	$1/N_c$	$\sigma_1^i\tau_1\cdot\tau_2$	$1/N_c$	1	−
σ_2^i	$1/N_c$	$\sigma_2^i\tau_1\cdot\tau_2$	$1/N_c$	1	−
$(\sigma_1\times\sigma_2)^k$	$1/N_c^2$	$(\sigma_1\times\sigma_2)^k\tau_1\cdot\tau_2$	1	1	+
$[\sigma_1^i\sigma_2^j]_2$	$1/N_c^2$	$[\sigma_1^i\sigma_2^j]_2\tau_1\cdot\tau_2$	1	2	+

Each of these spin-flavor structures then must be combined with tensors formed out of the momenta $\mathbf{p}_-$, $\mathbf{p}_+$ to form a T-even, P-even, rotationally invariant operator. In particular, the $S=2$ structure (16) must be contracted with a spatial tensor of rank two. Since at LO we have a local NN potential the only possible LO tensor is $\mathbf{p}_-^i\mathbf{p}_-^j$. Meanwhile, the second $(S=1)$ spin-flavor structure must be contracted with a three-vector. Parity invariance suggests $\mathbf{p}_-\times\mathbf{p}_+$ is the only possible candidate. However, $\mathbf{p}_-\times\mathbf{p}_+$ is odd under time reversal. And the constraint (9) means we cannot multiply by powers of the T-odd rotational scalar $\mathbf{p}_+\cdot\mathbf{p}_-$—at least not on-shell. Thus our $S=1$ spin-flavor structure cannot be multiplied by any combination of three-vectors that results in an overall P-even, T-even object. The operator $\epsilon^{ijk}G_1^{ja}G_2^{ka}$ therefore will not appear in the parity-conserving, time-reversal-non-violating NN force [44]. Finally, the first structure in Eq. (15) and the unit operator are the two leading-order $S=0, L=0$ operators.

The rotational scalars formed in this way may always be multiplied by an arbitrary scalar function of $\mathbf{p}_-^2$. Therefore, to leading order

$$U^{N_c} = N_c\big(U_S^1(\mathbf{p}_-^2)\mathbb{1} + U_S^2(\mathbf{p}_-^2)\sigma_1\cdot\sigma_2\tau_1\cdot\tau_2 + U_D^1(\mathbf{p}_-^2)[\mathbf{p}_-\mathbf{p}_-]_2\cdot[\sigma_1\sigma_2]_2\tau_1\cdot\tau_2\big), \qquad (17)$$

with

$$[A_iB_j]_2 \equiv A_iB_j + A_jB_i - \tfrac{2}{3}\delta_{ij}A\cdot B, \qquad (18)$$

the $L=2$ component of the tensor A_iB_j constructed from two vector quantities, and $U_S^{1,2}(\mathbf{p}_-^2)$, $U_D^1(\mathbf{p}_-^2)$ arbitrary $\mathcal{O}(1)$ scalar functions of $\mathbf{p}_-^2$. As discussed above, there are no $S=1$ terms at leading order.

Subleading corrections are associated with $1/N_c$-suppressed operators. Such suppression may occur for two

reasons. First, NN operators involving S and I, instead of G, will be reduced by factors of $1/N_c$, because of the N_c scaling of the nucleonic matrix elements of these operators. The second source of $1/N_c$ suppression is the appearance in expressions of the momentum $\mathbf{p}_+$. Time reversal and parity conservation conspire so the expansion is in $1/N_c^2$.

With these two results regarding $1/N_c$ suppression in hand, Kaplan and Manohar concluded that the following operators give contributions to the NN potential of $\mathcal{O}(1/N_c)$ (see also Table I) as follows:

$$\begin{aligned} U^{1/N_c} = {} & \delta^{(2)} U^{N_c} + N_c^{-1}\big(U_S^3 \mathbf{p}_+^2 \mathbb{1} + U_S^4 \sigma_1\cdot\sigma_2 + U_S^5 \tau_1\cdot\tau_2 \\ & + U_S^6 \mathbf{p}_+^2 \sigma_1\cdot\sigma_2 \tau_1\cdot\tau_2 + U_P^1 (\mathbf{p}_+\times\mathbf{p}_-)\cdot(\sigma_1+\sigma_2) \\ & + U_P^2 (\mathbf{p}_+\times\mathbf{p}_-)\cdot(\sigma_1+\sigma_2)\tau_1\cdot\tau_2 \\ & + U_D^2 [\mathbf{p}_-\mathbf{p}_-]_2\cdot[\sigma_1\sigma_2]_2 \\ & + U_D^3 [\mathbf{p}_+\mathbf{p}_+]_2\cdot[\sigma_1\sigma_2]_2 \tau_1\cdot\tau_2\big). \end{aligned} \tag{19}$$

At this order the leading-order operators appear again, as they can also be obtained by replacing one G^{ia}/N_c by $(S^i I^a)/N_c^2$ in the Hartree Hamiltonian. We denoted this contribution by $\delta^{(2)} U^{N_c}$ in the expression above. The spin-flavor structures that appear here (and in Table I) and momentum tensors with up to four momenta can also be read off from the results for the 3NF that will be presented later in Sec. III by eliminating the third nucleon and only keeping momentum structures that depend on $\mathbf{p}_\pm$. Here we only show the potential up to quadratic structures in momenta (modulo arbitrary functions of $\mathbf{p}_-$).

Comparing Table I with Eqs. (17) and (19), one can see that the spin-flavor structures proportional to $\sigma_1\times\sigma_2$ are missing because, as discussed for the LO case, they need to be multiplied by a T-even, P-even, $L=1$ momentum structure, which cannot be constructed in the NN case.

However, there is an additional constraint from permutation symmetry [44]. For example, σ_1, σ_2 appear only in the $\sigma_1+\sigma_2$ combination. The $\sigma_1-\sigma_2$ combination is excluded by permutation symmetry, as it is T-odd and parity-even and needs to be contracted with a vector built from $\mathbf{p}_+$, $\mathbf{p}_-$, where $\mathbf{p}_\pm$ are both odd under exchange of the nucleons 1, 2. For instance, if we would start from the general structure

$$U(\mathbf{p}_-^2)(\mathbf{p}_+\times\mathbf{p}_-)\sigma_1 + U'(\mathbf{p}_-^2)(\mathbf{p}_+\times\mathbf{p}_-)\sigma_2 \tag{20}$$

permutation symmetry imposes $U=U'=U_P^1$ so only the symmetric spin-flavor structure $\sigma_1+\sigma_2$ appears in Eq. (19). The $\sigma_1\times\sigma_2$ structure can also be eliminated by permutation symmetry.

In summary, to leading order $[\mathcal{O}(N_c)]$ there are two structures with $L=0$ and one with $L=2$. To subleading order $[\mathcal{O}(1/N_c)]$ and up to two momenta, there are four structures with $L=0$, two with $L=1$ and two with $L=2$.

This translates into definite scaling predictions for the different parts of the NN potential, which in the usual form is given by

$$\begin{aligned} V_{NN} = {} & V_C^0 + V_{SS}^0 S_1\cdot S_2 + V_{LS}^0 L\cdot S + V_T^0 S_{12} + V_Q^0 Q_{12} \\ & + \big(V_C^1 + V_{SS}^0 S_1\cdot S_2 + V_{LS}^1 L\cdot S + V_T^1 S_{12} \\ & + V_Q^1 Q_{12}\big)\tau_1\cdot\tau_2. \end{aligned} \tag{21}$$

Here L is the angular momentum operator, which is T-odd and P-even and in our notation is replaced via the Wigner-Eckart theorem by the $\mathbf{p}_+\times\mathbf{p}_-$ structure. The quadratic spin-orbit interaction Q_{12} involves four momenta in our notation and we did not include it in Eq. (19).

A comparison with "experiment" can be achieved by comparing with a successful phenomenological potential. This has been done in Ref. [37] using the Nijmegen potential [38]. The $1/N_c$ scaling of the different structures in Eq. (21) translates into a hierarchy for the functions used to parametrize the Nijmegen potential, which is well satisfied by their numerical values, as discussed in detail in Ref. [37].

Although in the two-nucleon case the operator structure of the interaction is simple enough to be obtained by explicit construction, as sketched above, at this point it is useful to discuss a more systematic way of counting the number of spin-flavor structures that can contribute, something that will prove very useful in the more involved three-nucleon case. The systematic classification can be done as follows.

The number of independent spin-flavor structures O_{IS} of isospin I and spin S that can contribute to the matrix element $\langle NN|O_{IS}|NN\rangle$ can be obtained by considering the decomposition of $\overline{\mathbf{R}}\otimes\mathbf{R}'$, with $\mathbf{R},\mathbf{R}'$ the irreducible representations of spin flavor for the two nucleons, so the matrix element is a scalar. To obtain the possible irreps $\mathbf{R}$, we decompose the tensor product of two-nucleon states, each nucleon transforming as the fundamental representation of SU(4),

$$\overset{\mathbf{4}}{\square} \equiv \{p\uparrow, p\downarrow, n\uparrow, n\downarrow\}. \tag{22}$$

The two-nucleon states are obtained as the decomposition of the tensor product $\mathbf{4}\otimes\mathbf{4}=\mathbf{6}\oplus\mathbf{10}$. In terms of Young tableaux

$$\overset{\mathbf{4}}{\square}\otimes\overset{\mathbf{4}}{\square} = \overset{\mathbf{6}}{\begin{array}{l}\square\\\square\end{array}} \oplus \overset{\mathbf{10}}{\square\square}. \tag{23}$$

As states and operators are labeled by their isospin and spin transformation properties, we decompose SU(4) irreps in $\mathrm{SU}(2)_I\times\mathrm{SU}(2)_S\subset\mathrm{SU}(4)$, labeled by $(2I+1, 2S+1)$. The result is [only SU(4) irreps are in bold]:

$$\overset{\mathbf{10}}{\square\square} = \left(\overset{3}{\square\square}, \overset{3}{\square\square}\right) \oplus \left(\overset{1}{\begin{array}{l}\square\\\square\end{array}}, \overset{1}{\begin{array}{l}\square\\\square\end{array}}\right), \tag{24}$$

$$\overset{\mathbf{6}}{\begin{array}{l}\square\\\square\end{array}} = \left(\overset{3}{\square\square}, \overset{1}{\begin{array}{l}\square\\\square\end{array}}\right) \oplus \left(\overset{1}{\begin{array}{l}\square\\\square\end{array}}, \overset{3}{\square\square}\right). \tag{25}$$

With this result in hand, we can determine the number, and type, of spin-flavor structures that occur in O_{IS}. We consider the decomposition of $\overline{\mathbf{R}}\otimes\mathbf{R}'$, with $\mathbf{R},\mathbf{R}'=\mathbf{6},\mathbf{10}$ into irreps of $\mathrm{SU}(2)_I\times\mathrm{SU}(2)_S$. We are interested in the pieces of the direct product that yield $I=0$ operators, which are

$$\sum_{\mathbf{R},\mathbf{R}'} \overline{\mathbf{R}}\otimes\mathbf{R}' \supset 4(0,0)\oplus 6(0,3)\oplus 2(0,5)+\ldots, \tag{26}$$

i.e., the direct product contains four independent isoscalar structures of $S = 0$, six of $S = 1$ and two of $S = 2$. Their explicit forms are $\mathbb{1}$, $\sigma_1 \cdot \sigma_2$, σ_1, σ_2, $\sigma_1 \times \sigma_2$ and $[\sigma_1\sigma_2]_2$, each of which can be multiplied by any of the two isospin invariants $\mathbb{1}$, $\tau_1 \cdot \tau_2$. The resulting spin-flavor structures are shown in Table I.

This finishes the review of the NN case. We proceed now to the construction of the $3N$ potential.

III. THE 3N POTENTIAL IN THE 1/N_c EXPANSION

In this section we will extend the analysis that we reviewed for the NN potential to the case of the $3N$ potential. The sum of all $3N \to 3N$ diagrams can be written in operator form as

$$V_{3N}^{A} = (1 + P_{12}P_{23} + P_{13}P_{23})(1 - P_{23})V. \tag{27}$$

The terms in parentheses in Eq. (27) thus generate the exchange diagrams necessitated by the identicality of the nucleons from the operator V, which itself is the sum of all direct diagrams (see, e.g., Fig. 1) and is the object that enters the Schrödinger equation in nuclear-physics computations. We will classify the structures that contribute to V and derive their scaling behavior with N_c.

We do this by first discussing the momenta involved and the possible momentum structures obtainable therefrom. We then derive the LO spin-flavor structures and count all possible spin-flavor structures. We finish with the explicit construction of the operators, including the spatial part.

A. Momenta and momentum structures

Throughout, we work in the $3N$ center-of-mass frame, where

$$\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = \mathbf{p}_1' + \mathbf{p}_2' + \mathbf{p}_3' = 0. \tag{28}$$

Any graph then can be expressed as a function of the Jacobi momenta $\mathbf{p}$ and $\mathbf{q}$,

$$\mathbf{p} = \mathbf{p}_1 - \mathbf{p}_2, \quad \mathbf{q} = \mathbf{p}_3 - (\mathbf{p}_1 + \mathbf{p}_2)/2, \tag{29}$$

so the 3NF can be written as a function of four three-momenta $\mathbf{p}$, $\mathbf{p}'$, $\mathbf{q}$, and $\mathbf{q}'$.

Conservation of energy yields the constraint

$$\mathbf{p}^2 + \tfrac{4}{3}\mathbf{q}^2 = \mathbf{p}'^2 + \tfrac{4}{3}\mathbf{q}'^2. \tag{30}$$

In terms of the momenta with well-defined properties under time reversal, $\mathbf{p}_\pm$ as in Eq. (8), and the analogous $\mathbf{q}_\pm = \mathbf{q}' \pm \mathbf{q}$, the constraint (30) becomes

$$\mathbf{p}_+ \cdot \mathbf{p}_- = -\tfrac{4}{3}\mathbf{q}_+ \cdot \mathbf{q}_-, \tag{31}$$

which will allow us to eliminate $\mathbf{q}_+ \cdot \mathbf{q}_-$ in favor of $\mathbf{p}_+ \cdot \mathbf{p}_-$. Analysis of the contributions to Fig. 1 shows that the presence of $\mathbf{p}_+$, $\mathbf{q}_+$ comes from relativistic corrections that introduce powers of $1/M_N$, so each power of either $\mathbf{p}_+$ or $\mathbf{q}_+$ is associated with a suppression factor of $1/N_c$. The LO momentum structures are $\mathcal{O}(N_c^0)$ and depend only on $\mathbf{p}_-$ and $\mathbf{q}_-$. They correspond to local potentials.

In fact, since all spin-flavor structures are built from P-even objects, parity invariance of V requires that the momentum structures appearing—at both leading and subleading orders in $1/N_c$—must contain an even number of momenta. In Table II we show the TP properties, L content, and order in $1/N_c$ of the $3N$-system momentum tensors which contain up to two momenta. Time-reversal-odd momentum structures appear only at subleading orders, as they must include at least either a $\mathbf{p}_+$ or a $\mathbf{q}_+$.

TABLE II. Tensors $\Pi^{\rm TP}$ constructed from up to two $3N$-system momenta, together with their T and P properties and their angular momentum (L) content. Note that in the NN system none of the tensors involving $\mathbf{q}_\pm$ are present. The $\pm$ signs in the subscripts are always to be read as correlated, so the last four entries in the table each contain two possible tensors. The last column shows the $1/N_c$ order at which the corresponding momentum structure appears.

T	P	$\Pi^{\rm TP}$	L	Order
+	−	$\mathbf{p}_-$	1	1
+	−	$\mathbf{q}_-$	1	1
−	−	$\mathbf{p}_+$	1	$1/N_c$
−	−	$\mathbf{q}_+$	1	$1/N_c$
−	+	$\mathbf{p}_+\mathbf{p}_-$	0,1,2	$1/N_c$
−	+	$\mathbf{q}_+\mathbf{q}_-$	0,1,2	$1/N_c$
−	+	$\mathbf{p}_\pm\mathbf{q}_\mp$	0,1,2	$1/N_c$, $1/N_c$
+	+	$\mathbf{p}_\pm\mathbf{q}_\pm$	0,1,2	$1/N_c^2$, 1
+	+	$\mathbf{p}_\pm\mathbf{p}_\pm$	0,2	$1/N_c^2$, 1
+	+	$\mathbf{q}_\pm\mathbf{q}_\pm$	0,2	$1/N_c^2$, 1

B. Leading spin-flavor structures

As in the NN case, leading-order spin-flavor structures are obtained from products of an arbitrary number of G's, with their indices contracted in order to get isoscalar operators of spin $S = 0, 1, 2, 3$, which are the only quantum numbers relevant for isospin conserving interactions in the $3N$ subspace. Any spatial indices associated with the spin tensor of rank 0, 1, 2, or 3 are then contracted with a momentum tensor of the same rank to form a singlet, so the interaction is invariant under rotations.

For example, the leading-order $S = 0$ $3N$ structures are obtained from

$$\begin{aligned} V_{L=0}^{N_c} \subset N_c \sum_{\substack{n_{12},n_{13},n_{23}\\ n_{123}}} v_{n_{12},n_{13}\ldots} \left(N_c^{-2} G_1^{ia} G_2^{ia}\right)^{n_{12}} \\ \times \left(N_c^{-2} G_1^{jb} G_3^{jb}\right)^{n_{13}} \left(N_c^{-2} G_2^{kc} G_3^{kc}\right)^{n_{23}} \\ \times \left(N_c^{-3} \epsilon^{lmr} \epsilon^{def} G_1^{ld} G_2^{me} G_3^{rf}\right)^{n_{123}} + \cdots, \end{aligned} \tag{32}$$

where the dots stand for terms with more complex index contractions. A general structure has the form $O_\alpha O_\beta O_\gamma$, with the greek index indicating the nucleon on which a particular O acts. As in the NN case, products like $G_1^{ia} G_1^{ja}$, where there is at least one index contracted between operators acting on the same nucleon, are not included. The structures shown in Eq. (32) still seem hard to reduce, but this can be achieved after taking matrix elements in the NNN subspace using Eq. (13) and Eq. (14), as we did in the NN case. The simple rule is again that, at leading order, an arbitrary product of G's can be reduced to a sum of $I = S$ operators, which for the N subspace reduce just to the unit operator and one G. So, as

TABLE III. Leading-order quark operators and their projection on nucleon spin-isospin structures. Structures are listed according to their spin content within the nucleonic space. α, β, γ are a permutation of 1,2,3, designating on which nucleon the spin and isospin operators act. The multiplicity indicates how many independent structures are generated by these permutations. The 17 leading-order structures are all parity even and time-reversal even.

Spin content	LO quark operator	$\sigma\tau$ projection	Multiplicity
$S=0$	$\mathbb{1}$	$\mathbb{1}$	1
	$N_c^{-2} G_\alpha^{ia} G_\beta^{ia}$	$\sigma_\alpha \cdot \sigma_\beta \tau_\alpha \cdot \tau_\beta$	3
	$N_c^{-3} \epsilon^{ijk}\epsilon^{abc} G_\alpha^{ia} G_\beta^{jb} G_\gamma^{kc}$	$(\sigma_\alpha \times \sigma_\beta)\cdot\sigma_\gamma(\tau_\alpha \times \tau_\beta)\cdot\tau_\gamma$	1
$S=1$	$N_c^{-2}\epsilon^{ijk} G_\alpha^{ia} G_\beta^{ja}$	$(\sigma_\alpha \times \sigma_\beta)\tau_\alpha \cdot \tau_\beta$	3
	$N_c^{-3}\epsilon^{abc} G_\alpha^{ia} G_\beta^{ib} G_\gamma^{kc}$	$(\sigma_\alpha \cdot \sigma_\beta)\sigma_\gamma(\tau_\alpha \times \tau_\beta)\cdot\tau_\gamma$	3
$S=2$	$N_c^{-2}[G_\alpha^{ia} G_\beta^{ja}]_2$	$[\sigma_\alpha\sigma_\beta]_2 \tau_\alpha \cdot \tau_\beta$	3
	$N_c^{-3}\epsilon^{abc}[(G_\alpha^{ia} G_\beta^{jb}\epsilon^{ijl})G_\gamma^{kc}]_2$	$[(\sigma_\alpha \times \sigma_\beta)\sigma_\gamma]_2(\tau_\alpha \times \tau_\beta)\cdot\tau_\gamma$	2
$S=3$	$N_c^{-3}\epsilon^{abc}[G_\alpha^{ia} G_\beta^{jb} G_\gamma^{kc}]_3$	$[\sigma_\alpha\sigma_\beta\sigma_\gamma]_3(\tau_\alpha \times \tau_\beta)\cdot\tau_\gamma$	1
			17

in the NN force, the LO structures are found by considering one-quark operators $\mathbb{1}$ and $N_c^{-1}G^{ia}$ acting on each nucleon. This gives the explicit $1/N_c$ suppression factors that come from the spin-flavor part. Then, within the N subspace, we replace G^{ia} by $\sigma^i\tau^a$. Bearing in mind that spin and isospin indices should be contracted with δ_{ij}, δ_{ab} or ϵ_{ijk}, ϵ_{abc} tensors, one straightforwardly obtains the leading spin-flavor structures shown in Table III.

The isospin structures are the unit operator, the three scalar products $\tau_\alpha \cdot \tau_\beta$, and a new structure that was not present in the NN case, the triple product $(\tau_\alpha \times \tau_\beta)\cdot\tau_\gamma$. It is important to notice that the triple product of τ is time-reversal odd, as under time reversal $(\tau^1, \tau^2, \tau^3) \to (\tau^1, -\tau^2, \tau^3)$. This is in contrast to $(\sigma^1, \sigma^2, \sigma^3) \to (-\sigma^1, -\sigma^2, -\sigma^3)$. The different transformation properties of the spin and isospin operators under time reversal just reflect the fact that under time reversal spins get flipped, while protons and neutrons retain their identity and are not exchanged.

The last column of Table III shows the multiplicity of each structure, obtained by running α, β, γ over all the permutations of $1, 2, 3$. For the spin-2 structures a nontrivial constraint reduces the multiplicity of the $[(\sigma_\alpha \times \sigma_\beta)\sigma_\gamma]_2 = \{[(\sigma_1 \times \sigma_2)\sigma_3]_2, [(\sigma_1 \times \sigma_3)\sigma_2]_2, [(\sigma_2 \times \sigma_3)\sigma_1]_2\}$ operator structure from three to two, because Eq. (B3), projected onto a symmetric and traceless rank-2 tensor, gives

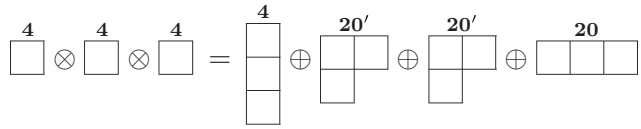
$$[(A \times B)C]_2 + [(B \times C)A]_2 + [(C \times A)B]_2 = 0. \tag{33}$$

There are 17 independent structures at leading order. They are all time-reversal even. Further details are given below, with the leading potential exhibited in Eqs. (39), (43), (45), and (48).

C. Counting all the spin-flavor structures

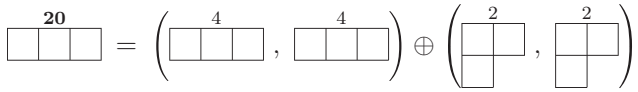
However, in order to enumerate all subleading structures, we find it important to first generalize our counting of spin-flavor structures using SU(4) irreps from the NN to the NNN case. In this way we determine the number of spin-flavor structures we expect to find once we consider all orders in N_c.

The number of NNN states is given by $\mathbf{4}\otimes\mathbf{4}\otimes\mathbf{4} = \mathbf{4}\oplus\mathbf{20'}\oplus\mathbf{20'}\oplus\mathbf{20}$,

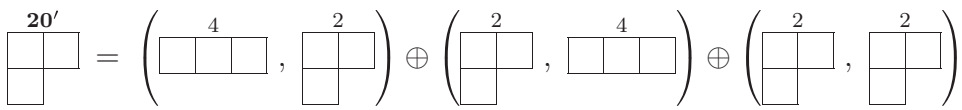


(34)

Decomposing these SU(4) irreps into $\mathrm{SU}(2)_I \times \mathrm{SU}(2)_S \subset \mathrm{SU}(4)$, we have [as above, the $\mathrm{SU}(2)_I \times \mathrm{SU}(2)_S$ irreps are labeled by $(2I+1, 2S+1)$ and only SU(4) irreps are in bold]:



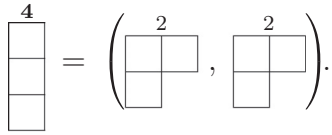
(35)



(36)

TABLE IV. $S = 0$ spin-isospin structures. The order given in the second column is relative to N_c. The third column indicates the behavior of each structure structure under time reversal, namely, even (+) or odd (–), and its multiplicity is given in the last column. Here $\alpha \neq \beta \neq \gamma$ label the nucleon on which each of the spin and isospin operators act. In the last line we give the total number of independent structures, obtained as the sum of $M_0^{(0)} = 5$, $M_0^{(1)} = 6$, $M_0^{(2)} = 12$, and $M_0^{(3)} = 2$.

Operator	Order	T	Multiplicity
$S_1^{(0)} = \mathbb{1}$	1	+	1
$S_{2-4}^{(0)} = \sigma_\alpha \cdot \sigma_\beta \tau_\alpha \cdot \tau_\beta$	1	+	3
$S_5^{(0)} = (\sigma_\alpha \times \sigma_\beta) \cdot \sigma_\gamma (\tau_\alpha \times \tau_\beta) \cdot \tau_\gamma$	1	+	1
$S_{1-3}^{(1)} = \sigma_\alpha \cdot \sigma_\beta (\tau_\alpha \times \tau_\beta) \cdot \tau_\gamma$	$1/N_c$	–	3
$S_{4-6}^{(1)} = (\sigma_\alpha \times \sigma_\beta) \cdot \sigma_\gamma \tau_\alpha \cdot \tau_\beta$	$1/N_c$	–	3
$S_{1-3}^{(2)} = \tau_\alpha \cdot \tau_\beta$	$1/N_c^2$	+	3
$S_{4-6}^{(2)} = \sigma_\alpha \cdot \sigma_\beta$	$1/N_c^2$	+	3
$S_{7-12}^{(2)} = \sigma_\alpha \cdot \sigma_\beta \tau_\beta \cdot \tau_\gamma$	$1/N_c^2$	+	6
$S_1^{(3)} = (\tau_\alpha \times \tau_\beta) \cdot \tau_\gamma$	$1/N_c^3$	–	1
$S_2^{(3)} = (\sigma_\alpha \times \sigma_\beta) \cdot \sigma_\gamma$	$1/N_c^3$	–	1
			25



(37)

The number of independent operators O_{IS} of isospin I and spin S that can contribute to $\langle NNN|O_{IS}|NNN\rangle$ can now be obtained by considering the decomposition of $\overline{\mathbf{R}} \otimes \mathbf{R}'$, with $\mathbf{R}, \mathbf{R}' = \mathbf{4}, \mathbf{20}', \mathbf{20}$ into irreps of $\mathrm{SU}(2)_I \times \mathrm{SU}(2)_S$. Notice that $\mathbf{20}'$ has to be considered twice. We are interested in $I = 0$ spin-flavor structures, and for those we find

$$\sum_{R,R'} \overline{\mathbf{R}} \otimes \mathbf{R}' \supset 25(0,0) \oplus 45(0,3) \oplus 25(0,5) \oplus 5(0,7) + \ldots . \tag{38}$$

So, there are 25 independent isoscalar structures of $S = 0$, 45 of $S = 1$, 25 of $S = 2$, and 5 of $S = 3$. There are thus 100 spin-flavor structures in total, 50 T-even and 50 T-odd. This provides an important check for the explicit construction of operators that we will describe in the next subsection. The results of that construction are shown in Tables IV–VII, and we indeed find a total of 100 structures.

D. Explicit construction of the three-nucleon operators

In the following sections we write down, successively, the $3N$ potential-energy operators which are built from $S = 0, 1, 2, 3$ spin-flavor structures. Since we seek rotational scalars, each spin-flavor structure is coupled to a momentum structure of equal rank. We therefore use the terms "$L = a$" and "$S = a$" interchangeably when referring to the operators that appear in V. We present explicit expressions up to $\mathcal{O}(1/N_c)$.

TABLE V. $S = 1$ spin-isospin structures, as in Table IV. In the last line we give the total number of independent structures, obtained as the sum of $M_1^{(0)} = 6$, $M_1^{(1)} = 21$, $M_1^{(2)} = 12$, and $M_1^{(3)} = 6$.

Operator	Order	T	Multiplicity
$P_{1-3}^{(0)} = (\sigma_\alpha \times \sigma_\beta)\tau_\alpha \cdot \tau_\beta$	1	+	3
$P_{4-6}^{(0)} = (\sigma_\alpha \cdot \sigma_\beta)\sigma_\gamma (\tau_\alpha \times \tau_\beta) \cdot \tau_\gamma$	1	+	3
$P_{1-3}^{(1)} = \sigma_\alpha$	$1/N_c$	–	3
$P_{4-9}^{(1)} = \sigma_\alpha \tau_\alpha \cdot \tau_\beta$	$1/N_c$	–	6
$P_{10-12}^{(1)} = (\sigma_\alpha \times \sigma_\beta)(\tau_\alpha \times \tau_\beta) \cdot \tau_\gamma$	$1/N_c$	–	3
$P_{13-15}^{(1)} = (\sigma_\alpha \cdot \sigma_\beta)\sigma_\gamma \tau_\alpha \cdot \tau_\beta$	$1/N_c$	–	3
$P_{16-21}^{(1)} = (\sigma_\alpha \cdot \sigma_\beta)\sigma_\gamma \tau_\beta \cdot \tau_\gamma$	$1/N_c$	–	6
$P_{1-3}^{(2)} = \sigma_\alpha (\tau_\alpha \times \tau_\beta) \cdot \tau_\gamma$	$1/N_c^2$	+	3
$P_{4-6}^{(2)} = (\sigma_\alpha \times \sigma_\beta)$	$1/N_c^2$	+	3
$P_{7-12}^{(2)} = (\sigma_\alpha \times \sigma_\beta)\tau_\beta \cdot \tau_\gamma$	$1/N_c^2$	+	6
$P_{1-3}^{(3)} = \sigma_\alpha \tau_\beta \cdot \tau_\gamma$	$1/N_c^3$	–	3
$P_{4-6}^{(3)} = (\sigma_\alpha \cdot \sigma_\beta)\sigma_\gamma$	$1/N_c^3$	–	3
			45

1. $L = S = 0$

A complete set of spin-flavor structures in the $S = 0$ sector is given by the $S_\xi^{(r)}$ listed in Table IV. The superscript (r) indicates the *relative order* in N_c at which the spin-flavor structure appears for the first time (i.e., its lowest order). This corresponds to $r = s + t$ in the Hartree Hamiltonian, Eq. (3), and essentially counts the number of subleading operators S and I that contribute to the structure. The resulting contribution to the $3N$ force is obtained after taking into account the overall factor of N_c in Eq. (3) and the momentum structure that

TABLE VI. $S = 2$ spin-isospin structures, as in Table IV. In the last line we give the total number of independent structures, obtained as the sum of $M_2^{(0)} = 5$, $M_2^{(1)} = 9$, $M_2^{(2)} = 9$, and $M_2^{(3)} = 2$.

Operator	Order	T	Multiplicity
$D_{1-3}^{(0)} = [\sigma_\alpha \sigma_\beta]_2 \tau_\alpha \cdot \tau_\beta$	1	+	3
$D_{4,5}^{(0)} = [(\sigma_\alpha \times \sigma_\beta)\sigma_\gamma]_2 (\tau_\alpha \times \tau_\beta) \cdot \tau_\gamma$	1	+	2
$D_{1-3}^{(1)} = [\sigma_\alpha \sigma_\beta]_2 (\tau_\alpha \times \tau_\beta) \cdot \tau_\gamma$	$1/N_c$	–	3
$D_{4,5}^{(1)} = [(\sigma_\alpha \times \sigma_\beta)\sigma_\gamma]_2 \tau_\alpha \cdot \tau_\beta$	$1/N_c$	–	2
$D_{6-9}^{(1)} = [(\sigma_\alpha \times \sigma_\beta)\sigma_\gamma]_2 \tau_\beta \cdot \tau_\gamma$	$1/N_c$	–	4
$D_{1-3}^{(2)} = [\sigma_\alpha \sigma_\beta]_2$	$1/N_c^2$	+	3
$D_{4-9}^{(2)} = [\sigma_\alpha \sigma_\beta]_2 \tau_\beta \cdot \tau_\gamma$	$1/N_c^2$	+	6
$D_{1,2}^{(3)} = [(\sigma_\alpha \times \sigma_\beta)\sigma_\gamma]_2$	$1/N_c^3$	–	2
			25

TABLE VII. $S = 3$ spin-isospin structures, as in Table IV. In the last line we give the total number of independent structures, obtained as the sum of $M_3^{(0)} = 1$, $M_3^{(1)} = 3$, $M_3^{(2)} = 0$, and $M_3^{(3)} = 1$.

Operator	Order	T	Multiplicity
$F_1^{(0)} = [\sigma_\alpha\sigma_\beta\sigma_\gamma]_3(\tau_\alpha \times \tau_\beta)\cdot\tau_\gamma$	1	+	1
$F_{1-3}^{(1)} = [\sigma_\alpha\sigma_\beta\sigma_\gamma]_3\tau_\alpha\cdot\tau_\beta$	$1/N_c$	−	3
$F_1^{(3)} = [\sigma_\alpha\sigma_\beta\sigma_\gamma]_3$	$1/N_c^3$	−	1
			5

combines with each spin-flavor structure to give a rotational scalar, time-reversal-even, and parity-even Hamiltonian. Each occurrence of a time-reversal-odd momentum $\mathbf{p}_+$, $\mathbf{q}_+$ costs an additional power of $1/N_c$.

The time-reversal-even spin-flavor structures at order N_c^0 and order $1/N_c^2$ can be straightforwardly incorporated into the potential. They only need to be multiplied by arbitrary scalar functions of the vectors $\mathbf{p}_-$ and $\mathbf{q}_-$. We denote the functions, which are all of $\mathcal{O}(N_c^0)$, $V_X^m(\mathbf{p}_-^2, \mathbf{q}_-^2, \mathbf{p}_-\cdot\mathbf{q}_-)$, where X runs over the different spin-flavor structures and m enumerates functions V corresponding to different momentum structures. Beyond the statement that they are $\mathcal{O}(N_c^0)$, the large-N_c expansion sheds no light on the behavior of these functions.

With this notation the $\mathcal{O}(N_c)$ (leading-order) potential is

$$V_{L=0}^{N_c} = N_c \sum_{\xi=1}^{M_0^{(0)}} V_{S_\xi}^1(\mathbf{p}_-^2, \mathbf{q}_-^2, \mathbf{p}_-\cdot\mathbf{q}_-) S_\xi^{(0)}, \tag{39}$$

with $M_0^{(0)} = 5$ the number of independent, leading-order, $S = 0$ spin-flavor structures (see Table IV or Table III). In fact, their contribution once the spatial part of the $3N$ state is taken into account is not completely independent, since the functions $V_{S_2}^1$, $V_{S_3}^1$, and $V_{S_4}^1$ are related to one another by permutation symmetry, i.e., the requirement that the total force must be symmetric under permutations of all particle labels. This constraint in the $3N$ case is, however, more complicated than in the NN case, and there seems to be no obvious simplification due to permutation symmetry.

There are thus five spin-flavor structures that contribute at leading order in N_c to the $L = 0$ part of the $3N$ potential: the identity, a $\sigma_\alpha\cdot\sigma_\beta\tau_\alpha\cdot\tau_\beta$ structure, where one of the three nucleons is not involved, and the structure $(\sigma_\alpha \times \sigma_\beta)\cdot\sigma_\gamma(\tau_\alpha \times \tau_\beta)\cdot\tau_\gamma$. Of these, the first two already occur in the NN potential, and, as already discussed above, without knowledge of the $\mathbf{q}_-$ dependence in V, we cannot separate their appearance here from the fact that they contribute to the energy of the NN pairs in the $3N$ system.

We now turn our attention to subleading corrections to the $L = 0$ $3N$ force. It at first appears that there are spin-flavor structures which generate contributions of relative order $1/N_c$. But in fact the resulting structures are all time-reversal odd. In consequence, they must be multiplied by a time-reversal-odd dot product in order to appear in the $L = 0$ component of the $3N$ potential. In contrast to the NN case such dot products exist in this system, e.g., $\mathbf{p}_+\cdot\mathbf{q}_-$. But all of the T-odd ones involve either $\mathbf{p}_+$ or $\mathbf{q}_+$. Thus the first subleading contribution is suppressed by two powers of $1/N_c^2$ relative to leading: one because of the matrix elements of the spin-flavor structures which appear and one because of the necessity for a $1/M_N$ factor in order to generate some nonlocality and introduce $\mathbf{p}_+$ or $\mathbf{q}_+$.

At relative order $1/N_c^2$ we also have the 12 structures $S_\xi^{(2)}$ shown in Table IV. In addition, the leading structures $S_\xi^{(0)}$ can reappear, now multiplied by two of the $1/N_c$ suppressed dot products or by one $1/N_c^2$ suppressed dot product of momenta. Using the energy-conservation and on-shell condition, Eq. (30), at $\mathcal{O}(1/N_c)$ we find three momentum structures of $\mathcal{O}(1/N_c)$ and three structures of $\mathcal{O}(1/N_c^2)$, all of which involve two momenta. With four momenta there are six new structures of order $\mathcal{O}(1/N_c^2)$.

Last, we observe that operators from the LO potential occur again at this order, as they can arise via the replacement of one G^{ia}/N_c by $S^i I^a/N_c^2$ in the Hartree Hamiltonian, as already discussed for the NN potential. We denote this contribution by $\delta^{(2)}V_{L=0}^{N_c}$ which stands for

$$\delta^{(2)}V_{L=0}^{N_c} = N_c^{-1}\sum_{\xi=1}^{M_0^{(0)}} V_{S_\xi,1/N_c^2}^1(\mathbf{p}_-^2, \mathbf{q}_-^2, \mathbf{p}_-\cdot\mathbf{q}_-) S_\xi^{(0)}, \tag{40}$$

where the explicit N_c factors ensure that the $V_{S_\xi,1/N_c^2}^1(\mathbf{p}_-^2, \mathbf{q}_-^2, \mathbf{p}_-\cdot\mathbf{q}_-)$ are of order $\mathcal{O}(N_c^0)$. Equations (39) and (40) can be combined, with the effect that the functions $V_{S_\xi}^1$ each have their own expansion in $1/N_c^2$.

The full $\mathcal{O}(1/N_c)$ piece of the $L = 0$ $3N$ potential is then

$$\begin{aligned} V_{L=0}^{1/N_c} &= \delta^{(2)}V_{L=0}^{N_c} + N_c\sum_{\xi=1}^{M_0^{(2)}} V_{S_\xi}^2 S_\xi^{(2)} \\ &+ N_c\sum_{\xi=1}^{M_0^{(1)}} \left(V_{S_\xi}^3\mathbf{p}_+\cdot\mathbf{p}_- + V_{S_\xi}^{4,5}\mathbf{p}_\pm\cdot\mathbf{q}_\mp\right) S_\xi^{(1)} \\ &+ N_c\sum_{\xi=1}^{M_0^{(0)}} \Big\{V_{S_\xi}^6\mathbf{p}_+^2 + V_{S_\xi}^7\mathbf{p}_+\cdot\mathbf{q}_+ + V_{S_\xi}^8\mathbf{q}_+^2 \\ &+ V_{S_\xi}^9(\mathbf{p}_+\cdot\mathbf{p}_-)^2 + V_{S_\xi}^{10,11}(\mathbf{p}_\pm\cdot\mathbf{q}_\mp)^2 \\ &+ V_{S_\xi}^{12,13}(\mathbf{p}_+\cdot\mathbf{p}_-)(\mathbf{p}_\pm\cdot\mathbf{q}_\mp) \\ &+ V_{S_\xi}^{14}(\mathbf{p}_+\cdot\mathbf{q}_-)(\mathbf{q}_+\cdot\mathbf{p}_-)\Big\} S_\xi^{(0)}. \end{aligned} \tag{41}$$

Here $M_0^{(2)} = 12$ and $M_0^{(1)} = 6$ are obtained summing over the multiplicities shown in Table IV. Note that here any momentum structure that involves more powers of the $\mathcal{O}(N_c^0)$ momenta $\mathbf{p}_-$, $\mathbf{q}_-$, is absorbed in the $\mathcal{O}(N_c^0)$ scalar functions $V_{S_\xi}^{2-14}(\mathbf{p}_-^2, \mathbf{q}_-^2, \mathbf{p}_-\cdot\mathbf{q}_-)$.

There is no correction at order $1/N_c^3$, due to the T-odd nature of the operators $S_\xi^{(3)}$ listed above and the restrictions of parity and time-reversal invariance regarding the vector dot products which can be considered. The correction of order $1/N_c^4$ can be constructed in analogy to the results for $1/N_c^2$ given in Eq. (41). For example, the terms involving the $M_0^{(3)} = 2$ operators $S_\xi^{(3)}$

are

$$N_c \sum_{\xi=1}^{M_0^{(3)}} \left(V_{S_\xi}^{15}\mathbf{p}_+ \cdot \mathbf{p}_- + V_{S_\xi}^{16,17}\mathbf{p}_\pm \cdot \mathbf{q}_\mp\right) S_\xi^{(3)}. \tag{42}$$

Meanwhile, there are many other terms at this order involving $\delta^{(4)} V_{L=0}^{N_c}$, $\delta^{(2)} V_{L=0}^{1/N_c}$ and various combinations of dot products of momenta and the operators $S_\xi^{(2)}$, $S_\xi^{(1)}$, and $S_\xi^{(0)}$. These are too numerous to list here, but it is clear that the expansion of the potential is in $1/N_c^2$, as was also the case for the NN potential.

Before proceeding to the construction of the $L = 1, 2, 3$ components of the potential it is important to explain why there are no cross products in the momentum structures of the $L = 0$ potential. Triple products like $(\mathbf{p}_- \times \mathbf{p}_+) \cdot \mathbf{q}_-$ are parity odd and should appear together with another triple product giving a structure of six momenta, which can be written in terms of scalar products alone. Quadruple products like $(\mathbf{p}_- \times \mathbf{p}_+) \cdot (\mathbf{p}_- \times \mathbf{p}_+)$ can also be reduced to the ones already present, as they involve two contracted epsilon tensors. The identities used for this purpose can be found in Appendix B; see, specifically, Eq. (B1).

In summary, for $L = 0$ there are five operators at LO, given in Eq. (39), which are built from T-even spin-isospin structures. At subleading order there are 57 additional operators which are built from T-even spin-isospin structures, as well as 18 operators involving T-odd structures. These are listed in Eq. (41). Up to $\mathcal{O}(1/N_c)$ we presented the explicit expressions for this total of 80 operators, of which 17 only depend on $\mathbf{p}_-$, $\mathbf{q}_-$ and correspond to a local potential.

2. $L = S = 1$

In contrast to the two-nucleon potential, the three-nucleon potential contains $S = L = 1$ terms at leading order. That is because vector spin-flavor structures can be constructed from two and three G's, see Table III, and then contracted with the P-even, T-even cross product $\mathbf{p}_- \times \mathbf{q}_-$. No such time-reversal-even cross product exists in the NN system at leading order. The leading-order $L = 1$ force is then

$$V_{L=1}^{N_c} = N_c \sum_{\xi=1}^{M_1^{(0)}} V_{P_\xi}^1 (\mathbf{p}_- \times \mathbf{q}_-) \cdot P_\xi^{(0)}. \tag{43}$$

Here $M_1^{(0)} = 6$ is the number of leading-order $S = 1$ structures, see Table V. Since $\mathbf{q}_-$ appears in the momentum structure of Eq. (43) they cannot occur in the NN force, and are unambiguously the result of $3N$ interactions. However, we note that, once again, the functions $V_{P_{1-3}}^1$ will be related to one another through permutation symmetry, as will the functions $V_{P_{4-6}}^1$.

Operators with matrix elements suppressed by $1/N_c$ are easily obtained; see Table V, where they are listed as $P_\xi^{(1)}$, with $\xi = 1, \ldots, M_1^{(1)}$ and $M_1^{(1)} = 21$. However, as is displayed in the table, these are all time-reversal odd. Thus they must be contracted with T-odd cross products, and this costs another power of $1/N_c$, since it mandates that $\mathbf{p}_+$ or $\mathbf{q}_+$ be involved. Thus the order of such contributions is $1/N_c^2$ relative to leading. At this order the T-even structures $P_\xi^{(2)}$ also appear, contracted with $\mathbf{p}_- \times \mathbf{q}_-$, the leading-order momentum structure. The $L = 1$ $3N$ force of order $1/N_c^2$ is therefore of the form

$$\begin{aligned}
V_{L=1}^{1/N_c} &= \delta^{(2)} V_{L=1}^{N_c} + N_c \sum_{\xi=1}^{M_1^{(2)}} V_{P_\xi}^2 (\mathbf{p}_- \times \mathbf{q}_-) \cdot P_\xi^{(2)} \\
&+ N_c \sum_{\xi=1}^{M_1^{(1)}} \Big\{ V_{P_\xi}^3 \mathbf{p}_+ \times \mathbf{p}_- + V_{P_\xi}^{4,5} \mathbf{p}_\pm \times \mathbf{q}_\mp \\
&+ V_{P_\xi}^6 \mathbf{q}_+ \times \mathbf{q}_- + \left(V_{P_\xi}^7 \mathbf{p}_+ \cdot \mathbf{p}_- + V_{P_\xi}^{8,9} \mathbf{p}_\pm \cdot \mathbf{q}_\mp\right) \\
&\times (\mathbf{p}_- \times \mathbf{q}_-) \Big\} \cdot P_\xi^{(1)} + N_c \sum_{\xi=1}^{M_1^{(0)}} \Big\{ V_{P_\xi}^{10} (\mathbf{p}_+ \times \mathbf{q}_+) \\
&+ \left(V_{P_\xi}^{11} \mathbf{p}_+^2 + V_{P_\xi}^{12} \mathbf{p}_+ \cdot \mathbf{q}_+ + V_{P_\xi}^{13} \mathbf{q}_+^2\right) (\mathbf{p}_- \times \mathbf{q}_-) \\
&+ \left(V_{P_\xi}^{14} \mathbf{p}_+ \cdot \mathbf{p}_- + V_{P_\xi}^{15,16} \mathbf{p}_\pm \cdot \mathbf{q}_\mp\right) (\mathbf{p}_+ \times \mathbf{p}_-) \\
&+ \left(V_{P_\xi}^{17} \mathbf{p}_+ \cdot \mathbf{p}_- + V_{P_\xi}^{18,19} \mathbf{p}_\pm \cdot \mathbf{q}_\mp\right) (\mathbf{p}_+ \times \mathbf{q}_-) \\
&+ \left(V_{P_\xi}^{20} \mathbf{p}_+ \cdot \mathbf{p}_- + V_{P_\xi}^{21,22} \mathbf{p}_\pm \cdot \mathbf{q}_\mp\right) (\mathbf{p}_- \times \mathbf{q}_+) \\
&+ \left(V_{P_\xi}^{23} \mathbf{p}_+ \cdot \mathbf{p}_- + V_{P_\xi}^{24,25} \mathbf{p}_\pm \cdot \mathbf{q}_\mp\right) (\mathbf{q}_+ \times \mathbf{q}_-) \Big\} \cdot P_\xi^{(0)}.
\end{aligned} \tag{44}$$

In this equation $M_1^{(2)} = 12$ and $M_1^{(1)} = 21$. Using two momenta there is only one structure of $\mathcal{O}(1)$. There are four structures of $\mathcal{O}(1/N_c)$ and one structure of $\mathcal{O}(1/N_c^2)$. With four momenta there are three structures at $\mathcal{O}(1/N_c)$ and 15 new structures of $\mathcal{O}(1/N_c^2)$. Triple products can be eliminated using the identity (B3).

Once again, the new spin-flavor structures that appear at $\mathcal{O}(1/N_c^3)$ are all T-odd. Thus they must be combined with $\mathbf{p}_- \times \mathbf{q}_+$, or one of the three other T-odd cross products involving a + vector, to yield something appropriate for inclusion in $V_{L=1}$. The overall result is then a contribution to the $3N$ force of relative order $1/N_c^4$. The expansion for $V_{L=1}$ is, like that for $V_{L=0}$, an expansion in $1/N_c^2$.

In summary, for $L = 1$ there are six operators at LO, given in Eq. (43), which are built from T-even spin-isospin structures. At subleading order there are 108 new operators involving T-even spin-isospin structures and 147 operators involving T-odd ones; see Eq. (44). To $\mathcal{O}(1/N_c)$ we presented the explicit expressions for these 261 operators, of which 18 involve only $\mathbf{p}_-$, $\mathbf{q}_-$, and so correspond to a local potential.

3. $L = S = 2$

The leading $3N$ spin-flavor structures with $S = 2$ are constructed from G's and $\mathbb{1}$'s, as shown in Table III. A Cartesian rank-2 tensor with $S = 2$, constructed from two vector quantities A^i and B^j, is symmetric and traceless in its two indices and will be denoted $[A_i B_j]_2$ [see Eq. (18)]. Subleading structures are obtained after introducing a growing number of S and I operators, following Eq. (3).

The complete set of resulting spin-flavor structures is displayed in Table VI. This time we have $M_2^{(0)} = 5$. The five LO structures must be contracted with $L = 2$ tensors constructed from $\mathbf{p}_-$ and $\mathbf{q}_-$ to obtain the LO contribution to the $3N$ potential.

We reiterate that Eq. (33) explains why there are only two structures $D^{(0)}_{4,5}$, instead of the multiplicity three in similar structures with three spin operators in the $L = 1$ case. Similar reductions in multiplicity occur for subleading $S = 2$ spin-flavor structures, too.

The leading force is then

$$V^{N_c}_{L=2} = N_c \sum_{\xi=1}^{M^{(0)}_2} \left\{ V^1_{D_\xi}[\mathbf{p}_-\mathbf{p}_-]_2 + V^2_{D_\xi}[\mathbf{p}_-\mathbf{q}_-]_2 + V^3_{D_\xi}[\mathbf{q}_-\mathbf{q}_-]_2 \right\} \cdot D^{(0)}_\xi, \tag{45}$$

with the sum over the $M^{(0)}_2 = 5$ structures as listed in Table VI. Note that the first three structures in the sum, i.e., $D^{(0)}_{1-3}$, contracted with the appropriate $\mathbf{p}_-$, already occur in the NN potential. Only $\mathbf{q}_-$ dependence in $V^1_{D_{1-3}}$ would reveal it is a "true" $3N$ force.

Using analogous arguments to those already discussed for the $L = 0$ and 1 cases, we obtain the subleading $L = S = 2$ contribution to the 3NF as follows:

$$\begin{aligned}
V^{1/N_c}_{L=2} = {} & \delta^{(2)} V^{N_c}_{L=2} + N_c \sum_{\xi=1}^{M^{(2)}_2} \left\{ V^4_{D_\xi}[\mathbf{p}_-\mathbf{p}_-]_2 + V^5_{D_\xi}[\mathbf{p}_-\mathbf{q}_-]_2 + V^6_{D_\xi}[\mathbf{q}_-\mathbf{q}_-]_2 \right\} \cdot D^{(2)}_\xi + N_c \sum_{\xi=1}^{M^{(1)}_2} \left\{ V^7_{D_\xi}[\mathbf{p}_+\mathbf{p}_-]_2 + V^{8,9}_{D_\xi}[\mathbf{p}_\pm\mathbf{q}_\mp]_2 \right. \\
& + V^{10}_{D_\xi}[\mathbf{q}_+\mathbf{q}_-]_2 + \left(V^{11}_{D_\xi}\mathbf{p}_+ \cdot \mathbf{p}_- + V^{12,13}_{D_\xi}\mathbf{p}_\pm \cdot \mathbf{q}_\mp\right)[\mathbf{p}_-\mathbf{p}_-]_2 + \left(V^{14}_{D_\xi}\mathbf{p}_+ \cdot \mathbf{p}_- + V^{15,16}_{D_\xi}\mathbf{p}_\pm \cdot \mathbf{q}_\mp\right)[\mathbf{p}_-\mathbf{q}_-]_2 \\
& \left. + \left(V^{17}_{D_\xi}\mathbf{p}_+ \cdot \mathbf{p}_- + V^{18,19}_{D_\xi}\mathbf{p}_\pm \cdot \mathbf{q}_\mp\right)[\mathbf{q}_-\mathbf{q}_-]_2 \right\} \cdot D^{(1)}_\xi + N_c \sum_{\xi=1}^{M^{(0)}_2} \left\{ V^{20}_{D_\xi}[\mathbf{p}_+\mathbf{p}_+]_2 + V^{21}_{D_\xi}[\mathbf{p}_+\mathbf{q}_+]_2 + V^{22}_{D_\xi}[\mathbf{q}_+\mathbf{q}_+]_2 \right. \\
& + \left(V^{23}_{D_\xi}\mathbf{p}^2_+ + V^{24}_{D_\xi}\mathbf{p}_+ \cdot \mathbf{q}_+ + V^{25}_{D_\xi}\mathbf{q}^2_+\right)[\mathbf{p}_-\mathbf{p}_-]_2 + \left(V^{26}_{D_\xi}\mathbf{p}^2_+ + V^{27}_{D_\xi}\mathbf{p}_+ \cdot \mathbf{q}_+ + V^{28}_{D_\xi}\mathbf{q}^2_+\right)[\mathbf{p}_-\mathbf{q}_-]_2 \\
& + \left(V^{29}_{D_\xi}\mathbf{p}^2_+ + V^{30}_{D_\xi}\mathbf{p}_+ \cdot \mathbf{q}_+ + V^{31}_{D_\xi}\mathbf{q}^2_+\right)[\mathbf{q}_-\mathbf{q}_-]_2 + \left(V^{32}_{D_\xi}\mathbf{p}_+ \cdot \mathbf{p}_- + V^{33,34}_{D_\xi}\mathbf{p}_\pm \cdot \mathbf{q}_\mp\right)[\mathbf{p}_+\mathbf{p}_-]_2 \\
& + \left(V^{35}_{D_\xi}\mathbf{p}_+ \cdot \mathbf{p}_- + V^{36,37}_{D_\xi}\mathbf{p}_\pm \cdot \mathbf{q}_\mp\right)[\mathbf{p}_+\mathbf{q}_-]_2 + \left(V^{38}_{D_\xi}\mathbf{p}_+ \cdot \mathbf{p}_- + V^{39,40}_{D_\xi}\mathbf{p}_\pm \cdot \mathbf{q}_\mp\right)[\mathbf{p}_-\mathbf{q}_+]_2 \\
& \left. + \left(V^{41}_{D_\xi}\mathbf{p}_+ \cdot \mathbf{p}_- + V^{42,43}_{D_\xi}\mathbf{p}_\pm \cdot \mathbf{q}_\mp\right)[\mathbf{q}_+\mathbf{q}_-]_2 \right\} \cdot D^{(0)}_\xi,
\end{aligned} \tag{46}$$

where $M^{(1)}_2 = 9$, $M^{(2)}_2 = 9$. In writing Eq. (46) we employed three tensors built from two momenta at $\mathcal{O}(1)$, four at $\mathcal{O}(1/N_c)$, and three at $\mathcal{O}(1/N_c^2)$; with four momenta there are 9 structures at $\mathcal{O}(1/N_c)$ and 21 at $\mathcal{O}(1/N_c^2)$. Of course, all structures are again multiplied by the usual arbitrary functions of $\mathbf{p}^2_-$, $\mathbf{q}^2_-$, and $\mathbf{p}_- \cdot \mathbf{q}_-$. As in the $L = S = 0$ and $L = S = 1$ cases, the next-to-next-to-leading-order contributions to V appear at relative order $1/N_c^4$, as a consequence of parity and time reversal.

In summary, for this $L = S = 2$ part of the $3N$ force there are 15 operators at LO, which are built from T-even spin-isospin structures. These are given in Eq. (45). There are 264 new operators at relative order $1/N_c^2$, 147 (117) of which are based on T-even (T-odd) spin-flavor structures; see Eq. (46). To $\mathcal{O}(1/N_c)$ we presented the explicit expressions for all of these 279 operators. Among these are 42 operators that only depend on $\mathbf{p}_-$, $\mathbf{q}_-$ and could appear in a local potential.

4. $L = S = 3$

These operators have no NN analog. The rank-3 Cartesian tensor with $L = 3$ that can be constructed from three vectors A, B, C is symmetric and traceless, namely

$$\begin{aligned}
[A_i B_j C_k]_3 = {} & A_i B_j C_k + A_j B_k C_i + A_k B_i C_j + A_j B_i C_k \\
& + A_i B_k C_j + A_k B_j C_i \\
& - \tfrac{2}{5}\delta_{ij}(A \cdot B C_k + A \cdot C B_k + B \cdot C A_k) \\
& - \tfrac{2}{5}\delta_{ik}(A \cdot B C_j + A \cdot C B_j + B \cdot C A_j) \\
& - \tfrac{2}{5}\delta_{jk}(A \cdot B C_i + A \cdot C B_i + B \cdot C A_i).
\end{aligned} \tag{47}$$

There is just one leading-order spin-flavor structure, three suppressed by $1/N_c$ and one suppressed by $1/N_c^3$, as shown in Table VII. Note that there are no $1/N_c^2$ structures in this case, because of isospin conservation, as three σ operators would need to appear along with one τ to generate a structure at that order. Table VII again shows that the suppressed structures are T-odd and so ultimately lead to contributions to the $3N$ force that are down by $1/N_c^2$ and $1/N_c^4$, respectively.

Because of parity conservation, we need at least four momenta to construct the $L = 3$ component of the potential. There are three $L = 3$, P-even, T-even, momentum tensors at $\mathcal{O}(1)$. Both $\mathbf{q}_-$ and $\mathbf{p}_-$ are needed to construct these, as, e.g., $\mathbf{q}_-\mathbf{q}_-\mathbf{q}_-\mathbf{q}_-$ only contains $L = 0, 2, 4$ components. The leading operator is

$$\begin{aligned}
V^{N_c}_{L=3} = {} & N_c \left\{ V^1_{F_1}[(\mathbf{p}_- \times \mathbf{q}_-)\mathbf{p}_-\mathbf{p}_-]_3 + V^2_{F_1}[(\mathbf{p}_- \times \mathbf{q}_-)\mathbf{p}_-\mathbf{q}_-]_3 \right. \\
& \left. + V^3_{F_1}[(\mathbf{p}_- \times \mathbf{q}_-)\mathbf{q}_-\mathbf{q}_-]_3 \right\} \cdot F^{(0)}_1.
\end{aligned} \tag{48}$$

The subleading $L = 3$ force contains $L = 3$, P-even T-odd tensors involving three $\mathbf{q}_-$ or $\mathbf{p}_-$ vectors, together with one $\mathbf{p}_+$ or $\mathbf{q}_+$, contracted with the three $F^{(1)}_{1-3}$ structures. Also appearing at this order are the terms which involve the leading spin-flavor structure, contracted with P-even T-even tensors in which two of the four vectors are $\mathbf{p}_+$ or $\mathbf{q}_+$. Meanwhile, the subleading correction to the LO structure $F^{(0)}_1$ corresponds to $s = t = 1$ in Eq. (3) and is given by $\delta^{(2)} V^{N_c}_{L=3}$.

Thus, finally we obtain

$$
\begin{aligned}
V_{L=3}^{1/N_c} &= \delta^{(2)} V_{L=3}^{N_c} \\
&+ N_c \sum_{\xi=1}^{M_3^{(1)}} \Big\{ V_{F_\xi}^4[(\mathbf{p}_+ \times \mathbf{p}_-)\mathbf{p}_-\mathbf{p}_-]_3 + V_{F_\xi}^5[(\mathbf{p}_+ \times \mathbf{p}_-)\mathbf{p}_-\mathbf{q}_-]_3 + V_{F_\xi}^6[(\mathbf{p}_+ \times \mathbf{p}_-)\mathbf{q}_-\mathbf{q}_-]_3 \\
&\quad + V_{F_\xi}^{7,8}[(\mathbf{p}_\pm \times \mathbf{q}_\mp)\mathbf{p}_-\mathbf{p}_-]_3 + V_{F_\xi}^{9,10}[(\mathbf{p}_\pm \times \mathbf{q}_\mp)\mathbf{p}_-\mathbf{q}_-]_3 + V_{F_\xi}^{11,12}[(\mathbf{p}_\pm \times \mathbf{q}_\mp)\mathbf{q}_-\mathbf{q}_-]_3 \\
&\quad + V_{F_\xi}^{13}[(\mathbf{q}_+ \times \mathbf{q}_-)\mathbf{p}_-\mathbf{p}_-]_3 + V_{F_\xi}^{14}[(\mathbf{q}_+ \times \mathbf{q}_-)\mathbf{p}_-\mathbf{q}_-]_3 + V_{F_\xi}^{15}[(\mathbf{q}_+ \times \mathbf{q}_-)\mathbf{q}_-\mathbf{q}_-]_3 \Big\} \cdot F_\xi^{(1)} \\
&+ N_c \Big\{ V_{F_1}^{16}[(\mathbf{p}_+ \times \mathbf{p}_-)\mathbf{p}_+\mathbf{p}_-]_3 + V_{F_1}^{17,18}[(\mathbf{p}_+ \times \mathbf{p}_-)\mathbf{p}_\pm\mathbf{q}_\mp]_3 + V_{F_1}^{19}[(\mathbf{p}_+ \times \mathbf{p}_-)\mathbf{q}_+\mathbf{q}_-]_3 \\
&\quad + V_{F_1}^{20,21}[(\mathbf{p}_\pm \times \mathbf{q}_\mp)\mathbf{p}_+\mathbf{p}_-]_3 + V_{F_1}^{22,23}[(\mathbf{p}_\pm \times \mathbf{q}_\mp)\mathbf{p}_+\mathbf{q}_-]_3 + V_{F_1}^{24,25}[(\mathbf{p}_\pm \times \mathbf{q}_\mp)\mathbf{q}_+\mathbf{p}_-]_3 \\
&\quad + V_{F_1}^{26,27}[(\mathbf{p}_\pm \times \mathbf{q}_\mp)\mathbf{q}_+\mathbf{q}_-]_3 + V_{F_1}^{28,29}[(\mathbf{q}_+ \times \mathbf{q}_-)\mathbf{p}_\pm\mathbf{q}_\mp]_3 + V_{F_1}^{30}[(\mathbf{q}_+ \times \mathbf{q}_-)\mathbf{q}_+\mathbf{q}_-]_3 \Big\} \cdot F_1^{(0)}, \qquad (49)
\end{aligned}
$$

where $M_3^{(1)} = 3$. There are 12 momentum structures at $\mathcal{O}(1/N_c)$ and 15 momentum structures at $\mathcal{O}(1/N_c^2)$. Here we used again Eq. (33) to reduce the number of momentum structures. The usual arguments show that the next correction, which involves the $F_1^{(3)}$ structure, is down by $1/N_c^4$ compared to the leading contribution, again because of the need to contract $F_1^{(3)}$ with a P-even, T-odd tensor.

In summary, for $L = 3$ there are three operators at LO built from T-even spin-isospin structures, given in Eq. (48), and 51 additional (15 corresponding to new T-even and 36 to T-odd structures) at relative order $1/N_c^2$, given in Eq. (49). We presented the explicit expressions for these 54 operators which occur up to $\mathcal{O}(1/N_c)$. Of these, three depend solely on $\mathbf{p}_-$, $\mathbf{q}_-$ and could be part of a local potential.

IV. SUMMARY AND DISCUSSION

We have classified all the spin-flavor structures that can contribute to the three-nucleon force (3NF) and power counted these structures in the $1/N_c$ expansion. The leading-order (LO) part of the 3NF is constructed from G^{ia}/N_c and the unit operator, since these are the quark operators that have nucleon matrix elements that are $\mathcal{O}(1)$. Isospin-invariant structures like

$$
\mathbb{1}_\alpha \mathbb{1}_\beta \mathbb{1}_\gamma, \quad N_c^{-2} G_\alpha^{ia} \mathbb{1}_\beta G_\gamma^{ja}, \quad N_c^{-3} \epsilon^{abc} G_\alpha^{ia} G_\beta^{jb} G_\gamma^{kc}, \qquad (50)
$$

with α, β, and γ labeling the three nucleons, are the leading contributions. Contraction of these structures with spatial tensors of the appropriate rank, built from the $\mathcal{O}(1)$ momenta $\mathbf{p}_-$ and $\mathbf{q}_-$, together with a re-expression in terms of the angular momentum content of these structures, and use of the reduction of G^{ia} to $\sigma^i \tau^a$ when restricted to the nucleon subspace, produces the LO force as follows:

$$
\begin{aligned}
V_{3N}^{N_c} &= N_c \sum_{\xi=1}^{M_0^{(0)}} V_{S_\xi}^1 S_\xi^{(0)} + N_c \sum_{\xi=1}^{M_1^{(0)}} V_{P_\xi}^1 (\mathbf{p}_- \times \mathbf{q}_-) \cdot P_\xi^{(0)} \\
&+ N_c \sum_{\xi=1}^{M_2^{(0)}} \Big\{ V_{D_\xi}^1 [\mathbf{p}_-\mathbf{p}_-]_2 + V_{D_\xi}^2 [\mathbf{p}_-\mathbf{q}_-]_2 + V_{D_\xi}^3 [\mathbf{q}_-\mathbf{q}_-]_2 \Big\} \cdot D_\xi^{(0)} \\
&+ N_c \Big\{ V_{F_1}^1[(\mathbf{p}_- \times \mathbf{q}_-)\mathbf{p}_-\mathbf{p}_-]_3 + V_{F_1}^2[(\mathbf{p}_- \times \mathbf{q}_-)\mathbf{p}_-\mathbf{q}_-]_3 \\
&+ V_{F_1}^3[(\mathbf{p}_- \times \mathbf{q}_-)\mathbf{q}_-\mathbf{q}_-]_3 \Big\} \cdot F_1^{(0)}. \qquad (51)
\end{aligned}
$$

Here the S, P, D, F spin-flavor structures are given in Tables IV, V, VI, and VII. The corresponding multiplicities are $M_{0,1,2,3}^{(0)} = 5, 6, 5, 1$. Including all the independent momentum structures, to leading order we have 29 operators distributed as 5, 6, 15 and 3 operators with $L = 0, 1, 2, 3$ respectively.

It follows straightforwardly that our LO force contains the structures present in the Fujita-Miyazawa three-nucleon potential, Eq. (2). Indeed, the only structure beyond the Fujita-Miyazawa result is the unit operator which is added to $V_{ijk}^{2\pi}$ in most modern implementations of the 3NF. Of course, models of the 3NFs contain specific predictions for the coefficient functions $V_{L_\xi}^m$. The large-N_c expansion can say nothing about these functions beyond the statement that they should be "natural," i.e., $\mathcal{O}(1)$; the insights from large-N_c reside in the statements regarding the overall size that different spin-isospin–momentum structures within the 3NF should have.

The LO 3NF contains spin-dependent forces, but it does not contain the spin-orbit forces that have been proposed as a solution to the A_y puzzle (see, e.g., Ref. [45]). The A_y puzzle is not straightforwardly resolved by $1/N_c$ power counting arguments.

Spin-orbit forces, together with several other operators, all of which we have tabulated in Sec. III, appear at $\mathcal{O}(1/N_c^2)$ compared to leading. We have also shown that the next-to-next-to-leading correction to the 3NF is at order $1/N_c^4$ relative to LO. The NNN force is therefore, like the NN force, an expansion in $1/N_c^2$. We have given explicit expressions for the 674 operators that appear in the $3N$ potential up to (overall) order $1/N_c$ in Eqs. (51), (41), (44), (46), and (49).

Many of these operators involve nonlocalities and time-reversal-odd momentum structures. For a local 3NF only time-reversal-even momentum structures involving $\mathbf{p}_-$ or $\mathbf{q}_-$ can occur. Such structures occur in both the leading and subleading 3NF but do not occur at higher orders, where the presence of at least one time-reversal-odd momentum is required. Taking into account the different momentum

structures which satisfy this constraint, at relative order $1/N_c^2$ we have 12, 12, and 27 operators with $L = 0, 1, 2$, respectively. No $L = 3$ local operator occurs at this relative order in the expansion. Combining these operators with the 29 LO operators yields a total of 80 operators that constitute the most general basis for a local 3NF. These operators can be easily read off from Eqs. (51), (41), (44), (46), and (49). In a recent paper [29] the authors needed a basis of 89 operators to obtain the most general contribution of a local 3NF. Their operator basis differs somewhat from ours, so a comparison is not immediate. An important subject for future investigation is the relation between the two sets of operators and a determination of the minimal basis of operators for a general, local 3NF.

We have not discussed the constraints imposed by the permutation group on our analysis. In the NN case such considerations resulted in the elimination of the spin structure $\sigma_1 - \sigma_2$. In the $3N$ case such constraints will impose relations between the different coefficient functions we have used in our expansion. Since the $3N$ coefficient functions depend on three rotational scalars, it seems unlikely that a general, permutation-group-based argument can be used to eliminate an operator structure from the 3NF—at least in the absence of additional assumptions about the coefficient functions themselves. A permutation-group analysis of the structures we have obtained would be a useful step towards understanding the particular $3N$ partial waves which the different operator structures we have obtained contribute to.

It would also be interesting to test whether adding the $1/N_c^2$ structures we have listed here to phenomenological 3NFs improves the description of few-nucleon scattering data and light-nuclear spectra. Recently developed three-body potentials like the lllinois force [18] or that derived from χPT [27–29] include several of these structures. Matching such 3NFs to our large-N_c expressions is appreciably more involved than in the NN case analyzed in Ref. [37], but nevertheless, they could be matched to the list of operators presented here. This would illuminate precisely which structures are present in these particular potentials and whether the relative size of the different contributions is well predicted by the $1/N_c$ expansion.

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APPENDIX A: AN EXPLICIT EXAMPLE OF OPERATOR REDUCTION

For completeness, it is worth discussing an explicit example of the operator reduction [Eq. (14)]. Consider the case of the product of two G's, which can be written as

$$G^{ia}G^{jb} = \tfrac{1}{2}\{G^{ia}, G^{jb}\} + \tfrac{1}{2}[G^{ia}, G^{jb}]. \tag{A1}$$

The commutator contains the antisymmetric terms in (ia) and (jb) and is suppressed by $1/N_c^2$. The symmetric part in (ia) and (jb) can be written as a tensor $W^{(ij),(ab)}$ which is symmetric in the spatial indices i, j and symmetric in the isospin indices a, b independently and a tensor $W^{[ij],[ab]}$ that is antisymmetric in the spatial and isospin indices taken separately,

$$\{G^{ia}, G^{jb}\} = W^{(ij),(ab)} + W^{[ij],[ab]}, \tag{A2}$$

where

$$W^{(ij),(ab)} = \tfrac{1}{2}\{G^{ia}, G^{jb}\} + \tfrac{1}{2}\{G^{ib}, G^{ja}\}, \tag{A3}$$

$$W^{[ij],[ab]} = \tfrac{1}{2}\{G^{ia}, G^{jb}\} - \tfrac{1}{2}\{G^{ib}, G^{ja}\} = \epsilon^{ijk}\epsilon^{abc}A^{kc}, \tag{A4}$$

with

$$A^{kc} = \tfrac{1}{4}\epsilon^{kij}\epsilon^{abc}\{G^{ia}, G^{jb}\}. \tag{A5}$$

Now we can use the following identity, see Ref. [39], to reduce the number of G's by 1,

$$\epsilon^{ijk}\epsilon^{abc}\{G^{ia}, G^{jb}\} = -(N_c + 2)G^{kc} + \tfrac{1}{2}\{S^k, I^c\}. \tag{A6}$$

The $W^{(ij),(ab)}$ tensor has $(I, S) = (0, 0), (2, 0), (0, 2), (2, 2)$ components. Only the $(0, 0)$ component contributes in the nucleon subspace. It is obtained by contracting the indices and is $W^{ij,ab}_{(0,0)} = \frac{2}{9}\delta^{ij}\delta^{ab}G^{kc}G^{kc}$. The SU(4) quadratic Casimir operator $C_2 = \frac{1}{2}S^2 + \frac{1}{2}I^2 + 2G^{kc}G^{kc}$ evaluated in the symmetric irrep S_N that corresponds to ground-state nucleons gives $C_2(S_{N_c}) = \frac{3}{8}N_c(N_c + 4)\mathbb{1}$ and shows explicitly that, to leading order, $G^{kc}G^{kc}$ can be replaced by the unit operator. The two terms in Eq. (A6) that enter in $W^{[ij],[ab]}$ correspond to the $(1, 1)$ component, and only the first one contributes to leading order.

We obtain

$$\langle N|N_c^{-2}G^{ia}G^{jb}|N\rangle = \langle N|\frac{N_c + 4}{48N_c}\delta^{ij}\delta^{ab}\mathbb{1} - \frac{N_c + 2}{8N_c}\epsilon^{ijk}\epsilon^{abc} \times \left(\frac{G^{kc}}{N_c}\right)|N\rangle + \mathcal{O}\big(N_c^{-2}\big), \tag{A7}$$

where this is an example of Eq. (14) that shows explicitly the tensor structure in the spatial and isospin indices.

APPENDIX B: USEFUL TENSOR IDENTITIES

Here we collect a few identities involving epsilon tensors that are used to simplify the number of spin-flavor and also momentum structures. The product of two epsilon tensors can be written as

$$\begin{aligned}\epsilon_{ijk}\epsilon_{lmn} &= \det\begin{bmatrix}\delta_{il} & \delta_{im} & \delta_{in}\\ \delta_{jl} & \delta_{jm} & \delta_{jn}\\ \delta_{kl} & \delta_{km} & \delta_{kn}\end{bmatrix}\\ &= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl})\\ &\quad + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}).\end{aligned} \tag{B1}$$

This expression is very useful for constructing an independent set of momentum structures. Another useful identity can be

obtained by contracting Eq. (B1) with ϵ_{ijp}, giving

$$\delta_{kp}\epsilon_{lmn} = \delta_{kn}\epsilon_{lmp} + \delta_{km}\epsilon_{lpn} + \delta_{kl}\epsilon_{pmn}, \tag{B2}$$

from where

$$(A \times B)^i C^j + (B \times C)^i A^j + (C \times A)^i B^j = (A \times B) \cdot C\delta^{ij} \tag{B3}$$

is obtained. This can be used to eliminate triple products from all our momentum structures and also to reduce the number of momentum or spin-flavor structures that contain a cross product.

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